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NONLINEAR PERTURBATIONS OF SOME NON-INVERTIBLE DIFFERENTIAL OPERATORS

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Dedicated to Patrick Habets and Jean Mawhin

Abstract. We consider perturbations, depending on a small parameter λ, of a non-invertible differential operator having a nonnegative spectrum. Given a pair of lower and upper solutions, belonging to the kernel of the differential operator, without any prescribed order, we prove the existence of a solution, when λ is sufficiently small. Our method of proof has the advantage of permitting a uniform choice of λ for a whole class of functions. Applications are given in a variety of situations, ranging from ODE problems to equations of parabolic type, or involving the p-Laplacian operator.

1. Introduction

In this paper we deal with problems of the type

$$Lu = Nu,$$

where L is a non-invertible differential operator, and N is, in some sense, a nonlinear perturbation. A typical situation is that of a linear operator L, with compact resolvent, having zero as its first eigenvalue, and of a nonlinear operator

$$N = N_\lambda,$$

depending continuously on a small parameter λ, with $N_0 = 0$. Problems of this type are sometimes referred to as "nonlinear eigenvalue problems" (see, e.g., [3]). More generally, we can also consider some type of nonlinear differential operators.

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To illustrate our results in a simple context, let us consider the following Neumann problem associated to an elliptic equation

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with regular boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is L^∞-Carathéodory, and λ is a real parameter.

Let us state the following theorem, which will be useful in order to compare our method with those used in the previous literature.

Theorem 1. *Assume that there exist two constants α, β such that*

$$f(x,\beta) \le 0 \le f(x,\alpha), \quad \text{a.e. in } \Omega. \tag{1.2}$$

Then, there is a $\bar\lambda > 0$ such that, for every $\lambda \in [0,\bar\lambda]$, problem (1.1) *has a solution.*

Notice that, in the above statement, α is a constant lower solution and β is an upper solution for (1.1). It is well known that a solution exists when α is a lower solution, β is an upper solution, and $\alpha \le \beta$ (see, e.g., [20, 15]). When α and β are not ordered in this way, some interesting results have been obtained, for λ not necessarily small, assuming a growth condition on the nonlinearity which prevents the interaction with the positive part of the spectrum. See [4, 29, 21, 22, 15, 17, 18, 14], for different kinds of equations.

Since λ is assumed to be small, Theorem 1 can be interpreted as a bifurcation type of result. In this framework, some general abstract theorems based on degree theory are already available, see e.g. [20, Theorem IV.2] or [19]. Roughly speaking, the main idea there is that, if the degree of some projected equation on the kernel is nonzero, it has to be nonzero also for a small perturbation. Assumption (1.2) in Theorem 1 could then be replaced by a more general integral condition, like

$$\int_\Omega f(x,\beta)\,dx < 0 < \int_\Omega f(x,\alpha)\,dx. \tag{1.3}$$

We propose here a different type of approach, based on lower and upper solutions, which has the advantage of showing that the choice of λ can be made uniformly for a whole class of functions f. For instance, we are able to prove the following generalization of Theorem 1.

Theorem 2. *Let $I \subseteq \mathbb{R}$ be an interval. Given a compact interval J, contained in the interior of I, there is a $\bar\lambda > 0$ with the following property: for* *every* $\lambda \in [0, \bar{\lambda}]$ *and every function* f *satisfying*

$$|f(x,u)| \leq 1 \quad \text{for a.e. } x \in \Omega \text{ and every } u \in I, \tag{1.4}$$

if there are two constants α, β *in* J *for which*

$$f(x,\beta) \leq 0 \leq f(x,\alpha), \quad \text{a.e. in } \Omega, \tag{1.5}$$

then problem (1.1) *has a solution* u, *with* $u(x) \in I$ *for every* $x \in \overline{\Omega}$.

The choice of the constant 1 in (1.4) is clearly irrelevant, since any positive constant can be absorbed by the parameter λ. Theorem 1 follows from Theorem 2 since, there, α and β are fixed, and one can choose as J the interval $[\min\{\alpha,\beta\}, \max\{\alpha,\beta\}]$ and as I any bounded open interval containing J.

The fact that $\bar{\lambda}$ can be chosen in such a uniform way is not evident from the above mentioned degree arguments, since it is not guaranteed that the associated degree remains nonzero for a small perturbation of the whole class of functions f considered in the statement. Indeed, we will prove the following result which shows that, contrarily to the situation encountered for Theorem 1, assumption (1.5) in Theorem 2 cannot be replaced by the integral condition (1.3).

Theorem 3. *Given two constants* α, β *with* $\alpha \neq \beta$, *for every* $\lambda > 0$ *there is a continuous function* $f_\lambda : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *such that*

$$|f_\lambda(x,u)| \leq 1 \qquad \text{for every } (x,u) \in \overline{\Omega} \times \mathbb{R},$$

and

$$\int_\Omega f_\lambda(x,\beta)\,dx < 0 < \int_\Omega f_\lambda(x,\alpha)\,dx,$$

for which the problem

$$\begin{cases} -\Delta u = \lambda f_\lambda(x,u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solutions.

Let us briefly sketch the proof of Theorem 2, which relies on some ideas developed in the framework of lower and upper solutions, cf. [2, 15, 17]. As mentioned above, the case $\alpha \leq \beta$ is well known. Therefore, let $\beta < \alpha$ be the constant upper and lower solutions. Modifying the function f outside the interval $[\beta-\varepsilon, \alpha+\varepsilon]$, for a small $\varepsilon > 0$, we can get a further pair of constant lower and upper solutions for the new problem, say $\beta-\delta$ and $\alpha+\delta$, for some $\delta > \varepsilon$. More precisely, $\beta - \delta$ is a strict lower solution, and $\alpha + \delta$ is a strict upper solution. So, degree theory applies, saying that the pair $(\beta-\delta, \alpha+\delta)$ is admissible and the associated degree's value is 1.

We now need to prove the following: *There is a solution between $\beta - \delta$ and $\alpha + \delta$, which enters the interval $[\beta, \alpha]$ somewhere.*

- If both the pairs $(\beta - \delta, \beta)$ and $(\alpha, \alpha + \delta)$ are admissible for the degree, then both the associated degree's values are 1 and, by the excision property, there must be a solution between $\beta - \delta$ and $\alpha + \delta$, which is neither between $\beta - \delta$ and β, nor between α and $\alpha + \delta$.

- Otherwise, assume for instance that the pair $(\beta - \delta, \beta)$ is not admissible for the degree. Since $\beta - \delta$ is a strict lower solution, there must be a solution between $\beta - \delta$ and β which touches β somewhere. Similarly, if the pair $(\alpha, \alpha + \delta)$ is not admissible, there must be a solution between α and $\alpha + \delta$ which touches α somewhere.

At this point, having a solution between $\beta - \delta$ and $\alpha + \delta$, which enters the interval $[\beta, \alpha]$ somewhere, we use some classical estimates and the fact that λ is small to show that this solution cannot go far from the interval $[\beta, \alpha]$ and is indeed contained in $[\beta - \varepsilon, \alpha + \varepsilon]$, thus being a solution of the original problem.

These ideas can be used in a large variety of situations, which do not need a variational structure. For instance, we can deal with the Neumann-periodic problem associated to a parabolic equation, or with the periodic problem for a second-order ODE including friction terms. Also, some nonlinear differential operators can be considered, as for instance the p-Laplacian operator. Moreover, we will show how to deal with a wider class of nonlinearities, not necessarily involving an explicit dependence on a parameter λ.

We will consider the following three examples, which can be deduced from the above arguments, after a suitable rescaling. Here, $\mathrm{p} > 1$ can be any exponent greater than 1.

I. The problem

$$\begin{cases} -\Delta u = |u|^{\mathrm{p}-1} u + e(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution, provided that $\|e\|_{L^\infty}$ is sufficiently small.

II. The problem

$$\begin{cases} -\Delta u = (u^+)^{\mathrm{p}} + e(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution, provided that $e(x) \leq 0$ for almost every $x \in \Omega$, and $\|e\|_{L^\infty}$ is small enough. Here, $u^+ = \max\{u, 0\}$.

III. The problem

$$\begin{cases} -\Delta u = u^{\mathrm{p}} - \lambda a(x) u^{\mathrm{q}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution, provided that $a(x)$ is continuous and positive, $\mathrm{q} < \mathrm{p}$, and $\lambda > 0$ is sufficiently small.

Our method, moreover, permits us to relate the smallness of the forcing term to the powers p, q. For instance, in the above examples I and II, if p is sufficiently large, any function e such that $\|e\|_{L^\infty} \leq \frac{1}{2^{\mathrm{p}}}$ will guarantee the existence of a solution.

Similar problems, with various kinds of boundary conditions, have been studied by many authors, mainly by the use of variational methods. See, e.g., [33, 6, 7, 32, 36, 11, 8, 5, 9, 24] for the Dirichlet boundary conditions and [23, 1, 35, 13, 12] for the Neumann problem. The main difficulty, in this framework, is the loss of compactness encountered at the critical exponent $\mathrm{p} = 2^* - 1 = \frac{N+2}{N-2}$, when $N \geq 3$.

The paper is organized as follows. In Section 2 we consider the Neumann-periodic problem associated to a parabolic equation

$$\begin{cases} \partial_t u - \Delta u = \lambda f(x, t, u, \nabla u) & \text{in } \Omega \times]0, T[, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times]0, T[, \\ u(x, 0) = u(x, T) & \text{in } \Omega, \end{cases}$$

assuming a Bernstein-Nagumo type of condition in order to deal with the dependence on ∇u. We provide in full detail the proof of our result for this case, which presents some more technical aspects, so that in the rest of the paper we will only need to show how to modify our arguments in order to deal with different kinds of problems.

A linear elliptic operator is considered in Section 3. The same arguments used for the parabolic equation are easily adapted to this case. In Section 4, a second-order ODE is studied, where more specific conditions of Bernstein-Nagumo type can be considered. Equations of Liénard or Rayleigh type are also treated there. We thus generalize a result stated in [31]. In Section 5, we deal with the p-Laplacian operator. Since a weaker notion of solution is adopted, a more detailed exposition of the proof is given in this case. For the reader's convenience, we provide in the Appendix a proof of the fundamental estimate used, in connection with the Bernstein-Nagumo condition, in the

ODE case treated in Section 4. The more general case of non-constant lower and upper solutions will be the object of a forthcoming paper.

2. The parabolic equation

Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Given $T > 0$, set $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$.

Define the elliptic differential operator

$$\mathcal{A}u := -\sum_{i,j=1}^{N} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} u.$$

Here, $a_{ij} \in C(\overline{Q}_T)$, $a_{ij} = a_{ji}$, $a_{ij}(x,0) = a_{ij}(x,T)$ in $\overline{\Omega}$, for $i,j = 1,\dots,N$, there exists $\bar{a} > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \bar{a}|\xi|^2, \quad \text{for every } (x,t,\xi) \in \overline{Q}_T \times \mathbb{R}^N,$$

and $a_i \in L^\infty(Q_T)$, for $i = 1,\dots,N$.

We consider the following Neumann-periodic problem

$$(P) \qquad \begin{cases} \partial_t u + \mathcal{A}u = f(x,t,u,\nabla_x u) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

Here, f is a L^r-Carathéodory function, for some $r > N+2$, ν is the unit outer normal to $\partial\Omega$ and $\partial_\nu u = \sum_{i=1}^N \partial_{x_i} u\, \nu_i$ is the normal derivative of u. If necessary, all functions defined on Q_T will be assumed to be extended by T-periodicity to $\Omega \times \mathbb{R}$.

Let $W^{2,1}_r(Q_T)$ be the space of functions u such that

$$u, \;\; \partial_t u, \;\; \partial_{x_i} u, \;\; \partial^2_{x_i x_j} u \;\; \in \;\; L^r(Q_T),$$

for $i,j = 1,\dots,N$, with the usual norm

$$\|u\|_{W^{2,1}_r} = \|u\|_{L^r} + \|\partial_t u\|_{L^r} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^r} + \sum_{i,j=1}^{N} \|\partial^2_{x_i x_j} u\|_{L^r}.$$

Recall that $W^{2,1}_r(Q_T)$ is compactly imbedded into $C^{1,0}(\overline{Q}_T)$.

We say that u is a *solution* of problem (P) if u belongs to $W^{2,1}_r(Q_T)$, it satisfies the differential equation almost everywhere in Q_T and the boundary and periodicity conditions pointwise. A function with these properties is usually called "strong solution" in the literature.

Let us define the class of functions $\mathcal{F}(I,\Lambda,K)$, where $I \subseteq \mathbb{R}$ is an interval, and Λ, K are some nonnegative constants. Its elements are the Carathéodory functions $f : Q_T \times I \times \mathbb{R}^N \to \mathbb{R}$ which satisfy the following Bernstein-Nagumo growth condition:

$$(BN) \qquad \begin{array}{l} |f(x,t,u,v)| \leq h(x,t) + K|v|^2, \\ \text{for a.e. } (x,t) \in Q_T \text{ and every } (u,v) \in I \times \mathbb{R}^N, \end{array}$$

for some $h \in L^r(Q_T)$, with $\|h\|_{L^r} \leq \Lambda$.

The following classical result, whose proof for our setting can be found in [18, Proposition III.1.4.], will be used in connection with the Bernstein-Nagumo condition, in order to get uniform estimates on the norm of the solutions.

Proposition 1 (Fundamental estimate)**.** *Given $M, \Lambda, K > 0$, there is a constant $C > 0$ such that, if $u \in W_r^{2,1}(Q_T)$ satisfies*

$$\begin{cases} |\partial_t u + \mathcal{A}u| \leq h(x,t) + K|\nabla_x u|^2, & \text{a.e. in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ |u| \leq M & \text{in } Q_T, \end{cases}$$

for some $h \in L^r(Q_T)$ with $\|h\|_{L^r} \leq \Lambda$, then $\|u\|_{W_r^{2,1}} \leq C$.

Let us now state our main result.

Theorem 4. *Let $I \subseteq \mathbb{R}$ be an interval. Given a compact interval J, contained in the interior of I, there is a $\Lambda > 0$ such that, for every $f \in \mathcal{F}(I,\Lambda,\Lambda)$, if there are two constants α, β in J for which*

$$f(x,t,\beta,0) \leq 0 \leq f(x,t,\alpha,0) \quad \text{a.e. in } Q_T,$$

then problem (P) has a solution u, with $u(x,t) \in I$ for every $(x,t) \in \overline{Q}_T$.

The proof relies on lower and upper solutions techniques which have been developed, among others, in [4, 29, 21, 22, 15, 17]. We need some preliminary considerations.

Definition 1. *A function $\alpha \in W_r^{2,1}(Q_T)$ is a* lower solution *of (P) if*

$$\begin{cases} \partial_t \alpha + \mathcal{A}\alpha \leq f(x,t,\alpha,\nabla_x \alpha) & \text{a.e. in } Q_T, \\ \partial_\nu \alpha \leq 0 & \text{on } \Sigma_T, \\ \alpha(x,0) \leq \alpha(x,T) & \text{in } \Omega. \end{cases}$$

The function α is a strict lower solution *if it is a lower solution and, for every solution u of (P) with $u \geq \alpha$, one has that $u > \alpha$ on $\overline{Q}_T$.*

Analogously, a function $\beta \in W_r^{2,1}(Q_T)$ *is an* upper solution *of (P) if*

$$\begin{cases} \partial_t \beta + \mathcal{A}\beta \geq f(x,t,\beta,\nabla_x \beta) & \text{a.e. in } Q_T, \\ \partial_\nu \beta \geq 0 & \text{on } \Sigma_T, \\ \beta(x,0) \geq \beta(x,T) & \text{in } \Omega. \end{cases}$$

The function β *is a* strict upper solution *if it is an upper solution and, for every solution* u *of (P) with* $u \leq \beta$*, one has that* $u < \beta$ *on* $\overline{Q}_T$*.*

Notice that the constants α and β appearing in the statement of Theorem 4 are lower and upper solutions, respectively, although they might not be strict.

Let us introduce the linear operator $L : D(L) \subset W_r^{2,1}(Q_T) \to L^r(Q_T)$, where

$$D(L) = \{u \in W_r^{2,1}(Q_T) : \partial_\nu u = 0 \text{ on } \Sigma_T \text{ and } u(x,0) = u(x,T) \text{ in } \Omega\},$$

defined by

$$Lu = \partial_t u + \mathcal{A}u.$$

The operator L is not invertible; 0 is its first eigenvalue, and $\ker L$ is made up of the constant functions. Let us fix a real number $\sigma > 0$. One can prove (cf. [26, Lemma 4.1 and Corollary 5.6], see also [18, Proposition I.1.3]) that $L+\sigma I$ is invertible, with a continuous inverse $(L+\sigma I)^{-1} : L^r(Q_T) \to W_r^{2,1}(Q_T)$. Let us also introduce the continuous nonlinear operator $N : C^{1,0}(\overline{Q}_T) \to L^r(Q_T)$, defined by $(Nu)(x,t) = f(x,t,u(x,t),\nabla_x u(x,t))$. Problem (P) can be written as

$$Lu = Nu,$$

which is equivalent to the fixed-point problem

$$u = \mathcal{S}u,$$

where the function $\mathcal{S} : C^{1,0}(\overline{Q}_T) \to C^{1,0}(\overline{Q}_T)$ is defined by

$$\mathcal{S}u = (L+\sigma I)^{-1}(N+\sigma I)u.$$

Since $W_r^{2,1}(Q_T)$ is compactly imbedded in $C^{1,0}(\overline{Q}_T)$, we have that $\mathcal{S}$ is completely continuous, so that we can use Leray-Schauder degree theory.

Definition 2. *Let* $\alpha, \beta : \overline{Q}_T \to \mathbb{R}$ *be two continuous functions. The pair* (α,β) *is said to be* admissible *for (P) if* $\alpha < \beta$ *and there is* $R > 0$ *with the following property: any solution* u *of (P) satisfying* $\alpha \leq u \leq \beta$ *is such that*

$$\alpha < u < \beta \quad \textit{and} \quad \|u\|_{C^{1,0}} < R.$$

Notice that, if (α,β) is an admissible pair, then the set

$$\mathcal{U}_{(\alpha,\beta)} := \{u \in C^{1,0}(\overline{Q}_T) : \alpha < u < \beta\},$$

is open and nonempty, and any fixed point of $\mathcal{S}$ contained in $\mathcal{U}_{(\alpha,\beta)}$ belongs to the ball B_R in $C^{1,0}(\overline{Q}_T)$. Hence, we can define

$$\deg\left(I-\mathcal{S},\mathcal{U}_{(\alpha,\beta)}\right) = \mathrm{d_{LS}}\left(I-\mathcal{S},\mathcal{U}_{(\alpha,\beta)}\cap B_R\right),$$

where $\mathrm{d_{LS}}$ denotes the Leray-Schauder degree.

The following classical result on lower and upper solutions has been proved in this setting in [18, Theorem II.2.1], although we state it here in a slightly more general form.

Proposition 2 (Well-ordered lower and upper solutions)**.** *Assume that α is a lower solution and β is an upper solution of (P) satisfying $\alpha \le \beta$. Then problem (P) has a solution u such that $\alpha \le u \le \beta$. If, moreover, (α,β) is an admissible pair, then*

$$\deg\left(I-\mathcal{S},\mathcal{U}_{(\alpha,\beta)}\right) = 1.$$

Proof of Theorem 4. Let $\delta > 0$ be such that $[\min J-\delta, \max J+\delta] \subseteq I$, and fix $\varepsilon \in (0,\frac{\delta}{2})$. We will prove the statement by taking $\Lambda = \frac{1}{m}$, with $m \ge 1$ a sufficiently large integer. By contradiction, assume there exist a function $f_m \in \mathcal{F}\left(I,\frac{1}{m},\frac{1}{m}\right)$, and two constants $\alpha_m,\beta_m \in J$, satisfying

$$f_m(x,t,\beta_m,0) \le 0 \le f_m(x,t,\alpha_m,0) \quad \text{a.e. in } Q_T,$$

for which the problem

$$(P_m) \qquad \begin{cases} \partial_t u + \mathcal{A}u = f_m(x,t,u,\nabla_x u) & \text{in } Q_T,\\ \partial_\nu u = 0 & \text{on } \Sigma_T,\\ u(x,0)=u(x,T) & \text{in } \Omega, \end{cases}$$

has no solution u, with $u(x,t) \in I$ for every $(x,t) \in \overline{Q}_T$. By Proposition 2, $\beta_m < \alpha_m$. Define the modified function

$$\tilde{f}_m(x,t,u,v) = \begin{cases} \frac{1}{m} & \text{if } u \le \beta_m - 2\varepsilon,\\ \frac{1}{m} + \frac{1}{\varepsilon}(f_m(x,t,u,v)-\frac{1}{m})(u-\beta_m+2\varepsilon), & \\ & \text{if } \beta_m-2\varepsilon \le u \le \beta_m-\varepsilon,\\ f_m(x,t,u,v) & \text{if } \beta_m-\varepsilon \le u \le \alpha_m+\varepsilon,\\ -\frac{1}{m} + \frac{1}{\varepsilon}(f_m(x,t,u,v)+\frac{1}{m})(\alpha_m+2\varepsilon-u), & \\ & \text{if } \alpha_m+\varepsilon \le u \le \alpha_m+2\varepsilon,\\ -\frac{1}{m} & \text{if } u \ge \alpha_m+2\varepsilon, \end{cases}$$

and consider the problem

$$(\tilde{P}_m) \qquad \begin{cases} \partial_t u + \mathcal{A}u = \tilde{f}_m(x,t,u,\nabla_x u) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

Notice that $\tilde{f}_m \in \mathcal{F}\left(\mathbb{R}, \frac{1}{m}(1+|Q_T|^{1/r}), \frac{1}{m}\right)$. Let us prove that, for the modified problem $(\tilde{P}_m)$,

$$\beta_m - \delta \text{ is a strict lower solution.}$$

Indeed, as $\tilde{f}_m(x,t,\beta_m-\delta,0) = \frac{1}{m}$, clearly $\beta_m - \delta$ is a lower solution. If it were not strict, there would be a solution u of $(\tilde{P}_m)$ with $u \geq \beta_m - \delta$, and a point $(x_0,t_0) \in \overline{Q}_T$ for which $u(x_0,t_0) = \beta_m - \delta$. Setting $v = u - \beta_m + \delta$, there is $\rho > 0$ such that

$$\partial_t v + \mathcal{A}v = \tfrac{1}{m} > 0 \qquad \text{a.e. in } (B(x_0,\rho)\cap\Omega) \times (t_0-\rho, t_0+\rho).$$

If $x_0 \in \Omega$, we take $\rho > 0$ small enough so that $B(x_0,\rho)$ is contained in Ω and, by the strong maximum principle (see, e.g., [18, Proposition I.1.1]), $v = 0$ in $B(x_0,\rho) \times (t_0-\rho, t_0]$, which is clearly impossible. On the other hand, if $x_0 \in \partial\Omega$, then $\partial_\mu v(x_0,t_0) < 0$ for every $\mu \in \mathbb{R}^N$ such that $\mu\cdot\nu > 0$, $\nu = \nu(x_0)$ being the outer normal at x_0, in contradiction with the Neumann boundary condition.

In the same way we can show that

$$\alpha_m + \delta \text{ is a strict upper solution.}$$

Hence, the pair $(\beta_m - \delta, \alpha_m + \delta)$ is admissible for $(\tilde{P}_m)$. Indeed, let u be a solution of $(\tilde{P}_m)$ with $\beta_m - \delta \leq u \leq \alpha_m + \delta$. Since $\tilde{f}_m \in \mathcal{F}(\mathbb{R}, 1+|Q_T|^{1/r}, 1)$, by Proposition 1 there is a constant $R > 0$ such that $\|u\|_{C^{1,0}} < R$. Since $\beta_m - \delta$ and $\alpha_m + \delta$ are strict, we also have that $\beta_m - \delta < u < \alpha_m + \delta$.

Let us introduce the operators $\tilde{N}_m : C^{1,0}(\overline{Q}_T) \to L^r(Q_T)$, defined by

$$(\tilde{N}_m u)(x,t) = \tilde{f}_m(x,t,u(x,t),\nabla_x u(x,t)),$$

and $\tilde{\mathcal{S}}_m : C^{1,0}(\overline{Q}_T) \to C^{1,0}(\overline{Q}_T)$, defined by

$$\tilde{\mathcal{S}}_m u = (L+\sigma I)^{-1}(\tilde{N}_m + \sigma I)u. \tag{2.1}$$

Here, σ is any fixed positive number. We now distinguish two cases.

Case 1. The pairs $(\beta_m - \delta, \beta_m)$ and $(\alpha_m, \alpha_m + \delta)$ are admissible for $(\tilde{P}_m)$. By Proposition 2, we have

$$\deg\left(I - \tilde{\mathcal{S}}_m, \mathcal{U}_{(\beta_m-\delta,\beta_m)}\right) = 1,$$

$$\deg\left(I-\tilde{\mathcal{S}}_m,\mathcal{U}_{(\alpha_m,\alpha_m+\delta)}\right)=1,$$
$$\deg\left(I-\tilde{\mathcal{S}}_m,\mathcal{U}_{(\beta_m-\delta,\alpha_m+\delta)}\right)=1.$$

By the excision property of the degree,

$$\deg\left(I-\tilde{\mathcal{S}}_m,\mathcal{U}_{(\beta_m-\delta,\alpha_m+\delta)}\setminus\overline{\mathcal{U}_{(\beta_m-\delta,\beta_m)}\cup\mathcal{U}_{(\alpha_m,\alpha_m+\delta)}}\right)=-1.$$

Hence, there is a solution u_m of $(\tilde{P}_m)$ such that

$$\beta_m-\delta<u_m<\alpha_m+\delta, \tag{2.2}$$

and, for some $(x_m,t_m)\in\overline{Q}_T$,

$$\beta_m\le u_m(x_m,t_m)\le\alpha_m. \tag{2.3}$$

Case 2. One or both of the pairs $(\beta_m-\delta,\beta_m)$ and $(\alpha_m,\alpha_m+\delta)$ are not admissible for $(\tilde{P}_m)$. Assume for instance $(\alpha_m,\alpha_m+\delta)$ is not admissible. Then, for every $R>0$ there is a solution u_m of $(\tilde{P}_m)$ with $\alpha_m\le u_m\le\alpha_m+\delta$ such that, either $\min(u_m-\alpha_m)=0$, or $\min(\alpha_m+\delta-u_m)=0$, or $\|u_m\|_{C^{1,0}}\ge R$. Since $\alpha_m+\delta$ is strict, we have $u_m<\alpha_m+\delta$. Since $\tilde{f}_m\in\mathcal{F}(\mathbb{R},1+|Q_T|^{1/r},1)$, by Proposition 1 the third possibility cannot occur for R large enough, so that $\min(u_m-\alpha_m)=0$. If instead $(\beta_m-\delta,\beta_m)$ is not admissible, analogously we find a solution u_m such that $\max(\beta_m-u_m)=0$.

In both cases, we thus find a solution u_m of $(\tilde{P}_m)$ satisfying (2.2) and (2.3).

Since $\tilde{f}_m\in\mathcal{F}(\mathbb{R},1+|Q_T|^{1/r},1)$, and (2.2) holds, by Proposition 1 there is a constant $C>0$ such that $\|u_m\|_{W_r^{2,1}}\le C$, for every m. Since $W_r^{2,1}(Q_T)$ is compactly imbedded in $C^{1,0}(\overline{Q}_T)$, there are a subsequence, still denoted by $(u_m)_m$, and a function $\overline{u}\in C^{1,0}(\overline{Q}_T)$ such that $u_m\to\overline{u}$ in $C^{1,0}(\overline{Q}_T)$. In particular, for every $(x,t)\in\overline{Q}_T$ we have $|\nabla_x u_m(x,t)|\le\bar{c}$, for some constant $\bar{c}>0$. As $\tilde{f}_m\in\mathcal{F}\left(\mathbb{R},\frac{1}{m}(1+|Q_T|^{1/r}),\frac{1}{m}\right)$, we have

$$\tilde{N}_m u_m=\tilde{f}_m(\cdot,u_m(\cdot),\nabla_x u_m(\cdot))\to 0\qquad\text{in } L^r(Q_T).$$

Thus,

$$\tilde{\mathcal{S}}_m u_m=(L+\sigma I)^{-1}(\tilde{N}_m u_m+\sigma u_m)\to(L+\sigma I)^{-1}(\sigma\overline{u})\qquad\text{in } C^{1,0}(\overline{Q}_T).$$

As $u_m=\tilde{\mathcal{S}}_m u_m$, we get

$$\overline{u}=(L+\sigma I)^{-1}(\sigma\overline{u});$$

i.e., $\overline{u}\in D(L)$ and $L\overline{u}=0$. Hence, $\overline{u}$ is constant. Since $u_m\to\overline{u}$ uniformly in $\overline{Q}_T$, for m large enough,

$$\operatorname{osc}(u_m)=\max u_m-\min u_m\le\varepsilon.$$

Recalling that u_m satisfies (2.3), it follows that

$$\beta_m - \varepsilon \le u_m \le \alpha_m + \varepsilon.$$

Thus, u_m is a solution to problem (P_m) and, since α_m, β_m are in J, by the choice of ε we have that $u_m(x,t) \in I$, for every $(x,t) \in \overline{Q}_T$. We, thus, get a contradiction, which ends the proof. □

Remark 1. For Λ small enough, the solution provided by Theorem 4 has a small oscillation. This can be proved by contradiction, as follows. Assume that there exist $\bar{\varepsilon} > 0$ and a sequence of functions $f_m \in \mathcal{F}\left(I, \frac{1}{m}, \frac{1}{m}\right)$ satisfying the assumptions of the theorem, and a sequence of solutions u_m to problem (P_m), with $u_m(x,t) \in I$ for every $(x,t) \in \overline{Q}_T$, for which $\operatorname{osc}(u_m) > \bar{\varepsilon}$. Arguing as in the last part of the proof of Theorem 4, for a subsequence we have that $(u_m)_m$ converges to a constant $\overline{u}$, thus getting the contradiction.

Remark 2. Besides the Neumann condition one could consider more general oblique derivative boundary conditions, like

$$\sum_{i=1}^{N} b_i(x,t)\partial_{x_i} u = 0 \qquad \text{on } \Sigma_T,$$

where $b_i \in C^{1,1/2}(\Sigma_T)$, $b_i(x,0) = b_i(x,T)$ on $\partial\Omega$, for $i = 1, \dots, N$ and there is $\bar{b} > 0$ such that, for all $(x,t) \in \Sigma_T$,

$$\sum_{i=1}^{N} b_i(x,t)\nu_i(x) \ge \bar{b}.$$

As a direct consequence of Theorem 4, we have the following.

Corollary 1. *Let $I \subseteq \mathbb{R}$ be an interval, and Λ, K be some fixed positive numbers. Given a compact interval J, contained in the interior of I, there is a $\bar{\lambda} > 0$ with the following property: for every $\lambda \in [0, \bar{\lambda}]$ and every $f \in \mathcal{F}(I, \Lambda, K)$, if there are two constants α, β in J for which*

$$f(x,t,\beta,0) \le 0 \le f(x,t,\alpha,0) \quad \text{a.e. in } Q_T, \tag{2.4}$$

then the problem

$$\begin{cases} \partial_t u + \mathcal{A}u = \lambda f(x,t,u,\nabla_x u) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases}$$

has a solution u, with $u(x,t) \in I$ for every $(x,t) \in \overline{Q}_T$.

As a particular case, we state the following.

Corollary 2. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $[a, b]$ is an interval such that $[a, b] \subseteq g(\mathbb{R})$. Then, there is a $\bar{\lambda} > 0$ such that, for every $\lambda \in [0, \bar{\lambda}]$ and every $e \in L^\infty(Q_T)$ such that*

$$e(x,t) \in [-b, -a], \qquad \text{a.e. in } Q_T, \tag{2.5}$$

the problem

$$\begin{cases} \partial_t u + \mathcal{A}u = \lambda(g(u) + e(x,t)) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{2.6}$$

has a solution.

Proof. Let α and β be such that $g(\alpha) = b$ and $g(\beta) = a$. By (2.5), α is a lower solution and β is an upper solution of (2.6). If $\alpha \le \beta$, the result is well known. Assume $\beta < \alpha$ and take $I = [\beta - 1, \alpha + 1]$. Then, writing $f(x,t,u) = g(u) + e(x,t)$, we have that $f \in \mathcal{F}(I, \Lambda, 0)$, where

$$\Lambda = \max\{|g(s)| : s \in I\} + \max\{|a|, |b|\}.$$

Corollary 1 then gives the conclusion. □

In particular, when the function g is unbounded from above and from below, any bounded function $e(x,t)$ will satisfy the assumptions.

We now show that, when $\mathcal{A}$ is taken to be the Laplacian operator, with negative sign, the assumption (2.4) in Corollary 1 cannot be replaced by some integral condition, not even by taking the interval I to be equal to the whole line $\mathbb{R}$.

Theorem 5. *Given two constants α, β with $\alpha \neq \beta$, for every $\lambda > 0$ there is a continuous function $f_\lambda : \overline{Q}_T \times \mathbb{R} \to \mathbb{R}$ such that*

$$|f_\lambda(x,t,u)| \le 1 \qquad \text{for every } (x,t,u) \in \overline{Q}_T \times \mathbb{R}, \tag{2.7}$$

and

$$\int_{Q_T} f_\lambda(x,t,\beta)\, dx\, dt < 0 < \int_{Q_T} f_\lambda(x,t,\alpha)\, dx\, dt, \tag{2.8}$$

for which the problem

$$\begin{cases} \partial_t u - \Delta u = \lambda f_\lambda(x,t,u) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{2.9}$$

has no solutions.

Proof. Let $\zeta : \overline{Q}_T \to \mathbb{R}$ be a non-constant continuous function such that $-1 \le \zeta(x,t) \le 1$, for every $(x,t) \in \overline{Q}_T$. We denote by $\bar\zeta$ its mean value on Q_T. Fix $\lambda > 0$ and consider the function

$$f_\lambda(x,t,u) = \frac{1}{4}\Big(\cos\Big(\frac{\pi}{\beta-\alpha}(u-\alpha)\Big) + \zeta(x,t) + e_\lambda\Big),$$

where e_λ is a real number in the interval $(-1-\bar\zeta,\, 1-\bar\zeta)$, to be fixed. Notice that this function f_λ satisfies (2.7) and (2.8). Let us now consider the set S_λ of those e_λ for which problem (2.9) has a solution. Integrating the equation over Q_T immediately shows that S_λ is contained in $[-1-\bar\zeta,\, 1-\bar\zeta]$. (Here we exploit the choice $\mathcal{A} = -\Delta$.) More precisely, since $\zeta(x,t)$ is non-constant,

$$S_\lambda \subseteq \, (-1-\bar\zeta,\, 1-\bar\zeta).$$

We now show that S_λ is a closed set. After doing so, it is sufficient to take $e_\lambda \in \, (-1-\bar\zeta,\, 1-\bar\zeta) \setminus S_\lambda$ to have the proof concluded.

Let $(e_{\lambda,n})_n$ be a sequence in S_λ converging to some $\hat e_\lambda$. Let $u_{\lambda,n}$ be corresponding solutions of (2.9), with $e_\lambda = e_{\lambda,n}$. By the periodicity of the cosine function, it is possible to choose them in such a way that their mean values $\bar u_{\lambda,n}$ satisfy $|\bar u_{\lambda,n}| \le |\beta - \alpha|$.

Let us prove that the sequence $(\|u_{\lambda,n}\|_{L^\infty})_n$ is bounded. By contradiction, assume that, for a subsequence, $\|u_{\lambda,n}\|_{L^\infty} \to \infty$. Set $v_{\lambda,n} = u_{\lambda,n}/\|u_{\lambda,n}\|_{L^\infty}$. Then $v_{\lambda,n}$ satisfies

$$\begin{cases} \partial_t v_{\lambda,n} - \Delta v_{\lambda,n} = \lambda \|u_{\lambda,n}\|_{L^\infty}^{-1} f_\lambda(x,t,\|u_{\lambda,n}\|_{L^\infty} v_{\lambda,n}) & \text{in } Q_T,\\ \partial_\nu v_{\lambda,n} = 0 & \text{on } \Sigma_T,\\ v_{\lambda,n}(x,0) = v_{\lambda,n}(x,T) & \text{in } \Omega. \end{cases} \tag{2.10}$$

Since $\|v_{\lambda,n}\|_{L^\infty} = 1$ and $(\|\partial_t v_{\lambda,n} - \Delta v_{\lambda,n}\|_{L^\infty})_n$ is bounded, by Proposition 1 we have that also $(\|v_{\lambda,n}\|_{W_r^{2,1}})_n$ is bounded. Hence, there is a subsequence, still denoted by $(v_{\lambda,n})_n$, and a function $v_\lambda \in W_r^{2,1}(Q_T)$ such that $v_{\lambda,n}$ converges to v_λ, weakly in $W_r^{2,1}(Q_T)$, and uniformly. Hence, $\|v_\lambda\|_{L^\infty} = 1$, and passing to the limit in (2.10) shows that v_λ has to be in the kernel of the differential operator. So, v_λ is constant, and we have two possibilities: either $v_\lambda \equiv 1$, or $v_\lambda \equiv -1$. Assume, for instance, $v_\lambda \equiv 1$. Then,

$$u_{\lambda,n}(x,t) = \|u_{\lambda,n}\|_{L^\infty} v_{\lambda,n}(x,t) \to +\infty,$$

uniformly in (x,t), in contradiction with the fact that $(\bar u_{\lambda,n})_n$ is bounded. A similar contradiction is obtained assuming $v_\lambda \equiv -1$.

Having proved that $(\|u_{\lambda,n}\|_{L^\infty})_n$ is bounded, since $(\|\partial_t u_{\lambda,n} - \Delta u_{\lambda,n}\|_{L^\infty})_n$ is also bounded, by Proposition 1 we have that $(\|u_{\lambda,n}\|_{W_r^{2,1}})_n$ is bounded, as well. Hence, there is a subsequence, still denoted by $(u_{\lambda,n})_n$, and a function $\hat{u}_\lambda \in W_r^{2,1}(Q_T)$ such that $u_{\lambda,n}$ converges to $\hat{u}_\lambda$, weakly in $W_r^{2,1}(Q_T)$, and uniformly. Passing to the limit, it is readily seen that $\hat{u}_\lambda$ solves (2.9), with $e_\lambda = \hat{e}_\lambda$. Hence, $\hat{e}_\lambda$ belongs to S_λ, and the proof is concluded. □

Remark 3. As a counterpart of Theorem 5, we recall that, from some abstract theorems in degree theory (cf. [20, 19]), it is possible to prove that, given a L^r-Carathéodory function $f : \overline{Q}_T \times \mathbb{R} \to \mathbb{R}$ for which

$$\int_{Q_T} f(x,t,\beta)\,dx\,dt < 0 < \int_{Q_T} f(x,t,\alpha)\,dx\,dt, \tag{2.11}$$

there is a $\bar{\lambda} > 0$ such that, for every $\lambda \in [0, \bar{\lambda}]$, the problem

$$\begin{cases} \partial_t u - \Delta u = \lambda f(x,t,u) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases}$$

has a solution. Theorem 5 states that such a constant $\bar{\lambda}$ cannot be taken uniformly for all the functions f satisfying (2.11), not even restricting the choice of f in a bounded set of continuous functions.

Let us now provide two examples where Theorem 4 applies.

Corollary 3. *Let $\zeta \in L^\infty(Q_T)$ be a function bounded below by a positive constant. Given $\mathrm{p} > 1$, the problem*

$$\begin{cases} \partial_t u + \mathcal{A}u = \zeta(x,t)|u|^{\mathrm{p}-1}u + e(x,t) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{2.12}$$

has a solution, provided that $e \in L^\infty(Q_T)$ and $\|e\|_{L^\infty}$ is sufficiently small. More precisely, for every given $\delta > 0$ there is a $\bar{\Lambda}_\delta > 0$ such that, for every $\mathrm{p} > 1$, if

$$\|e\|_{L^\infty} \le \left(\frac{\bar{\Lambda}_\delta}{(1+\delta)^{\mathrm{p}}} \right)^{\frac{\mathrm{p}}{\mathrm{p}-1}}, \tag{2.13}$$

then (2.12) has a solution.

Proof. Fix $\delta > 0$ and consider the problem

$$\begin{cases} \partial_t w + \mathcal{A}w = \lambda(\zeta(x,t)|w|^{\mathrm{p}-1}w + \tilde{e}(x,t)) & \text{in } Q_T, \\ \partial_\nu w = 0 & \text{on } \Sigma_T, \\ w(x,0) = w(x,T) & \text{in } \Omega, \end{cases} \tag{2.14}$$

where λ is a positive constant and $\tilde{e}$ is such that $\|\tilde{e}\|_{L^\infty} \leq \operatorname{ess\,inf} \zeta$. Take $J = [-1, 1]$ and $I = [-1-\delta, 1+\delta]$. Let $\Lambda = \Lambda_\delta > 0$ be as in the statement of Theorem 4. Setting $f_\lambda(x,t,w) = \lambda(\zeta(x,t)|w|^{\mathrm{p}-1}w + \tilde{e}(x,t))$, we have

$$f_\lambda(x,t,-1) \leq 0 \leq f_\lambda(x,t,1),$$

for almost every $(x,t) \in Q_T$ and, for every $w \in I$,

$$|f_\lambda(x,t,w)| \leq \lambda(\|\zeta\|_{L^\infty}(1+\delta)^{\mathrm{p}} + \operatorname{ess\,inf} \zeta) \leq 2\lambda\|\zeta\|_{L^\infty}(1+\delta)^{\mathrm{p}}. \tag{2.15}$$

By Theorem 4, with $r = +\infty$, if

$$2\lambda\|\zeta\|_{L^\infty}(1+\delta)^{\mathrm{p}} \leq \Lambda_\delta, \tag{2.16}$$

then (2.14) has a solution w such that $w(x,t) \in I$, for every $(x,t) \in \overline{Q}_T$. Set

$$\bar{\Lambda}_\delta = \frac{\Lambda_\delta \min\{\operatorname{ess\,inf} \zeta, 1\}}{2\|\zeta\|_{L^\infty}}. \tag{2.17}$$

Consider now (2.12), assume (2.13) with $\bar{\Lambda}_\delta > 0$ as above, and fix

$$\lambda = \frac{\Lambda_\delta}{2\|\zeta\|_{L^\infty}(1+\delta)^{\mathrm{p}}}, \tag{2.18}$$

so that (2.16) is satisfied. Setting $w = \lambda^{\frac{1}{1-\mathrm{p}}}u$, we see that (2.12) is equivalent to (2.14), with $\tilde{e}(x,t) = \lambda^{\frac{\mathrm{p}}{1-\mathrm{p}}}e(x,t)$. Since, using (2.13), (2.17) and (2.18),

$$\|\tilde{e}\|_{L^\infty} = \lambda^{\frac{\mathrm{p}}{1-\mathrm{p}}}\|e\|_{L^\infty} \leq \lambda^{\frac{\mathrm{p}}{1-\mathrm{p}}}\left(\frac{\bar{\Lambda}_\delta}{(1+\delta)^{\mathrm{p}}}\right)^{\frac{\mathrm{p}}{\mathrm{p}-1}} = \min\{\operatorname{ess\,inf} \zeta, 1\}^{\frac{\mathrm{p}}{\mathrm{p}-1}} \leq \operatorname{ess\,inf} \zeta,$$

we have that (2.14) is solvable. Therefore, (2.12) is solvable, as well. □

Remark 4. Notice that, taking $\delta = \frac{1}{2}$, condition (2.13) is satisfied if p is sufficiently large and $\|e\|_{L^\infty} \leq \frac{1}{2^{\mathrm{p}}}$.

In a similar way, we have the following result, where we use the notation

$$u^+ = \max\{u, 0\}.$$

Corollary 4. *Let $\zeta \in L^\infty(Q_T)$ be a function bounded below by a positive constant. Given $\mathrm{p} > 1$, the problem*

$$\begin{cases} \partial_t u + \mathcal{A}u = \zeta(x,t)(u^+)^{\mathrm{p}} + e(x,t) & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{2.19}$$

has a solution, provided that $e \in L^\infty(Q_T)$, $e(x,t) \leq 0$ for almost every $(x,t) \in Q_T$, and $\|e\|_{L^\infty}$ is sufficiently small. More precisely, for every given

$\delta > 0$ *there is a* $\bar{\Lambda}_\delta > 0$ *such that, for every* $\mathrm{p} > 1$, *if* $e(x,t) \le 0$ *for almost every* $(x,t) \in Q_T$ *and* (2.13) *holds, then* (2.19) *has a solution.*

Proof. As in the proof of Corollary 3, first consider the problem

$$\begin{cases} \partial_t w + \mathcal{A}w = \lambda(\zeta(x,t)(w^+)^{\mathrm{p}} + \tilde{e}(x,t)) & \text{in } Q_T, \\ \partial_\nu w = 0 & \text{on } \Sigma_T, \\ w(x,0) = w(x,T) & \text{in } \Omega, \end{cases} \tag{2.20}$$

where $\tilde{e}$ is such that $-\operatorname{ess\,inf} \zeta \le \tilde{e}(x,t) \le 0$, for almost every $(x,t) \in Q_T$. Once $\delta > 0$ is fixed, take $J = [0,1]$ and $I = [-\delta, 1+\delta]$. Setting $f_\lambda(x,t,w) = \lambda(\zeta(x,t)(w^+)^{\mathrm{p}} + \tilde{e}(x,t))$, we have

$$f_\lambda(x,t,0) \le 0 \le f_\lambda(x,t,1),$$

and (2.15) holds, for almost every $(x,t) \in Q_T$ and every $w \in I$. One then concludes as in the proof of Corollary 3. □

As a further example, let us consider the problem

$$\begin{cases} \partial_t u + \mathcal{A}u = \zeta_1(x,t)u^{\mathrm{p}} - \lambda\zeta_2(x,t)u^{\mathrm{q}} & \text{in } Q_T, \\ u > 0 & \text{in } Q_T, \\ \partial_\nu u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases} \tag{2.21}$$

Corollary 5. *Assume that* $\zeta_1, \zeta_2 \in L^\infty(Q_T)$ *are bounded below by a positive constant,* $\mathrm{p} > 1$, *and* $\mathrm{q} < \mathrm{p}$. *Then, there is a* $\bar{\lambda} > 0$ *such that, for every* $\lambda \in [0, \bar{\lambda}]$, *problem* (2.21) *has a solution.*

Proof. Let us first show that, for $\hat{\lambda} > 0$ small enough, there is a positive solution of

$$\begin{cases} \partial_t w + \mathcal{A}w = \hat{\lambda}(\zeta_1(x,t)w^{\mathrm{p}} - \zeta_2(x,t)w^{\mathrm{q}}) & \text{in } Q_T, \\ w > 0 & \text{in } Q_T, \\ \partial_\nu w = 0 & \text{on } \Sigma_T, \\ w(x,0) = w(x,T) & \text{in } \Omega. \end{cases} \tag{2.22}$$

Let α, β be defined by

$$\alpha = \Big(\frac{\operatorname{ess\,sup} \zeta_2}{\operatorname{ess\,inf} \zeta_1}\Big)^{\frac{1}{\mathrm{p}-\mathrm{q}}}, \qquad \beta = \Big(\frac{\operatorname{ess\,inf} \zeta_2}{\operatorname{ess\,sup} \zeta_1}\Big)^{\frac{1}{\mathrm{p}-\mathrm{q}}}.$$

Then $0 < \beta \le \alpha$, and

$$\zeta_1(x,t)\beta^{\mathrm{p}} - \zeta_2(x,t)\beta^{\mathrm{q}} \le 0 \le \zeta_1(x,t)\alpha^{\mathrm{p}} - \zeta_2(x,t)\alpha^{\mathrm{q}},$$

for almost every $(x,t) \in Q_T$. Applying Corollary 1, we obtain a positive solution w of (2.22), for $\hat{\lambda} > 0$ sufficiently small.

It is now easy to see that the rescaled function $u = \hat\lambda^{\frac{1}{p-1}} w$ is a solution of (2.21), with $\lambda = \hat\lambda^{\frac{p-q}{p-1}}$. □

Remark 5. The same result as in Corollary 5 holds if $p < 1$ and $q > p$, with natural modifications needed in the proof.

3. The elliptic equation

In this section, we consider the Neumann problem

$$\begin{cases} -\sum_{i,j=1}^{N} a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^{N} a_i(x)\partial_{x_i} u = f(x,u,\nabla u) \ \text{ in } \Omega, \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ with a C^2-boundary, $a_{ij} \in C(\overline{\Omega})$, $a_{ij} = a_{ji}$, for $i, j = 1, \dots, N$, there exists $\bar a > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \bar a|\xi|^2, \quad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N,$$

and $a_i \in L^\infty(\Omega)$, for $i = 1, \dots, N$. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is assumed to be L^r-Carathéodory, for some $r > N$.

We say that u is a *solution* of problem (3.1) if u belongs to $W^{2,r}(\Omega)$ and it satisfies the differential equation almost everywhere in Ω and the boundary condition pointwise. (Here and in the sequel we use the standard notation for the Sobolev space $W^{2,r}(\Omega)$, which should not be confused with the notation used in the previous section.)

Let us define the class of functions $\mathcal{F}(I, \Lambda, K)$, where $I \subseteq \mathbb{R}$ is an interval, and Λ, K are some nonnegative constants. Its elements are the Carathéodory functions $f : \Omega \times I \times \mathbb{R}^N \to \mathbb{R}$ which satisfy the following Bernstein-Nagumo growth condition:

$$(BN) \qquad \begin{array}{l} |f(x,u,v)| \le h(x) + K|v|^2, \\ \text{for a.e. } x \in \Omega \text{ and every } (u,v) \in I \times \mathbb{R}^N, \end{array}$$

for some $h \in L^r(\Omega)$, with $\|h\|_{L^r} \le \Lambda$.

Theorem 6. *Let $I \subseteq \mathbb{R}$ be an interval. Given a compact interval J, contained in the interior of I, there is a $\Lambda > 0$ such that, for every $f \in \mathcal{F}(I, \Lambda, \Lambda)$, if there are two constants α, β in J for which*

$$f(x,\beta,0) \le 0 \le f(x,\alpha,0) \quad \text{a.e. in } \Omega,$$

then problem (3.1) *has a solution u, with $u(x) \in I$ for every $x \in \overline{\Omega}$.*

The proof follows the same lines as the one given in the previous section for the parabolic case. The analogue of Proposition 2 has been first proved by Amann [2] in the case of classical solutions, and can be found in [15] for our setting, while the analogue of Proposition 1 is proved in [34, Lemma 5.10].

The remarks and the corollaries at the end of Section 2 clearly hold in this situation, as well.

Corollary 6. *Given* $\mathrm{p} > 1$ *and a function* $\zeta \in L^\infty(\Omega)$*, with* $\operatorname{ess\,inf} \zeta > 0$*, the problem*

$$\begin{cases} -\Delta u = \zeta(x)|u|^{\mathrm{p}-1}u + e(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

has a solution, provided that $e \in L^\infty(\Omega)$ *and* $\|e\|_{L^\infty}$ *is sufficiently small. More precisely, given* $\zeta \in L^\infty(\Omega)$*, with* $\operatorname{ess\,inf} \zeta > 0$*, for every given* $\delta > 0$ *there is a* $\bar{\Lambda}_\delta > 0$ *such that, for every* $\mathrm{p} > 1$*, if*

$$\|e\|_{L^\infty} \leq \Big(\frac{\bar{\Lambda}_\delta}{(1+\delta)^{\mathrm{p}}}\Big)^{\frac{\mathrm{p}}{\mathrm{p}-1}}, \tag{3.3}$$

then (3.2) *has a solution.*

When $N \geq 3$, problems like (3.2) are usually considered in the literature under the hypothesis that $\mathrm{p} \leq \frac{N+2}{N-2} = 2^*-1$, where 2^* is the critical exponent for the Sobolev imbedding. See, e.g., the recent paper [12] and the references therein. We do not require such a restriction, provided that $\|e\|_{L^\infty}$ is small.

Corollary 7. *Given* $\mathrm{p} > 1$ *and a function* $\zeta \in L^\infty(\Omega)$*, with* $\operatorname{ess\,inf} \zeta > 0$*, the problem*

$$\begin{cases} -\Delta u = \zeta(x)(u^+)^{\mathrm{p}} + e(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

has a solution, provided that $e \in L^\infty(\Omega)$*,* $e(x) \leq 0$ *for almost every* $x \in \Omega$*, and* $\|e\|_{L^\infty}$ *is sufficiently small. More precisely, given* $\zeta \in L^\infty(\Omega)$*, with* $\operatorname{ess\,inf} \zeta > 0$*, for every given* $\delta > 0$ *there is a* $\bar{\Lambda}_\delta > 0$ *such that, for every* $\mathrm{p} > 1$*, if* $e(x) \leq 0$ *for almost every* $x \in \Omega$*, and* (3.3) *holds, then* (3.4) *has a solution.*

Problems like (3.4) have been studied in [36] and [23], assuming, when $N \geq 3$, that $\mathrm{p} < \frac{N}{N-2}$. We do not require such a restriction, provided that $\|e\|_{L^\infty}$ is small.

Corollary 8. *Assume that* $\zeta_1, \zeta_2 \in L^\infty(\Omega)$ *are bounded below by a positive constant,* $\mathrm{p} > 1$*, and* $\mathrm{q} < \mathrm{p}$*. Then, there is a* $\bar\lambda > 0$ *such that, for every* $\lambda \in [0, \bar\lambda]$*, problem*

$$\begin{cases} -\Delta u = \zeta_1(x)u^{\mathrm{p}} - \lambda\zeta_2(x)u^{\mathrm{q}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

has a solution.

Similar problems were considered, e.g., in [28] when $N \geq 3$ and $\mathrm{p} < 2^* - 1$, and in [35, 13] for $\mathrm{p} = 2^* - 1$. If $1 < \mathrm{q} < \mathrm{p}$, problem (3.5) has a concave-convex nonlinearity and recalls a similar Dirichlet problem introduced in [5] (see also [10]). However, differently from our situation, lower and upper solutions are found there with the usual ordering $\alpha \leq \beta$. Using this fact, multiplicity results are proved by the use of variational methods, provided that $\mathrm{p} \leq 2^* - 1$ when $N \geq 3$. We do not know whether in our case multiplicity results could be obtained.

The non-existence result obtained in Theorem 5 for the parabolic equation has its counterpart even in this situation, as stated in Theorem 3, with essentially the same proof.

4. Second-order ODE's

In this section, we consider either the Neumann problem

$$-u'' + a(x)u' = f(x, u, u'), \qquad u'(0) = u'(1) = 0, \tag{4.1}$$

or the periodic problem

$$-u'' + a(x)u' = f(x, u, u'), \qquad u(0) = u(1), \quad u'(0) = u'(1). \tag{4.2}$$

Here, $a \in L^\infty(0,1)$, and the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory.

We say that u is a *solution* of problem (4.1), or (4.2), if u belongs to $W^{2,1}(0,1)$ and it satisfies the differential equation almost everywhere in $(0,1)$ and the boundary conditions pointwise.

Let us define the class of functions $\mathcal{F}(I, \Lambda, K)$, where $I \subseteq \mathbb{R}$ is an interval, and Λ, K are some nonnegative constants. Its elements are the Carathéodory functions $f : [0,1] \times I \times \mathbb{R} \to \mathbb{R}$ which satisfy the following Bernstein-Nagumo growth condition:

$$(BN) \qquad \begin{array}{l} |f(x,u,v)| \leq h(x) + K|v|^2, \\ \text{for a.e. } x \in (0,1) \text{ and every } (u,v) \in I \times \mathbb{R}, \end{array}$$

for some $h \in L^1(0,1)$, with $\|h\|_{L^1} \leq \Lambda$.

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an interval. Given a compact interval J, contained in the interior of I, there is a $\Lambda > 0$ such that, for every $f \in \mathcal{F}(I,\Lambda,\Lambda)$, if there are two constants α, β in J for which*

$$f(x,\beta,0) \le 0 \le f(x,\alpha,0) \quad \text{a.e. in } (0,1),$$

then problems (4.1) *and* (4.2) *have a solution u, with $u(x) \in I$ for every $x \in [0,1]$.*

The proof of the theorem uses the same arguments as in the parabolic case, with some small modifications, due to the fact that, now, $r = N = 1$. The analogue of Proposition 2 is rather standard in this setting (see, e.g., [14], with different kinds of Bernstein-Nagumo conditions). The analogue of Proposition 1 is reported in the Appendix, and provides a uniform estimate in the $W^{2,1}$-norm. Some more care has to be used in the choice of the spaces, since $W^{2,1}(0,1)$ is continuously imbedded in $C^1([0,1])$, but not compactly imbedded. Taking advantage of (BN), one can choose to define the operators $\tilde{N}_m$ on the space $W^{1,2}(0,1)$ instead of $C^1([0,1])$, taking values in $L^1(0,1)$. Since $\tilde{N}_m$ maps bounded sets into bounded sets and

$$W^{2,1}(0,1) \subset\subset W^{1,2}(0,1) \subset\subset C([0,1]),$$

the corresponding operators $\tilde{\mathcal{S}}_m : W^{1,2}(0,1) \to W^{1,2}(0,1)$, defined by (2.1), are now completely continuous, and the remaining part of the proof is analogous to that of the parabolic case.

The remarks and the corollaries at the end of Section 2 clearly hold in this situation, as well. Theorem 7 generalizes a result by Pournaki and Razani [30, 31], where the existence of periodic solutions with small period has been proved. A simple change of the time variable permits us to fix the period and reduce to our case.

Remark 6. Assuming f to be L^r-Carathéodory, it is possible to introduce variants of the Bernstein-Nagumo condition, involving only one-sided growth restrictions, cf. [14]. For example, instead of $\mathcal{F}(I,\Lambda,K)$, we can introduce the class of functions $\mathcal{G}(I,\Lambda,\varphi)$, where $I \subseteq \mathbb{R}$ is an interval, Λ is a nonnegative constant, and $\varphi : [0,+\infty) \to (0,+\infty)$ is a continuous function such that

$$\int_1^{+\infty} \frac{s^{1-\frac{1}{r}}}{\varphi(s)}\, ds = +\infty.$$

Its elements are the L^r-Carathéodory functions $f : [0,1] \times I \times \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x,u,v) \le \Lambda\, \varphi(|v|),$$

for every $(x, u, v) \in [0, 1] \times I \times \mathbb{R}$. A comprehensive treatment can be found in [14]. Notice, however that, when $r = 1$, the two classes $\mathcal{F}(I, \Lambda, K)$ and $\mathcal{G}(I, \Lambda, \varphi)$ are not contained in each other.

Remark 7. The left-hand side of the differential equation may also contain some type of nonlinear lower-order terms. For example, we can deal with a Liénard type equation

$$-u'' + \eta_1(u)u' = f(x, u, u'),$$

or a Rayleigh type equation

$$-u'' + \eta_2(u') = f(x, u, u'),$$

with $\eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}$ continuous, provided that η_2 has at most a quadratic growth and $\eta_2(0) = 0$. (For simplicity, we did not write the term $a(x)u'$.) Indeed, if $f = 0$, the only solutions of the Neumann or the periodic problems associated to these equations are the constants, as can be seen by multiplying in the first equation by u, in the second one by u'', and integrating. This fact permits us to show, as in the proof of Theorem 4, that the solutions have a small oscillation, when $f \in \mathcal{F}(I, \Lambda, \Lambda)$, or $f \in \mathcal{G}(I, \Lambda, \varphi)$, with Λ sufficiently small. Notice also that, as shown in [14, Proposition I.4.1], the quadratic growth condition on η_2 can be avoided, at least when f belongs to $\mathcal{F}(I, \Lambda, 0)$.

To conclude this section, let us now state a version of the non-existence result obtained for the parabolic equation in the case of the periodic problem.

Theorem 8. *Given two constants α, β with $\alpha \neq \beta$, for every $\lambda > 0$ there is a continuous function $f_\lambda : [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that*

$$|f_\lambda(x, u)| \leq 1 \qquad \text{for every } (x, u) \in [0, 1] \times \mathbb{R},$$

and

$$\int_0^1 f_\lambda(x, \beta)\, dx < 0 < \int_0^1 f_\lambda(x, \alpha)\, dx,$$

for which the problem

$$-u'' = \lambda f_\lambda(x, u), \qquad u(0) = u(1), \quad u'(0) = u'(1),$$

has no solutions.

5. THE p-LAPLACIAN

In this section, we consider the Neumann problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ with a C^2-boundary, Δ_p is the p-Laplacian, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $p > 1$, and the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is L^∞-Carathéodory.

In this case, the function f does not depend on the gradient of u, so that problem (5.1) has a variational structure. We consider a weaker notion of solution than in the previous sections. A function u is a *solution* of problem (5.1) if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w = \int_\Omega f(x,u)w,$$

for every $w \in W^{1,p}(\Omega)$. Known regularity results (see, e.g., [25]) imply that such a solution is in $C^{1,\sigma}(\overline{\Omega})$ for some $\sigma > 0$.

Theorem 9. *Let $I \subseteq \mathbb{R}$ be an interval. Given a compact interval J, contained in the interior of I, there is a $\Lambda > 0$ such that, for every function f satisfying*

$$|f(x,u)| \le \Lambda \quad \textit{for a.e. } x \in \Omega \textit{ and every } u \in I,$$

if there are two constants α, β in J for which

$$f(x,\beta) \le 0 \le f(x,\alpha) \quad \text{a.e. in } \Omega, \tag{5.2}$$

then problem (5.1) *has a solution u, with $u(x) \in I$ for every $x \in \overline{\Omega}$.*

Proof. Denote by $\mathcal{R} : L^\infty(\Omega) \to C(\overline{\Omega})$ the completely continuous operator which associates to any $h \in L^\infty(\Omega)$ the unique solution v of the problem

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = h & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

We see that (5.1) is equivalent to the fixed-point problem

$$u = \mathcal{T}(u),$$

where $\mathcal{T} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is the completely continuous operator defined by

$$\mathcal{T}(u) = \mathcal{R}(f(\cdot,u) + |u|^{p-2}u).$$

First of all, we consider the case when f and α, β satisfy (5.2), with $\alpha \le \beta$. Even if this is the classical situation (see e.g. [15] for the Dirichlet problem), we nevertheless want to recall the proof, for the reader's convenience. In this case, we will need no restriction on Λ.

If $\alpha = \beta$ we obviously have a constant solution. If $\alpha < \beta$, define

$$\gamma(u) = \begin{cases} \alpha & \text{if } u \le \alpha, \\ u & \text{if } \alpha \le u \le \beta, \\ \beta & u \ge \beta, \end{cases}$$

and consider the modified problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f(x, \gamma(u)) + |\gamma(u)|^{p-2}\gamma(u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

We see that the modified problem (5.3) is equivalent to the fixed-point problem

$$u = \widehat{\mathcal{T}}(u),$$

where $\widehat{\mathcal{T}} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is the completely continuous operator defined by

$$\widehat{\mathcal{T}}(u) = \mathcal{R}(f(\cdot, \gamma(u)) + |\gamma(u)|^{p-2}\gamma(u)).$$

Since the image of $\widehat{\mathcal{T}}$ is contained in a ball B_R, by Schauder's Theorem, $\widehat{\mathcal{T}}$ has a fixed point in B_R, and

$$d_{LS}(I - \widehat{\mathcal{T}},\, B_R) = 1.$$

Moreover, if u is any solution of (5.3), taking $(\alpha - u)^+$ and $(u - \beta)^+$ as test functions in (5.3) we see that $\alpha \le u \le \beta$. Hence, u actually solves (5.1), thus concluding the proof in the case $\alpha < \beta$.

Notice that, in this setting, the pair (α, β) is said to be *admissible* for (5.1) if $\alpha < \beta$ and any solution u of (5.1) satisfying $\alpha \le u \le \beta$ is such that $\alpha < u < \beta$. Define

$$\mathcal{U}_{(\alpha,\beta)} = \{u \in C(\overline{\Omega}) : \alpha < u < \beta\},$$

which is bounded and open in $C(\overline{\Omega})$. Since $\mathcal{T}$ and $\widehat{\mathcal{T}}$ coincide on $\overline{\mathcal{U}}_{(\alpha,\beta)}$, by the excision property of the degree, we have that, if (α, β) is admissible,

$$d_{LS}\left(I - \mathcal{T}, \mathcal{U}_{(\alpha,\beta)}\right) = d_{LS}\left(I - \widehat{\mathcal{T}}, \mathcal{U}_{(\alpha,\beta)}\right) = d_{LS}\left(I - \widehat{\mathcal{T}},\, B_R\right) = 1.$$

As for Theorem 4, we will now provide the proof by contradiction, by taking $\Lambda = \frac{1}{m}$, with $m \ge 1$ a sufficiently large integer. Let $\delta > 0$ be such that $[\min J - \delta, \max J + \delta] \subseteq I$, and fix $\varepsilon \in (0, \frac{\delta}{2})$. By contradiction, assume that there exist a function f_m such that

$$|f_m(x, u)| \le \frac{1}{m} \quad \text{for a.e. } x \in \Omega \text{ and every } u \in I,$$

and two constants $\alpha_m, \beta_m \in J$ satisfying

$$f_m(x, \beta_m) \leq 0 \leq f_m(x, \alpha_m) \quad \text{a.e. in } \Omega,$$

for which the problem

$$\begin{cases} -\Delta_p u = f_m(x, u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

has no solution u, with $u(x) \in I$ for every $x \in \overline{\Omega}$. By the above, $\beta_m < \alpha_m$. Define the modified function

$$\tilde{f}_m(x,u) = \begin{cases} \frac{1}{m} & \text{if } u \leq \beta_m - 2\varepsilon, \\ \frac{1}{m} + \frac{1}{\varepsilon}(f_m(x,u) - \frac{1}{m})\,(u - \beta_m + 2\varepsilon) & \text{if } \beta_m - 2\varepsilon \leq u \leq \beta_m - \varepsilon, \\ f_m(x,u) & \text{if } \beta_m - \varepsilon \leq u \leq \alpha_m + \varepsilon, \\ -\frac{1}{m} + \frac{1}{\varepsilon}(f_m(x,u) + \frac{1}{m})\,(\alpha_m + 2\varepsilon - u) & \text{if } \alpha_m + \varepsilon \leq u \leq \alpha_m + 2\varepsilon, \\ -\frac{1}{m} & \text{if } u \geq \alpha_m + 2\varepsilon, \end{cases}$$

and consider the problem

$$\begin{cases} -\Delta_p u = \tilde{f}_m(x, u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Notice that, as $\Lambda = \frac{1}{m}$, we have $|\tilde{f}_m(x,u)| \leq \frac{1}{m}$ for every $u \in \mathbb{R}$, and

$$\tilde{f}_m(x, \alpha_m + \delta) < 0 < \tilde{f}_m(x, \beta_m - \delta) \quad \text{a.e. in } \Omega.$$

If u is a solution of (5.5), taking $(\beta_m - 2\varepsilon - u)^+$ as test function we see that $u \geq \beta_m - 2\varepsilon$, so that $u > \beta_m - \delta$. Similarly, taking $(u - \alpha_m - 2\varepsilon)^+$ as test function, one has $u < \alpha_m + \delta$. Thus, the pair $(\beta_m - \delta, \alpha_m + \delta)$ is admissible for (5.5).

Using the above information on the degree, we now proceed exactly as in Section 2 to prove the existence of a solution u_m of (5.5) such that

$$\beta_m - \delta < u_m < \alpha_m + \delta, \tag{5.6}$$

and, for some $x_m \in \overline{\Omega}$,

$$\beta_m \leq u_m(x_m) \leq \alpha_m. \tag{5.7}$$

(See (2.2) and (2.3).) We can now write $u_m = \mathcal{R}(h_m)$, where

$$h_m(x) = \tilde{f}_m(x, u_m(x)) + |u_m(x)|^{p-2} u_m(x).$$

By (5.6), there is a constant $c > 0$ such that

$$\|h_m\|_{L^\infty} \leq c, \qquad \text{for every } m.$$

Since $\mathcal{R}$ is completely continuous, there are a subsequence, still denoted by $(u_m)_m$, and a function $\overline{u} \in C(\overline{\Omega})$ such that $u_m \to \overline{u}$ uniformly in $\overline{\Omega}$. As $|\tilde{f}_m(x, u_m(x))| \leq \frac{1}{m}$, for almost every $x \in \Omega$,

$$\big\| \, h_m - |\overline{u}|^{p-2}\overline{u} \, \big\|_{L^\infty} \to 0,$$

so that $\overline{u} = \mathcal{R}(|\overline{u}|^{p-2}\overline{u})$; i.e.

$$\begin{cases} -\Delta_p \overline{u} = 0 & \text{in } \Omega, \\ \partial_\nu \overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, $\overline{u}$ is constant. Since $u_m \to \overline{u}$ uniformly in $\overline{\Omega}$, for m sufficiently large we have that $\operatorname{osc} u_m \leq \varepsilon$, and hence, by (5.7),

$$\beta_m - \varepsilon \leq u_m \leq \alpha_m + \varepsilon.$$

Thus, u_m is a solution of (P_m), and $u_m(x) \in I$, for every $x \in \overline{\Omega}$, a contradiction. □

To conclude, one can easily verify that the remarks and the corollaries in Sections 2 and 3 can be adapted to this situation, as well as the non-existence result stated in Theorem 3.

6. Appendix

For the reader's convenience we now give the proof of the following analogue of Proposition 1, which was needed in Section 4 in the framework of ordinary differential equations. The main difference from the case treated in [34] is that, now, $r = 1$.

Proposition 3. *Given $M, \Lambda, K > 0$, there is a constant $C > 0$ such that, if $u \in W^{2,1}(0,1)$ satisfies*

$$\begin{cases} |u''| \leq h(x) + K|u'|^2, & \text{a.e. in } (0,1), \\ |u| \leq M & \text{in } [0,1], \end{cases} \tag{6.1}$$

for some $h \in L^1(0,1)$ with $\|h\|_{L^1} \leq \Lambda$, then

$$\|u\|_{W^{2,1}} \leq C.$$

Proof. Taking if necessary the positive part of h, we can assume that $h \geq 0$ almost everywhere in $(0,1)$. We first consider the case when $h \in L^\infty(0,1)$ and define

$$\varphi(s) = \|h\|_{L^\infty} + 1 + Ks^2.$$

Since $-M \le u \le M$, by Lagrange's theorem, we can find $\xi \in (0,1)$ such that $|u'(\xi)| \le 2M$. If $\|u'\|_{L^\infty} \le 2M$, using (6.1), we have

$$\|u\|_{W^{2,\infty}} = \|u\|_{L^\infty} + \|u'\|_{L^\infty} + \|u''\|_{L^\infty} \le 3M + (\|h\|_{L^\infty} + 4KM^2).$$

Otherwise, let $[x_1, x_2] \subseteq [0,1]$ be an interval such that

$$|u'(x)| > 2M \quad \text{for every } x \in (x_1, x_2),$$

and, either $|u'(x_1)| = 2M$, or $|u'(x_2)| = 2M$. Let $i = 1,2$ be the index for which $|u'(x_i)| = 2M$. Then, considering separately the cases when u' is positive or negative on $[x_1, x_2]$, one easily verifies that, for every $x \in [x_1, x_2]$,

$$\int_{2M}^{|u'(x)|} \frac{s}{\varphi(s)} ds \le \Big| \int_{x_i}^{x} \frac{|u'(t)|\,|u''(t)|}{\varphi(|u'(t)|)} dt \Big| \le \Big| \int_{x_i}^{x} |u'(t)| dt \Big| = |u(x) - u(x_i)| \le 2M.$$

Since $\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty$, there is a constant $R > 0$, depending only on $\|h\|_{L^\infty}$, K, and M, such that

$$|u'(x)| \le R \quad \text{for every } x \in [x_1, x_2].$$

This proves that $\|u'\|_{L^\infty} \le R$. Using (6.1), we then have

$$\|u\|_{W^{2,\infty}} \le M + R + (\|h\|_{L^\infty} + KR^2),$$

and the proof is completed in this case.

Assume now $h \in L^1(0,1)$ and, following [34], set

$$e(x) = \frac{-u''(x) + u(x)}{1 + h(x) + K|u'(x)|^2}.$$

Then, since $h \ge 0$ almost everywhere in $(0,1)$, $e \in L^\infty(0,1)$, with $\|e\|_{L^\infty} \le 1 + M$.

Let $v \in W^{2,1}(0,1)$ be the solution of

$$\begin{cases} -v'' + v = e(x)(1 + h(x)), \\ \quad v(0) = v(1), \quad v'(0) = v'(1). \end{cases}$$

Then, there is a constant $\bar{c} > 0$ such that

$$\|v\|_{C^1} \le \bar{c}\|e(1+h)\|_{L^1} \le \bar{c}(1 + M)(1 + \|h\|_{L^1}).$$

Let now $w = u - v$. Then

$$-w'' + w = Ke(x)|u'(x)|^2,$$

so that, by (6.1),

$$\begin{aligned} |w''(x)| &\le |w(x)| + K|e(x)|\,|u'(x)|^2 \\ &\le |u(x)| + |v(x)| + K|e(x)|(2|v'(x)|^2 + 2|w'(x)|^2) \end{aligned}$$

$$\leq C_1 + C_2|w'(x)|^2,$$

for almost every $x \in (0,1)$, where

$$C_1 = M + \bar{c}(1+M)(1+\|h\|_{L^1}) + 2K(1+M)(\bar{c}(1+M)(1+\|h\|_{L^1}))^2$$

and $C_2 = 2K(1+M)$. Moreover,

$$|w(x)| \leq |u(x)| + |v(x)| \leq M + \bar{c}(1+M)(1+\|h\|_{L^1}),$$

for every $x \in [0,1]$.

Therefore, w satisfies the same assumptions as u with a constant function $h = C_1$, and, by the first part of the proof, there is a constant $R > 0$ depending only on M, C_1, and C_2, such that $\|w'\|_{L^\infty} \leq R$. Then,

$$\|u'\|_{L^\infty} \leq \|w'\|_{L^\infty} + \|v'\|_{L^\infty} \leq R + \bar{c}(1+M)(1+\|h\|_{L^1}).$$

By (6.1), we have

$$\|u''\|_{L^1} \leq \|h\|_{L^1} + K[R + \bar{c}(1+M)(1+\|h\|_{L^1})]^2,$$

which concludes the proof. □

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