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TRAVELING WAVES FOR THE WHITHAM EQUATION

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Abstract. The existence of traveling waves for the original Whitham equation is investigated. This equation combines a generic nonlinear quadratic term with the exact linear dispersion relation of surface water waves of finite depth. It is found that there exist small-amplitude periodic traveling waves with sub-critical speeds. As the period of these traveling waves tends to infinity, their velocities approach the limiting long-wave speed c_0. It is also shown that there can be no solitary waves with velocities much greater than c_0. Finally, numerical approximations of some periodic traveling waves are presented. It is found that there is a periodic wave of greatest height $\sim 0.642h_0$. Periodic traveling waves with increasing wavelengths appear to converge to a solitary wave.

1. Introduction

The study of waves on the surface of a fluid has been a source of intriguing mathematical problems for a long time. When studying such waves, viscosity is often neglected, so that the governing equations are the nonlinear Euler equations, supplemented by a set of nonlinear boundary conditions at the unknown fluid surface. This set of equations is commonly known as the water-wave problem. Of special interest is the study of permanent progressive waves, such as solitary or traveling periodic waves. These waves which are also called steady waves propagate without changing their shape over time.

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An interesting line of research on this topic was initiated by the discovery of the solitary wave by John Scott Russell [21]. His observations and experiments gave an impetus to finding a mathematical formulation capable of describing such waves. The Korteweg-de Vries (KdV) equation

$$\eta_t + c_0\,\eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x + \tfrac{1}{6}c_0 h_0^2\,\eta_{xxx} = 0 \tag{1.1}$$

is a simplified model equation for waves at the water surface which includes the essential effects of nonlinearity and dispersion [6, 16]. Balancing these two effects is the basic mechanism behind the existence of both solitary-wave solutions and periodic traveling waves. Equation (1.1) is given in dimensional form, $c_0 := \sqrt{gh_0}$ is the limiting long-wave speed, h_0 denotes the undisturbed water depth (assuming a flat bottom), and g is the gravitational constant. The function $\eta(x,t)$ describes the deflection of the fluid surface from the rest position at a point x at time t. As explained in [12], the equation is a valid approximation describing the evolution of surface water waves in the case when the waves are long compared to the undisturbed depth h_0 of the fluid, and the average amplitude of the waves is small when compared to h_0. In addition, transverse effects are assumed to be weak.

The success of the KdV equation in describing traveling waves and the discovery of its completely integrable Hamiltonian structure has led to an intense study of this equation for the last four decades. The mathematical theory for the KdV equation has reached a very advanced level, with a solid theory of well-posedness in place, and a sound understanding of the stability properties of solitary and traveling waves [1, 2, 4, 5, 14, 19]. However, as a model for water waves, the KdV equation has some shortcomings concerning the propagation of shorter waves. The linear phase velocity in the KdV equation is given by

$$c(\xi) = c_0 - \tfrac{1}{6}c_0 h_0^2 \xi^2, \tag{1.2}$$

where $\xi = \frac{2\pi}{\lambda}$ is the wave number, and λ is the wavelength. This is a second-order approximation to the phase velocity

$$c(\xi) = \tfrac{\omega}{\xi} = \sqrt{\tfrac{g \tanh \xi h_0}{\xi}} \tag{1.3}$$

of the linearized water-wave problem. The latter expression for $c(\xi)$ appears when the full water-wave problem is linearized around the vanishing (irrotational) solution, and solutions of the form $\exp(ix\xi - i\omega t)$ are sought [12, 25]. As is visible in Figure 1, the linearized KdV equation does not give a faithful representation of the full dispersion relation even for intermediate values of the wave number ξ. This problem with the KdV equation as a model for

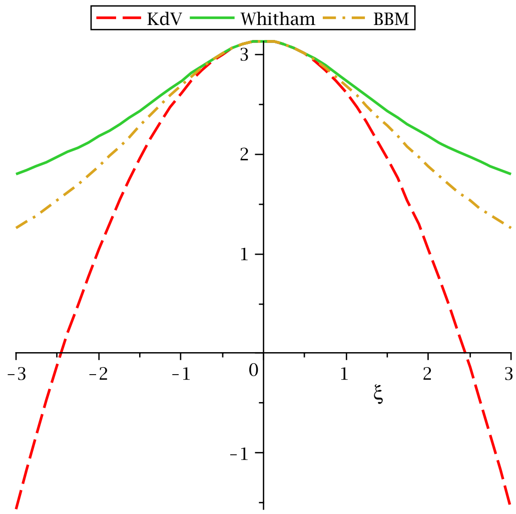

FIGURE 1. Comparison of linear wave speeds $c = \omega/\xi$ for the KdV, Whitham, and BBM equations. Here $g = 9.81$ and $h_0 = 1$. The maximum of all three graphs is at $c_0 = \sqrt{gh_0}$.

water waves was recognized early on, and has been remedied somewhat by the introduction of the regularized long-wave (BBM) equation

$$\eta_t + c_0\,\eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x - \tfrac{1}{6}h_0^2\,\eta_{xxt} = 0 \tag{1.4}$$

by Peregrine [20] and Benjamin, Bona and Mahoney [3]. The linear phase velocity of (1.4) is given by

$$c(\xi) = \frac{c_0}{1 + \frac{1}{6}h_0^2\xi^2}, \tag{1.5}$$

which is qualitatively closer to (1.3) than (1.2). A comprehensive review of these modeling issues was given in [3].

Also recognizing the problems of the KdV equation as a model equation for water waves, Whitham introduced what is now called the Whitham equation [24]. The idea was to use the exact form of the phase velocity (1.3) instead of a second-order approximation like (1.2) or (1.5). The equation proposed by Whitham has the form

$$\eta_t + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x + K_{h_0} * \eta_x = 0, \tag{1.6}$$

where the convolution is in the x-variable. The equation is written in dimensional form, with $\eta(x,t)$ representing the deflection of the surface from rest, just as in the KdV equation. The convolution kernel is given by

$$K_{h_0} := \mathcal{F}^{-1}\Big(\sqrt{\tfrac{g \tanh h_0 \xi}{\xi}}\Big), \tag{1.7}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform to be defined by (2.1) in Section 2.

One of the reasons that (1.6) was advocated by Whitham in [24] is the possibility of wave breaking and peaking. Wave breaking describes a situation in which the spatial derivative of the function η becomes unbounded in finite time, while η itself remains bounded. The term peaking means that a wave forms a sharp crest or peak, such as a stagnation point in the full water-wave problem. To analyze such properties, Whitham also proposed the investigation of the kernel

$$K = \tfrac{\pi}{4} \exp\left(-\tfrac{\pi}{2}|x|\right), \tag{1.8}$$

in (1.6), instead of the kernel K_{h_0}. The kernel K matches the asymptotic behavior of K_{h_0} [25], and has certain mathematical advantages over (1.7), such as not having a singularity at the origin. Moreover, this approximation gives rise to a differential equation, the so-called Burgers-Poisson equation [10]. The properties of (1.8) were exploited by Seliger [22], who presented a formal but ingenious argument that for this simplified kernel wave breaking is possible. A full analysis of this fact was later given by Constantin and Escher in [9]. A proof of wave breaking for equation (1.6) for a class of kernels, including (1.7), has also been provided by Naumkin and Shishmarev [18].

As regards peaking, it is clear that traveling-wave solutions to the KdV and regularized long-wave equation are always smooth. However, as was shown by a formal argument in [24], the Whitham equation features traveling waves with a cusp. In this respect, the Whitham equation captures the peaking phenomenon of the Stokes waves for the full water-wave problem.

Interest in breaking, peaking and other phenomena connected with (1.6) has spawned a large amount of mathematical work. The monograph by Naumkin and Shishmarev [18] is devoted entirely to equations like (1.6). While this work is mainly focused on problems of time evolution, traveling solutions of the equation (1.6) with the kernel (1.8) were studied in [11] and [26]. In spite of these advances, it appears that the question whether the original Whitham equation (with the kernel K_{h_0}) admits a nontrivial traveling-wave solution has remained open. In the present article, an answer to this question is provided as the existence of periodic traveling waves is established. While no proof of the existence of solitary waves is given, their existence is indicated by numerical experiment.

The plan of the paper is as follows. In Section 3, we make use of the Crandall-Rabinowitz local bifurcation theorem to prove the existence of small-amplitude periodic traveling waves. A similar treatment was outlined by Gabov in [11], but for the exact kernel (1.7) no proof was given. In Section 4, we prove *a priori* continuity of bounded traveling-wave solutions. Section 5 is on non-existence. It is shown that for large velocities there can be no continuous solitary-wave solutions of the Whitham equation. In Section 6, we compute numerical approximations of both periodic traveling waves and solitary waves. According to our computations in Section 6, there is a highest wave, which appears not to be smooth at the center, and which has a waveheight of approximately $0.642h_0$.

2. Preliminaries

In this article, the standard notation of mathematical analysis is used. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the set of measurable real-valued functions of a real variable whose p^{th} powers are Lebesgue integrable over a subset $\Omega \subseteq \mathbb{R}$. If $f \in L^p(\Omega)$, its norm is given by $\|f\|^p_{L^p(\Omega)} := \int_\Omega |f|^P \, dx$. The space $L^\infty(\Omega)$ consists of all measurable, essentially bounded functions with norm $\|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega} |f(x)|$. We define the Fourier transform $\mathcal{F}$ of a function $f \in L^1(\mathbb{R})$ by

$$\mathcal{F}f(\xi) := \int_{-\infty}^{\infty} f(x) \exp(-ix\xi) \, dx,$$

and the inverse Fourier transform $\mathcal{F}^{-1}$ by

$$\mathcal{F}^{-1}f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \exp(ix\xi) \, d\xi, \tag{2.1}$$

for any $\hat{f} \in L^1(\mathbb{R})$. We shall also use the notation $\hat{f} := \mathcal{F}f$. The Fourier coefficients of $2L$-periodic functions on $\mathbb{R}$ are defined by

$$\hat{f}_k := \int_{-L}^{L} f(x) \exp\left(-ix\tfrac{k\pi}{L}\right) dx.$$

We write

$$f(x) \sim \frac{1}{2L} \sum_{k\in\mathbb{Z}} \hat{f}_k \exp\left(ix\tfrac{k\pi}{L}\right)$$

to indicate that, under certain conditions on f, this infinite trigonometric series converges to f pointwise, uniformly, or in norm. For example, if

$f \in L^p((-L,L))$, $p > 1$, then the Carleson–Hunt theorem [13] guarantees that the series converges to $f(x)$ almost everywhere. If, in addition, $f(x)$ is an even function, the series can be written as

$$f(x) \sim \frac{1}{2L}\hat{f}_0 + \frac{1}{L}\sum_{k=1}^{\infty} \hat{f}_k \cos\left(x\tfrac{k\pi}{L}\right) = \frac{1}{L}{\sum_{k=0}^{\infty}}' \hat{f}_k \cos\left(x\tfrac{k\pi}{L}\right),$$

where the prime indicates that the first term of the sum is multiplied by $1/2$.

Next, we turn to recording some elementary properties of the Whitham kernel, K_{h_0}, and its Fourier transform. It can be seen that the function $\sqrt{g(\tanh h_0\xi)/\xi}$ is real analytic, even, and strictly decreasing on $(0,\infty)$. Moreover, $\sqrt{g(\tanh h_0\xi)/\xi}$ takes the following limits:

$$\lim_{\xi\to 0}\sqrt{\tfrac{g\tanh h_0\xi}{\xi}} = \sqrt{gh_0}, \qquad \lim_{\xi\to\infty}\sqrt{\tfrac{g\tanh h_0\xi}{\xi}} = 0.$$

Consequently,

$$\int_{-\infty}^{\infty} K_{h_0}(x)\,dx = \sqrt{gh_0},$$

and

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \sqrt{gh_0}\left\|\mathcal{F}^{-1}\left(\sqrt{\tfrac{\tanh\xi}{\xi}}\right)\right\|_{L^1(\mathbb{R})}. \tag{2.2}$$

Thus, it can be shown that $K_{h_0} \in L^1(\mathbb{R})$ in the following way. The substitution of variables $y = x\xi$ and partial integration shows that the growth of $\mathcal{F}^{-1}(\sqrt{(\tanh\xi)/\xi})$ is of order $x^{-1/2}$ as $x \to 0$ (for a rigorous proof of this fact, cf. Section 4).

Since the function $\sqrt{(\tanh\xi)/\xi}$ is analytic, the inverse Fourier transform has rapid decay. Thus, splitting the integral according to

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \int_{|x|\le 1} |K_{h_0}(x)|\,dx + \int_{|x|\ge 1} |K_{h_0}(x)|\,dx,$$

it is plain that K_{h_0} has finite $L^1(\mathbb{R})$-norm. In fact, this argument establishes more generally that $K_{h_0} \in L^p(\mathbb{R})$ for $1 \le p < 2$.

Since the existence of traveling waves is in view, we make the usual ansatz $\eta(x,t) = \phi(x-ct)$, with $c > 0$ being the propagation speed of a right-going traveling wave. Using this form, the equation (1.6) transforms into

$$-c\,\phi' + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\phi\,\phi' + K_{h_0} * \phi' = 0,$$

which may be integrated to

$$-c\,\phi + \tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi^2 + K_{h_0} * \phi = B, \tag{2.3}$$

for some real constant B. For solutions $\phi \in L^2(\mathbb{R})$, it appears that the convolution $K_{h_0} * \phi$ is in $L^2(\mathbb{R})$ since K_{h_0} is in $L^1(\mathbb{R})$. Therefore, the left-hand side must vanish as $|x| \to \infty$, and we shall consider here only the case when $B = 0$. The scaling $\phi \mapsto \frac{3}{4}\frac{c_0}{h_0\,c}\phi$ then yields the normalized weak Whitham equation

$$\phi = \phi^2 + \tfrac{1}{c} K_{h_0} * \phi. \tag{2.4}$$

3. Existence of periodic traveling waves

Theorem 3.1. *For a given $L > 0$ and a given depth $h_0 > 0$, there exists a local bifurcation curve of, $2L$-periodic, even and continuous solutions of the weak Whitham equation (2.4). Those solutions are perturbations in the direction of $\cos(\pi x/L)$, and their wave speed at the bifurcation point is determined by the full dispersion relation*

$$c^* = \sqrt{\frac{gL \tanh\left(h_0\pi/L\right)}{\pi}}. \tag{3.1}$$

In particular, as $L \to \infty$ we have $c^* \to \sqrt{gh_0}$.

We shall make use of the Crandall-Rabinowitz bifurcation theorem [15, Section I.5], which we state in a form suitable for our purposes. Here, and elsewhere, D_c is the Fréchet derivative with respect to c.

Lemma 3.2. *Let W be a Banach algebra, $c \in I := (0, \sqrt{gh_0})$ a parameter, and let $\mathcal{L} : W \to W$ be the Fréchet derivative at 0 with respect to u of the Whitham map*

$$u \mapsto u - \tfrac{1}{c} K_{h_0} * u - u^2. \tag{3.2}$$

Suppose that $\mathcal{L}$ and $D_c\mathcal{L}$ exist and are continuous from $W \to W$, and that for some $c^* \in I$ the following conditions hold:

i) $\dim\ker(\mathcal{L}) = 1$;
ii) $W = \ker(\mathcal{L}) \oplus \operatorname{ran}(\mathcal{L})$;
iii) $(D_c\mathcal{L})\ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = 0$.

Then there exist $\varepsilon > 0$ and a continuous bifurcation curve $\{(c_s, \phi_s) \colon |s| < \varepsilon\}$ with $c_s|_{s=0} = c^*$, such that ϕ_0 is the vanishing solution of (2.4), and $\{\phi_s\}_s$ is a family of nontrivial solutions with corresponding wave speeds $\{c_s\}_s$. Moreover, we have

$$\operatorname{dist}(\phi_s, \ker(\mathcal{L})) = o(s) \quad \text{in } W.$$

Remark 3.3. We remark that our method works equally well for the generalized Whitham equation

$$\eta_t + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta^p\,\eta_x + K_{h_0} * \eta_x = 0,$$

whenever $1 \le p \in \mathbb{Z}$. In that case, the Whitham map becomes $u \mapsto u - \frac{1}{c}K_{h_0} * u - u^p$. Since the linearization around the vanishing solution is the same for this map as for (3.2), all that is needed to check is the continuity of the full map in W. As we shall see in the proof of Theorem 3.1, our choice of W is an algebra, so that continuity is evident.

Before we turn to the proof, let us explain how the convolution operator $K_{h_0}*$ acts on periodic functions. Suppose then that $f \in L^\infty(\mathbb{R})$ is periodic and even. Since K_{h_0} is in $L^1(\mathbb{R})$, we can write the integral

$$\begin{aligned}\int_{-\infty}^{\infty} K_{h_0}(x-y)f(y)\,dy &= \sum_{k=-\infty}^{\infty} \int_{-L}^{L} K_{h_0}(x-y+2kL)f(y)\,dy \\ &= \int_{-L}^{L} \Big(\sum_{k=-\infty}^{\infty} K_{h_0}(x-y+2kL) \Big) f(y)\,dy =: \int_{-L}^{L} A(x-y)f(y)\,dy.\end{aligned}$$

Inspection of the definition of $A(x)$ shows that it is $2L$-periodic, even, and continuous on $[-L,L]\setminus\{0\}$. Moreover, a straightforward proof using Minkowski's inequality shows that $A(x)$ belongs to $L^p(-L,L)$, for $1 \le p < 2$. Therefore, according to the Carleson–Hunt theorem [13], $A(x)$ can be approximated pointwise by its Fourier series. Thus, we have

$$A(x) = \frac{1}{L} \sum_{k=0}^{\infty}{}' \hat{A}_k \cos\left(\tfrac{k\pi x}{L}\right) \;\; a.e.,$$

where the Fourier coefficients of A are given by

$$\begin{aligned}\hat{A}_k &= \int_{-L}^{L} \sum_{j=-\infty}^{\infty} K_{h_0}(x+2jL) \exp\left(-\tfrac{ixk\pi}{L}\right) dx \\ &= \sum_{j=-\infty}^{\infty} \int_{-L}^{L} K_{h_0}(x+2jL) \exp\left(-\tfrac{i(x+2jL)k\pi}{L}\right) dx \\ &= \int_{-\infty}^{\infty} K_{h_0}(x) \exp\left(-\tfrac{ixk\pi}{L}\right) dx = \hat{K}_{h_0}(\tfrac{k\pi}{L}).\end{aligned} \tag{3.3}$$

Thus, it appears that the periodic problem is given by the same multiplier as the problem on the line, and we have the representation

$$K_{h_0} * f(x) = \frac{1}{L} \sum_{k=0}^{\infty}{}' \hat{f}_k \hat{A}_k \cos\left(\tfrac{k\pi x}{L}\right) = \frac{1}{L} \sum_{k=0}^{\infty}{}' \hat{f}_k \hat{K}_{h_0}(\tfrac{k\pi}{L}) \cos\left(\tfrac{k\pi x}{L}\right). \tag{3.4}$$

Proof of Theorem 3.1. Looking for a traveling solution we consider first the linearized equation

$$\mathcal{L}\psi := \psi - \tfrac{1}{c} K_{h_0} * \psi = 0.$$

For $\psi \in L^\infty(\mathbb{R})$ we see that

$$\hat{\psi}\left(1 - \tfrac{1}{c}\sqrt{\tfrac{g \tanh h\xi}{\xi}}\right) = 0.$$

This makes sense in the setting of distributions. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing functions (see [23]). Then $\frac{1}{c}\widehat{K_{h_0} * \psi}$, $\hat{\psi}$ and $\frac{1}{c}\hat{K}_{h_0}$ all exist in $\mathcal{S}'(\mathbb{R})$. Since $1 - \sqrt{g \tanh(h\xi)/\xi}$ is in $L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$, the product of $\hat{\psi}$ and this function is well defined acting on functions in $\mathcal{S}(\mathbb{R})$. The convolution theorem [23, Section 4.3] then implies that $\frac{1}{c}\widehat{K_{h_0} * \psi}(v) = \frac{1}{c}(\hat{\psi}\hat{K}_{h_0})(v)$ for any $v \in \mathcal{S}(\mathbb{R})$.

Now, if $c < \sqrt{gh_0}$ the support of $\hat{\psi}$ is contained in $\{\pm\xi_0\}$, where $\xi_0 := \xi_0(c, h_0)$ is the unique positive root of $g \tanh h_0\xi = c^2\xi$; if $c = \sqrt{gh_0}$ then $\operatorname{supp}(\hat{\psi}) \subseteq \{0\}$; and if $c > \sqrt{gh_0}$ it follows that $\hat{\psi}(\xi) = 0$ for all ξ.

The nontrivial even solutions of the linear problem are thus given by

$$\begin{cases} \psi(x) = C, & c = \sqrt{gh_0}, \\ \psi(x) = C\cos(\xi_0 x), & c < \sqrt{gh_0}, \end{cases} \tag{3.5}$$

where $C \in \mathbb{R}$ can be any nonzero constant. Note that the constant solutions different from zero are nonphysical, and therefore discarded in this analysis. We want to find even periodic small amplitude solutions by bifurcating from a curve of trivial flows. For this purpose, fix the depth h_0 and the half wavelength $L > 0$. The speed $c > 0$ shall be our bifurcation parameter. It is clear from (3.5) that, in any real linear space of $2L$-periodic and even functions,

$$\dim\ker(\mathcal{L}) = 1$$

if and only if $\xi_0 = k\pi/L$, $k \in \mathbb{Z}^+$. Settling for the lowest mode, $k = 1$, gives a unique c as in (3.1), which from now on will be presupposed as our candidate for c^* as in Lemma 3.2.

Looking for even, continuous, and periodic solutions, we introduce the commuting Banach algebra

$$W := \left\{ u(x) = \frac{1}{L}\sum_{k=0}^{\infty}{}' \hat{u}_k \cos\left(\tfrac{k\pi x}{L}\right) : \|u\| := \frac{1}{L}\sum_{k=0}^{\infty}{}' |\hat{u}_k| < \infty \right\}.$$

Note that each member of W is uniformly continuous on all of $\mathbb{R}$. We shall consider the Whitham equation as the map (3.2) from W, and it will be shown that it is a continuous map into W. As shown in (3.4) the periodic problem is given by the same multiplier as the problem on the line. In effect,

$$\mathcal{L}u \sim \frac{1}{L} \sum_{k=0}^{\infty}{}' \hat{u}(k) \left(1 - \tfrac{1}{c}\hat{A}(k)\right) \cos\left(\tfrac{k\pi x}{L}\right) \tag{3.6}$$

holds almost everywhere on $[-L, L]$. By the Riemann-Lebesgue lemma [17, page 133] $\hat{A}(k) \to 0$ as $k \to \infty$, so the right-hand side is in W, hence continuous, and

$$\|\mathcal{L}u\| \leq \left(1 + \tfrac{1}{c} \max_k \{\hat{A}(k)\}\right) \|u\|,$$

so that $\mathcal{L}\colon W \to W$ is continuous. Since also the left-hand side is continuous, (3.6) is an equality, which in its turn implies that the full nonlinear Whitham map $u \mapsto \mathcal{L}u - u^2$ is a continuous endomorphism on W, since this is an algebra. The fact that $\ker(\mathcal{L}) = \operatorname{span}_{\mathbb{R}} (\cos(\pi x/L))$ corresponds to

$$\hat{A}(1) = c, \qquad \text{and} \qquad \hat{A}(k) \neq c, \quad k \neq 1. \tag{3.7}$$

To show that $\operatorname{codim} \operatorname{ran}(\mathcal{L})$ is one dimensional, consider a given $u \in W$. Take $u^\perp \in W$ with $\widehat{u^\perp}(1) = 0$, and $\widehat{u^\perp}(k) = \hat{u}(k)$ for $k \neq 1$. Then the function

$$v(x) := \frac{1}{L} \sum_{k=0}^{\infty}{}' \frac{\widehat{u^\perp}(k)}{1 - \frac{1}{c}\hat{A}(k)} \cos\left(\tfrac{k\pi x}{L}\right)$$

is well defined and belongs to W (this can be seen from (3.3), but it also follows from the Riemann-Lebesgue lemma in combination with (3.7)). Indeed, $v(x) = \mathcal{L}^{-1} u^\perp(x)$. Consequently,

$$u(x) = \mathcal{L}v + \tfrac{\hat{u}(1)}{L} \cos\left(\tfrac{k\pi x}{L}\right),$$

so that $W = \ker(\mathcal{L}) \oplus \operatorname{ran}(\mathcal{L})$. The derivative with respect to the bifurcation parameter c is

$$(D_c\mathcal{L})u = -(D_c \tfrac{1}{c} K_{h_0}) * u = \frac{1}{c^2} K_{h_0} * u.$$

Hence—by exactly the same arguments as above—we have that

$$(D_c\mathcal{L})u = \frac{1}{Lc^2} \sum_{k=0}^{\infty}{}' \hat{u}(k)\hat{A}(k) \cos\left(\tfrac{k\pi x}{L}\right)$$

is bounded as a map on W. In particular,

$$(D_c\mathcal{L}) \ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = \ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = 0. \qquad \square$$

4. Continuity of bounded solutions

We present here a regularity result for traveling solutions of the weak Whitham equation (2.4). Notice that, while the solutions found in Theorem 3.1 are continuous by construction, the following result yields *a priori* continuity for a general class of solutions.

Theorem 4.1. *Let ϕ be a solution of* (2.4) *such that $\|\phi\|_{L^\infty(\mathbb{R})} < \frac{1}{2}$. Then ϕ is continuous.*

Proof. Without loss of generality we pursue the analysis for

$$k(\xi) := \sqrt{\tanh(\xi)/\xi}.$$

In view of the fact that $D_\xi \tanh \xi = 1 - \tanh^2 \xi \in \mathcal{S}(\mathbb{R})$, it follows from the Leibniz rule that

$$D_\xi^n k(\xi) \in \mathcal{O}\big(\xi^{-1/2-n}\big) \quad \text{as} \quad |\xi| \to \infty.$$

Hence, we use partial integration to rewrite

$$K(x) := \frac{1}{2\pi} \int k(\xi) \exp(ix\xi)\, d\xi = \frac{1}{(-ix)^n} \int k^{(n)}(\xi) \exp(ix\xi)\, d\xi,$$

for any $x \neq 0$, $n \in \mathbb{Z}^+$. Consequently, we have well-defined derivatives of all orders away from the origin,

$$D_x^j \int k(\xi) \exp(ix\xi)\, d\xi \in \mathcal{O}(|x|^{j-n}) \quad \text{as} \quad |x| \to \infty.$$

For any fixed j, we may choose n as large as required to obtain that $K(x)$ is smooth away from the origin, and all its derivatives have rapid decay at infinity. Consider then

$$K*\phi(x) = I_1(x)+I_2(x) := \int_{|x-z|\le 1} K(x-z)\phi(z)\, dz + \int_{|x-z|\ge 1} K(x-z)\phi(z)\, dz.$$

Since $|K(x-z)|$ has rapid decay and ϕ is bounded, it follows from an application of the dominated convergence theorem that $I_2(x)$ is continuous. By the change of variables $\xi \mapsto s := (x-z)\xi$ we have that

$$\begin{aligned} I_1(x) &= \frac{1}{2\pi} \iint_{|z-x|\le 1} k(\xi) \exp\left(i(x-z)\xi\right) \phi(z)\, d\xi\, dz \\ &= \frac{1}{2\pi} \int_{|z-x|\le 1} \frac{x-z}{|x-z|^{3/2}} \Big(\int \sqrt{\frac{\tanh\left(s|x-z|^{-1}\right)}{s}} \exp\left(is\right) ds \Big) \phi(z)\, dz. \end{aligned}$$

Likewise, the inner integral can be divided into two parts,

$$\int \sqrt{\frac{\tanh sy}{s}} \exp{(is)}\, ds = I_i(y) + I_{ii}(y)$$
$$:= \int_{|s|\le 1} \sqrt{\frac{\tanh sy}{s}} \exp{(is)}\, ds + \int_{|s|\ge 1} \sqrt{\frac{\tanh sy}{s}} \exp{(is)}\, ds,$$

where we have used the shorthand $y := |x-z|^{-1}$. It is clear that $|\tanh sy| \le 1$, so that $I_i(y)$ is well defined. Its integrand is furthermore bounded by $|s|^{-1/2}$, uniformly for all y. Now, for I_{ii}, using partial integration, we obtain that

$$\int_{|s|\ge 1} \sqrt{\frac{\tanh{(sy)}}{s}} \exp{(is)}\, ds = -2\sin(1)\sqrt{\tanh(y)}$$
$$+ \frac{1}{2i}\int_{|s|\ge 1} \frac{\left(sy\tanh^2{(sy)} - sy + \tanh{(sy)}\right)}{s^{3/2}\sqrt{\tanh{(|s|y)}}} \exp{(is)}\, ds$$
$$= -2\sin(1)\sqrt{\tanh(y)} + \frac{1}{2i}\int_{|s|\ge 1} \frac{f\,(sy)}{s^{3/2}} \exp{(is)}\, ds,$$

where $f(\tau) := (\tau\tanh^2\tau - \tau + \tanh\tau)/\sqrt{|\tanh\tau|}$. It is immediate that the boundary term is bounded by 2, and it can be seen that f is uniformly bounded with $\|f\|_\infty = 1$.

Since ϕ is bounded, this implies that, if $x_n \to x$, then there is a uniformly integrable bound, $C|x-z|^{-1/2}(|s|^{1/2}+|s|^{3/2})^{-1}$, for the integrands of $I_1(x_n)$. It thus follows from dominated convergence that $I_1(x)$ is continuous, and hence, $K*\phi(x)$ is. Using (2.4), we see that

$$|\phi(x)-\phi(y)| = \frac{|K*\phi(x) - K*\phi(y)|}{1-\phi(x)-\phi(y)} \le \frac{|K*\phi(x) - K*\phi(y)|}{1-2\|\phi\|_{L^\infty(\mathbb{R})}}, \tag{4.1}$$

and hence, ϕ is continuous. Here, we have used the assumption that $\|\phi\|_{L^\infty(\mathbb{R})} < \frac{1}{2}$.

5. Nonexistence of a class of solitary waves

The Whitham equation was designed to incorporate both wave breaking and dispersion. However, if the depth $h_0 > 0$ is small when compared to the wave speed, then the dispersion term is small, and moreover, dispersion is very weak compared to the KdV-equation. As a result, for large velocities c, there are no traveling waves.

Theorem 5.1. *There are no nontrivial, nonnegative, bounded travelling-wave solutions of the Whitham equation with* $\inf \phi \le \frac{1}{2}$, *satisfying*

$$c > \kappa \sqrt{g h_0}, \tag{5.1}$$

where $\kappa = 2\left(\sqrt{2}+1\right) \left\| \mathcal{F}^{-1}\left(\sqrt{(\tanh \xi)/\xi}\right)\right\|_{L^1(\mathbb{R})}$.

Remark 5.2. Note that the condition (5.1) means that there are no solitary waves with velocities much larger than the critical long wave speed $\sqrt{g h_0}$. Using the estimate

$$1 = \left\| \mathcal{F}\mathcal{F}^{-1}\left(\sqrt{\frac{\tanh \xi}{\xi}}\right)\right\|_{L^\infty(\mathbb{R})} \le \left\| \mathcal{F}^{-1}\left(\sqrt{\frac{\tanh \xi}{\xi}}\right)\right\|_{L^1(\mathbb{R})},$$

the value of κ appearing in the statement of the theorem may be estimated below by $2\left(\sqrt{2}+1\right)$.

Proof of Theorem 5.1. The proof proceeds by contradiction. Suppose that there exists a nontrivial bounded solution ϕ to (2.4). Then the following inequalities must hold:

$$\begin{aligned} \left(\|\phi\|_{L^\infty(\mathbb{R})} - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\right) \|\phi\|_{L^\infty(\mathbb{R})} &\le \|\phi\|_{L^\infty(\mathbb{R})} \\ &\le \left(\|\phi\|_{L^\infty(\mathbb{R})} + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\right) \|\phi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

so that

$$1 - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \le \|\phi\|_{L^\infty(\mathbb{R})} \le 1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}, \tag{5.2}$$

in view of the fact that $\sup \phi > 0$. Note first that

$$\phi^2(x) \ge \phi(x) - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})},$$

for all x. This is a simple consequence of (2.4). For the desired contradiction it is thus enough to show that there is some x, such that

$$\phi^2(x) < \phi(x) - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})},$$

or in other words

$$\phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})} < 0. \tag{5.3}$$

An application of (5.2) yields that

$$\begin{aligned} &\phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})} \\ &\quad \le \phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} (1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}), \end{aligned}$$

and we set out to examine the right-hand side,

$$F(\phi) := \phi^2 - \phi + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} (1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}).$$

Observe that $F(\phi)$ is negative whenever

$$\tfrac{1}{2}\big(1-\sqrt{2-(2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}+1)^2}\big)<\phi<\tfrac{1}{2}\big(1+\sqrt{2-(2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}+1)^2}\big). \quad (5.4)$$

The left- and right-hand sides of (5.4) are real if $\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \frac{1}{2}(\sqrt{2}-1)$. Taking the scaling and (2.2) into consideration, that follows from the requirement (5.1). Therefore, under that assumption, we have that

$$\tfrac{1}{2}\Big(1-\sqrt{2-(2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}+1)^2}\Big)<\tfrac{1}{2}<1-\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\leq\|\phi\|_{L^\infty(\mathbb{R})},$$

in view of (5.2). Since ϕ is continuous with $\inf\phi \leq \frac{1}{2}$, there is thus an x, such that both inequalities in (5.4) are satisfied. As a result, we have obtained that (5.3) holds, reaching the desired contradiction. □

6. Numerical Approximation

For the numerical approximation of periodic traveling waves of the Whitham equation, a spectral projection is used. As above, the undisturbed depth h_0 and the wavelength L are fixed, and the speed c is used as the bifurcation parameter. For the purpose of approximating periodic solutions of (2.3), a Fourier method is optimal. To define the Fourier-collocation projection, define the subspace $S_N = \mathrm{span}_{\mathbb{C}}\,\{\exp(ikx) : k \in \mathbb{Z},\ -N/2 \leq k \leq N/2-1\}$ of $L^2(0,2\pi)$. The collocation points are defined to be $x_j = \frac{2\pi j}{N}$ for $j = 0,1,\dots,N-1$. For the spectral projection, we use the equation (2.3) with $B = 0$. According to Theorem 3.1, the equation is defined on the interval $[-L,L]$ (or on $[0,2L]$ by periodicity), whereas the discrete Fourier transform to be used is most conveniently defined on $[0,2\pi]$. Therefore, the scaling $\phi(x)\to\phi(ax)$ is used, where $a=\frac{L}{\pi}$. Special attention has to be paid to the operator K_{h_0}. A straightforward calculation shows that

$$\big(K_{h_0}*u\big)(ax)=\sqrt{a}K_{h_0/a}*\big(u(a\,\cdot)\big)(x). \quad (6.1)$$

Therefore, the rescaled equation for 2π-periodic solutions is

$$-c\,\phi+\tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi^2+\sqrt{a}K_{h_0/a}*\phi=0.$$

The discretized form of this equation is

$$-c\,\phi_N+\tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi_N^2+\sqrt{a}[K_{h_0/a}]_N\phi_N=0, \quad (6.2)$$

which is enforced at the collocation points x_j. If ϕ_N is written in terms of its discrete Fourier coefficients $\tilde{\phi}_N(k)$ as

$$\phi_N(x)=\sum_{-N/2\leq k\leq N/2-1}\tilde{\phi}_N(k)\exp(ikx),$$

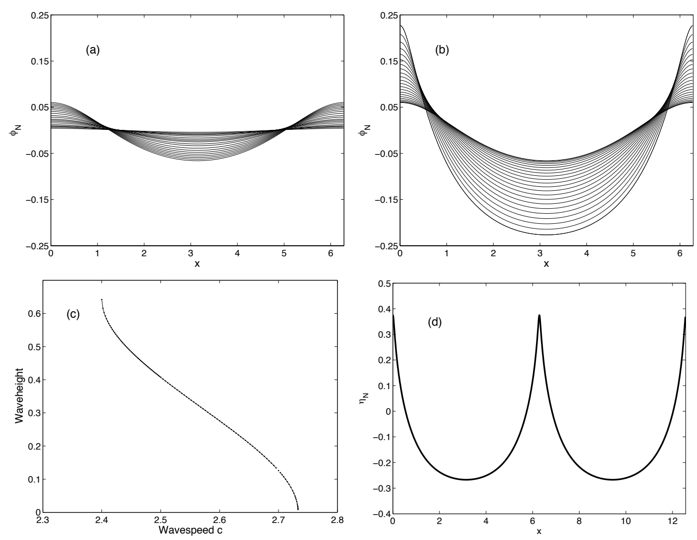

FIGURE 2. (a) and (b) Part of a branch of solutions of (6.2) with $h_0 = 1$ and $L = \pi$. Note that the highest wave is not shown here. (c) Amplitude vs. wave speed. (d) Two wavelengths of the (nearly) highest wave.

the operator $[K_{h_0/a}]_N$ can be evaluated using the formula

$$[K_{h_0/a}]_N \phi_N(x) = \sqrt{\tfrac{gh_0}{a}}\,\tilde{\phi}_N(0) + \sum_{\substack{1-N/2\le k\le N/2-1\\ k\neq 0}} \sqrt{\tfrac{g}{k}\tanh k(h_0/a)}\,\tilde{\phi}_N(k)\exp(ikx).$$

Thus, the operator $[K_{h_0/a}]_N$ is the truncation at the $N/2$-th Fourier mode of the operator given by the periodic convolution with $K_{h_0/a}$. Note that this formulation includes the truncation of the Fourier mode $\tilde{\phi}_N(-N/2)$ which otherwise can lead to instabilities in the computation. The equation (6.2) is treated pseudospectrally. That is, multiplication is carried out in physical space, while the term involving $K_{h_0/a}$ is evaluated using the discrete Fourier transform. To make sure that the computed functions are approximate traveling waves for the Whitham equation, we have also used a dynamic integrator for the time-dependent Whitham equation. Even after integrating for up to 10000 periods, the shape of the traveling wave is nearly unaltered, and the main error incurred by the time integration is a phase error.

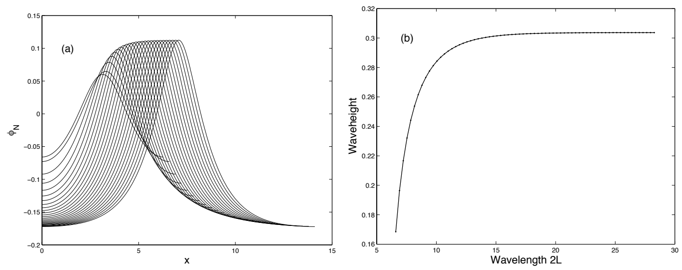

FIGURE 3. (a) Approximate traveling waves for the Whitham equation with $h_0 = 1$ and $c = 2.733$, and with increasing wavelength. (b) Amplitude as a function of wavelength.

In Figure 2, a branch of traveling-wave solutions is shown. Here, the wavelength is chosen to be 2π, and the depth is $h_0 = 1$. Note that, in this case, the wave number is $k = \frac{2\pi}{2L} = 1$, and, therefore, the phase velocity of a linear wave is given by $\sqrt{g \tanh(h_0)} \sim 2.7334$. It can be seen in panel (c) of Figure 2 that, as the amplitude of the traveling wave approaches zero, the velocity approaches the linear wave speed. Note also that only a part of the branch is shown in panels (a) and (b). Two wavelengths of the highest wave we were able to compute are shown in panel (d). The maximal waveheight of this branch appears to be ~ 0.642. This solution seems to nearly have a cusp, a fact already noted by Whitham [24]. Since a Fourier-collocation method is used, it is implicitly assumed that the solutions are smooth, and it is not possible to find the very highest wave predicted by Whitham. A possible method for finding the highest wave is outlined in [7], where a scheme based on Lagrange polynomials is used, and the highest point on the wave is treated as a boundary condition. However, the Whitham equation as it appears here was not treated in [7].

Finally, we want to explore the possible convergence of periodic traveling waves to solitary waves as the wavelength increases. In Figure 3, a family of approximate traveling waves is shown in the case when the wavelength L in increasing, while h_0 and c are held constant.

Note that amplitude grows initially, but seems to level off to an approximate value of 0.145. As Figure 4 shows, even though the wavelength L keeps increasing, the shape of the traveling waves does not change very much if a certain threshold is passed. The numerical evidence suggests that these waves converge to a solitary wave. Generally, a solitary wave is assumed to

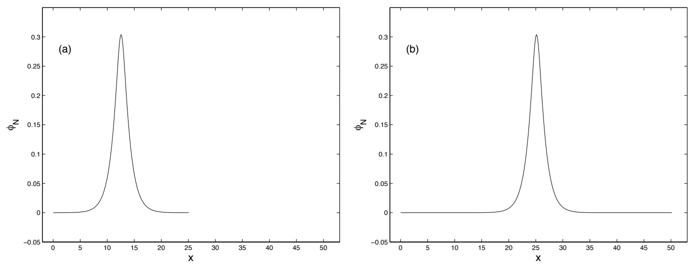

FIGURE 4. Approximate traveling waves for the Whitham equation with $h_0 = 1$ after a Galilean shift as in (6.3). (a) $L = 4\pi$, $c = 3.5640$, $\gamma = 0.4320$, and $B = 3.7802e - 05$. (b) $L = 8\pi$, $c = 3.5642$, $\gamma = 0.4321$, and $B = 1.4706e - 09$.

decay to zero at infinity. For the limiting solitary wave suggested in Figure 3, this can be achieved by a Galilean transformation of the form

$$\phi \to \phi + \gamma \quad \text{and} \quad c \to c + 2\gamma. \tag{6.3}$$

This introduces a nonzero constant B in equation (2.3). However, it can be seen that the constant levels off to zero as the amplitudes of the sequence of traveling waves approaches the asymptotic value as shown in Figure 4 (b).

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