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# ENERGY DECAY AND PROPAGATION IN A VISCOELASTIC FLUID WITH MEMORY

Carlo Alberto Bosello and Barbara Lazzari
Dept. of Mathematics, University of Bologna
5 Piazza di Porta S. Donato, 40126 Bologna, Italy

(Submitted by: Viorel Barbu)

**Abstract.** A non-compressible viscoelastic fluid with memory is considered and the well posedness of the associated dynamical problem is studied. The exponential decay of the energy is obtained under suitable hypotheses on the memory kernel. Furthermore, a domain of dependence inequality is established, thereby showing the hyperbolic nature of the problem.

## 1. Introduction

In this paper we consider isotropic, homogeneous, incompressible, linear viscoelastic fluids for which the constitutive equation for the symmetric stress tensor is a local functional of the relative history of the infinitesimal strain $\mathbf{E} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$, where $\mathbf{u}$ is the displacement vector. The *relative history* of the strain at time $t$ is defined as

$$\mathbf{E}_r^t(\mathbf{x}, s) = \mathbf{E}^t(\mathbf{x}, s) - \mathbf{E}(\mathbf{x}, t),\ \ s > 0,$$

where $\mathbf{E}^t(\mathbf{x}, s) = \mathbf{E}(\mathbf{x}, t - s)$ is its past history.

More precisely, let $\Omega$ be an open bounded subset of $R^3$ with smooth boundary $\partial\Omega$ occupied by an incompressible viscoelastic fluid. For any $\mathbf{x} \in \Omega$ and time $t \geq 0$, the linearized constitutive equation for the stress tensor is (see [1])

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{I} + 2\int_0^\infty \mu'(\mathbf{x}, s)\left[\mathbf{E}(\mathbf{x}, t - s) - \mathbf{E}(\mathbf{x}, t)\right] \mathrm{d}s, \tag{1.1}$$

where $p$ is the pressure, $\mathbf{I}$ is the identity second-order tensor and $\mu'$ a memory kernel. By posing

$$\tilde{\mathbf{T}}_E(\mathbf{x}, \mathbf{E}_r^t) = 2\int_0^\infty \mu'(\mathbf{x}, s)\mathbf{E}_r^t(\mathbf{x}, s)\,\mathrm{d}s,$$

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we can rewrite (1.1) as

$$\mathbf{T}(\mathbf{x},t) = -p(\mathbf{x},t)\mathbf{I} + \tilde{\mathbf{T}}_E(\mathbf{x},\mathbf{E}_r^t), \tag{1.2}$$

where $\tilde{\mathbf{T}}_E$ is a functional of the relative history of the strain called *extra stress*. Moreover, we shall write

$$\mathbf{T}_E(\mathbf{x},t) = \tilde{\mathbf{T}}_E(\mathbf{x},\mathbf{E}_r^t) \tag{1.3}$$

for the extra stress at time $t$ and, from now on, any dependence on $\mathbf{x}$ will be understood but not written whenever no ambiguity arises.

By (1.3), the extra stress at time $t$ is known if so is the relative history of the deformation gradient up to time $t$. On the other hand, as has been observed in [2], identifying the state of the material with $\mathbf{E}_r^t$ does not yield a minimal state representation, since two distinct histories $\mathbf{E}_{1r}^t$ and $\mathbf{E}_{2r}^t$ for which

$$\int_0^\infty \mu'(s)\mathbf{E}_{1r}^t \,\mathrm{d}s = \int_0^\infty \mu'(s)\mathbf{E}_{2r}^t \,\mathrm{d}s$$

determine the same extra stress. This leads to defining (see [2]) a new history functional

$$\breve{\mathbf{I}}^t(\tau,\mathbf{E}_r^t) = -2\int_0^\infty \mu'(s+\tau)\mathbf{E}_r^t(s)\,\mathrm{d}s,$$

so that we are able to rewrite the extra stress at time $t$ as

$$\mathbf{T}_E(t) = -\breve{\mathbf{I}}^t(0). \tag{1.4}$$

In this paper, we give some results about the well posedness of the dynamical problem using some results from semigroup theory. We also treat the questions of exponential decay and hyperbolicity. To this end, we use a free energy functional to define a norm in the space of solutions of the problem.

Free energies for materials with memory have been investigated by many authors over the years. It was recognized that free energies are not in general uniquely defined for materials with memory. In fact, the free energies associated to a material with memory form a bounded, convex set (see, for example [3]) with a maximum and a minimum element. By virtue of general theorems, the minimum free energy has been shown to be equal to the maximum recoverable work [4], [5]. Explicit formulae for the maximum and various intermediate free energies have been given for linear models. We refer the reader to [6] and to the references therein for a general treatment of free energy functionals for linear viscoelastic fluids. In this work, we use a particular free energy considered in [7] which allows us to prove our results for a large class of initial data. As for the hyperbolicity of the problem, in the last section we shall prove a domain of dependence estimate giving the value of the velocity of propagation.

## 2. Setup of the problem

We study a linear isotropic and incompressible fluid occupying a bounded open set $\Omega$ of $R^3$ with smooth boundary $\partial\Omega$.

The memory kernel $\mu'$ is such that the shear relaxation function

$$\mu(s) = -\int_s^\infty \mu'(\xi)\,\mathrm{d}\xi$$

is in $L^1(R^+)$.

Observe that, since we are studying incompressible fluids, the velocity vector $\mathbf{v}$ is a solenoidal vector and we have $\mathbf{T}\cdot\dot{\mathbf{E}} = \mathbf{T}_E\cdot\dot{\mathbf{E}}$.

As has been amply remarked (see, for instance, [8]) in the context of dynamical problems in linear viscoelasticity, by means of the free energy functional a natural norm can be defined in the space of solutions. Since the presence of viscosity implies dissipation, in agreement with the laws of thermodynamics we shall give the following definition of *free energy density* and of *free energy*.

**Definition 2.1.** A *free energy density* is a state functional $\tilde{\psi}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}(\breve{\mathbf{I}}^t) = -\breve{\mathbf{I}}^t(0)\cdot\dot{\mathbf{E}} - \tilde{D}_\psi(\breve{\mathbf{I}}^t) \leq \mathbf{T}(t)\cdot\nabla\mathbf{v}(t), \tag{2.1}$$

where the functional $\tilde{D}_\psi$, called the *internal dissipation*, is non-negative. The quantity

$$\tilde{\psi}_\Omega(\breve{\mathbf{I}}^t) = \int_\Omega \tilde{\psi}(\breve{\mathbf{I}}^t)\,\mathrm{d}\mathbf{x}$$

shall be called the *free energy* associated with $\tilde{\psi}$.

While referring to [6] for a more extensive treatment of free energies, in this paper, under the assumption that

$$\mu'(s) < 0,\ \mu''(s) \geq 0, \quad \forall s \geq 0, \tag{2.2}$$

we shall only consider the following particular free energy density introduced in [7]:

$$\psi(t) = \tilde{\psi}(\breve{\mathbf{I}}^t) = -\frac{1}{4}\int_0^\infty \frac{1}{\mu'(\tau)}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\,\mathrm{d}\tau$$

and the associated free energy

$$\psi_\Omega(t) = -\frac{1}{4}\int_\Omega \int_0^\infty \frac{1}{\mu'(\tau)}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\, \mathrm{d}\tau\, \mathrm{dx}.$$

Observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\breve{\mathbf{I}}^t(\tau) = \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau) - 2\mu(\tau)\dot{\mathbf{E}}(t), \tag{2.3}$$

a straightforward computation shows that

$$\dot{\psi}(t) = \mathbf{T}(t)\cdot\nabla\mathbf{v}(t) - \frac{1}{4}\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\, \mathrm{d}\tau,$$

so that the dissipation is given by

$$\begin{aligned}\tilde{D}_\psi(\breve{\mathbf{I}}^t) &= \frac{1}{4}\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\, \mathrm{d}\tau \\ &= -\frac{1}{4}\frac{1}{\mu'(0)}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(0)\Big|^2 + \frac{1}{4}\int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\, \mathrm{d}\tau \geq 0, \end{aligned} \tag{2.4}$$

where $\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(0)$ denotes the partial derivative evaluated at $\tau = 0$.

The introduction of the free energy allows us to define a norm in the history space in a natural way, thereby turning it into a Hilbert space (see [3])

$$\mathcal{H}_\psi = \Big\{\breve{\mathbf{I}}^t : R^+ \longrightarrow \text{ Sym}, \breve{\mathbf{I}}^t \text{ measurable}\,; \tilde{\psi}_\Omega(\breve{\mathbf{I}}^t) < \infty\Big\}.$$

Now, observe that, recalling (1.4),

$$\mathbf{T}_E(t) = \int_0^\infty \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\, \mathrm{d}\tau,$$

so that

$$|\mathbf{T}_E(t)|^2 \leq \int_0^\infty -\mu'(\tau)\, \mathrm{d}\tau \int_0^\infty -\frac{1}{\mu'(\tau)}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\, \mathrm{d}\tau \leq 4\mu(0)\psi(t). \tag{2.5}$$

It follows that, if a past history is such that $\psi_\Omega(t)$ is finite, then $\mathbf{T}_E(t) \in L^2(\Omega)$ for almost all $t$. Moreover, if $\varepsilon > 0$, by (2.5) we have

$$|\mathbf{T}_E(t)\mathbf{v}(t)| \leq \frac{1}{2\varepsilon}\,|\mathbf{T}_E(t)|^2 + \frac{\varepsilon}{2}|\mathbf{v}(t)|^2 \leq \frac{2\mu(0)}{\varepsilon}\psi(t) + \frac{\varepsilon}{2}|\mathbf{v}(t)|^2,$$

hence, by choosing $\varepsilon = \sqrt{2\mu(0)}$, we get

$$|\mathbf{T}_E(t)\mathbf{v}(t)| \leq \sqrt{2\mu(0)}\big(\tfrac{1}{2}|\mathbf{v}(t)|^2 + \psi(t)\big). \tag{2.6}$$

Within the above setting, we want to study the initial boundary-value problem with Dirichlet conditions associated with (1.1) and we can write it using (1.2) and (1.4) as follows:

$$\dot{\mathbf{v}}(\mathbf{x},t) = \nabla\cdot\mathbf{T}(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = -\nabla\cdot\big[p(\mathbf{x},t)\mathbf{I} + \breve{\mathbf{I}}^t(0)\big] + \mathbf{f}(\mathbf{x},t), \qquad (2.7)$$

$$\nabla\cdot\mathbf{v}(\mathbf{x},t) = 0, \quad \mathbf{v}(\mathbf{x},t)\big|_{\partial\Omega} = 0, \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \qquad (2.8)$$

where, in writing (2.7), the (constant) density has been taken to be equal to one and $\mathbf{f}$ is the body force. Henceforth, we shall refer to (2.7)–(2.8) as to problem P.

## 3. Semigroup approach

In this section we shall prove some preliminary results necessary to ensure applicability of semigroup theory to the asymptotic behavior of problem P. Such results also imply the well posedness of the problem.

Let us define the *energy density* at time $t$ as

$$e(t) = \frac{1}{2}|\mathbf{v}(t)|^2 + \tilde{\psi}(\breve{\mathbf{I}}^t),$$

namely, the sum of kinetic and free energy densities, so that the energy at time $t$ is

$$\mathcal{E}(t) = \int_\Omega e(t)\,\mathrm{d}\mathbf{x} = \frac{1}{2}\int_\Omega |\mathbf{v}(t)|^2\,\mathrm{d}\mathbf{x} + \tilde{\psi}_\Omega(\breve{\mathbf{I}}^t). \qquad (3.1)$$

The natural setting in which to look for existence and uniqueness of solutions is the *admissible states space* $\mathcal{K}$, consisting of those pairs $\sigma(t) = (\mathbf{v}(t), \breve{\mathbf{I}}^t)$ having finite total energy. Thus, we introduce the space $J(\Omega) = \{\mathbf{v} \in C_0^\infty(\Omega) : \nabla\cdot\mathbf{v} = 0\}$ and we denote by $\overset{\circ}{H}{}_0^1(\Omega)$ the closure of $J(\Omega)$ in the $H^1$ norm. We now set $\mathcal{K} = \overset{\circ}{H}{}_0^1(\Omega) \times \mathcal{H}_\psi$ and we endow $\mathcal{K}$ with an inner product $\langle\cdot,\cdot\rangle$

$$\langle\sigma_1,\sigma_2\rangle = \int_\Omega \left[\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{1}{2}\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\breve{\mathbf{I}}_1^t(\tau)\cdot\frac{\partial}{\partial\tau}\breve{\mathbf{I}}_2^t(\tau)\mathrm{d}\tau\right]\mathrm{d}\mathbf{x},$$

where $\sigma_i = (\mathbf{v}_i, \breve{\mathbf{I}}_i^t)$, for $i = 1,2$, so that the norm of $\sigma$ corresponds to the mechanical energy of the system in state $\sigma$, namely $\langle\sigma,\sigma\rangle = \|\sigma\|^2 = 2\mathcal{E}(\sigma)$ and, recalling (2.3), we can rewrite problem P as an abstract Cauchy problem as follows:

$$\dot{\sigma}(t) = \mathbf{A}\sigma(t) + F(t), \quad \sigma(0) = \sigma_0, \qquad (3.2)$$

where $F = (\mathbf{f}, 0)$, $\sigma_0 = (\mathbf{v}_0, \breve{\mathbf{I}}^0)$ and,

$$\mathbf{A}\sigma(t) = \Big( -\nabla\cdot\Big(p\mathbf{I} + \breve{\mathbf{I}}^t(0)\Big), \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau) - 2\mu(\tau)\dot{\mathbf{E}}(t)\Big).$$

We shall denote by $\mathcal{D}(\mathbf{A})$ the domain of operator $\mathbf{A}$, namely $\mathcal{D}(\mathbf{A}) = \{\sigma \in \mathcal{K} : \mathbf{A}\sigma \in \mathcal{K}\}$

It is possible to state the following theorem establishing the well posedness of problem $P$ in $\Omega \times \mathrm{R}^+$.

**Theorem 3.1.** *If $F \in W^{1,p}_{\mathrm{loc}}(R^+; L^2(\Omega))$ and $\sigma_0 \in \mathcal{D}(\mathbf{A})$, then problem* (3.2) *admits one and only one strict solution $\sigma \in C^1(R^+; \mathcal{K}) \cap C(R^+; \mathcal{D}(\mathbf{A}))$.*

In order to prove the above theorem, making use of the results obtained by Da Prato - Sinestrari [9], it is sufficient to prove that $\mathbf{A}$ generates a $C_0$-semigroup or, by the Lumer-Phillips theorem, that both $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ are strongly dissipative and that $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\widetilde{\mathbf{A}})$ are dense in $\mathcal{K}$.

We have

$$\begin{aligned}
\langle \mathbf{A}\sigma(t), \sigma(t)\rangle &= -\int_\Omega \Big[\nabla\cdot\Big(p\mathbf{I} + \breve{\mathbf{I}}^t(0)\Big)\cdot\mathbf{v}(t)\Big]\,\mathrm{d}\mathbf{x} \\
&\quad -\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big(\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau) - 2\mu(\tau)\dot{\mathbf{E}}(t)\Big)\cdot\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \breve{\mathbf{I}}^t(0)\cdot\nabla\mathbf{v}(t)\,\mathrm{d}\mathbf{x} + \int_\Omega\int_0^\infty \dot{\mathbf{E}}(t)\cdot\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x} \\
&\quad -\frac{1}{4}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \breve{\mathbf{I}}^t(0)\cdot\Big[\nabla\mathbf{v}(t) - \dot{\mathbf{E}}(t)\Big]\,\mathrm{d}\mathbf{x} - \int_\Omega D_\psi(t)\,\mathrm{d}\mathbf{x} = -\int_\Omega D_\psi(t)\,\mathrm{d}\mathbf{x} \le 0,
\end{aligned}$$

by (2.4) and the symmetry of $\breve{\mathbf{I}}^t(\tau)$, so that $\mathbf{A}$ is dissipative.

We now proceed to look for the adjoint of $\mathbf{A}$. We claim that

$$\mathcal{D}(\mathbf{A}) = \mathcal{D}(\widetilde{\mathbf{A}}), \tag{3.3}$$

and that

$$\widetilde{\mathbf{A}}\widetilde{\sigma}(t) = \Big(\nabla\cdot(\widetilde{p}\mathbf{I} + \widetilde{\breve{\mathbf{I}}}^t(0)), \frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau) - 2\mu(\tau)\dot{\widetilde{\mathbf{E}}}(t) + g(\widetilde{\breve{\mathbf{I}}}^t)(\tau)\Big), \tag{3.4}$$

where

$$\frac{\partial}{\partial\tau}g\big(\widetilde{\breve{\mathbf{I}}}^t\big)(\tau) = \mu'(\tau)\frac{\partial}{\partial\tau}\Big(\frac{H(\tau)}{\mu'(\tau)}\Big)\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau), \tag{3.5}$$

and $H$ is the Heaviside function.

In order to prove (3.3) and (3.4), let us now compute $\langle \mathbf{A}\sigma, \widetilde{\sigma}\rangle$, where $\sigma = (\mathbf{v}, \breve{\mathbf{I}}^t)$ and $\widetilde{\sigma} = (\widetilde{\mathbf{v}}, \widetilde{\breve{\mathbf{I}}}^t)$ both belong to $\mathcal{D}(\mathbf{A})$.

$$\langle \mathbf{A}\sigma(t), \widetilde{\sigma}(t)\rangle = -\int_\Omega \nabla\cdot(p\mathbf{I} + \breve{\mathbf{I}}^t(0)) \cdot \widetilde{\mathbf{v}}(t)\,\mathrm{d}\mathbf{x}$$
$$-\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big[\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau) - 2\mu(\tau)\dot{\mathbf{E}}(t)\Big]\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}.$$

Recalling that $\nabla\cdot\widetilde{\mathbf{v}} = 0$ and using the divergence theorem to transform the first integral, we get

$$\langle \mathbf{A}\sigma(t), \widetilde{\sigma}(t)\rangle = \int_\Omega \breve{\mathbf{I}}^t(0)\cdot\nabla\widetilde{\mathbf{v}}(t)\,\mathrm{d}\mathbf{x} - \int_\Omega \dot{\mathbf{E}}(t)\cdot\widetilde{\breve{\mathbf{I}}}^t(0)\,\mathrm{d}\mathbf{x}$$
$$-\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\Big[\frac{\partial}{\partial\tau}\Big(\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\Big) - \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\frac{\partial^2\widetilde{\breve{\mathbf{I}}}^t(\tau)}{\partial\tau^2}\Big]\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}.$$

Now, by the symmetry of $\breve{\mathbf{I}}^t$ we can write the first term as follows:

$$\int_\Omega \breve{\mathbf{I}}^t(0)\cdot\nabla\widetilde{\mathbf{v}}\,\mathrm{d}\mathbf{x} = -\int_\Omega\int_0^\infty \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\dot{\widetilde{\mathbf{E}}}(t)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x},$$

and observing that $\widetilde{\breve{\mathbf{I}}}^t$ is also symmetric and $\nabla\cdot\mathbf{v} = 0$, applying again the divergence theorem the second term becomes

$$\int_\Omega \dot{\mathbf{E}}(t)\cdot\widetilde{\breve{\mathbf{I}}}^t(0)\,\mathrm{d}\mathbf{x} = \int_\Omega \widetilde{\breve{\mathbf{I}}}^t(0)\cdot\nabla\mathbf{v}(t)\,\mathrm{d}\mathbf{x} = -\int_\Omega \nabla\cdot\Big(\widetilde{\breve{\mathbf{I}}}^t(0) + \widetilde{p}\mathbf{I}\Big)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},$$

so that

$$\langle \mathbf{A}\sigma(t), \widetilde{\sigma}(t)\rangle = \int_\Omega\int_0^\infty \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\dot{\widetilde{\mathbf{E}}}(t)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla\cdot\Big(\widetilde{\breve{\mathbf{I}}}^t(0) + \widetilde{p}\mathbf{I}\Big)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_\Omega \frac{1}{\mu'(0)}\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(0)\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(0)\,\mathrm{d}\mathbf{x}$$
$$-\frac{1}{2}\int_\Omega\int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\frac{\partial^2}{\partial\tau^2}\widetilde{\breve{\mathbf{I}}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}.$$

By grouping terms containing $\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)$ and recalling (3.5) we obtain

$$\langle \mathbf{A}\sigma, \widetilde{\sigma}\rangle = \int_\Omega \mathbf{v}\cdot\Big[\nabla\cdot(\widetilde{\breve{\mathbf{I}}}^t(0) + \widetilde{p}\mathbf{I})\Big]\,\mathrm{d}\mathbf{x}$$

$$+\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\cdot\frac{\partial}{\partial\tau}\Big[\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)-2\mu(\tau)\dot{\widetilde{\mathbf{E}}}(t)+g\big(\widetilde{\breve{\mathbf{I}}}^t\big)(\tau)\Big]\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$=\big\langle\sigma,\widetilde{\mathbf{A}}\widetilde{\sigma}\big\rangle,$$

therefore, (3.4) holds.

Now, observe that, for $\widetilde{\sigma}\in\mathcal{D}(\mathbf{A})$, we have

$$\Big\langle\widetilde{\mathbf{A}}\widetilde{\sigma}(t),\widetilde{\sigma}(t)\Big\rangle=\int_\Omega\nabla\cdot(\widetilde{p}\mathbf{I}+\widetilde{\breve{\mathbf{I}}}^t(0))\cdot\widetilde{\mathbf{v}}(t)\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial}{\partial\tau}\Big[\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)-2\mu(\tau)\dot{\widetilde{\mathbf{E}}}(t)+g\big(\widetilde{\breve{\mathbf{I}}}^t\big)(\tau)\Big]\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$=\frac{1}{2}\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\frac{\partial^2}{\partial\tau^2}\widetilde{\breve{\mathbf{I}}}^t(\tau)\cdot\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_\Omega\int_0^\infty \frac{\partial}{\partial\tau}\Big(\frac{H(\tau)}{\mu'(\tau)}\Big)\Big|\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\Big|^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$=\frac{1}{4}\int_\Omega\frac{1}{\mu'(0)}\Big|\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(0)\Big|^2\,\mathrm{d}\mathbf{x}-\frac{1}{4}\int_\Omega\int_0^\infty\frac{\mu''(\tau)}{[\mu'(\tau)]^2}\Big|\frac{\partial}{\partial\tau}\widetilde{\breve{\mathbf{I}}}^t(\tau)\Big|^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}$$
$$=-D_\psi(\widetilde{\breve{\mathbf{I}}}^t)\le 0,$$

so that $\widetilde{\mathbf{A}}$ is dissipative too, which concludes our proof.

**Remark 3.1.** This section's results continue to hold independently of which free energy is used to define the norm in the admissible state space. See, for instance, [10].

## 4. Exponential decay

While, in general, dissipation due to memory effects implies asymptotic stability, it does not ensure the exponential decay of the energy. In this section we introduce a condition on the memory kernel that is sufficient to establish such exponential decay.

In addition to assuming that $\mu$ satisfy (2.2), we shall require that, for all $s\ge 0$, there exists a positive constant $k$ such that

$$k\mu''(s)+\mu'(s)>0. \tag{4.1}$$

As a consequence of (4.1) we have

$$D_\psi(t)\ge\frac{1}{k}\psi(t)-\frac{1}{4}\frac{1}{\mu'(0)}\Big|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\Big|^2. \tag{4.2}$$

The main result of this section is the following.

**Theorem 4.1.** *Let $\sigma$ be a solution to* (3.2) *with* $\mathbf{f}=0$ *and with initial data* $\sigma_0 \in \mathcal{D}(\mathbf{A})$. *If* (4.1) *holds, then there exist two positive constants* $c_1$ *and* $c_2$ *such that*

$$\mathcal{E}(t) \leq c_2 e^{-c_1 t}\mathcal{E}(0).$$

In order to prove Theorem 4.1, we introduce the following functional:

$$\mathcal{L}_{t_0}(t) = (t+t_0)\mathcal{E}(t) + \frac{1}{2\mu(0)}\int_\Omega \mathbf{v}(t)\cdot \breve{\mathbf{J}}^t(0)\,\mathrm{d}\mathbf{x}, \tag{4.3}$$

where $t_0 \geq 0$ and $\breve{\mathbf{J}}^t$ is an auxiliary history such that $\breve{\mathbf{J}}^t(s) \in \overset{\circ}{H}{}^1_0(\Omega)$ for all $s \geq 0$ and

$$\frac{1}{2}\left[\nabla\breve{\mathbf{J}}^t(s) + (\nabla\breve{\mathbf{J}}^t(s))^T\right] = \breve{\mathbf{I}}^t(s) \quad \forall s \geq 0. \tag{4.4}$$

As a consequence of (2.5), (2.3) and (4.4), and of Poincare's and Korn's inequalities, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\breve{\mathbf{J}}^t(\tau) = \frac{\partial}{\partial\tau}\breve{\mathbf{J}}^t(\tau) - 2\mu(\tau)\mathbf{v}(t) \tag{4.5}$$

$$\begin{aligned}\int_\Omega \mathbf{v}(t)\cdot\frac{\partial}{\partial\tau}\breve{\mathbf{J}}^t(\tau)\,\mathrm{d}\mathbf{x} &\leq \frac{1}{4\mu(0)}\left\|\frac{\partial}{\partial\tau}\breve{\mathbf{J}}^t(\tau)\right\|_\Omega^2 + \mu(0)\left\|\mathbf{v}(t)\right\|_\Omega^2 \\ &\leq \frac{C_\Omega}{4\mu(0)}\left\|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\right\|_\Omega^2 + \mu(0)\left\|\mathbf{v}(t)\right\|_\Omega^2\end{aligned} \tag{4.6}$$

$$\frac{1}{2\mu(0)}\left|\int_\Omega \mathbf{v}(t)\cdot\breve{\mathbf{J}}^t(0)\,\mathrm{d}\mathbf{x}\right| \leq C\mathcal{E}(t), \tag{4.7}$$

where $\|\cdot\|_\Omega$ denotes the norm in $L^2(\Omega)$ and $C_\Omega$ and $C$ are suitable positive constants depending on the domain $\Omega$.

**Lemma 4.1.** *Let* $\mathbf{f}=0$ *in* (2.7) *and let* (4.1) *hold. Then, for* $t_0$ *large enough and for all positive* $t$, $\mathcal{L}_{t_0}(t)$ *is a monotonically non-increasing function.*

**Proof.** It suffices to show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{t_0}(t) \leq 0$. Recalling (2.1) and (3.1), using (4.5) and the fact that $\sigma$ is a solution of problem P and applying the divergence theorem we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{t_0}(t) &= \mathcal{E}(t) + (t+t_0)\Big[\dot{\psi}_\Omega(t) + \int_\Omega \dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)\,\mathrm{d}\mathbf{x}\Big] \\ &\quad + \frac{1}{2\mu(0)}\int_\Omega\Big[\dot{\mathbf{v}}(t)\cdot\breve{\mathbf{J}}^t(0) + \mathbf{v}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\breve{\mathbf{J}}^t(0)\Big]\,\mathrm{d}\mathbf{x} \\ &= \mathcal{E}(t) - (t+t_0)\int_\Omega D_\psi(t)\,\mathrm{d}\mathbf{x}\end{aligned}$$

$$+\frac{1}{2\mu(0)}\int_\Omega\Big[-\nabla\cdot\Big(p\mathbf{I}+\breve{\mathbf{I}}^t(0)\Big)\cdot\breve{\mathbf{J}}^t(0)+\mathbf{v}(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\breve{\mathbf{J}}^t(0)\Big]\,\mathrm{d}\mathbf{x}$$

$$=\mathcal{E}(t)-(t+t_0)\int_\Omega D_\psi(t)\,\mathrm{d}\mathbf{x}+\frac{1}{2\mu(0)}\left\|\breve{\mathbf{I}}^t(0)\right\|_\Omega^2-\|\mathbf{v}(t)\|_\Omega^2$$

$$+\frac{1}{2\mu(0)}\int_\Omega\frac{\partial}{\partial\tau}\breve{\mathbf{J}}^t(\tau)\cdot\mathbf{v}(t)\,\mathrm{d}\mathbf{x}.$$

Now, recalling (2.5), (3.1), (4.2) and (4.6) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{t_0}(t)\le-\frac{t+t_0-3k}{k}\psi_\Omega(t)+\frac{1}{4\mu'(0)}\Big(t+t_0+\frac{C_\Omega\mu'(0)}{2[\mu(0)]^2}\Big)\left\|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)\right\|_\Omega^2$$

so that, taking

$$t_0>\max\left\{3k,-\frac{C_\Omega\mu'(0)}{2[\mu(0)]^2}\right\},$$

the thesis follows.

We can now proceed to give the following.

**Proof of Theorem 4.1.** Let $T>0$. Then (4.3) and (4.7) imply that

$$\mathcal{L}_{t_0}(T)-\mathcal{L}_{t_0}(0)\ge(T+t_0-C)\mathcal{E}(T)-(t_0+C)\mathcal{E}(0). \tag{4.8}$$

On the other hand, Lemma (4.1) implies that

$$\mathcal{L}_{t_0}(T)-\mathcal{L}_{t_0}(0)=\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{t_0}(t)\,\mathrm{d}t\le 0, \tag{4.9}$$

so that, by combining (4.8) and (4.9), we get

$$\mathcal{E}(T)\le\frac{t_0+C}{T+t_0-C}\mathcal{E}(0). \tag{4.10}$$

Estimate (4.10) ensures the exponential decay of $\mathcal{E}(t)$ thanks to the semigroup properties proved in Section 3 (see, for instance, Theorem 4.1 in [11]).

## 5. Domain of dependence

We conclude by proving a domain of dependence inequality for the solutions of problem P. For an analogous result for compressible fluids see [12].

Given a vector $\mathbf{w}$ let $\mathbf{w}^S$ and $\mathbf{w}^I$ be, respectively, its solenoidal and irrotational components in the Helmholtz decomposition $\mathbf{w}=\mathbf{w}^S+\mathbf{w}^I$, and let $S(\mathbf{x}_0,r)$ denote the ball of center $\mathbf{x}_0$ and radius $r$.

**Theorem 5.1.** *Let $\sigma(t) = (\mathbf{v}(t), \breve{\mathbf{I}}^t)$ be a solution of problem P. Let $\mathbf{x}_0 \in \Omega$, $r > 0$, and $T > 0$ be fixed. Then*

$$\int_{\Omega\cap S(\mathbf{x}_0,r)} e(\sigma(T))\,\mathrm{d}\mathbf{x} \leq \int_{\Omega\cap S\left(\mathbf{x}_0,r+\sqrt{2\mu(0)}T\right)} e(\sigma_0)\,\mathrm{d}\mathbf{x}$$
$$+ \int_0^T \int_{\Omega\cap S\left(\mathbf{x}_0,r+\sqrt{2\mu(0)}(T-t)\right)} \mathbf{f}^S(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{5.1}$$

**Proof.** Let $\delta > 0$ and let $\varphi_\delta \in C^\infty(R)$ be such that

$$0 \leq \varphi_\delta \leq 1, \tag{5.2}$$

$$\varphi_\delta(s) = \begin{cases} 1 & s \leq -\delta \\ 0 & s > \delta \end{cases}, \tag{5.3}$$

$$\varphi_\delta' \leq 0 \tag{5.4}$$

and define $\varphi(\mathbf{x},t) = \varphi_\delta\left(|\mathbf{x}-\mathbf{x}_0| - r - c(T-t)\right)$, where $c$ is a positive constant. It follows that

$$\nabla\varphi(\mathbf{x},t) = \varphi_\delta'(\mathbf{x},t)\nabla|\mathbf{x}-\mathbf{x}_0| \tag{5.5}$$

$$\dot\varphi(\mathbf{x},t) = c\varphi_\delta'(\mathbf{x},t). \tag{5.6}$$

Now, we introduce the weighted energy

$$\mathcal{E}_\varphi(t) = \int_\Omega \Big[\frac{1}{2}\,|\mathbf{v}(\mathbf{x},t)|^2 + \psi(\mathbf{x},t)\Big]\varphi(\mathbf{x},t)\,\mathrm{d}\mathbf{x},$$

and, using (2.7), (2.1) and (5.6), we compute the time derivative of $\mathcal{E}_\varphi(t)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varphi(t) &= \int_\Omega \Big[\mathbf{v}(\mathbf{x},t)\cdot\dot{\mathbf{v}}(\mathbf{x},t) + \dot\psi(\mathbf{x},t)\Big]\,\varphi(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \Big[\frac{1}{2}|\mathbf{v}(\mathbf{x},t)|^2 + \psi(\mathbf{x},t)\Big]\dot\varphi(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&= \int_\Omega \mathbf{v}(\mathbf{x},t)\cdot(\nabla\cdot\mathbf{T}(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t))\varphi(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega \left(\mathbf{T}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) - D_\psi(\mathbf{x},t)\right)\varphi(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega c\Big[\frac{1}{2}\,|\mathbf{v}(\mathbf{x},t)|^2 + \psi(\mathbf{x},t)\Big]\varphi_\delta'(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\leq -\int_\Omega \nabla\varphi(\mathbf{x},t)\cdot\mathbf{T}(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} + \int_\Omega \varphi(\mathbf{x},t)\mathbf{f}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\Omega c\Big[\frac{1}{2}|\mathbf{v}(\mathbf{x},t)|^2 + \psi(\mathbf{x},t)\Big]\varphi_\delta'(\mathbf{x},t)\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where, in the last step, we applied the divergence theorem and used (2.4) and (5.2). Therefore, because of incompressibility, and using (1.2) and (5.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varphi(t) \leq \int_\Omega \varphi(\mathbf{x},t)\left(\mathbf{f}(\mathbf{x},t) - \nabla p(\mathbf{x},t)\right)\cdot \mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$$

$$-\int_\Omega \Big[\mathbf{T}_E(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\cdot\nabla|\mathbf{x}-\mathbf{x}_0| - c\Big(\frac{1}{2}|\mathbf{v}(\mathbf{x},t)|^2 + \psi(\mathbf{x},t)\Big)\Big]\varphi'_\delta(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

Now, choosing $c = \sqrt{2\mu(0)}$ and noting that $\nabla|\mathbf{x}-\mathbf{x}_0| = 1$, by (5.4) and (2.6) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varphi(t) \leq \int_\Omega \varphi(\mathbf{x},t)\left(\mathbf{f}(\mathbf{x},t) - \nabla p(\mathbf{x},t)\right)\cdot \mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x},$$

so that, decomposing (2.7) into a solenoidal and an irrotational term, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varphi(t) \leq \int_\Omega \varphi(\mathbf{x},t)\mathbf{f}^S(\mathbf{x},t)\cdot \mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}, \tag{5.7}$$

and integrating (5.7) in time from 0 to $T$ gives

$$\mathcal{E}_\varphi(T) \leq \mathcal{E}_\varphi(0) + \int_0^T\int_\Omega \varphi(\mathbf{x},t)\mathbf{f}^S(\mathbf{x},t)\cdot \mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{5.8}$$

Now, by (5.3) and (5.2), $\varphi_\delta(-s)$ tends boundedly to the Heaviside function as $\delta \to 0$, therefore $\varphi(\mathbf{x},t)$ tends to the characteristic function of $S\left(\mathbf{x}_0, r + \sqrt{2\mu(0)}(T-t)\right)$, taking the limit.

**Remark 5.1.** The quantity $\sqrt{2\mu(0)}$ is clearly the velocity of propagation of energy in the medium.

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