Differential and Integral Equations Volume 22, Numbers 9-10 (2009), 881–900

CLOSURE OF SMOOTH MAPS IN $W^{1,p}(B^3; S^2)$

Augusto C. Ponce and Jean Van Schaftingen
Université Catholique de Louvain, Département de Mathématique
Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

To Jean Mawhin and Patrick Habets with esteem and affection.

Abstract. For every $2 < p < 3$, we show that $u \in W^{1,p}(B^3; S^2)$ can be strongly approximated by maps in $C^\infty(\overline{B}^3; S^2)$ if, and only if, the distributional Jacobian of u vanishes identically. This result was originally proved by Bethuel-Coron-Demengel-Hélein, but we present a different strategy which is motivated by the $W^{2,p}$-case.

1. Introduction

Let B^3 be the unit ball and S^2 be the unit sphere of $\mathbb{R}^3$. Given $1 \le p < \infty$, consider

$$W^{1,p}(B^3; S^2) = \Big\{ u \in W^{1,p}(B^3; \mathbb{R}^3) : u(x) \in S^2 \text{ a.e.} \Big\}. \tag{1.1}$$

Although $W^{1,p}(B^3; S^2)$ is not a vector space, it inherits the usual distance from $W^{1,p}(B^3; \mathbb{R}^3)$; namely,

$$\|u - v\|_{W^{1,p}} = \|u - v\|_{L^p} + \|\nabla u - \nabla v\|_{L^p} \quad \forall u, v \in W^{1,p}(B^3; S^2). \tag{1.2}$$

Using standard extension and convolution arguments, it is easy to see that every $u \in W^{1,p}(B^3; S^2)$ can be approximated by maps $\varphi \in C^\infty(\overline{B}^3; \mathbb{R}^3)$ with respect to the $W^{1,p}$-distance. If we assume in addition that $p > 3$, then by Morrey's estimates such approximations converge uniformly to u, and we can project this sequence back to S^2 to obtain an approximation in $C^\infty(\overline{B}^3; S^2)$. Although Morrey's estimates are no longer true in the critical case $p = 3$, this argument still works as a consequence of the theory of vanishing mean oscillation (VMO) functions.

Theorem 1.1 (Schoen-Uhlenbeck [10]). *Let $p \ge 3$. Then $C^\infty(\overline{B}^3; S^2)$ is dense in $W^{1,p}(B^3; S^2)$.*

The reader may wonder what happens if $1 \le p < 3$. It turns out that such a conclusion is still true if $1 \le p < 2$, but surprisingly it fails if $2 \le p < 3$.

AMS Subject Classifications: 46E35, 46T30, 58D15.

Theorem 1.2 (Bethuel-Zheng [4]). *Let* $1 \le p < 3$. *Then,* $C^\infty(\overline{B}^3; S^2)$ *is dense in* $W^{1,p}(B^3; S^2)$ *if and only if* $1 \le p < 2$.

The reason for the lack of density in the case $2 \le p < 3$ is the existence of "topological singularities" of maps in $W^{1,p}(B^3; S^2)$. For instance, given a smooth map $g : S^2 \to S^2$, let

$$u(x) = g\big(\tfrac{x}{|x|}\big) \quad \forall x \in B^3 \setminus \{0\}. \tag{1.3}$$

In this case, $u \in W^{1,p}(B^3; S^2)$ for every $2 \le p < 3$, but u cannot be strongly approximated by smooth maps $\varphi : \overline{B}^3 \to S^2$ in $W^{1,p}$ if $\deg g \neq 0$.

Indeed, assume by contradiction that there exists a sequence (φ_n) in $C^\infty(B^3; S^2)$ strongly converging to u in $W^{1,p}(B^3; S^2)$. By Fubini's theorem, for almost every $r > 0$,

$$\varphi_n \to u \quad \text{strongly in } W^{1,p}(\partial B_r; S^2).$$

If $2 < p < 3$, then by Morrey's estimates $\varphi_n \to u$ uniformly on ∂B_r for any such r (note that ∂B_r has dimension 2) and thus

$$\deg(\varphi_n|_{\partial B_r}) \to \deg(u|_{\partial B_r}) = \deg g. \tag{1.4}$$

Since, for every $n \ge 1$, $\deg(\varphi_n|_{\partial B_r}) = 0$, this would imply that $\deg g = 0$, which is a contradiction. When $p = 2$, by continuity of the degree under VMO-convergence (see [8]) assertion (1.4) still holds and we can conclude as before.

In the above example, u has a topological singularity at 0. This raises the question of how to find such singularities for a general map $u \in W^{1,p}(B^3; S^2)$. Their location and strength can be detected using a simple yet powerful tool introduced by Brezis-Coron-Lieb [7]: the distributional Jacobian "Jac."

Given $p \ge 2$ and a map $u \in W^{1,p}(B^3; S^2)$, consider the vector field

$$D(u) = \big(u \cdot u_{x_2} \wedge u_{x_3}, u \cdot u_{x_3} \wedge u_{x_1}, u \cdot u_{x_1} \wedge u_{x_2}\big), \tag{1.5}$$

where $u_{x_i} \in L^p(B^3; \mathbb{R}^3)$ denotes the partial derivative of u in the weak sense. Since $u \in W^{1,2} \cap L^\infty$, we have $D(u) \in L^1(B^3; \mathbb{R}^3)$. We then define the distributional Jacobian as

$$\operatorname{Jac}(u) = \tfrac{1}{3} \operatorname{div} D(u) \quad \text{in } \mathcal{D}'(B^3); \tag{1.6}$$

more precisely,

$$\langle \operatorname{Jac}(u), \zeta \rangle = -\frac{1}{3} \int_{B^3} D(u) \cdot \nabla \zeta \quad \forall \zeta \in C_0^\infty(B^3).$$

For instance, if u is smooth (in which case there are no singularities), then one has

$$\operatorname{Jac}(u) = u_{x_1} \cdot u_{x_2} \wedge u_{x_3} = 0.$$

On the other hand, if u is given by (1.3), then

$$\operatorname{Jac}(u) = \frac{4\pi}{3}(\deg g)\,\delta_0,$$

where δ_0 denotes the Dirac mass at the origin.

Since smooth maps are not dense in $W^{1,p}(B^3;S^2)$ when $2 \leq p < 3$, one should be able to identify those maps in $W^{1,p}(B^3;S^2)$ which can be approximated by functions in $C^\infty(\overline{B}^3;S^2)$. It turns out that the only obstruction to density of smooth maps is of a topological nature.

Theorem 1.3 (Bethuel [1])**.** *Let $u \in W^{1,2}(B^3;S^2)$. Then, there exists a sequence $(\varphi_n) \subset C^\infty(\overline{B}^3;S^2)$ such that*

$$\varphi_n \to u \quad \text{strongly in } W^{1,2}, \tag{1.7}$$

if and only if

$$\operatorname{Jac}(u) = 0 \quad \text{in } \mathcal{D}'(B^3). \tag{1.8}$$

The counterpart of Theorem 1.3 in the case $2 < p < 3$ is the following.

Theorem 1.4 (Bethuel-Coron-Demengel-Hélein [3])**.** *Let $u \in W^{1,p}(B^3;S^2)$, where $2 < p < 3$. Then, there exists a sequence $(\varphi_n) \subset C^\infty(\overline{B}^3;S^2)$ such that*

$$\varphi_n \to u \quad \text{strongly in } W^{1,p}, \tag{1.9}$$

if and only if

$$\operatorname{Jac}(u) = 0 \quad \text{in } \mathcal{D}'(B^3). \tag{1.10}$$

Although Theorem 1.4 is usually attributed to Bethuel [1], such a result was never mentioned in [1]. Actually, Bethuel's proof of Theorem 1.3 uses a removing dipole technique and strongly relies on the fact that $p = 2$. The proof of Theorem 1.4, instead, is based on a different strategy from another work of Bethuel [2].

More generally, we consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$. The distributional Jacobian still makes sense for maps in $W^{1,p}(\Omega;S^{N-1})$ as long as $p \geq N-1$. The counterparts of Theorems 1.3 and 1.4 are presented in the theorem below.

Theorem 1.5 (Bethuel [1], Bethuel-Coron-Demengel-Hélein [3])**.** *Let* $u \in W^{1,p}(\Omega; S^{N-1})$*, where* $N-1 \le p < N$*. Then, there exists a sequence* $(\varphi_n) \subset C^\infty(\overline{\Omega}; S^{N-1})$ *satisfying* (1.9) *if and only if*

$$\operatorname{Jac}(u) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.11}$$

In addition, one can estimate the $W^{1,N-1}$-distance between any given map $u \in W^{1,N-1}(\Omega; S^{N-1})$ and the class of smooth maps in terms of $L(u)$, the length of the minimal connection of u (see definition (2.4) below).

Theorem 1.6 (Bethuel [1])**.** *If* $u \in W^{1,N-1}(\Omega; S^{N-1})$*, then*

$$\inf\Big\{ \|\nabla u - \nabla\varphi\|_{L^{N-1}} \; ; \; \varphi \in C^\infty(\overline{\Omega}; S^{N-1}) \Big\} \le C\big(L(u)\big)^{\frac{1}{N-1}}. \tag{1.12}$$

The main goal of this paper is to use a different strategy from [1, 3] to prove Theorem 1.5 for $N-1 < p < N$. An advantage of our approach is that it can be adapted to higher-order Sobolev spaces and in particular to $W^{2,p}$; see [6]. As a by-product we also prove the following new counterpart of Theorem 1.6 when $N-1 < p < N$.

Theorem 1.7. *If* $N-1 < p < N$*, then for every* $u \in W^{1,p}(\Omega; S^{N-1})$*,*

$$\inf\Big\{ \|u - \varphi\|_{W^{1,p}} \; ; \; \varphi \in C^\infty(\overline{\Omega}; S^{N-1}) \Big\} \le C\|\nabla u\|_{L^p(A)}, \tag{1.13}$$

for some open set $A \subset \Omega$ *such that*

$$|A|^{1/p} \le C\, L(u) \|\nabla u\|_{L^p(\Omega)}. \tag{1.14}$$

We now explain the main idea in the proof of Theorem 1.7. We first cover the domain Ω with finitely many balls $\big(B_r(x_i)\big)_{i \in I}$, and then we modify u on $B_r(x_i)$ according to whether

$$\int_{B_{2r}(x_i)} |\nabla u|^p < \lambda r^{N-p} \quad \text{or} \quad \int_{B_{2r}(x_i)} |\nabla u|^p \ge \lambda r^{N-p}, \tag{1.15}$$

for some parameter $\lambda > 0$ suitably chosen. In the first case, we call $B_r(x_i)$ a *good ball*, otherwise $B_r(x_i)$ is a *bad ball*. This type of condition was introduced in a remarkable work of Bethuel [2].

If $B_r(x_i)$ is a good ball and $\lambda > 0$ is sufficiently small, then most of the values of $u(B_r(x_i))$ lie in a small geodesic disk of S^{N-1}. In this case, a projection into this disk and a convolution allow us to replace u on $B_r(x_i)$ by a smooth map. In contrast, if $B_r(x_i)$ is a bad ball, then $u|_{\partial B_r(x_i)}$ need not be contained in a small geodesic disk, but if the radius r is larger than the length of the minimal connection $L(u)$, we can slightly decrease the radius r if necessary so that $u|_{\partial B_r(x_i)}$ is homotopic to a constant. In this case, using an idea of Bethuel-Zheng [4], it is possible to use such a homotopy to replace u by a smooth map, while keeping the energy on $B_r(x_i)$ under control.

The detailed constructions on good and bad balls are presented in Sections 4 and 5 below. In the next section, we define the distributional Jacobian for maps in $W^{1,p}(\Omega; S^{N-1})$ with $p \geq N-1$, and we explain some of its main properties. In Section 7, we prove Theorems 1.5 and 1.7.

2. The distributional Jacobian

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Given a map $u \in W^{1,N-1}(\Omega; \mathbb{R}^N) \cap L^\infty$, we consider the L^1-vector field

$$D(u) = (D_1, \dots, D_N), \tag{2.1}$$

where

$$D_j = \det\left[u_{x_1}, \dots, u_{x_{j-1}}, u, u_{x_{j+1}}, \dots, u_{x_N}\right]. \tag{2.2}$$

We then associate to the map u the distribution

$$\operatorname{Jac}(u) = \frac{1}{N} \operatorname{div} D(u); \tag{2.3}$$

more precisely,

$$\langle \operatorname{Jac}(u), \zeta \rangle = -\frac{1}{N} \sum_{j=1}^{N} \int_\Omega D_j \zeta_{x_j} \quad \forall \zeta \in C_0^\infty(\Omega).$$

Given $u \in W^{1,N-1}(\Omega; S^{N-1})$, we define the length of the minimal connection of u as

$$L(u) = \frac{1}{\omega_N} \sup_{\substack{\zeta \in C_0^\infty(\Omega) \\ \|\nabla \zeta\|_{L^\infty} \leq 1}} \langle \operatorname{Jac}(u), \zeta \rangle, \tag{2.4}$$

where ω_N denotes the measure of the unit ball $B_1 \subset \mathbb{R}^N$. The reason for calling $L(u)$ the length of the minimal connection of u comes from the geometric meaning of $L(u)$ (see equations (2.7) and (2.9) below). If u is smooth, then

$$\operatorname{Jac}(u) = \det\left[u_{x_1}, \dots, u_{x_N}\right] = 0.$$

More generally, if u is smooth except at finitely many points $a_1, \dots, a_k \in \Omega$, then (see e.g. [7])

$$\operatorname{Jac}(u) = \omega_N \sum_{i=1}^{k} d_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\Omega), \tag{2.5}$$

where $d_i = \deg(u, a_i)$ denotes the degree of u with respect to any small sphere centered at a_i. Since we are not making any additional assumption about u on $\partial\Omega$, it may happen that $\sum_{i=1}^{k} d_i \neq 0$. However, by using points from $\partial\Omega$ one can always rewrite (2.5) as

$$\operatorname{Jac}(u) = \omega_N \sum_{i=1}^{\tilde{k}} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } \mathcal{D}'(\Omega), \tag{2.6}$$

where $p_1, \ldots, p_{\tilde{k}}, n_1, \ldots, n_{\tilde{k}} \in \overline{\Omega}$ (note that points on $\partial\Omega$ are harmless from the point of view of test functions with compact support in Ω). In particular, one always has $L(u) \leq \sum_{i=1}^{\tilde{k}} |p_i - n_i|$. Brezis-Coron-Lieb [7] proved that these points can be chosen and rearranged so that

$$L(u) = \sum_{i=1}^{\tilde{k}} |p_i - n_i|. \tag{2.7}$$

For a general map $u \in W^{1,N-1}(\Omega; S^{N-1})$, not necessarily with finitely many singularities, one has the following characterizations of $\operatorname{Jac}(u)$ and $L(u)$.

Theorem 2.1 (Bourgain-Brezis-Mironescu [5]). *Given $u \in W^{1,N-1}(\Omega; S^{N-1})$, there exist sequences of points $(p_i), (n_i) \subset \overline{\Omega}$ such that $\sum_{i=1}^{\infty} |p_i - n_i| < \infty$ and*

$$\operatorname{Jac}(u) = \omega_N \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.8}$$

Moreover,

$$L(u) = \inf\Big\{ \sum_{i=1}^{\infty} |p_i - n_i| \ ; \ (p_i), (n_i) \subset \overline{\Omega} \textit{ satisfy } (2.8) \Big\}. \tag{2.9}$$

In contrast with the case of finitely many singularities, the infimum in (2.9) need not be achieved in general; see [9].

We end this section by showing the well-known fact that $L(u)$ is continuous with respect to the strong convergence in $W^{1,N-1}(\Omega; S^{N-1})$.

Proposition 2.1. *Let $(u_n) \subset W^{1,N-1}(\Omega; S^{N-1})$ be a sequence such that $u_n \to u$ in $W^{1,N-1}$. Then*

$$L(u_n) \to L(u). \tag{2.10}$$

Proof. Note that for every $u, v \in W^{1,N-1}(\Omega; \mathbb{R}^N) \cap L^\infty$ we have

$$\big|\langle \operatorname{Jac}(u) - \operatorname{Jac}(v), \zeta\rangle\big| \leq \|D(u) - D(v)\|_{L^1} \|\nabla\zeta\|_{L^\infty} \quad \forall \zeta \in C_0^\infty(\Omega). \tag{2.11}$$

Thus, a standard argument gives

$$\big|L(u) - L(v)\big| \leq \|D(u) - D(v)\|_{L^1}. \tag{2.12}$$

If (u_n) is a sequence converging strongly to u in $W^{1,N-1}$, then by dominated convergence $D(u_n) \to D(u)$ in L^1 and the conclusion holds. □

3. A Fubini-type argument

In Sections 4–5, we present the main ingredients in the proof of Theorem 1.7. The construction in those sections relies on an argument based on Fubini's theorem which we shall explain below. But first, given $1 \leq p < \infty$, let us introduce the following class of functions:

$$\mathcal{R}^{1,p}(\Omega) = \left\{ v \in W^{1,p}(\Omega; S^{N-1}) \,\middle|\, \begin{array}{l} \text{there exist } a_1, \ldots, a_k \in \Omega \text{ such that} \\ v \text{ is smooth in } \overline{\Omega} \setminus \{a_1, \ldots, a_k\} \end{array} \right\}. \tag{3.1}$$

For later use, given $v \in \mathcal{R}^{1,p}(\Omega)$, we denote by $S(v)$ the set of points of Ω where v is not smooth (by definition this set is finite).

As we have already explained, smooth maps are not dense in $W^{1,p}(\Omega; S^{N-1})$ if $N - 1 \leq p < N$. However, we have the following.

Theorem 3.1 (Bethuel-Zheng [4])**.** *If $N-1 \leq p < N$, then $\mathcal{R}^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega; S^{N-1})$.*

This result is particularly useful since it reduces the problem of studying maps in $W^{1,p}(\Omega; S^{N-1})$ into a problem where all maps have finitely many singularities. This is for instance one of the main ingredients in the proof of Theorem 2.1 above. For the sake of Theorem 1.7, one could avoid Theorem 3.1, but the proof becomes less transparent.

We show, in this section, that if $v \in \mathcal{R}^{1,p}(\Omega)$ and if $B_r(x_0)$ is a sufficiently large ball contained in Ω, then it is possible to find a sphere $\partial B_s(x_0)$ such that $v|_{\partial B_s(x_0)}$ is homotopic to a constant.

Lemma 3.1. *Let $N - 1 \leq p < N$. Given $v \in \mathcal{R}^{1,p}(\Omega)$, let $r > 0$ be such that*

$$r > 4L(v). \tag{3.2}$$

Then, for every $x_0 \in \Omega$ with $B_{2r}(x_0) \subset \Omega$ there exists $s \in \left(\frac{3r}{2}, 2r\right)$ such that

$$\deg\big(v|_{\partial B_s(x_0)}\big) = 0 \quad \textit{and} \quad \|\nabla v\|_{L^p(\partial B_s(x_0))} \leq \frac{C}{r^{1/p}} \|\nabla v\|_{L^p(B_{2r}(x_0))}; \tag{3.3}$$

moreover, there exists $\psi \in C_0^\infty(B_{2r}(x_0))$ such that $\psi = 1$ on $B_s(x_0)$ and

$$\big|\langle \mathrm{Jac}\,(v), (1-\psi)\zeta\rangle\big| \leq L(v)\|\nabla\zeta\|_{L^\infty(\Omega)} \quad \forall \zeta \in C_0^\infty(\Omega). \tag{3.4}$$

Proof. By scaling and translation we can assume that $r = 1$ and $x_0 = 0$. Let $p_1, \ldots, p_{\tilde{k}}$ and $n_1, \ldots, n_{\tilde{k}}$ in $\overline{\Omega}$ be such that

$$\mathrm{Jac}\,(v) = \omega_N \sum_{i=1}^{\tilde{k}} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } \mathcal{D}'(\Omega), \tag{3.5}$$

and

$$L(v) = \sum_{i=1}^{\tilde{k}} |p_i - n_i|. \tag{3.6}$$

Denote by $[p_i, n_i]$ the segment joining p_i to n_i. Let

$$T = \Big\{ t \in \big(\tfrac{3}{2}, 2\big) : \partial B_t \cap [p_i, n_i] = \emptyset \quad \forall i \in \big\{1, \ldots, \tilde{k}\big\} \Big\}.$$

Since $L(v) < 1/4$, it follows from the area formula that $|T| > 1/4$. On the other hand, by Fubini's theorem,

$$\int_T dt \int_{\partial B_t} |\nabla v|^p \leq \int_{B_2} |\nabla v|^p.$$

Thus, there exists $s \in T$ such that

$$\int_{\partial B_s} |\nabla v|^p \leq 4 \int_{B_2} |\nabla v|^p. \tag{3.7}$$

Moreover, since $s \in T$, the number of points p_i and n_i inside the ball B_s (including multiplicities) are equal, thus, $\deg\,(v|_{\partial B_s}) = 0$. It remains to show (3.4). To prove this we use the fact that ∂B_s does not intersect any of the segments $[p_i, n_i]$. Thus, for some $\varepsilon > 0$ small, the annulus $B_{s+\varepsilon} \setminus B_s$ does not intersect any of those segments. Let $\psi \in C_0^\infty(B_2)$ be such that $\psi = 1$ in B_s. Denoting by $I \subset \{1, \ldots, \tilde{k}\}$ the set of indices such that $[p_i, n_i]$ is not in $B_{s+\varepsilon}$, we then have

$$\langle \mathrm{Jac}\,(v), (1 - \psi)\zeta \rangle = \sum_{i \in I} \big[\zeta(p_i) - \zeta(n_i)\big] \quad \forall \zeta \in C_0^\infty(\Omega),$$

and thus

$$\big|\langle \mathrm{Jac}\,(v), (1 - \psi)\zeta \rangle\big| \leq L(v) \|\nabla \zeta\|_{L^\infty(\Omega)} \quad \forall \zeta \in C_0^\infty(\Omega). \qquad \square$$

4. REPLACING u ON BAD BALLS

Given $\lambda > 0$ and a ball $B_r(x_0)$ such that $B_{2r}(x_0) \subset \Omega$, we say that $B_r(x_0)$ is a bad ball for a map $v \in W^{1,p}(\Omega; S^{N-1})$ if

$$\int_{B_{2r}(x_0)} |\nabla v|^p \geq \lambda r^{N-p}. \tag{4.1}$$

We explain below how to replace v by a smooth map on bad balls. This construction is possible if the radius r is large enough compared to the length of the minimal connection $L(v)$. At this stage, the choice of the parameter $\lambda > 0$ plays no role whatsoever in the proof.

Proposition 4.1. *Let $N-1 < p < N$. If $B_r(x_0)$ is a bad ball for $v \in \mathcal{R}^{1,p}(\Omega)$ and if*

$$r > 4L(v), \tag{4.2}$$

then one can find $w \in \mathcal{R}^{1,p}(\Omega)$ such that

- (B_1) *w is smooth in $B_r(x_0)$;*
- (B_2) *$w = v$ in $\Omega \setminus B_{2r}(x_0)$;*
- (B_3) *$L(w) \leq L(v)$ and $S(w) \subset S(v)$;*
- (B_4) *$\|w - v\|_{L^p(\Omega)} \leq Cr\|\nabla w - \nabla v\|_{L^p(B_{2r}(x_0))}$;*
- (B_5) *$\|\nabla w - \nabla v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(B_{2r}(x_0))}$.*

Proof. We shall use a strategy similar to the proof of [2, Lemma 1].
We may assume that $\|\nabla v\|_{L^p(B_{2r}(x_0))} > 0$, for otherwise v is constant in $B_{2r}(x_0)$ and there is nothing to prove. By scaling and translation, we may also suppose that $r = 1$ and $x_0 = 0$. Since r satisfies (4.2), by Lemma 3.1 there exists $s \in \left(\frac{3}{2}, 2\right)$ such that

$$\deg\left(v|_{\partial B_s}\right) = 0 \quad \text{and} \quad \|\nabla v\|_{L^p(\partial B_s)} \leq C\|\nabla v\|_{L^p(B_2)}. \tag{4.3}$$

Let

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega \setminus B_s, \\ v\left(s\frac{x}{|x|}\right) & \text{if } x \in B_s. \end{cases}$$

Then, $\tilde{v} \in \mathcal{R}^{1,p}(\Omega)$, $\tilde{v}$ is continuous in $B_s \setminus \{0\}$ and, by the choice of s,

$$\|\nabla \tilde{v}\|_{L^p(B_s)} = C\|\nabla v\|_{L^p(\partial B_s)} \leq C\|\nabla v\|_{L^p(B_2)}.$$

Using the triangle inequality, we then get

$$\begin{aligned} \|\nabla \tilde{v} - \nabla v\|_{L^p(\Omega)} &= \|\nabla \tilde{v} - \nabla v\|_{L^p(B_s)} \\ &\leq \|\nabla \tilde{v}\|_{L^p(B_s)} + \|\nabla v\|_{L^p(B_s)} \leq \widetilde{C}\|\nabla v\|_{L^p(B_2)}. \end{aligned} \tag{4.4}$$

Note that $\tilde{v}$ is continuous in a neighborhood of ∂B_s but $\tilde{v}$ is not necessarily smooth there. By convolution and projection we may modify $\tilde{v}$ to make it smooth near ∂B_s. For this reason, we shall henceforth suppose that we do have $\tilde{v} \in \mathcal{R}^{1,p}(\Omega)$.

By (4.4), the map $\tilde{v}$ satisfies (B_5) but $\tilde{v}$ need not satisfy (B_1) because of its possible singularity at 0. We now use the fact that $\deg(v|_{\partial B_s}) = 0$ to remove that singularity. Indeed, by the Hopf theorem, $v|_{\partial B_s}$ is homotopic to a constant. One can thus find a continuous homotopy $H : [0,1] \times \partial B_s \to S^{N-1}$ such that $H(t,\cdot) = p_0$ if $0 \le t \le \frac{1}{3}$ for some $p_0 \in S^{N-1}$ and $H(t,\cdot) = v|_{\partial B_s}$ if $\frac{2}{3} \le t \le 1$. Making a convolution of H and projecting the resulting map back to S^{N-1}, one can even assume that H belongs to $C^\infty\big([0,1] \times \partial B_s; S^{N-1}\big)$ (recall that H was just assumed to be continuous and needed not be even in $W^{1,p}$). Since H is constant on $[0, \frac{1}{3}]$, for every $0 < \varepsilon < t$, the map

$$w_\varepsilon(x) = \begin{cases} \tilde{v}(x) & \text{if } x \in \Omega \setminus B_\varepsilon, \\ H\big(\frac{|x|}{\varepsilon}, x\big) & \text{if } x \in B_\varepsilon, \end{cases}$$

belongs to $W^{1,p}(\Omega; S^{N-1})$ and is continuous in B_s. Since $w_\varepsilon \to \tilde{v}$ strongly in $W^{1,p}$ as $\varepsilon \to 0$, we can take $\varepsilon > 0$ sufficiently small so that

$$\|\nabla w_\varepsilon - \nabla \tilde{v}\|_{L^p(\Omega)} \le \|\nabla v\|_{L^p(B_2)}. \tag{4.5}$$

Combining (4.4)–(4.5) we deduce that w_ε also satisfies (B_5). Since $w_\varepsilon = v$ outside the ball B_2, by Poincaré's inequality,

$$\|w - v\|_{L^p(\Omega)} = \|w - v\|_{L^p(B_2)} \le C\|\nabla w - \nabla v\|_{L^p(B_2)},$$

and, thus, (B_4) also holds. In order to check property (B_3) we can use (3.4). Indeed, since w_ε is smooth on B_s, $\operatorname{Jac}(w_\varepsilon) = 0$ on B_s. Thus, if $\psi \in C_0^\infty(B_2)$ denotes the function given by Lemma 3.1, then

$$\langle \operatorname{Jac}(w_\varepsilon), \zeta\rangle = \langle \operatorname{Jac}(w_\varepsilon), \psi\zeta\rangle + \langle \operatorname{Jac}(w_\varepsilon), (1-\psi)\zeta\rangle = \langle \operatorname{Jac}(w_\varepsilon), (1-\psi)\zeta\rangle,$$

for every $\zeta \in C_0^\infty(\Omega)$. Since $v = w_\varepsilon$ on $\Omega \setminus B_s$ and $\psi = 1$ on B_s, it follows that

$$\langle \operatorname{Jac}(w_\varepsilon), \zeta\rangle = \langle \operatorname{Jac}(v), (1-\psi)\zeta\rangle.$$

Taking the supremum over all test functions ζ with $\|\nabla\zeta\|_{L^\infty} \le 1$, we deduce from (3.4) that $L(w_\varepsilon) \le L(v)$, which is the desired inequality. □

Remark 4.1. Strictly speaking, in the previous proof we have not used the fact that $B_r(x_0)$ was a bad ball, but we do it now. In fact, since $B_r(x_0)$ is a bad ball,

$$|B_{2r}(x_0)|^{1/p} = \big(\omega_N(2r)^N\big)^{1/p} = \Big(\frac{2^N\omega_N}{\lambda}\Big)^{1/p} r(\lambda r^{N-p})^{1/p}$$
$$\leq Cr\|\nabla v\|_{L^p(B_{2r}(x_0))},$$

where the constant $C > 0$ depends on the choice of λ. We can thus rewrite property (B_5) in the way it will be used in the proof of Theorem 1.7:

(B_5') $\|\nabla w - \nabla v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(A)}$, for some open set $A \subset B_{2r}(x_0)$ such that $|A|^{1/p} \leq Cr\|\nabla v\|_{L^p(B_{2r}(x_0))}$.

5. Replacing u on good balls

Given $\lambda > 0$ and a ball $B_r(x_0)$ such that $B_{2r}(x_0) \subset \Omega$, we say that $B_r(x_0)$ is a good ball for a map $v \in W^{1,p}(\Omega; S^{N-1})$ if

$$\int_{B_{2r}(x_0)} |\nabla v|^p < \lambda r^{N-p}. \tag{5.1}$$

In this section we explain how to replace v by a smooth map on good balls. This construction strongly relies on a suitable choice of the parameter λ.

Proposition 5.1. *Let $N - 1 < p < N$. There exists $\lambda = \lambda(N, p) > 0$ such that if $B_r(x_0)$ is a good ball for $v \in \mathcal{R}^{1,p}(\Omega)$ and if*

$$r > 4L(v), \tag{5.2}$$

then one can find $w \in \mathcal{R}^{1,p}(\Omega)$ such that

- (G_1) *w is smooth in $B_r(x_0)$;*
- (G_2) *$w = v$ in $\Omega \setminus B_{2r}(x_0)$;*
- (G_3) *$L(w) \leq L(v)$ and $S(w) \subset S(v)$;*
- (G_4) *$\|w - v\|_{L^p(\Omega)} \leq Cr\|\nabla w - \nabla v\|_{L^p(B_{2r}(x_0))}$;*
- (G_5) *$\|\nabla w - \nabla v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(A)}$, for some open set $A \subset B_{2r}(x_0)$ such that $|A|^{1/p} \leq Cr\|\nabla v\|_{L^p(B_{2r}(x_0))}$.*

Proof. We can assume that

$$\|\nabla v\|_{L^p(B_{2r}(x_0))} > 0, \tag{5.3}$$

for otherwise v is constant in $B_r(x_0)$ and the conclusion is obvious. By scaling and translation, we may also assume that $r = 1$ and $x_0 = 0$. Since r satisfies (5.2), by Lemma 3.1 there exists $s \in \big(\frac{3}{2}, 2\big)$ such that

$$\deg(v|_{\partial B_s}) = 0 \quad \text{and} \quad \|\nabla v\|_{L^p(\partial B_s)} \leq C\|\nabla v\|_{L^p(B_2)}. \tag{5.4}$$

Since $p > N-1$, it follows from Morrey's estimates that $v|_{\partial B_s}$ is a continuous function and there exists $\lambda_1 > 0$ (depending only on N and p) such that, if

$$\|\nabla v\|_{L^p(\partial B_s)} \le \lambda_1, \tag{5.5}$$

then $v(\partial B_s)$ is a subset of S^{N-1} of diameter at most $1/3$. We then choose λ so that

$$C\lambda^{1/p} = \lambda_1,$$

where C is the constant in (5.4). We denote by $D_{1/3}(\xi_0)$ a closed geodesic disk of S^{N-1} of radius $1/3$ containing $v(\partial B_s)$ and centered at ξ_0. Let $\Phi : S^{N-1} \to S^{N-1}$ be a smooth function such that $\Phi(x) = x$, for all $x \in D_{2/3}(\xi_0)$, $\|\Phi'\|_{L^\infty} \le 2$ and

$$\Phi(S^{N-1}) \subset D_1(\xi_0). \tag{5.6}$$

Let

$$\tilde{v} = \begin{cases} v & \text{in } \Omega \setminus B_s, \\ \Phi \circ v & \text{in } B_s. \end{cases} \tag{5.7}$$

Then, $\tilde{v} \in \mathcal{R}^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla v - \nabla \tilde{v}|^p = \int_{B_s} |1 - \Phi'(v)|^p |\nabla v|^p \le C \int_U |\nabla v|^p, \tag{5.8}$$

where

$$A = \Big\{ x \in B_s \setminus S(v) : v(x) \notin D_{2/3}(\xi_0) \Big\}. \tag{5.9}$$

Since v is continuous on $B_s \setminus S(v)$, A is an open set. We now show that

$$|A|^{1/p} \le C \|\nabla v\|_{L^p(B_2)}. \tag{5.10}$$

For this purpose, consider the function

$$f(x) = \big[3\, d(v(x), \xi_0) - 1\big]^+ \quad \forall x \in B_s,$$

where d denotes the geodesic distance in S^{N-1}. Note that

$$f \ge 1 \text{ on } A, \quad f = 0 \text{ on } \partial B_s \quad \text{and} \quad |\nabla f| \le 3|\nabla v| \text{ a.e.}$$

Thus, by Chebyshev's and Poincaré's inequalities,

$$|A| \le \int_{B_s} |f|^p \le C \int_{B_s} |\nabla f|^p \le 3^p C \int_{B_s} |\nabla v|^p \le 3^p C \int_{B_2} |\nabla v|^p, \tag{5.11}$$

which gives (5.10). Although $\tilde{v}$ need not be continuous in B_1, its image is contained in a geodesic disk of S^{N-1}. A standard argument allows us to replace $\Phi \circ v$ by a function which is smooth in B_1.

We present a detailed proof for the convenience of the reader. We first take a family of nonnegative smooth mollifiers $(\rho_\varepsilon) \subset C_0^\infty(\mathbb{R}^N)$ and $\zeta \in C_0^\infty(B_{3/2})$ such that $\operatorname{supp} \zeta \subset B_{3/2}$ and $\zeta = 1$ on B_1. Consider

$$v_\varepsilon = (1-\zeta)\Phi(v) + \zeta\big[\rho_\varepsilon * \Phi(v)\big] \quad \text{in } B_s. \tag{5.12}$$

Denote by V the convex hull in $\mathbb{R}^N$ of the geodesic disk $D_1(\xi_0)$. By (5.6) we have

$$\Phi(v(x)) \in V \quad \text{and} \quad \big[\rho_\varepsilon * \Phi(v)\big](x) \in V \quad \forall x \in B_s.$$

Thus,

$$v_\varepsilon(x) \in V \quad \forall x \in B_s.$$

On the other hand, we have $|y| \geq 1/2$ for every $y \in V$. Therefore,

$$|v_\varepsilon(x)| \geq \frac{1}{2} \quad \forall x \in B_s. \tag{5.13}$$

In particular,

$$\frac{v_\varepsilon}{|v_\varepsilon|} \to \Phi(v) \quad \text{in } W^{1,p}. \tag{5.14}$$

Take $\varepsilon > 0$ sufficiently small so that

$$\int_{B_s} \Big|\nabla\Big(\frac{v_\varepsilon}{|v_\varepsilon|}\Big) - \nabla\big(\Phi(v)\big)\Big|^p \leq \int_A |\nabla v|^p. \tag{5.15}$$

If the integral in the right-hand side vanishes, take $A \subset B_1$ to be any open set of measure at most $\|\nabla v\|_{L^p(B_2)}^p$ for which the right-hand side is not zero; this is possible in view of (5.3). Let w be the function given by

$$w = \begin{cases} v & \text{in } \Omega \setminus B_s, \\ \dfrac{v_\varepsilon}{|v_\varepsilon|} & \text{in } B_s. \end{cases} \tag{5.16}$$

This function satisfies (B_5) and, by Poincaré's inequality, also satisfies (B_4). The proof of the inequality $L(w) \leq L(v)$ follows the same lines as in the previous lemma. Indeed, since the image of $\frac{v_\varepsilon}{|v_\varepsilon|}$ is contained in a small geodesic disk, all singularities of w in B_s have degree zero. Thus, $\operatorname{Jac}(w) = 0$ in B_s. Thus, if $\psi \in C_0^\infty(B_2)$ denotes the function given by Lemma 3.1, then, for every $\zeta \in C_0^\infty(\Omega)$,

$$\langle \operatorname{Jac}(w_\varepsilon), \zeta\rangle = \langle \operatorname{Jac}(v), (1-\psi)\zeta\rangle,$$

which implies $L(w) \leq L(v)$. □

6. Replacing u on balls near the boundary

The reader probably has noticed that even though the constructions performed on bad balls and on good balls are different, the conclusions of Propositions 4.1 and 5.1 —taking into account Remark 4.1— are the same. The goal of this section is two-fold: to merge both statements and to take into account the possibility of performing the same construction on balls which need not be entirely contained in Ω.

Note that the underlying notions of good balls and bad balls can be adapted to balls which are not entirely contained in Ω in a straightforward way. Actually, there are essentially two types of balls $B_r(x_0)$ one should really take care of: those such that $B_{2r}(x_0) \subset \Omega$, which have been studied in Sections 4 and 5 above, and those such that $x_0 \in \partial\Omega$, which will be our main concern in the proof below. Indeed, the general construction can be always reduced to one of these types.

Proposition 6.1. *Let $N-1 < p < N$. There exists $\delta = \delta(\Omega) > 0$ such that if $v \in \mathcal{R}^{1,p}(\Omega)$ and if*

$$\delta > r > 4L(v), \tag{6.1}$$

then for every $x_0 \in \overline{\Omega}$ there exists $w \in \mathcal{R}^{1,p}(\Omega)$ such that

- (M_1) *w is smooth in $B_r(x_0) \cap \overline{\Omega}$;*
- (M_2) *$w = v$ in $\Omega \setminus B_{8r}(x_0)$;*
- (M_3) *$L(w) \leq L(v)$ and $S(w) \subset S(v)$;*
- (M_4) *$\|w - v\|_{L^p(\Omega)} \leq Cr\|\nabla w - \nabla v\|_{L^p(B_{8r}(x_0)\cap\Omega)}$;*
- (M_5) *$\|\nabla w - \nabla v\|_{L^p(\Omega)} \leq C\|\nabla v\|_{L^p(A)}$, for some open set $A \subset B_{8r}(x_0) \cap \Omega$ such that $|A|^{1/p} \leq Cr\|\nabla v\|_{L^p(B_{8r}(x_0)\cap\Omega)}$.*

Proof. If $B_{2r}(x_0) \subset \Omega$, the conclusion follows from Proposition 4.1 (and Remark 4.1) or from Proposition 5.1 depending on whether $B_r(x_0)$ is a bad ball or a good ball. We may then restrict ourselves to the case where $B_{2r}(x_0) \cap \partial\Omega \neq \emptyset$. We shall reduce the problem to a situation where the ball is centered at some point of $\partial\Omega$. Indeed, since $B_{2r}(x_0) \cap \partial\Omega \neq \emptyset$, there exists $y_0 \in \partial\Omega$ such that $|y_0 - x_0| < 2r$ and, thus,

$$B_r(x_0) \subset B_{3r}(y_0) \quad \text{and} \quad B_{6r}(y_0) \subset B_{8r}(x_0).$$

It thus suffices to construct a map $w \in \mathcal{R}^{1,p}(\Omega)$ such that w is smooth in $B_{3r}(y_0) \cap \overline{\Omega}$, $w = v$ in $\Omega \setminus B_{6r}(y_0)$ and satisfies (M_3)–(M_5).

In what follows, we assume that $B_{6r}(y_0) \cap \partial\Omega$ is flat and thus $B_{6r}(y_0) \cap \Omega$ coincides with a half-ball. By a translation and a scaling argument, we may suppose that $y_0 = 0$ and $r = \frac{1}{3}$. By the Fubini-type argument of Lemma 3.1, one finds $s \in \left(\frac{3}{2}, 2\right)$ such that

$$\|\nabla v\|_{L^p(\partial B_s \cap \Omega)} \le C\|\nabla v\|_{L^p(B_2)}, \tag{6.2}$$

and ∂B_s does not intersect any of the segments $[p_i, n_i]$, where the points p_i and n_i denote the singularities of v arranged so as to satisfy (3.6).

If B_1 is a *bad ball* for v, in the sense that

$$\int_{B_2 \cap \Omega} |\nabla v|^p \ge \lambda,$$

for some parameter $\lambda > 0$ to be chosen later on, then we proceed as in the proof of Proposition 4.1 and define

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \overline{\Omega} \setminus B_s, \\ v\left(s\frac{x}{|x|}\right) & \text{if } x \in B_s \cap \overline{\Omega}, \end{cases}$$

which is continuous except possibly at 0 and satisfies

$$\|\nabla \tilde{v} - \nabla v\|_{L^p(\Omega)} \le C\|\nabla v\|_{L^p(B_2 \cap \Omega)}. \tag{6.3}$$

Since $u|_{\partial B_s \cap \Omega}$ is necessarily homotopic to a constant map (recall that $\partial B_s \cap \Omega$ is a half-sphere, which is topologically trivial), one can remove that singularity at 0 as in Proposition 4.1 without losing property (6.3). Thus, we get a map $w \in \mathcal{R}^{1,p}(\Omega)$ which is now smooth on $B_s \cap \overline{\Omega}$ and

$$\|\nabla w - \nabla v\|_{L^p(\Omega)} \le C\|\nabla v\|_{L^p(B_2 \cap \Omega)}.$$

Since B_1 was assumed to be a bad ball, as in Remark 4.1 we have

$$|B_2 \cap \Omega|^{1/p} \le C_\lambda \|\nabla v\|_{L^p(B_2 \cap \Omega)},$$

and thus w satisfies (M_5) with $A = B_{6r}(y_0)$ (which corresponds to B_2 after translation and scaling). Property (M_4) just follows from Poincaré's inequality. Finally, since ∂B_s does not intersect any of the segments $[p_i, n_i]$, one deduces that $L(w) \le L(v)$. Thus, w satisfies all the required properties.

On the other hand, if B_1 is a *good ball* for v, in the sense that

$$\int_{B_2 \cap \Omega} |\nabla v|^p < \lambda,$$

then in view of (6.2), $\|\nabla v\|_{L^p(\partial B_s \cap \Omega)} < C\lambda^{1/p}$. Therefore, by Morrey's estimates we can fix some $\lambda > 0$ sufficiently small (depending on N and p) so that $v(\partial B_s \cap \Omega)$ is contained in a small geodesic disk of S^{N-1}. One can then proceed exactly as in the proof of Proposition 5.1 by taking a family of convolutions (ρ_ε) supported in $B_1 \cap \Omega$; this way the function v_ε remains well defined and the conclusion follows.

We now deal with the case where $B_{6r}(y_0) \cap \partial\Omega$ is not necessarily flat. By choosing $\delta > 0$ sufficiently small (depending on Ω) it is possible to find a diffeomorphism Φ such that the image of $B_{3r}(y_0) \cap \Omega$ is contained in the half-ball B_{3r}^+ and the image of $B_{6r}(y_0) \cap \Omega$ contains the half-ball B_{6r}^+. We can then apply the previous construction to the map $v \circ \Phi^{-1}$. The proof of the proposition is complete. □

7. PROOFS OF THEOREMS 1.5 AND 1.7

Proof of Theorem 1.7. Let us assume momentarily that we have proved (1.13) for maps $u \in \mathcal{R}^{1,p}(\Omega)$. We show that this implies a similar estimate for every

$$u \in W^{1,p}(\Omega; S^{N-1}).$$

Indeed, given $u \in W^{1,p}(\Omega; S^{N-1})$ we consider two separate cases, depending on whether $L(u) = 0$ or $L(u) > 0$. We first assume that $L(u) = 0$. Taking a sequence $(u_n) \subset \mathcal{R}^{1,p}(\Omega)$ such that $u_n \to u$ strongly in $W^{1,p}$, then by continuity of the length of the minimal connection,

$$L(u_n) \to L(u) = 0.$$

By (1.13) applied to u_n and Lebesgue's dominated convergence theorem,

$$\inf \Big\{ \|u_n - \varphi\|_{W^{1,p}} \ ; \ \varphi \in C^\infty(\overline{\Omega}; S^{N-1}) \Big\} \to 0.$$

Therefore, there exists a sequence $(\varphi_n) \subset C^\infty(\overline{\Omega}; S^{N-1})$ such that $\varphi_n \to u$ strongly in $W^{1,p}$. Hence, u satisfies (1.13) with $A = \emptyset$. On the other hand, if $L(u) > 0$, then we first take an open set $A_1 \subset \Omega$ such that

$$\|\nabla u\|_{L^p(A_1)} > 0 \quad \text{and} \quad |A_1|^{1/p} \le L(u) \|\nabla u\|_{L^p(\Omega)},$$

and then, by Theorem 2.1, one can choose $v \in \mathcal{R}^{1,p}(\Omega)$ such that

$$\|u - v\|_{W^{1,p}(\Omega)} \le \|\nabla u\|_{L^p(A_1)}.$$

We may also assume that v satisfies

$$L(v) \le 2L(u) \quad \text{and} \quad \|\nabla v\|_{L^p(\Omega)} \le 2\|\nabla u\|_{L^p(\Omega)}.$$

Since by assumption estimate (1.13) holds for v, there exists

$$\varphi \in C^\infty(\overline{\Omega}; S^{N-1})$$

such that

$$\|v - \varphi\|_{W^{1,p}(\Omega)} \le 2C \|\nabla v\|_{L^p(A_2)},$$

where $A_2 \subset \Omega$ is an open set satisfying $|A_2| \leq CL(v)\|\nabla v\|_{L^p(\Omega)}$. We then have

$$\begin{aligned}\|u-\varphi\|_{W^{1,p}} &\leq \|u-v\|_{W^{1,p}} + \|v-\varphi\|_{W^{1,p}} \\ &\leq \|\nabla u\|_{L^p(A_1)} + C\|\nabla v\|_{L^p(A_2)} \\ &\leq \|\nabla u\|_{L^p(A_1)} + C\big(\|\nabla u\|_{L^p(A_2)} + \|\nabla u - \nabla v\|_{L^p(A_2)}\big) \\ &\leq \|\nabla u\|_{L^p(A_1)} + C\big(\|\nabla u\|_{L^p(A_2)} + \|\nabla u\|_{L^p(A_1)}\big) \\ &\leq (1+2C)\|\nabla u\|_{L^p(A_1\cup A_2)},\end{aligned}$$

where

$$\begin{aligned}|A_1\cup A_2|^{1/p} \leq |A_1|^{1/p} + |A_2|^{1/p} &\leq L(u)\|\nabla u\|_{L^p(\Omega)} + CL(v)\|\nabla v\|_{L^p(\Omega)} \\ &\leq L(u)\|\nabla u\|_{L^p(\Omega)} + 4CL(u)\|\nabla u\|_{L^p(\Omega)} \\ &= (1+4C)L(u)\|\nabla u\|_{L^p(\Omega)}.\end{aligned}$$

Thus, u also satisfies an estimate of the type (1.13).

In view of the above it suffices to establish (1.13) for maps $u \in \mathcal{R}^{1,p}(\Omega)$. Let $\delta > 0$ be the quantity given by Proposition 6.1, depending only on Ω. We consider two separate cases.

Case 1. $4L(u) < \delta$. Let $r > 0$ be such that $4L(u) < r < \delta$. We can cover Ω with balls $(B_r(x_i))_{i\in I}$ in such a way that, for every $i \in I$, $x_i \in \overline{\Omega}$ and each ball $B_{8r}(x_i)$ intersects at most θ balls $B_{8r}(x_j)$, where θ depends only on the dimension N. We can thus split the set of indices I as $I = I_1 \cup \cdots \cup I_{\theta+1}$ so that for any $i = 1, \ldots, \theta+1$ and any distinct indices $j_1, j_2 \in I_i$ we have $B_{8r}(x_{j_1}) \cap B_{8r}(x_{j_2}) = \emptyset$.

Starting from $u_0 = u$, we construct maps $u_1, \ldots, u_{\theta+1} \in W^{1,p}(\Omega; S^{N-1})$ inductively as follows. Given $k \geq 0$ and u_k we apply Proposition 6.1 to the map u_k and to each ball $B_r(x_i)$ with $i \in I_{k+1}$ until we exhaust I_{k+1}; denote by u_{k+1} the map obtained by this procedure. Since the balls $\big(B_{8r}(x_i)\big)_{i\in I_{k+1}}$ are disjoint, by properties (M_4)–(M_5) we have

$$\|u_{k+1} - u_k\|_{L^p(\Omega)} \leq Cr\|\nabla u_{k+1} - \nabla u_k\|_{L^p(\Omega)} \tag{7.1}$$

$$\|\nabla u_{k+1} - \nabla u_k\|_{L^p(\Omega)} \leq C\|\nabla u_k\|_{L^p(E_k)}, \tag{7.2}$$

for some open set $E_k \subset \Omega$ such that $|E_k|^{1/p} \leq Cr\|\nabla u_k\|_{L^p(\Omega)}$; E_k is the union of all sets A arising from Proposition 6.1.

By induction, it follows from (7.1)–(7.2) that for every $k = 1, \ldots, \theta+1$ we have

$$\|u_k - u\|_{L^p(\Omega)} \leq C_k r\|\nabla u\|_{L^{2p}(\Omega)} \tag{7.3}$$

$$\|\nabla u_k - \nabla u\|_{L^p(\Omega)} \le C_k \|\nabla u\|_{L^p(F_k)}, \tag{7.4}$$

where $F_k = \bigcup_{j=0}^{k-1} E_j$. We first prove (7.4). Since the conclusion is clear if $k = 1$, we may assume that (7.3) holds for some $k \ge 1$. We then have

$$\begin{aligned}
\|\nabla u_{k+1} - \nabla u\|_{L^p(\Omega)} &\le \|\nabla u_{k+1} - \nabla u_k\|_{L^p(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega)} \\
\text{(by (7.2))} \quad &\le C\|\nabla u_k\|_{L^p(E_k)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega)} \\
\text{(by triangle inequality)} \quad &\le C\|\nabla u\|_{L^p(E_k)} + (1+C)\|\nabla u_k - \nabla u\|_{L^p(\Omega)} \\
\text{(by induction)} \quad &\le C\|\nabla u\|_{L^p(E_k)} + (1+C)C_k\|\nabla u\|_{L^p(F_{k-1})} \\
&\le \big[C + (1+C)C_k\big]\|\nabla u\|_{L^p(F_{k-1}\cup E_k)}.
\end{aligned}$$

This establishes (7.4). Combining (7.1) and (7.4), one gets (7.3). Note, in addition, that the set F_k satisfies $|F_k|^{1/p} \le C_k r\|\nabla u\|_{L^p(\Omega)}$. Indeed, proceeding by induction we have

$$\begin{aligned}
|F_{k+1}|^{1/p} &\le |F_k|^{1/p} + |E_k|^{1/p} \\
\text{(by estimate on } |E_k|) &\le |F_k|^{1/p} + Cr\|\nabla u_k\|_{L^p(\Omega)} \\
\text{(by induction)} &\le C_k r\|\nabla u\|_{L^p(\Omega)} + Cr\|\nabla u_k\|_{L^p(\Omega)} \\
\text{(by triangle inequality)} &\le C_k r\|\nabla u\|_{L^p(\Omega)} + Cr\big(\|\nabla u\|_{L^p(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega)}\big) \\
\text{(by (7.4))} &\le C_k r\|\nabla u\|_{L^p(\Omega)} + Cr\big(\|\nabla u\|_{L^p(\Omega)} + C_k\|\nabla u\|_{L^p(F_k)}\big) \\
&\le (C_k + C(1+C_k))r\|\nabla u\|_{L^p(\Omega)},
\end{aligned}$$

which gives the estimate for the sets $|F_k|$. Since the balls $(B_r(x_i))_{i\in I}$ cover Ω and we have swept away all the singularities of u from these balls, the map $u_{\theta+1}$ is smooth. We have, thus, obtained for every $r > 4L(u)$ a map $\varphi_r \in C^\infty(\overline{\Omega}; S^{N-1})$, namely $u_{\theta+1}$, such that

$$\|\varphi_r - u\|_{L^p(\Omega)} \le Cr\|\nabla u\|_{L^{2p}(\Omega)} \tag{7.5}$$

$$\|\nabla\varphi_r - \nabla u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(A_r)}, \tag{7.6}$$

where $A_r \subset \Omega$ is an open set such that $|A_r|^{1/p} \le Cr\|\nabla u\|_{L^p(\Omega)}$. If $L(u) = 0$, it follows from dominated convergence that $\varphi_r \to u$ strongly in $W^{1,p}$ and thus (1.13) holds with $A = \emptyset$. Otherwise, $L(u) > 0$, in which case we can take $r \approx 4L(u)$.

Case 2. $4L(u) \ge \delta$. We show the conclusion holds by taking $A = \Omega$. Indeed, by an easy variant of Poincaré's inequality, there exists $\alpha_u \in S^{N-1}$ such that

$$\|u - \alpha_u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \tag{7.7}$$

where the constant $C > 0$ does not depend on u; thus,

$$\|u - \alpha_u\|_{W^{1,p}(\Omega)} \leq (1 + C)\|\nabla u\|_{L^p(\Omega)}.$$

On the other hand, by Hölder's inequality,

$$L(u) \leq |\Omega|^{1-\frac{N-1}{p}} \|\nabla u\|^{N-1}_{L^p(\Omega)}.$$

Thus,

$$L(u)\|\nabla u\|_{L^p(\Omega)} \geq \frac{\big(L(u)\big)^{\frac{N}{N-1}}}{|\Omega|^{\frac{1}{N-1}-\frac{1}{p}}} \geq \frac{(\delta/4)^{\frac{N}{N-1}}}{|\Omega|^{\frac{1}{N-1}}} |\Omega|^{1/p} = C_0|\Omega|^{1/p}, \tag{7.8}$$

where $C_0 > 0$ is a constant depending on N and Ω.

In both cases, we have obtained estimate (1.13). The proof of the theorem is complete. □

Proof of Theorem 1.5. The implication ($\Leftarrow$) follows from Theorem 1.7 if $N - 1 < p < N$ or from Theorem 1.6 if $p = N - 1$. To prove the converse, let $(\varphi_n) \subset C^\infty(\overline{\Omega}; S^{N-1})$ be a sequence such that

$$\varphi_n \to u \quad \text{strongly in } W^{1,p}.$$

For every $n \geq 1$, we have $\operatorname{Jac}(\varphi_n) = 0$; thus, $L(\varphi_n) = 0$. In view of Proposition 2.1, this implies $L(u) = 0$ or, equivalently, $\operatorname{Jac}(u) = 0$ in $\mathcal{D}'(\Omega)$. □

Acknowledgments. The authors would like to thank P. Bousquet for interesting discussions and the Department of Mathematics of the Université François Rabelais (Tours, France), where part of this work was carried out. The second author was supported by the Fonds de la Recherche scientifique–FNRS and by the Fonds spécial de Recherche.

References

[1] F. Bethuel, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 7 (1990), 269–286.

[2] F. Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math., 167 (1991), 153–206.

[3] F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein, *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, Nematics (Orsay, 1990), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 332 (1991), 15–23.

[4] F. Bethuel and X.M. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal., 80 (1988), 60–75.

[5] J. Bourgain, H. Brezis, and P. Mironescu, *$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation*, Publ. Math. Inst. Hautes Études Sci., 99 (2004), 1–115.

[6] P. Bousquet, A.C. Ponce, and J. Van Schaftingen, *A case of density in* $W^{2,p}(M;N)$, C. R. Math. Acad. Sci. Paris, 346 (2008), 735–740.
[7] H. Brezis, J.-M. Coron, and E.H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys., 107 (1986), 649–705.
[8] H. Brezis and L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) 1 (1995), 197–263.
[9] A.C. Ponce, *On the distributions of the form* $\sum_i(\delta_{p_i} - \delta_{n_i})$, J. Funct. Anal., 210 (2004), 391–435.
[10] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom., 18 (1983), 253–268.