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LOCAL WELL-POSEDNESS FOR QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS AND THE "GOOD" BOUSSINESQ EQUATION

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Abstract. The Cauchy problem for 1-D nonlinear Schrödinger equations with quadratic nonlinearities are considered in the spaces $H^{s,a}$ defined by $\|f\|_{H^{s,a}} = \|(1+|\xi|)^{s-a}|\xi|^a\widehat{f}\|_{L^2}$, and sharp local well-posedness and ill-posedness results are obtained in these spaces for nonlinearities including the term $u\bar{u}$. In particular, when $a = 0$ the previous well-posedness result in H^s, $s > -1/4$, given by Kenig, Ponce and Vega (1996), is improved to $s \geq -1/4$. This also extends the result in $H^{s,a}$ by Otani (2004). The proof is based on an iteration argument similar to that of Kenig, Ponce and Vega, with a modification of the spaces of the Fourier restriction norm. Our result is also applied to the "good" Boussinesq equation and yields local well-posedness in $H^s \times H^{s-2}$ with $s > -1/2$, which is an improvement of the previous result given by Farah (2009).

1. Introduction

We consider the Cauchy problem of the following quadratic nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t - \partial_x^2)u = N(u), & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the unknown function u is complex valued and the nonlinearity is quadratic,

$$N(u,v) = \alpha_1 uv + \alpha_2 \bar{u}\bar{v} + \alpha_3 u\bar{v}, \quad N(u) = N(u,u), \tag{1.2}$$

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with complex constants $\alpha_1, \alpha_2, \alpha_3$. Our aim is to prove the local well-posedness of (1.1) with initial data of low regularity.

We first recall some known results of local well-posedness for (1.1) with the initial data given in Sobolev spaces $H^s(\mathbb{R})$. For $s \geq 0$, an iterative argument with Strichartz-type estimates imply well-posedness without distinguishing u^2, $\bar{u}^2$ and $u\bar{u}$ (and other quadratic nonlinearities such as $|u|u$); see [4]. This result is, however, far from the regularity $s = -3/2$ corresponding to the scale invariance of the equation, $u(t,x) \mapsto \lambda^{-2}u(\lambda^{-2}t, \lambda^{-1}x)$, $\lambda > 0$.

To study the KdV equation and the nonlinear Schrödinger equation under low regularities, Bourgain [2] introduced the Fourier restriction norm method, which is the iteration method in the function spaces of the Fourier restriction norm, $X^{s,b}$. For the nonlinear Schrödinger equations, this norm is defined as

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - \xi^2\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}},$$

where $\langle\cdot\rangle = 1 + |\cdot|$ and $\widetilde{u}$ is the Fourier transform of u with respect to t and x. This method was further developed by Kenig, Ponce and Vega [7] to yield the local well-posedness of (1.1) in H^s for $s > -3/4$ when $\alpha_3 = 0$ and for $s > -1/4$ when $\alpha_3 \neq 0$. In their proof of well-posedness, the following bilinear estimate was essentially used:

$$\|N(u,v)\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

However, it was shown by them, and by Nakanishi, Takaoka and Tsutsumi [14], that this estimate fails for $s \leq -3/4$ if $N(u,v) = uv$ or $\bar{u}\bar{v}$ and for $s \leq -1/4$ if $N(u,v) = u\bar{v}$, which means that we cannot improve the above well-posedness result with the standard Fourier restriction norm method (see [13] and [16], for well-posedness in the Besov space).

To overcome this difficulty, we have to take a real look at each nonlinear structure; note that the Fourier restriction norm is based on the linear part of the equation. In this direction, Bejenaru and Tao [1] introduced a modified Fourier restriction norm for the nonlinearity u^2 and improved the local well-posedness to $s \geq -1$ in this case. They focused on a support property of solutions of (1.1), namely, the fact that the support of $\widetilde{u}$ is almost in $\{(\tau,\xi) \in \mathbb{R}^2 : \tau \geq 0\}$ when $N(u) = \alpha_1 u^2$ and u satisfies (1.1). When $N(u) = \alpha_2\bar{u}^2$, although this property does not hold and the problem is much more complicated, modification in a similar spirit is still useful to achieve the same regularity threshold ([8]). We remark that each of these modified norms is not sufficient for the combined nonlinearity $N(u) = \alpha_1 u^2 + \alpha_2\bar{u}^2$, but a further refinement allows us to treat these two nonlinearities simultaneously for $s \geq -1$ ([9]).

The case of $u\bar{u}$ is totally different from the other cases. In fact, it is easily verified by observing the energy flow from high frequency parts to low frequency parts that the first nonlinear (quadratic) term appearing in the iteration scheme does not continuously depend on the data, which shows that the data-to-solution map fails to be C^2 as a map from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ when $s < -1/4$. This fact actually defeats any attempt to obtain well-posedness in H^s, $s < -1/4$, by the direct iteration as used for the other nonlinearities. This kind of argument was originally given for the KdV initial-value problem ([3], [18]) and is familiar in the field of nonlinear dispersive equations. We will show similar ill-posedness results later.

To treat lower regularities, we consider well-posedness in the following function space:

$$H^{s,a} = \{f \in \mathcal{Z}'(\mathbb{R}) : \|f\|_{H^{s,a}} < \infty\}, \qquad \|f\|_{H^{s,a}} = \|\langle\xi\rangle^{s-a}|\xi|^a \widehat{f}\|_{L^2_\xi},$$

where $\widehat{f}$ is the Fourier transform of f with respect to x and $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : D^\alpha \mathcal{F} f(0) = 0 \,\text{for every multi-index}\, \alpha\}.$$

For details of $\mathcal{Z}(\mathbb{R}^n)$, see e.g. page 237 in [17].

In this setting, Otani [15] proved that (1.1) is locally well-posed in $H^{s,a}$ for $-1/2 > s-a > -3/4$ and $1/4 > a > 0$ when $\alpha_3 = 0$ and for $0 > s-a > -1/4$ and $1/4 > a > 0$ when $\alpha_3 \neq 0$. The Fourier restriction norm corresponding to $H^{s,a}$ is defined as

$$\|u\|_{X^{s,a,b}} = \|\langle\xi\rangle^{s-a}|\xi|^a \langle\tau-\xi^2\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}, \tag{1.3}$$

and the following bilinear estimate is crucial in this case:

$$\|N(u,v)\|_{X^{s,a,b-1}} \leq C\|u\|_{X^{s,a,b}}\|v\|_{X^{s,a,b}}. \tag{1.4}$$

Unfortunately, this estimate still fails for $s < -3/8$, for any $a, b \in \mathbb{R}$ when $\alpha_3 \neq 0$. Let us show this in the case $N(u,v) = u\bar{v}$. We first recall the example given by Kenig, Ponce and Vega [7]. Put

$$f_N(\tau,\xi) = \begin{cases} 1, & \text{for}\, |\tau-\xi^2| \leq 1,\, |\xi - N| \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $\widetilde{u}_N(\tau,\xi) = \widetilde{v}_N(\tau,\xi) = f_N(\tau,\xi)$ for sufficiently large N. Then, it follows that $\|u_N\|_{X^{s,a,b}} = \|v_N\|_{X^{s,a,b}} \sim N^s$ and $\widetilde{u}_N * \widetilde{\bar{v}}_N(\tau,\xi) \gtrsim 1_A(\tau,\xi)$, where A is the rectangle of dimension $N \times N^{-1}$ centered at the origin with longest side pointing in the $(2N,1)$ direction. The left-hand side of (1.4) is bounded from below by

$$\Big(\int\Big[\int_{1/2}^{1}\langle\tau\rangle^{2(b-1)}1_A(\tau,\xi)\,d\xi\Big]\,d\tau\Big)^{1/2}\sim N^{b-1}.$$

Therefore, (1.4) fails when $b>2s+1$. We next put $\widetilde{u}_N(\tau,\xi)=f_N(\tau,\xi-2),\widetilde{v}(\tau,\xi)=f_N(\tau,\xi)$. Then, we have $\|u_N\|_{X^{s,a,b}}\sim N^{s+b},\|v_N\|_{X^{s,a,b}}\sim N^s$ and $\widetilde{u}_N*\widetilde{\bar{v}}_N(\tau,\xi)\gtrsim 1_A(\tau,\xi-1)$. The left-hand side of (1.4) is bounded from below by

$$\Big(\int\Big[\int_{0}^{1}1_A(\tau,\xi-1)\,d\tau\Big]\,d\xi\Big)^{1/2}\sim N^{-1/2}.$$

Therefore, (1.4) fails when $b<-2s-1/2$. Since $(-\infty,2s+1]\cap[-2s-1/2,\infty)=\emptyset$ if $s<-3/8$, the estimate (1.4) requires $s\ge-3/8$.

To go further away, we modify the weight function of (1.3) and introduce the $Z^{s,a}$ norm, which is the main idea of this paper:

$$\|u\|_{Z^{s,a}}=\|w_{s,a}\widetilde{u}\|_{L^2_{\tau,\xi}},$$

where $w_{s,a}(\tau,\xi)$ satisfies

$$w_{s,a}(\tau,\xi)\sim\min\{\langle\xi\rangle^{s-a}|\xi|^a\langle\tau-\xi^2\rangle,\langle\xi\rangle^{s+\sigma-a}|\xi|^a\langle\tau-\xi^2\rangle^{1-\sigma}\},$$

for some $\sigma\in(0,1)$. See (2.1) and (4.1) for the precise definition of $w_{s,a}$.

We will prove refined bilinear estimates with respect to the $Z^{s,a}$ norm in Section 3 (see Proposition 3.1) and obtain the following well-posedness result.

Theorem 1.1. *Let $|a|<1/2$ and $s\ge-(2a+1)/4$. Then,* (1.1) *with $\alpha_1,\alpha_2,\alpha_3\in\mathbb{C}$ is locally well-posed in $H^{s,a}$.*

We remark that if we take $a=0$ the above result is in H^s with $s\ge-1/4$, which improves the previous one given by Kenig, Ponce and Vega [7] for the endpoint $s=-1/4$. This should be compared with the failure of the crucial bilinear estimate in $X^{-1/4,b}$ shown by Nakanishi, Takaoka and Tsutsumi [14]. As they mentioned in [14], the lack of the bilinear estimate in one space does not necessarily imply ill-posedness; there may be a chance that we can recover it by changing the space.

On the other hand, we obtain the following ill-posedness result when s and a do not satisfy the condition in Theorem 1.1. In this sense, the condition on s and a in Theorem 1.1 is optimal.

Theorem 1.2. *Assume that the nonlinearity $N(u)$ includes the term of $u\bar{u}$ (i.e., $\alpha_3 \neq 0$).*

(i) Let $|a| < 1/2$ and $s < -(2a+1)/4$. Then, there exists $t_0 > 0$ such that, for any $t \in (0, t_0]$, even if the flow map for (1.1)

$$S(t) : H^{s,a}(\mathbb{R}) \ni u_0 \longmapsto u(t) \in H^{s,a}(\mathbb{R})$$

is defined up to the time t, $S(t)$ can not be continuous at the origin.

(ii) Let $|a| \geq 1/2$. Then, there exists $t_0 > 0$ such that, for any $t \in (0, t_0]$, even if the flow map as in (i) is defined, $S(t)$ cannot be C^2 at the origin.

Remark 1.1. Since $H^s \subset H^{s,a}$ when $a \geq 0$, by Theorem 1.1 we have the existence of a solution evolving in $H^{s,-(4s+1)/2}$ for $u_0 \in H^s$ with $-1/4 > s > -1/2$. In this case, however, it is not clear whether such a solution remains in H^s. Even if it is true, we know from the above theorem that the flow map is not continuous as a map from H^s to itself.

We next consider a physical example:

$$\begin{cases} (\partial_t^2 - \partial_x^2 + \partial_x^4)u + \partial_x^2(u^2) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where the unknown function u is real or complex valued. (1.5) was referred to as the "good" Boussinesq equation in [10], and the solitary-wave interaction mechanism for (1.5) was studied in [11], [12].

As mentioned below, the local well-posedness of complex-valued (1.5) in $H^s \times H^{s-2}$ can be reduced to that of a system of quadratic nonlinear Schrödinger equations in H^s. So, the result for complex-valued (1.5) in $H^s \times H^{s-2}$, $s > -1/4$ is obtained from the result by Kenig, Ponce and Vega in [7]. Farah [5] also proved that for the data $u_0 = \phi$, $u_1 = \psi_x$ with $(\phi, \psi) \in H^s \times H^{s-1}$, $s > -1/4$ in a slightly different manner.

Note that (1.5) can be rewritten as

$$\begin{cases} (i\partial_t + 1 - \partial_x^2)(-i\partial_t + 1 - \partial_x^2)u = (1 - \partial_x^2)u - \partial_x^2(u^2), \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x). \end{cases}$$

We put

$$v = \frac{-i\partial_t + 1 - \partial_x^2}{1 - \partial_x^2} u = u - i(1 - \partial_x^2)^{-1}\partial_t u,$$

$$\bar{w} = \frac{i\partial_t + 1 - \partial_x^2}{1 - \partial_x^2} u = u + i(1 - \partial_x^2)^{-1}\partial_t u,$$

then the Cauchy problem (1.5) transforms into

$$\begin{cases}(i\partial_t - \partial_x^2)v = -v + (v+\bar{w})/2 + \omega^2(v+\bar{w})^2/4,\\ (i\partial_t - \partial_x^2)w = -w + (\bar{v}+w)/2 + \omega^2(\bar{v}+w)^2/4,\\ v(0,x) = v_0(x),\ w(0,x) = w_0(x),\end{cases} \tag{1.6}$$

where

$$\omega = \mathcal{F}_\xi^{-1}\sqrt{\xi^2/(1+\xi^2)}\mathcal{F}_x,$$
$$v_0 = u_0 - i(1-\partial_x^2)^{-1}u_1, \quad w_0 = \bar{u}_0 - i(1-\partial_x^2)^{-1}\bar{u}_1.$$

It is clear that the transform $(u_0,u_1)\mapsto(v_0,w_0)$; $H^s\times H^{s-2}\to H^s\times H^s$ is Lipschitz continuous. In addition, we can recover (1.5) from a solution (v,w) to (1.6) by putting $u=(v+\bar{w})/2$. We see the relation $-2i\partial_t u = (1-\partial_x^2)(v-\bar{w})$, which implies that the transform

$$(v,\ w) \longmapsto u\,;$$
$$C([0,T]:H^s)\times C([0,T]:H^s) \longrightarrow C([0,T]:H^s)\cap C^1([0,T]:H^{s-2}),$$

is also Lipschitz. Consequently, the local well-posedness of (1.6) in $H^s\times H^s$ yields that of (1.5) in $H^s\times H^{s-2}$. The same holds if we replace H^s with $H^{s,a}$.

If u is real valued, then $v=w$ holds. Therefore, the local well-posedness of real-valued (1.5) in $H^{s,a}\times H^{s-2,a}$ is replaced by that of

$$\begin{cases}(i\partial_t - \partial_x^2)v = -v + (v+\bar{v})/2 + \omega^2(v+\bar{v})^2/4,\\ v(0,x) = v_0(x),\end{cases} \tag{1.7}$$

in $H^{s,a}$. It turns out that the first and the second terms of the right-hand side of (1.7) are easy to estimate if considered as part of the nonlinearity (see the discussion following Proposition 4.3 below). Therefore, the problem can be reduced to the local well-posedness of (1.1) with $N(u)=\omega^2(u^2+\bar{u}^2+2u\bar{u})/4$. Since ω is the Fourier multiplier on L^2, we can neglect it and obtain the local well-posedness of (1.7) in $H^{s,a}$, or that of real-valued (1.5) in $H^{s,a}\times H^{s-2,a}$, under the same assumption as in Theorem 1.1. Obviously, (1.6) can be treated in the same manner as long as we apply the Fourier restriction norm method, so we also have the same result for complex-valued (1.5).

Corollary 1.3. *Let $|a|<1/2$ and $s\geq -(2a+1)/4$. Then, real- or complex-valued (1.5) is locally well-posed in $H^{s,a}\times H^{s-2,a}$.*

Note that the operator ω satisfies $|\widehat{\omega}|\sim|\xi|$ for $|\xi|<1$. Employing this property (see Proposition 4.4), we can improve the result for $a=0$ in this corollary to $s > -1/2$. This is in strong contrast to the ordinary quadratic nonlinear Schrödinger equation, which is ill-posed below $H^{-1/4}$ in the sense of Theorem 1.2.

Theorem 1.4. *Let $s > -1/2$. Then, real- or complex-valued* (1.5) *is locally well-posed in $H^s \times H^{s-2}$.*

We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. Implicit constants may depend on $s, s_1, s_2, a, b, b_1, b_2$ and b_3. The rest of this paper is planned as follows. In Section 2, we give some notation and preliminary lemmas. In Section 3, we prove bilinear estimates. In Section 4, we prove Theorems 1.1, 1.2, Corollary 1.3 and Theorem 1.4.

2. Notation and Preliminary lemmas

Throughout this section and the next, we assume $s < 0$. Note that the result in the case $s \geq 0$ follows from that for $s < 0$; see Section 4 for details.

Put

$$P_1 = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^2| \leq |\xi|/4 \text{ } and \text{ } |\xi| \geq 1\},$$
$$P_2 = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^2| \geq |\xi|/4 \text{ } or \text{ } |\xi| < 1\},$$

and

$$w_{s,a}(\tau, \xi) = \begin{cases} \langle \xi \rangle^s \langle \tau - \xi^2 \rangle, & (\tau, \xi) \in P_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in P_2, \end{cases} \tag{2.1}$$

for $s < 0$. A simple calculation shows that

$$w_{s,a}(\tau, \xi) \sim \min\{\langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle, \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s}\}, \tag{2.2}$$

for $s < 0$ and $a \in \mathbb{R}$, where the implicit constant depends only on s and a.

For $s < 0$ and $a \in \mathbb{R}$, we define function spaces $Z^{s,a}$, $Y^{s,a}$ and $Z_I^{s,a}$ as follows:

$$Z^{s,a} = \{u \in \mathcal{Z}'(\mathbb{R}^2) : \|u\|_{Z^{s,a}} < \infty\}, \quad Y^{s,a} = \{u \in \mathcal{Z}'(\mathbb{R}^2) : \|u\|_{Y^{s,a}} < \infty\},$$
$$Z_I^{s,a} = \{u \in \mathcal{Z}'(\mathbb{R}^2) : \|u\|_{Z_I^{s,a}} < \infty\},$$

where

$$\|u\|_{Z^{s,a}} = \|w_{s,a}\widetilde{u}\|_{L^2_{\tau,\xi}}, \quad \|u\|_{Y^{s,a}} = \|\int \langle \xi \rangle^{s-a} |\xi|^a |\widetilde{u}| \, d\tau\|_{L^2_\xi},$$
$$\|u\|_{Z_I^{s,a}} = \inf\{\|v\|_{Z^{s,a}} | u = v \, \text{for} \, t \in I\}.$$

Note that from (2.1) and (2.2) we have

$$\|u\|_{Z^{s,a}} \sim \|u_1\|_{X^{s,a,1}} + \|u_2\|_{X^{1/2,a,1/2+s}} \lesssim \min\{\|u\|_{X^{s,a,1}},\ \|u\|_{X^{1/2,a,1/2+s}}\}$$

for $s < 0$ and $a \in \mathbb{R}$, where $\widetilde{u_j} = \widetilde{u}|_{P_j}$, $j = 1, 2$ and $\|u\|_{X^{s,a,b}}$ is defined by (1.3). We also define

$$Q_1 = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau + \xi^2| \le |\xi|/4 \textit{ and } |\xi| \ge 1\},$$
$$Q_2 = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau + \xi^2| \ge |\xi|/4 \textit{ or } |\xi| < 1\},$$
$$w'_{s,a}(\tau, \xi) = \begin{cases} \langle \xi \rangle^{s} \langle \tau + \xi^2 \rangle, & (\tau, \xi) \in Q_1, \\ \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s}, & (\tau, \xi) \in Q_2, \end{cases}$$

and $\|u\|_{\bar{Z}^{s,a}} = \|w'_{s,a} \widetilde{u}\|_{L^2_{\tau,\xi}}$. Note that $w_{s,a}(\tau, \xi) = w'_{s,a}(-\tau, -\xi)$ and $\|\bar{u}\|_{Z^{s,a}} = \|u\|_{\bar{Z}^{s,a}}$.

The following lemmas are basic tools of the Fourier restriction norm method.

Lemma 2.1. *Let $0 \le p \le q$ and $p + q > 1$. Then the following estimate holds for all $a, b \in \mathbb{R}$:*

$$\int \langle \tau - a \rangle^{-p} \langle \tau - b \rangle^{-q} \, d\tau \lesssim \langle a - b \rangle^{-r}$$

where $r = p - [1 - q]_+$. (We recall that $[\lambda]_+ = \lambda$ if $\lambda > 0$, $= \varepsilon > 0$ if $\lambda = 0$ and $= 0$ if $\lambda < 0$.)

For the proof of this lemma, see Lemma 4.2 in [6].

For a subset $\Omega \subset \mathbb{R}^4$, we define the characteristic function χ_Ω as follows:

$$\chi_\Omega(\tau, \xi, \tau_1, \xi_1) = \begin{cases} 1, & \text{for } (\tau, \xi, \tau_1, \xi_1) \in \Omega \\ 0, & \text{for } (\tau, \xi, \tau_1, \xi_1) \notin \Omega \end{cases}$$

and put

$$\widetilde{B_\Omega(u, v)} := \int_{\mathbb{R}^2} \chi_\Omega \widetilde{u}(\tau - \tau_1, \xi - \xi_1) \widetilde{v}(\tau_1, \xi_1) \, d\tau_1 d\xi_1.$$

We can show the next two lemmas by using duality, applying the Cauchy-Schwarz inequality twice and the Hölder inequality.

Lemma 2.2. *If*

$$\sup_{\tau, \xi} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) \, d\tau_1 d\xi_1 \lesssim 1$$

or

$$\sup_{\tau_1, \xi_1} \int_{\mathbb{R}^2} \chi_\Omega w_1^{-2}(\tau, \xi) w_2^{-2}(\tau - \tau_1, \xi - \xi_1) w_3^{-2}(\tau_1, \xi_1) \, d\tau d\xi \lesssim 1$$

holds for non-negative measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\|w_1^{-1}\widetilde{B_\Omega(u,v)}\|_{L^2_{\tau,\xi}} \lesssim \|w_2\widetilde{u}\|_{L^2_{\tau,\xi}}\|w_3\widetilde{v}\|_{L^2_{\tau,\xi}}.$$

Lemma 2.3. *If*

$$\sup_{\xi}\int_{\mathbb{R}^3}\chi_\Omega w_1^{-2}(\tau,\xi)w_2^{-2}(\tau-\tau_1,\xi-\xi_1)w_3^{-2}(\tau_1,\xi_1)\,d\tau_1 d\xi_1 d\tau \lesssim 1$$

or

$$\sup_{\xi_1}\int_{\mathbb{R}^3}\chi_\Omega w_1^{-2}(\tau,\xi)w_2^{-2}(\tau-\tau_1,\xi-\xi_1)w_3^{-2}(\tau_1,\xi_1)\,d\tau d\xi d\tau_1 \lesssim 1$$

holds for non-negative measurable functions w_1, w_2 and w_3 on $\mathbb{R}^2$, then we have

$$\|\int w_1^{-1}|\widetilde{B_\Omega(u,v)}|\,d\tau\|_{L^2_\xi} \lesssim \|w_2\widetilde{u}\|_{L^2_{\tau,\xi}}\|w_3\widetilde{v}\|_{L^2_{\tau,\xi}}.$$

Let $\widehat{P_l f} = \widehat{f}|_{|\xi|<1}$ and $\langle\cdot,\cdot\rangle_{L^2}$ be the inner product in L^2. The following lemma is a variant of the Sobolev inequality.

Lemma 2.4. (*i*) *Let $b_1+b_2+b_3>1/2, b_1\ge 0, b_2\ge 0, b_3\ge 0$ and $\alpha,\beta,\gamma\in\mathbb{R}$. Then, we have*

$$\langle fg,h\rangle_{L^2_x} \lesssim \|\langle\xi-\alpha\rangle^{b_1}\widehat{f}\|_{L^2_\xi}\|\langle\xi-\beta\rangle^{b_2}\widehat{g}\|_{L^2_\xi}\|\langle\xi-\gamma\rangle^{b_3}\widehat{h}\|_{L^2_\xi}, \tag{2.3}$$

where the implicit constant depends only on b_1, b_2 and b_3.

(*ii*) *Let $s_1+s_2+s_3>1/2, s_1+s_2\ge 0, s_2+s_3\ge 0$ and $s_3+s_1\ge 0$. Then, we have*

$$\langle fg,h\rangle_{L^2_x} \lesssim \|\langle\xi\rangle^{s_1}\widehat{f}\|_{L^2_\xi}\|\langle\xi\rangle^{s_2}\widehat{g}\|_{L^2_\xi}\|\langle\xi\rangle^{s_3}\widehat{h}\|_{L^2_\xi}, \tag{2.4}$$

where the implicit constant depends only on s_1, s_2 and s_3.

(*iii*) *Let $-1/2<a<1/2$. Then, we have*

$$\langle(P_l f)g,h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi}\|\widehat{g}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}, \tag{2.5}$$

$$\langle(P_l f)(P_l g),h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi}\||\xi|^{-a}\widehat{g}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}, \tag{2.6}$$

$$\langle(P_l f)(P_l g),P_l h\rangle_{L^2_x} \lesssim \||\xi|^a\widehat{f}\|_{L^2_\xi}\||\xi|^a\widehat{g}\|_{L^2_\xi}\||\xi|^{-a}\widehat{h}\|_{L^2_\xi}, \tag{2.7}$$

where the implicit constants depend only on a.

Proof. By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\begin{aligned}\langle fg,h\rangle_{L^2_x} \sim \langle \widehat{f}*\widehat{g},\widehat{h}\rangle_{L^2_\xi} &\lesssim \|\widehat{f}\|_{L^{p_1}_\xi}\|\widehat{g}\|_{L^{p_2}_\xi}\|\widehat{h}\|_{L^{p_3}_\xi}\\ &\lesssim \|\langle\xi-\alpha\rangle^{-b_1}\|_{L^{q_1}_\xi}\|\langle\xi-\beta\rangle^{-b_2}\|_{L^{q_2}_\xi}\|\langle\xi-\gamma\rangle^{-b_3}\|_{L^{q_3}_\xi}\\ &\quad\times\|\langle\xi-\alpha\rangle^{b_1}\widehat{f}\|_{L^2_\xi}\|\langle\xi-\beta\rangle^{b_2}\widehat{g}\|_{L^2_\xi}\langle\xi-\gamma\rangle^{b_3}\widehat{h}\|_{L^2_\xi},\end{aligned}$$

for any $1\le p_j\le 2$ and $2\le q_j\le\infty$ satisfying $1/p_1+1/p_2+1/p_3=2$ and $1/q_j+1/2=1/p_j$. Since $b_1+b_2+b_3>1/2$ and $1/q_1+1/q_2+1/q_3=1/2$, we can take q_j such that $q_j>1/b_j$ for $b_j>0$ and $q_j=\infty$ for $b_j=0$. Thus, we obtain (2.3).

For the proof of (2.4), we can assume $s_1\ge s_2\ge s_3$ without loss of generality. Since the case $s_3\ge 0$ follows from (2.3) and the case $s_2<0$ is excluded by the assumption $s_2+s_3\ge 0$, we only need to show the case $s_2\ge 0>s_3$. By using the triangle inequality $\langle\xi\rangle\le\langle\xi_1\rangle+\langle\xi-\xi_1\rangle$ and the Plancherel theorem, we have

$$\begin{aligned}\langle fg,h\rangle_{L^2_x} \sim&\Big\langle \int \widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)\,d\xi_1,\widehat{h}(\xi)\Big\rangle_{L^2_\xi}\\ \lesssim&\Big\langle \int \widehat{f}(\xi-\xi_1)\langle\xi_1\rangle^{-s_3}\widehat{g}(\xi_1)\,d\xi_1,\langle\xi\rangle^{s_3}\widehat{h}(\xi)\Big\rangle_{L^2_\xi}\\ &+\Big\langle \int \langle\xi-\xi_1\rangle^{-s_3}\widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)\,d\xi_1,\langle\xi\rangle^{s_3}\widehat{h}(\xi)\Big\rangle_{L^2_\xi}.\end{aligned}$$

Therefore, this case also follows from (2.3).

By the Plancherel theorem, the Hölder inequality and the Young inequality, we have

$$\langle (P_lf)g,h\rangle_{L^2_x}\sim\langle\widehat{P_lf}*\widehat{g},\widehat{h}\rangle_{L^2_\xi}\lesssim\|\widehat{P_lf}\|_{L^1_\xi}\|\widehat{g}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}.$$

Since $\|\widehat{P_lf}\|_{L^1_\xi}\le\||\xi|^{-a}\|_{L^2_\xi(-1,1)}\||\xi|^a\widehat{f}\|_{L^2_\xi}\lesssim\||\xi|^a\widehat{f}\|_{L^2_\xi}$, we obtain (2.5).

For the proof of (2.6), we can assume $a\ge 0$ without loss of generality. From (2.5), we have

$$\langle (P_lf)(P_lg),h\rangle_{L^2_x}\lesssim\||\xi|^a\widehat{f}\|_{L^2_\xi}\|\widehat{P_lg}\|_{L^2_\xi}\|\widehat{h}\|_{L^2_\xi}.$$

Since $\|\widehat{P_lg}\|_{L^2_\xi}\le\||\xi|^{-a}\widehat{g}\|_{L^2_\xi}$, we obtain (2.6).

For the proof of (2.7), we have

$$\langle (P_lf)(P_lg),P_lh\rangle_{L^2_x}\sim\Big\langle\int\widehat{P_lf}(\xi-\xi_1)\widehat{P_lg}(\xi_1)\,d\xi_1,\widehat{P_lh}(\xi)\Big\rangle_{L^2_\xi},$$

from the Plancherel theorem. If $a \geq 0$, (2.7) is reduced to (2.5) by the estimate $\max\{|\xi-\xi_1|^a, |\xi_1|^a\} \gtrsim |\xi|^a$. For the case $a<0$, (2.7) follows from

$$\langle (P_l f)(P_l g), P_l h\rangle_{L_x^2} \lesssim \|\widehat{P_l f}\|_{L_\xi^2}\|\widehat{P_l g}\|_{L_\xi^2}\|\widehat{P_l h}\|_{L_\xi^1}$$

and $\|\widehat{P_l f}\|_{L_\xi^2} \leq \||\xi|^a \widehat{f}\|_{L_\xi^2}$, $\|\widehat{P_l h}\|_{L_\xi^1} \leq \||\xi|^a\|_{L_\xi^2(-1,1)}\||\xi|^{-a}\widehat{h}\|_{L_\xi^2} \lesssim \||\xi|^{-a}\widehat{h}\|_{L_\xi^2}$. □

From this lemma, we obtain the following space-time estimates.

Proposition 2.5. *Let $b_1+b_2+b_3>1/2, b_1\geq 0, b_2\geq 0, b_3\geq 0$ and $\mu_1,\mu_2,\mu_3\in\{\pm 1\}$.*
(i) In addition, we assume that $s_1+s_2+s_3>1/2, s_1+s_2\geq 0, s_2+s_3\geq 0$ and $s_3+s_1\geq 0$. Then, we have

$$\begin{aligned}&\langle fg, h\rangle_{L_{t,x}^2}\\ &\lesssim \|\langle\xi\rangle^{s_1}\langle\tau-\mu_1\xi^2\rangle^{b_1}\widetilde{f}\|_{L_{\tau,\xi}^2}\|\langle\xi\rangle^{s_2}\langle\tau-\mu_2\xi^2\rangle^{b_2}\widetilde{g}\|_{L_{\tau,\xi}^2}\|\langle\xi\rangle^{s_3}\langle\tau-\mu_3\xi^2\rangle^{b_3}\widetilde{h}\|_{L_{\tau,\xi}^2}.\end{aligned} \tag{2.8}$$

(ii) In addition, we assume $-1/2<a<1/2$. Then, we have

$$\begin{aligned}&\langle (P_l f)g, h\rangle_{L_{t,x}^2}\\ &\lesssim \||\xi|^a\langle\tau-\mu_1\xi^2\rangle^{b_1}\widetilde{f}\|_{L_{\tau,\xi}^2}\|\langle\tau-\mu_2\xi^2\rangle^{b_2}\widetilde{g}\|_{L_{\tau,\xi}^2}\|\langle\tau-\mu_3\xi^2\rangle^{b_3}\widetilde{h}\|_{L_{\tau,\xi}^2},\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\langle (P_l f)(P_l g), h\rangle_{L_{t,x}^2}\\ &\lesssim \||\xi|^a\langle\tau-\mu_1\xi^2\rangle^{b_1}\widetilde{f}\|_{L_{\tau,\xi}^2}\||\xi|^{-a}\langle\tau-\mu_2\xi^2\rangle^{b_2}\widetilde{g}\|_{L_{\tau,\xi}^2}\|\langle\tau-\mu_3\xi^2\rangle^{b_3}\widetilde{h}\|_{L_{\tau,\xi}^2},\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\langle (P_l f)(P_l g), P_l h\rangle_{L_{t,x}^2}\\ &\lesssim \||\xi|^a\langle\tau-\mu_1\xi^2\rangle^{b_1}\widetilde{f}\|_{L_{\tau,\xi}^2}\||\xi|^{a}\langle\tau-\mu_2\xi^2\rangle^{b_2}\widetilde{g}\|_{L_{\tau,\xi}^2}\||\xi|^{-a}\langle\tau-\mu_3\xi^2\rangle^{b_3}\widetilde{h}\|_{L_{\tau,\xi}^2}.\end{aligned} \tag{2.11}$$

Proof. Fix $\xi,\xi_1\in\mathbb{R}$. Then, from (2.3), we have

$$\begin{aligned}&\int \widetilde{f}(\tau_1,\xi_1)\widetilde{g}(\tau-\tau_1,\xi-\xi_1)\widetilde{h}(\tau,\xi)\,d\tau_1 d\tau\\ \lesssim&\|\langle\cdot-\mu_1\xi_1^2\rangle^{b_1}\widetilde{f}(\cdot,\xi_1)\|_{L_\tau^2}\|\langle\cdot-\mu_2(\xi-\xi_1)^2\rangle^{b_2}\widetilde{g}(\cdot,\xi-\xi_1)\|_{L_\tau^2}\\ &\times\|\langle\cdot-\mu_3\xi^2\rangle^{b_3}\widetilde{h}(\cdot,\xi)\|_{L_\tau^2},\end{aligned}$$

where the implicit constant does not depend on ξ, ξ_1. Therefore, the left-hand side of (2.8) is bounded by

$$\int \|\langle\cdot - \mu_1\xi_1^2\rangle^{b_1}\widetilde{f}(\cdot,\xi_1)\|_{L^2_\tau}\|\langle\cdot - \mu_2(\xi-\xi_1)^2\rangle^{b_2}\widetilde{g}(\cdot,\xi-\xi_1)\|_{L^2_\tau} \times \|\langle\cdot - \mu_3\xi^2\rangle^{b_3}\widetilde{h}(\cdot,\xi)\|_{L^2_\tau}\,d\xi_1 d\xi,$$

which is bounded by the right-hand side of (2.8) by (2.4). In the same manner, (2.9)–(2.11) follow from (2.3), (2.5)–(2.7). □

Let $U(t)$ be the propagator of the linear Schrödinger equation, namely, $\widehat{U}(t) = \exp(it\xi^2)$, and $\varphi(t)$ be a smooth cut-off function satisfying $\varphi(t) = 1$ for $|t| < 1$ and $= 0$ for $|t| > 2$. We give linear estimates for homogeneous and inhomogeneous Schrödinger equations.

Proposition 2.6. *Let $s < 0$, $a \in \mathbb{R}$ and $u = \varphi(t)U(t)u_0$. Then, we have*

$$\|u\|_{Z^{s,a}} + \|u\|_{L^\infty_t(H^{s,a}_x)} \lesssim \|u_0\|_{H^{s,a}}.$$

Proof. Recall that $\|u\|_{Z^{s,a}} \lesssim \|u\|_{X^{s,a,1}}$. By the Cauchy-Schwarz inequality, we also have

$$\|u\|_{L^\infty_t(H^{s,a}_x)} = c\sup_t \|\langle\xi\rangle^{s-a}|\xi|^a\int_{\mathbb{R}} e^{it\tau}\widetilde{u}(\tau,\xi)\,d\tau\|_{L^2_\xi} \lesssim \|u\|_{X^{s,a,1}}.$$

Now the claim follows from the relation

$$\|\varphi(t)U(t)u_0\|_{X^{s,a,b}} = \|\varphi(t)u_0\|_{H^b_t H^{s,a}_x} = \|\varphi\|_{H^b}\|u_0\|_{H^{s,a}}. \qquad \square$$

Proposition 2.7. *Let $-1/2 \le s < 0$, $a \in \mathbb{R}$ and*

$$u = \varphi(t)\int_0^t U(t-t')F(t')\,dt'.$$

Then, we have

$$\|u\|_{Z^{s,a}} + \|u\|_{L^\infty_t(H^{s,a}_x)} \lesssim \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{F}\|_{Z^{s,a}} + \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{F}\|_{Y^{s,a}}.$$

Proof. For the estimate of $\|u\|_{Z^{s,a}}$, we divide $\widetilde{F} = \widetilde{F}|_{P_1} + \widetilde{F}|_{P_2} =: \widetilde{F}_1 + \widetilde{F}_2$, and $u = u_1 + u_2$, correspondingly. Note that

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{F}\|_{Z^{s,a}} \sim \|F_1\|_{X^{s,a,0}} + \|F_2\|_{X^{1/2,a,s-1/2}}.$$

We consider the following subdivision of u_2:

$$\mathcal{F}_x u_2(t,\xi) = \varphi(t)\int_0^t e^{i(t-t')\xi^2} c\int e^{it'\tau}(\widetilde{F}_{2,+} + \widetilde{F}_{2,-})(\tau,\xi)\,d\tau\,dt',$$

where $\widetilde{F}_{2,\pm} := \widetilde{F}_2|_{|\tau-\xi^2|\gtrless 1}$, and then

$$= c\varphi(t)\int \frac{e^{it\tau}}{i(\tau-\xi^2)}\widetilde{F}_{2,+}(\tau,\xi)\,d\tau - c\varphi(t)\int \frac{e^{it\xi^2}}{i(\tau-\xi^2)}\widetilde{F}_{2,+}(\tau,\xi)\,d\tau$$
$$+\ c\int \frac{e^{it\tau}\varphi(t)-e^{it\xi^2}\varphi(t)}{i(\tau-\xi^2)}\widetilde{F}_{2,-}(\tau,\xi)\,d\tau =: \mathcal{F}_x(u_{21}+u_{22}+u_{23}).$$

Following the proof of Lemma 2.1 in [6] we can verify that

$$\begin{aligned}
\|u_1\|_{Z^{s,a}} &\lesssim \|u_1\|_{X^{s,a,1}} \lesssim \|F_1\|_{X^{s,a,0}},\\
\|u_{21}\|_{Z^{s,a}} &\lesssim \|u_{21}\|_{X^{1/2,a,s+1/2}} \lesssim \|F_{2,+}\|_{X^{1/2,a,s-1/2}},\\
\|u_{22}\|_{Z^{s,a}} &\lesssim \|u_{22}\|_{X^{s,a,1}} \lesssim \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{F}_{2,+}\|_{Y^{s,a}}, \quad \text{and}\\
\|u_{23}\|_{Z^{s,a}} &\lesssim \|u_{23}\|_{X^{s,a,1}} \lesssim \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{F}_{2,-}\|_{Y^{s,a}},
\end{aligned}$$

which implies the desired estimate. Finally, the estimate of $\|u\|_{L_t^\infty(H_x^{s,a})}$ follows from

$$\mathcal{F}_x u(t,\xi) = c\varphi(t)e^{it\xi^2}\int \frac{e^{it(\tau-\xi^2)}-1}{i(\tau-\xi^2)}\widetilde{F}(\tau,\xi)\,d\tau,$$

and the estimate

$$\left|\frac{e^{it(\tau-\xi^2)}-1}{i(\tau-\xi^2)}\right| \lesssim \langle t\rangle\langle\tau-\xi^2\rangle^{-1}. \qquad \square$$

We next consider the scaling property. (1.1) with (1.2) is invariant under the following scaling:

$$u_\lambda(t,x) = \lambda^{-2}u(\lambda^{-2}t,\lambda^{-1}x), \qquad \lambda>0;$$

namely, if u satisfies (1.1) with (1.2), then u_λ satisfies the same equation.

Proposition 2.8. *Let $-1/2\le s<0$ and $a\in\mathbb{R}$. For $\lambda>1$, we have*

$$\|u_\lambda(0,\cdot)\|_{H^{s,a}} \lesssim \lambda^{-3/2-\min\{a,s\}}\|u(0,\cdot)\|_{H^{s,a}}, \tag{2.12}$$
$$\|u_\lambda\|_{Z^{s,a}} \lesssim \lambda^{-1/2-\min\{a,s\}}\|u\|_{Z^{s,a}}. \tag{2.13}$$

Proof. Since $\widehat{u}_\lambda(t,\xi) = \lambda^{-1}\widehat{u}(\lambda^{-2}t,\lambda\xi)$, we have

$$\begin{aligned}
\|u_\lambda(0,x)\|_{H^{s,a}} &= \|\langle\xi\rangle^{s-a}|\xi|^a\lambda^{-1}\widehat{u}(0,\lambda\xi)\|_{L_\xi^2},\\
&= \|\langle\lambda^{-1}\xi\rangle^{s-a}|\lambda^{-1}\xi|^a\lambda^{-3/2}\widehat{u}(0,\xi)\|_{L_\xi^2}.
\end{aligned}$$

Therefore, (2.12) follows from

$$\langle\lambda^{-1}\xi\rangle^{s-a}|\lambda^{-1}\xi|^{a}\lambda^{-3/2} \lesssim \lambda^{-3/2-\min\{a,s\}}\langle\xi\rangle^{s-a}|\xi|^{a}.$$

Since $\widetilde{u}_\lambda(\tau,\xi)=\lambda\widetilde{u}(\lambda^2\tau,\lambda\xi)$, we have

$$\begin{aligned}\|u_\lambda(t,x)\|_{Z^{s,a}} &= \|w_{s,a}(\tau,\xi)\lambda\widetilde{u}(\lambda^2\tau,\lambda\xi)\|_{L^2_{\tau,\xi}},\\ &= \|w_{s,a}(\lambda^{-2}\tau,\lambda^{-1}\xi)\lambda^{-1/2}\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}.\end{aligned}$$

Recall (2.2). Then, we have

$$\begin{aligned}&w_{s,a}(\lambda^{-2}\tau,\lambda^{-1}\xi)\sim \min\{\langle\lambda^{-1}\xi\rangle^{s-a}|\lambda^{-1}\xi|^{a}\langle\lambda^{-2}(\tau-\xi^2)\rangle,\\ &\qquad\qquad \langle\lambda^{-1}\xi\rangle^{1/2-a}|\lambda^{-1}\xi|^{a}\langle\lambda^{-2}(\tau-\xi^2)\rangle^{1/2+s}\}\\ &\lesssim \min\{\lambda^{-\min\{a,s\}}\langle\xi\rangle^{s-a}|\xi|^{a}\langle\tau-\xi^2\rangle, \lambda^{-\min\{a,1/2\}}\langle\xi\rangle^{1/2-a}|\xi|^{a}\langle\tau-\xi^2\rangle^{1/2+s}\}\\ &\lesssim \lambda^{-\min\{a,s\}}w_{s,a}(\tau,\xi),\end{aligned}$$

which yields (2.13). □

3. Bilinear estimates

This section will be devoted to the proof of the main bilinear estimates.

Proposition 3.1. *Let $|a|<1/2$ and $0>s\geq -(2a+1)/4$. Then the following estimates hold:*

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{u\bar{v}}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{Z^{s,a}}, \tag{3.1}$$

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{u\bar{v}}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{Z^{s,a}}. \tag{3.2}$$

Moreover, the same estimates hold with $u\bar{v}$ replaced by uv or $\bar{u}\bar{v}$.

Proof. Note that $0>s>-1/2$. We first consider (3.1), which is equivalent to

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{uv}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}.$$

Put

$$\Omega_{i,j,k}=\{(\tau,\xi,\tau_1,\xi_1):(\tau,\xi)\in P_i,(\tau-\tau_1,\xi-\xi_1)\in P_j,(\tau_1,\xi_1)\in Q_k\}$$

for $i,j,k=1$ or 2. Then, we have

$$B_{\mathbb{R}^4}(u,v)=\sum_{i,j,k}B_{\Omega_{i,j,k}}(u,v).$$

Therefore, we only need to show

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}, \tag{3.3}$$

with $\Omega = \Omega_{i,j,k}$ for $i,j,k = 1$ or 2. Moreover, put

$$M_1 = \max\{|\tau-\xi^2|,\, |\tau-\tau_1-(\xi-\xi_1)^2|,\, |\tau_1+\xi_1^2|\};$$

then we have the following algebraic property:

$$M_1 \geq (|\tau-\xi^2| + |\tau-\tau_1-(\xi-\xi_1)^2| + |\tau_1+\xi_1^2|)/3 \geq 2|\xi\xi_1|/3,$$

which plays an important role in our proof.

(a-1) We prove that $\Omega_{1,1,1}$ is empty. If $M_1 = |\tau-\xi^2|$ and $(\tau,\xi)\in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi|/4$. Therefore, we have $|\xi_1| \leq 3/8$, which contradicts $(\tau_1,\xi_1)\in Q_1$. If $M_1 = |\tau_1+\xi_1^2|$ and $(\tau_1,\xi_1)\in Q_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi_1|/4$. Therefore, we have $|\xi| \leq 3/8$, which contradicts $(\tau,\xi)\in P_1$. If $M_1 = |\tau-\tau_1+(\xi-\xi_1)^2|$ and $(\tau-\tau_1,\xi-\xi_1)\in P_1$, then $2|\xi\xi_1|/3 \leq M_1 \leq |\xi-\xi_1|/4 \leq \max\{|\xi|,|\xi_1|\}/2$. Therefore, we have $|\xi|\leq 3/4$ or $|\xi_1|\leq 3/4$, which contradicts $(\tau,\xi)\in P_1$ and $(\tau_1,\xi_1)\in Q_1$. Thus, we obtain (3.3) with $\Omega=\Omega_{1,1,1}$.

(a-2) (3.3) with $\Omega=\Omega_{2,1,1}$ is equivalent to

$$\begin{aligned}&\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,1,1}}(u,v)}\|_{L^2_{\tau,\xi}}\\ &\lesssim \|\langle\xi\rangle^s\langle\tau-\xi^2\rangle\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}.\end{aligned} \tag{3.4}$$

We divide $\Omega_{2,1,1}$ into two parts:

$$A_1 = \{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,1,1} : |\xi|<1\},\quad A_2 = \{(\tau,\xi,\tau_1,\xi_1)\in\Omega_{2,1,1} : |\xi|\geq 1\}.$$

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_1 can be reduced to

$$\sup_{\tau_1,\xi_1}\int \frac{\chi_{A_1}\langle\xi_1\rangle^{-2s}|\xi|^{2a}\langle\xi-\xi_1\rangle^{-2s}}{\langle\tau_1+\xi_1^2\rangle^2\langle\tau-\xi^2\rangle^{1-2s}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^2}\,d\tau d\xi \lesssim 1.$$

Since $\langle M_1\rangle \gtrsim \langle\xi\xi_1\rangle$ and $\langle\xi_1\rangle\sim\langle\xi-\xi_1\rangle\sim|\xi_1|$, from Lemma 2.1, the left-hand side is bounded by

$$\int\frac{|\xi|^{2a}|\xi_1|^{-4s}}{\langle M_1\rangle^{1-2s}}\,d\xi \lesssim \int\frac{|\xi\xi_1|^{-4s-1}}{\langle\xi\xi_1\rangle^{1-2s}}|\xi_1|\,d\xi \lesssim \int\frac{|p|^{-4s-1}}{\langle p\rangle^{1-2s}}\,dp \lesssim 1.$$

Here, we put $p=\xi\xi_1$ and used $2a\geq -4s-1$ and $1-2s>-4s$.

From Lemma 2.2, (3.4) with $\Omega_{2,1,1}$ replaced by A_2 can be reduced to

$$\sup_{\tau,\xi} \int \frac{\chi_{A_2}|\xi|\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\tau-\xi^2\rangle^{1-2s}\langle\tau_1+\xi_1^2\rangle^2\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^2}\, d\tau_1 d\xi_1 \lesssim 1.$$

In the same manner as (a-1), it follows that $M_1 = \langle\tau-\xi^2\rangle \sim \langle\xi\xi_1\rangle$ from $(\tau-\tau_1,\xi-\xi_1) \in P_1, (\tau_1,\xi_1) \in Q_1$ and $|\xi| \geq 1$. Therefore, from Lemma 2.1, the left-hand side is bounded by

$$\int \frac{\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\xi\xi_1\rangle^{1-2s}\langle\tau-\xi^2+2\xi\xi_1\rangle^2}|\xi|\, d\xi_1 \lesssim \int \frac{\langle p\rangle^{-4s}}{\langle p\rangle^{1-2s}\langle\tau-\xi^2+2p\rangle^2}\, dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $1-2s \geq -4s$.

(a-3) (3.3) with $\Omega = \Omega_{1,2,1}$ is equivalent to

$$\|\langle\xi\rangle^s \widetilde{B_{\Omega_{1,2,1}}(u,v)}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}. \tag{3.5}$$

We divide $\Omega_{1,2,1}$ into two parts:

$$A_1 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{1,2,1} : |\xi-\xi_1| < 1\},$$
$$A_2 = \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{1,2,1} : |\xi-\xi_1| \geq 1\}.$$

Since $\langle\xi\rangle \sim \langle\xi_1\rangle$ and $\langle\xi-\xi_1\rangle \sim 1$ in A_1, (3.5) with $\Omega_{1,2,1}$ replaced by A_1 can be reduced to

$$\|\widetilde{(P_l u)v}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.9) in Proposition 2.5.

Since $\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle \gtrsim \langle\xi-\xi_1\rangle$ in A_2, (3.5) with $\Omega_{1,2,1}$ replaced by A_2 can be reduced to

$$\|\langle\xi\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.8) in Proposition 2.5.

(a-4) (3.3) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$\begin{aligned} &\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,2,1}}(u,v)}\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.6}$$

We divide $\Omega_{2,2,1}$ into four parts:

$$\begin{aligned}
A_1 &= \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} : |\xi| < 1, |\xi-\xi_1| < 1\},\\
A_2 &= \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} : |\xi| < 1, |\xi-\xi_1| \geq 1\},\\
A_3 &= \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} : |\xi| \geq 1, |\xi-\xi_1| < 1\},\\
A_4 &= \{(\tau,\xi,\tau_1,\xi_1) \in \Omega_{2,2,1} : |\xi| \geq 1, |\xi-\xi_1| \geq 1\}.
\end{aligned}$$

Since $\langle\xi\rangle \sim \langle\xi_1\rangle \sim \langle\xi-\xi_1\rangle \sim 1$ in A_1, (3.6) with $\Omega_{2,2,1}$ replaced by A_1 can be reduced to

$$\||\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s} P_l\widetilde{\{(P_l u)v\}}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from the duality argument and (2.10) in Proposition 2.5.

Since $\langle\xi\rangle \sim 1$ and $\langle\xi_1\rangle \sim \langle\xi-\xi_1\rangle$ in A_2, (3.6) with $\Omega_{2,2,1}$ replaced by A_2 can be reduced to

$$\||\xi|^a\langle\tau-\xi^2\rangle^{-1/2+s} \widetilde{P_l(uv)}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2+s}\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.9) in Proposition 2.5.

Since $|\tau-\xi^2| \gtrsim |\xi| \sim |\xi_1| \gtrsim 1$, $\langle\xi-\xi_1\rangle \sim 1$ in A_3, (3.6) with $\Omega_{2,2,1}$ replaced by A_3 can be reduced to

$$\|\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \||\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.9) in Proposition 2.5.

Since $|\xi| \sim |\xi_1| \gtrsim 1$, $\langle\tau-\xi^2\rangle \gtrsim \langle\xi\rangle$ and $\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle \gtrsim \langle\xi-\xi_1\rangle$ in A_4, (3.6) with $\Omega_{2,2,1}$ replaced by A_4 can be reduced to

$$\|\langle\xi\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

(a-5) We can prove (3.3) with $\Omega = \Omega_{1,1,2}$ in the same manner as (a-3).

(a-6) We can prove (3.3) with $\Omega = \Omega_{2,1,2}$ in the same manner as (a-4).

(a-7) Since $\langle\xi\rangle^s \leq \langle\tau-\xi^2\rangle^s$ in $\Omega_{1,2,2}$, (3.3) with $\Omega = \Omega_{1,2,2}$ can be reduced to

$$\begin{aligned}
&\|\langle\tau-\xi^2\rangle^s \widetilde{B_{\Omega_{1,2,2}}(u,v)}\|_{L^2_{\tau,\xi}}\\
&\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau-\xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau+\xi^2\rangle^{1/2+s}\widetilde{v}\|_{L^2_{\tau,\xi}}.
\end{aligned} \tag{3.7}$$

We divide $\Omega_{1,2,2}$ into three parts:

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} : |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\},$$
$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} : |\xi - \xi_1| \geq 1/2, |\xi| < 1/2\},$$
$$A_3 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{1,2,2} : |\xi - \xi_1| \geq 1/2, |\xi| \geq 1/2\}.$$

(3.7) with $\Omega_{1,2,2}$ replaced by A_1 or A_2 follows from (2.9) in Proposition 2.5 and (3.7) with $\Omega_{1,2,2}$ replaced by A_3 follows from (2.8) in Proposition 2.5.

(a-8) (3.3) with $\Omega = \Omega_{2,2,2}$ is equivalent to

$$\begin{aligned} &\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau - \xi^2\rangle^{-1/2+s}\widetilde{B_{\Omega_{2,2,2}}(u,v)}\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau - \xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau + \xi^2\rangle^{1/2+s}\widetilde{v}\|_{L^2_{\tau,\xi}}. \end{aligned} \tag{3.8}$$

We divide $\Omega_{2,2,2}$ into seven parts:

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| < 1, |\xi - \xi_1| < 1/2, |\xi_1| < 1/2\},$$
$$A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| < 1, |\xi - \xi_1| \geq 1/2, |\xi_1| < 1/2\},$$
$$A_3 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| < 1, |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\},$$
$$A_4 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| < 1, |\xi - \xi_1| \geq 1/2, |\xi_1| \geq 1/2\},$$
$$A_5 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| \geq 1, |\xi - \xi_1| \geq 1/2, |\xi_1| < 1/2\},$$
$$A_6 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| \geq 1, |\xi - \xi_1| < 1/2, |\xi_1| \geq 1/2\},$$
$$A_7 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,2,2} : |\xi| \geq 1, |\xi - \xi_1| \geq 1/2, |\xi_1| \geq 1/2\}.$$

(3.8) with $\Omega_{2,2,2}$ replaced by A_1 follows from (2.11) in Proposition 2.5, (3.8) with $\Omega_{2,2,2}$ replaced by A_2 or A_3 follows from (2.10) in Proposition 2.5 and (3.8) with $\Omega_{2,2,2}$ replaced by A_4 or A_5 or A_6 follows from (2.9) in Proposition 2.5. Since $\langle\xi\rangle^{1/2-a}|\xi|^a\langle\tau - \xi^2\rangle^{-1/2+s} \leq \langle\tau - \xi^2\rangle^s$ in A_7, (3.8) with $\Omega_{2,2,2}$ replaced by A_7 can be reduced to

$$\|\langle\tau - \xi^2\rangle^s\widetilde{uv}\|_{L^2_{\tau,\xi}} \lesssim \|\langle\xi\rangle^{1/2}\langle\tau - \xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^{1/2}\langle\tau + \xi^2\rangle^{1/2+s}\widetilde{v}\|_{L^2_{\tau,\xi}},$$

which follows from (2.8) in Proposition 2.5.

We next consider (3.2), which is equivalent to

$$\|\mathcal{F}^{-1}\langle\tau - \xi^2\rangle^{-1}\widetilde{uv}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}.$$

From the Cauchy-Schwarz inequality in τ, we have

$$\|\mathcal{F}^{-1}\langle\tau - \xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Y^{s,a}} \lesssim \|\mathcal{F}^{-1}\langle\tau - \xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{X^{s,a}},$$

for $\Omega = \Omega_{1,j,k}$ with $j, k = 1$ or 2, so we only need to show

$$\|\mathcal{F}^{-1}\langle \tau - \xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{\bar{Z}^{s,a}}, \tag{3.9}$$

for $\Omega = \Omega_{2,j,k}$ with $j, k = 1$ or 2.

(b-1) We divide $\Omega_{2,1,1}$ into two parts:

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} : |\xi| < 1\}, \quad A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} : |\xi| \geq 1\}.$$

Since $\langle \xi_1\rangle \sim \langle \xi - \xi_1\rangle \sim |\xi_1| \gtrsim 1$ in A_1, from Lemma 2.3, (3.9) with $\Omega = A_1$ can be reduced to

$$\sup_{\xi_1} \int \frac{\chi_{A_1}|\xi|^{2a}|\xi_1|^{-4s}}{\langle \tau - \xi^2\rangle^2 \langle \tau_1 + \xi_1^2\rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2\rangle^2}\, d\tau_1 d\tau d\xi \lesssim 1.$$

Since $\langle M_1\rangle \gtrsim \langle \xi\xi_1\rangle$, from Lemma 2.1, the left-hand side is bounded by

$$\int \frac{|\xi|^{2a}|\xi_1|^{-4s}}{\langle M_1\rangle^2}\, d\xi \lesssim \int \frac{|\xi\xi_1|^{-4s-1}}{\langle \xi\xi_1\rangle^2}|\xi_1|\, d\xi \lesssim \int \frac{|p|^{-4s-1}}{\langle p\rangle^2}\, dp \lesssim 1.$$

Here, we put $p = \xi\xi_1$ and used $2a \geq -4s - 1$, $2 > -4s$.

From Lemma 2.3, (3.9) with $\Omega = A_2$ can be reduced to

$$\sup_{\xi_1} \int \frac{\chi_{A_2}\langle \xi_1\rangle^{-2s}\langle \xi\rangle^{2s}\langle \xi - \xi_1\rangle^{-2s}}{\langle \tau - \xi^2\rangle^2 \langle \tau_1 + \xi_1^2\rangle^2 \langle \tau - \tau_1 - (\xi - \xi_1)^2\rangle^2}\, d\tau_1 d\xi d\tau \lesssim 1.$$

In this case, the left-hand side is bounded by

$$\int \frac{\langle \xi_1\rangle^{-2s}\langle \xi\rangle^{2s}(\langle \xi_1\rangle^{-2s} + \langle \xi\rangle^{-2s})}{\langle \xi\xi_1\rangle^2} d\xi \lesssim \int \frac{\langle \xi_1\rangle^{-4s}\langle \xi\rangle^{2s}}{\langle \xi\xi_1\rangle^2}\, d\xi + \int \frac{\langle \xi_1\rangle^{-2s}}{\langle \xi\xi_1\rangle^2} d\xi \lesssim 1.$$

(b-2) (3.9) with $\Omega = \Omega_{2,2,1}$ is equivalent to

$$\begin{aligned} &\Big\| \int \langle \xi\rangle^{s-a}|\xi|^a \langle \tau - \xi^2\rangle^{-1}|\widetilde{B_{\Omega_{2,2,1}}(u,v)}|\, d\tau \Big\|_{L^2_\xi} \\ &\lesssim \|\langle \xi\rangle^{1/2-a}|\xi|^a\langle \tau - \xi^2\rangle^{1/2+s}\widetilde{u}\|_{L^2_{\tau,\xi}} \|\langle \xi\rangle^s \langle \tau + \xi^2\rangle \widetilde{v}\|_{L^2_{\tau,\xi}}, \end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-4) because the left-hand side is bounded by

$$\|\langle \xi\rangle^{s-a}|\xi|^a\langle \tau - \xi^2\rangle^{-1/2+\varepsilon}\widetilde{B_{\Omega_{2,2,1}}(u,v)}\|_{L^2_{\tau,\xi}},$$

for any $\varepsilon > 0$.

(b-3) For $\Omega = \Omega_{2,1,2}$, we can prove the estimate in the same manner as (b-2).

(b-4) For $\Omega = \Omega_{2,2,2}$, we only need to show

$$\begin{aligned}&\| \int \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1} |\widetilde{B_{\Omega_{2,2,2}}(u,v)}| \, d\tau \|_{L^2_\xi} \\ &\lesssim \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{1/2+s} \widetilde{u} \|_{L^2_\xi} \| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau + \xi^2 \rangle^{1/2+s} \widetilde{v} \|_{L^2_\xi},\end{aligned}$$

which follows from Proposition 2.5 in the same manner as (a-8) because the left-hand side is bounded by

$$\| \langle \xi \rangle^{s-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+\varepsilon} \widetilde{B_{\Omega_{2,2,2}}(u,v)} \|_{L^2_{\tau,\xi}},$$

for any $\varepsilon > 0$.

We next consider (3.1) and (3.2) with $u\bar{v}$ replaced by uv. (3.1) with $u\bar{v}$ replaced by uv is equivalent to

$$\| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{uv} \|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}} \|v\|_{Z^{s,a}}.$$

Put $\Omega_{i,j,k} = \{(\tau, \xi, \tau_1, \xi_1) : (\tau, \xi) \in P_i, (\tau - \tau_1, \xi - \xi_1) \in P_j, (\tau_1, \xi_1) \in P_k\}$ for $i, j, k = 1$ or 2. We only need to show

$$\| \mathcal{F}^{-1} \langle \tau - \xi^2 \rangle^{-1} \widetilde{B_\Omega(u,v)} \|_{Z^{s,a}} \lesssim \|u\|_{Z^{s,a}} \|v\|_{Z^{s,a}}, \tag{3.10}$$

for $\Omega = \Omega_{i,j,k}$ with $i, j, k = 1$ or 2. Put $M_2 = \max\{|\tau - \xi^2|, |\tau - \tau_1 - (\xi - \xi_1)^2|, |\tau_1 - \xi_1^2|\}$. Then, we have the following algebraic property:

$$M_2 \geq (|\tau - \xi^2| + |\tau - \tau_1 + (\xi - \xi_1)^2| + |\tau_1 - \xi_1^2|)/3 \geq 2|(\xi - \xi_1)\xi_1|/3.$$

Here, we consider only the case $\Omega = \Omega_{2,1,1}$ because the proof for the other cases is almost the same as above. (3.10) with $\Omega = \Omega_{2,1,1}$ is equivalent to

$$\begin{aligned}&\| \langle \xi \rangle^{1/2-a} |\xi|^a \langle \tau - \xi^2 \rangle^{-1/2+s} \widetilde{B_{\Omega_{2,1,1}}(u,v)} \|_{L^2_{\tau,\xi}} \\ &\lesssim \| \langle \xi \rangle^s \langle \tau - \xi^2 \rangle \widetilde{u} \|_{L^2_{\tau,\xi}} \| \langle \xi \rangle^s \langle \tau - \xi^2 \rangle \widetilde{v} \|_{L^2_{\tau,\xi}}.\end{aligned} \tag{3.11}$$

We divide $\Omega_{2,1,1}$ into two parts:

$$A_1 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} : |\xi| < 1\}, \quad A_2 = \{(\tau, \xi, \tau_1, \xi_1) \in \Omega_{2,1,1} \,|\, |\xi| \geq 1\}.$$

From Lemma 2.2, (3.11) with $\Omega_{2,1,1}$ replaced by A_1 can be reduced to

$$\sup_{\tau_1, \xi_1} \int \frac{\chi_{A_1} \langle \xi_1 \rangle^{-2s} |\xi|^{2a} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \tau_1 - \xi_1^2 \rangle^2 \langle \tau - \xi^2 \rangle^{1-2s} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^2} \, d\tau d\xi \lesssim 1.$$

Since $\langle M_2\rangle \gtrsim \langle(\xi-\xi_1)\xi_1\rangle \sim \langle\xi-\xi_1\rangle\langle\xi_1\rangle$, from Lemma 2.1, the left-hand side is bounded by

$$\int_{|\xi|<1} \frac{|\xi|^{2a}\langle\xi-\xi_1\rangle^{-2s}\langle\xi_1\rangle^{-2s}}{\langle M_2\rangle^{1-2s}}\,d\xi \lesssim \int_{|\xi|<1}|\xi|^{2a} \lesssim 1.$$

Here, we used $2a>-1$ and $1-2s\geq -2s$.

From Lemma 2.2, (3.11) with $\Omega_{2,1,1}$ replaced by A_2 can be reduced to

$$\sup_{\tau,\xi}\int \frac{\chi_{A_2}\langle\xi\rangle\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\tau_1-\xi_1^2\rangle^2\langle\tau-\xi^2\rangle^{1-2s}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^2}\,d\tau_1 d\xi_1 \lesssim 1.$$

In the same manner as (a-1), it follows that $M_2=\langle\tau-\xi^2\rangle\sim\langle(\xi-\xi_1)\xi_1\rangle$ from $(\tau-\tau_1,\xi-\xi_1)\in P_1, (\tau_1,\xi_1)\in P_1$ and $|\xi|\geq 1$. Since $\langle\xi\rangle\lesssim\langle\xi-\xi_1\rangle\langle\xi_1\rangle$, from Lemma 2.1, the left-hand side is bounded by

$$\begin{aligned}&\int \frac{\langle\xi_1\rangle^{1-2s}\langle\xi-\xi_1\rangle^{1-2s}}{\langle(\xi-\xi_1)\xi_1\rangle^{1-2s}\langle\tau-\xi^2+2(\xi-\xi_1)\xi_1\rangle^2}\,d\xi_1\\ \lesssim &\int \frac{1}{\langle\tau-\xi^2+2(\xi-\xi_1)\xi_1\rangle^2}\,d\xi_1 \lesssim 1.\end{aligned}$$

(3.2) with $u\bar{v}$ replaced by uv is equivalent to

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{B_\Omega(u,v)}\|_{Y^{s,a}} \lesssim \|u\|_{Z^{s,a}}\|v\|_{Z^{s,a}}, \tag{3.12}$$

for $\Omega=\mathbb{R}^4$. Here, we consider only the case $\Omega=\Omega_{2,1,1}$ because the proof for the other cases is almost the same as above. We divide $\Omega_{2,1,1}$ into A_1 and A_2 again.

From Lemma 2.3, (3.12) with $\Omega=A_1$ can be reduced to

$$\sup_{\xi_1}\int \frac{\chi_{A_1}|\xi|^{2a}\langle\xi-\xi_1\rangle^{-2s}\langle\xi_1\rangle^{-2s}}{\langle\tau-\xi^2\rangle^2\langle\tau_1-\xi_1^2\rangle^2\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^2}\,d\tau_1 d\tau d\xi \lesssim 1,$$

which is verified similarly to above using Lemma 2.1.

From Lemma 2.3, (3.12) with $\Omega=A_2$ can be reduced to

$$\sup_{\xi_1}\int \frac{\chi_{A_2}\langle\xi_1\rangle^{-2s}\langle\xi\rangle^{2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\tau-\xi^2\rangle^2\langle\tau_1+\xi_1^2\rangle^2\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^2}\,d\tau_1 d\xi d\tau \lesssim 1.$$

Since $\langle M_2\rangle=\langle\tau-\xi^2\rangle\sim\langle(\xi-\xi_1)\xi_1\rangle\sim\langle\xi-\xi_1\rangle\langle\xi_1\rangle$, from Lemma 2.1, the left-hand side is bounded by

$$\int \frac{\langle\xi_1\rangle^{-2s}\langle\xi\rangle^{2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle M_2\rangle^2}\,d\xi \lesssim \int \frac{\langle\xi\rangle^{2s}}{\langle(\xi-\xi_1)\xi_1\rangle^{2+2s}}\,d\xi \lesssim 1.$$

In the same manner, we can easily show (3.1) and (3.2) with $u\bar{v}$ replaced by $\bar{u}\bar{v}$ by using the following algebraic property:

$$M_3 \geq (|\tau-\xi^2| + |\tau-\tau_1+(\xi-\xi_1)^2| + |\tau_1+\xi_1^2|)/3 \geq |\xi^2+(\xi-\xi_1)^2+\xi_1^2|/3,$$

where $M_3 = \max\{|\tau-\xi^2|, |\tau-\tau_1+(\xi-\xi_1)^2|, |\tau_1+\xi_1^2|\}$. Therefore, we omit the proof. □

4. Proof of Theorems

For $s \geq 0$ and $|a| < 1/2$, we define the spaces $Z^{s,a}$ by

$$\|u\|_{Z^{s,a}} = \|w_{s,a}\widetilde{u}\|_{L^2_{\tau,\xi}}, \qquad w_{s,a} = \langle\xi\rangle^{s-s_a} w_{s_a,a}, \tag{4.1}$$

where $s_a = -(2a+1)/4 \in (-1/2, 0)$, and use the same definition for the spaces $Y^{s,a}$,

$$\|u\|_{Y^{s,a}} = \Big\| \int \langle\xi\rangle^{s-a}|\xi|^a|\widetilde{u}|\,d\tau \Big\|_{L^2_\xi}.$$

It follows easily that these spaces satisfy the estimates in Propositions 2.6 and 2.7, because the operator $U(t)$ commutes with the spatial derivative and then the estimates for $s \geq 0$ are reduced to those for $s = s_a$. Similarly, for $s \geq 0$ and $|a| < 1/2$ we obtain the same bilinear estimates as Proposition 3.1 from those for $s = s_a$ and the property $\langle\xi\rangle^{s-s_a} \lesssim \langle\xi-\xi_1\rangle^{s-s_a}\langle\xi_1\rangle^{s-s_a}$ for $s \geq s_a$. Moreover, Proposition 2.8 also holds for $s \geq 0$. Using these estimates, we can treat $s \geq 0$ and $s < 0$ in the same way.

We remark that if we define the space $Z^{s,a}$ also in the case $s < 0$ by (4.1), these spaces will be sufficient for establishing Theorem 1.1 and Corollary 1.3, but they do not have the properties

$$\|\mathcal{F}^{-1}\widetilde{u}|_{|\xi|\leq 1}\|_{Z^{s,a}} \sim \|\mathcal{F}^{-1}|\xi|^a\widetilde{u}|_{|\xi|\leq 1}\|_{Z^{s,0}},$$
$$\|\mathcal{F}^{-1}\widetilde{u}|_{|\xi|\geq 1}\|_{Z^{s,a}} \sim \|\mathcal{F}^{-1}\widetilde{u}|_{|\xi|\geq 1}\|_{Z^{s,0}},$$

which are needed for Theorem 1.4 (see the proof of Proposition 4.4 below).

We first prove Theorem 1.1. Precisely speaking, we have the following well-posedness result. Put $B_r(X) := \{f \in X : \|f\|_X \leq r\}$ for a Banach space X.

Proposition 4.1. *Let $|a| < 1/2$, $s \geq -(2a+1)/4$ and $r > 1$.*

(Existence) For any $u_0 \in B_r(H^{s,a})$, there exist $T \sim r^{-4/(3+2\min\{a,s\})}$ and $u \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ satisfying the integral form of (1.1) on $[0,T]$:

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')N(u(t'))\,dt'. \tag{4.2}$$

Moreover, the data-to-solution map $u_0 \in B_r(H^{s,a}) \mapsto u \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ *is Lipschitz continuous.*

(*Uniqueness*) *Assume that* $u, v \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ *satisfy* (4.2). *Then,* $u = v$ *on* $t \in [0,T]$.

Proof. We first prove the existence of the solution by a fixed-point argument. C_1 is a sufficiently large constant to be determined later. From Proposition 2.8 with $\lambda^{3/2+\min\{a,s\}} = C_1 r$ and the scaling invariance of (4.2), we only need to prove that for any $u_0 \in B_{C_1^{-1}}(H^{s,a})$ there exists $u \in Z^{s,a} \cap C(\mathbb{R} : H^{s,a})$ satisfying

$$u = M(u), \quad M(u) := \varphi(t)U(t)u_0 - i\varphi(t)\int_0^t U(t-t')N(u(t'))\,dt'. \quad (4.3)$$

From Propositions 2.6, 2.7 and 3.1, we have

$$\|M(u)\|_{Z^{s,a}} + \|M(u)\|_{C(\mathbb{R}:H^{s,a})} \le C_2\big(\|u_0\|_{H^{s,a}} + \|u\|^2_{Z^{s,a}}\big). \quad (4.4)$$

We take C_1 satisfying $C_1 > 4C_2^2$. Then, M is a map from $B_{2C_1^{-1}C_2}(Z^{s,a} \cap C(\mathbb{R} : H^{s,a}))$ to itself. In the same manner, we have

$$\begin{aligned}&\|M(u) - M(v)\|_{Z^{s,a}} + \|M(u) - M(v)\|_{C(\mathbb{R}:H^{s,a})}\\ \le& C_2\|u-v\|_{Z^{s,a}}(\|u\|_{Z^{s,a}} + \|v\|_{Z^{s,a}}) \le 4C_1^{-1}C_2^2\|u-v\|_{Z^{s,a}}.\end{aligned}$$

Therefore, M is a contraction map on $B_{2C_1^{-1}C_2}(Z^{s,a} \cap C(\mathbb{R} : H^{s,a}))$ and we obtain the solution of (4.3). By the standard argument, we can prove that the data-to-solution map is Lipschitz continuous.

We next prove the uniqueness. We assume $u, v \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ satisfy (4.2) on $[0,T]$. Put $K = \|u+v\|_{Z^{s,a}_{[0,T]}}$ and $T_\lambda = \lambda^2 T$. By the scaling invariance and Proposition 2.8, u_λ, v_λ satisfy (4.2) on $[0,T_\lambda]$ and $\|u_\lambda + v_\lambda\|_{Z^{s,a}_{[0,T_\lambda]}} \lesssim \lambda^{-1/2-\min\{a,s\}}K$.

For any functions $u'_\lambda, v'_\lambda \in \mathscr{Z}'(\mathbb{R}^2)$ such that $u'_\lambda = u_\lambda, v'_\lambda = v_\lambda$ on $[0,1]$, we have

$$\begin{aligned}\big\|u_\lambda - v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}} \le& C\Big\|\int_0^t U(t-t')(N(u_\lambda) - N(v_\lambda))\,dt'\big|_{[0,1]}\Big\|_{Z^{s,a}_{[0,1]}}\\ \le& C\Big\|\varphi\int_0^t U(t-t')(N(u'_\lambda) - N(v'_\lambda))\,dt'\Big\|_{Z^{s,a}}.\end{aligned}$$

From Propositions 2.7, 3.1, we have

$$C\Big\|\varphi\int_0^t U(t-t')(N(u'_\lambda) - N(v'_\lambda))\,dt'\Big\|_{Z^{s,a}} \le C\|u'_\lambda - v'_\lambda\|_{Z^{s,a}}\|u'_\lambda + v'_\lambda\|_{Z^{s,a}}.$$

Thus,

$$\big\|u_\lambda - v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}} \le C\big\|u_\lambda - v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}}\big\|u_\lambda + v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}}.$$

For sufficiently large λ, we have $T_\lambda \ge 1$ and

$$C\big\|u_\lambda + v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}} \le C\|u_\lambda + v_\lambda\|_{Z^{s,a}_{[0,T_\lambda]}} \le CK\lambda^{-1/2-\min\{a,s\}} \le 1/2.$$

Therefore, we obtain

$$\big\|u_\lambda - v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}} \le 1/2\big\|u_\lambda - v_\lambda\big|_{[0,1]}\big\|_{Z^{s,a}_{[0,1]}};$$

that is, $u(t) = v(t)$ on $[0, \lambda^{-2}]$. Repeating this process, we obtain $u(t) = v(t)$ on $[0, T]$. □

We next prove Corollary 1.3 and Theorem 1.4. As mentioned in the introduction, we can treat (1.6) in the same manner as (1.7). So, we prove only the real-valued case for simplicity. Precisely speaking, we show the following well-posedness results.

Proposition 4.2. *Let $|a| < 1/2$, $s \ge -(2a+1)/4$ and $r > 1$.*

(*Existence*) *For any $v_0 \in B_r(H^{s,a})$, there exist $T \sim r^{-4/(3+2\min\{a,s\})}$ and $u \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ satisfying the integral form of (1.7) on $[0,T]$:*

$$v(t) = U(t)v_0 - i\int_0^t U(t-t')N^B(v(t'))\,dt', \tag{4.5}$$

where $N^B(v) = -v + (v+\bar{v})/2 + \omega^2(v+\bar{v})^2/4$. Moreover, the data-to-solution map $u_0 \in B_r(H^{s,a}) \mapsto u \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ is Lipschitz continuous.

(*Uniqueness*) *Assume that $u, v \in Z^{s,a}_{[0,T]} \cap C([0,T] : H^{s,a})$ satisfy (4.5). Then, $u = v$ on $t \in [0,T]$.*

Proposition 4.3. *Let $-1/4 > s > -1/2$, $a = 0$ and $r > 1$. Then, the same existence and uniqueness results in Proposition 4.2 hold for the existence time $T \sim r^{-2/(2+3s)}$.*

The proof of Proposition 4.2 is similar to that of Proposition 4.1. The only difference is the absence of the scaling invariance for (1.7). Put

$$v_\lambda(t,x) = \lambda^{-2}v(\lambda^{-2}t, \lambda^{-1}x), \quad v_{0\lambda} = \lambda^{-2}v_0(\lambda^{-1}x).$$

We see that v solves (1.7) on the time interval $[0,T]$ if and only if v_λ solves

$$\begin{cases} (i\partial_t - \partial_x^2)v_\lambda = -\lambda^{-2}v_\lambda + \lambda^{-2}(v_\lambda + \bar{v}_\lambda)/2 + \omega_\lambda^2(v_\lambda + \bar{v}_\lambda)^2/4, \\ v_\lambda(0,x) = v_{0\lambda}(x), \end{cases}$$

on $[0, \lambda^2 T]$, where $\omega_\lambda = \mathcal{F}_\xi^{-1}\sqrt{\lambda^2\xi^2/(1+\lambda^2\xi^2)}\mathcal{F}_x$. Therefore, we try to solve the integral equation

$$v_\lambda(t) = U(t)v_{0\lambda} - i\int_0^t U(t-t')N_\lambda^B(v_\lambda(t'))\,dt',$$
$$N_\lambda^B(v_\lambda) = -\lambda^{-2}v_\lambda + \lambda^{-2}(v_\lambda+\bar v_\lambda)/2 + \omega_\lambda^2(v_\lambda+\bar v_\lambda)^2/4,$$

for small rescaled initial data $v_{0\lambda}\in H^{s,a}$, applying a fixed-point argument to the equation

$$v_\lambda = M_\lambda^B(v_\lambda), \quad M_\lambda^B(v_\lambda) := \varphi(t)U(t)v_{0\lambda} - i\varphi(t)\int_0^t U(t-t')N_\lambda^B(v_\lambda(t'))\,dt'.$$

We note that the bound on the size of the data which ensure the contractiveness of M_λ^B may depend on λ.

It follows from the definition of the spaces and the Cauchy-Schwarz inequality that

$$\begin{aligned}&\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{v_\lambda}\|_{Z^{s,a}} + \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{\bar v_\lambda}\|_{Z^{s,a}}\\ +&\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{v_\lambda}\|_{Y^{s,a}} + \|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{\bar v_\lambda}\|_{Y^{s,a}}\\ \lesssim&\|\langle\xi\rangle^{s-a}|\xi|^a\widetilde{v_\lambda}\|_{L^2_{\tau,\xi}} \le \|v_\lambda\|_{Z^{s,a}},\end{aligned}$$

for $|a|<1/2$ and $s\in\mathbb{R}$, so linear terms in $N_\lambda^B(v_\lambda)$ can be easily estimated. Using Propositions 2.6, 2.7, 3.1 and the fact $|\widehat{\omega_\lambda^2}|\le 1$, we have

$$\|M_\lambda^B(v_\lambda)\|_{Z^{s,a}} + \|M_\lambda^B(v_\lambda)\|_{C(\mathbb{R}:H^{s,a})} \le C_2\big(\|v_{0\lambda}\|_{H^{s,a}} + \lambda^{-2}\|v_\lambda\|_{Z^{s,a}} + \|v_\lambda\|_{Z^{s,a}}^2\big)$$

instead of (4.4), where the constant C_2 does not depend on λ. Following the argument for Proposition 4.1, we verify that M_λ^B is a contraction map on $B_{3C_1^{-1}C_2}(Z^{s,a}\cap C(\mathbb{R}:H^{s,a}))$ for $v_{0\lambda}\in B_{C_1^{-1}}(H^{s,a})$ with $C_1 > 9C_2^2$ and $\lambda^2 > 3C_2$ (the bound for the data does not depend on λ in this case). Uniqueness also follows from a similar argument, and we obtain Proposition 4.2.

The proof of Proposition 4.3 requires a variant of Proposition 3.1.

Proposition 4.4. *Let $-1/4 > s > -1/2$ and $\lambda\ge 1$. Then, the following estimates hold:*

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{\omega_\lambda^2 u\bar v}\|_{Z^{s,0}} \lesssim \lambda^{a_s}\|u\|_{Z^{s,0}}\|v\|_{Z^{s,0}}, \tag{4.6}$$

$$\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{\omega_\lambda^2 u\bar v}\|_{Y^{s,0}} \lesssim \lambda^{a_s}\|u\|_{Z^{s,0}}\|v\|_{Z^{s,0}}, \tag{4.7}$$

where a_s is defined by $s = -(2a_s+1)/4$ and $1/2 > a_s > 0$. Moreover, the same estimates hold with $u\bar v$ replaced by uv or $\bar u\bar v$.

Proof. Since $|\widehat{\omega_\lambda^2}| \le |\widehat{\omega_\lambda^{a_s}}| \le \min\{\lambda^{a_s}|\xi|^{a_s},\, 1\}$ and $\lambda^{a_s} \ge 1$, we have

$$\begin{aligned}&\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{\omega_\lambda^2 u\bar{v}}\|_{Z^{s,0}}\\ \lesssim&\lambda^{a_s}\|\mathcal{F}^{-1}|\xi|^{a_s}\langle\tau-\xi^2\rangle^{-1}\widetilde{u\bar{v}}|_{|\xi|\le 1}\|_{Z^{s,0}}+\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{u\bar{v}}|_{|\xi|\ge 1}\|_{Z^{s,0}}\\ \lesssim&\lambda^{a_s}\|\mathcal{F}^{-1}\langle\tau-\xi^2\rangle^{-1}\widetilde{u\bar{v}}\|_{Z^{s,a_s}}.\end{aligned}$$

Obviously, it follows that $\|u\|_{Z^{s,a_s}} \le \|u\|_{Z^{s,0}}$ and $\|v\|_{Z^{s,a_s}} \le \|v\|_{Z^{s,0}}$. Therefore, we have (4.6) from (3.1). In the same manner, we have (4.7) from (3.2). □

Proof of Proposition 4.3. We first observe that the map M_λ^B is contractive on a ball in $Z^{s,0} \cap C(\mathbb{R} : H^s)$ if λ is sufficiently large and $v_{0\lambda} \in B_{C_3^{-1}\lambda^{-a_s}}(H^s)$ for some large constant C_3 to be determined later (the bound on the data depends on λ in this case). From Propositions 2.6, 2.7 and 4.4, we have

$$\begin{aligned}&\|M_\lambda^B(v_\lambda)\|_{Z^{s,0}}+\|M_\lambda^B(v_\lambda)\|_{C(\mathbb{R}:H^s)}\le C_4\big(\|v_{0\lambda}\|_{H^s}+\lambda^{-2}\|v_\lambda\|_{Z^{s,0}}+\lambda^{a_s}\|v_\lambda\|_{Z^{s,0}}^2\big)\\ &\le C_4(C_3^{-1}\lambda^{-a_s}+3C_3^{-1}C_4\lambda^{-2-a_s}+9C_3^{-2}C_4^2\lambda^{-a_s})\\ &=C_3^{-1}C_4\lambda^{-a_s}(1+3C_4\lambda^{-2}+9C_3^{-1}C_4^2),\end{aligned}$$

for $v_\lambda \in B_{3C_3^{-1}C_4\lambda^{-a_s}}(Z^{s,0} \cap C(\mathbb{R} : H^s))$, and

$$\begin{aligned}&\|M_\lambda^B(v_\lambda)-M_\lambda^B(w_\lambda)\|_{Z^{s,0}}+\|M_\lambda^B(v_\lambda)-M_\lambda^B(w_\lambda)\|_{C(\mathbb{R}:H^s)}\\ \le&C_4\big(\lambda^{-2}\|v_\lambda-w_\lambda\|_{Z^{s,0}}+\lambda^{a_s}\|v_\lambda+w_\lambda\|_{Z^{s,0}}\|v_\lambda-w_\lambda\|_{Z^{s,0}}\big)\\ \le&(C_4\lambda^{-2}+6C_3^{-1}C_4^2)\|v_\lambda-w_\lambda\|_{Z^{s,0}},\end{aligned}$$

for $v_\lambda, w_\lambda \in B_{3C_3^{-1}C_4\lambda^{-a_s}}(Z^{s,0} \cap C(\mathbb{R} : H^s))$. Therefore, $\lambda^2 > 3C_4$ and $C_3 > 9C_4^2$ will be sufficient.

To solve the original Cauchy problem (1.7), we rescale the data $v_0 \in B_r(H^s)$ so that $v_{0\lambda} \in B_{C_3^{-1}\lambda^{-a_s}}(H^s)$. From Proposition 2.8, this is possible by choosing λ so that $\lambda^{-3/2-s}r \lesssim \lambda^{-a_s} \Leftrightarrow \lambda \gtrsim r^{1/(2+3s)}$.

Finally, uniqueness follows from Proposition 4.2 because we have the uniqueness in $Z^{s,a_s}_{[0,T]} \cap C([0,T] : H^{s,a_s})$ and $Z^{s,0}_{[0,T]} \cap C([0,T] : H^s)$ is embedded in this space. □

We finally prove Theorem 1.2.

The C^2 differentiability of the flow map $S(t)$ implies boundedness of the quadratic term appearing in the iteration scheme:

$$\|\mathcal{Q}(u_0)(t)\|_{H^{s,a}} \lesssim \|u_0\|_{H^{s,a}}^2, \quad \mathcal{Q}(u_0) = -i\int_0^t U(t-t')N(U(t')u_0)\,dt', \quad (4.8)$$

so it is sufficient for the case $|a| \geq 1/2$ to exhibit some counterexamples to disprove (4.8). For the details of this reduction, see [3], [18], and also [5].

We will also use unboundedness of the quadratic term $\mathcal{Q}$ to prove the discontinuity of the flow map in the case $|a| < 1/2$, $s < -(2a+1)/4$. This kind of ill-posedness also appeared in [1], in which Bejenaru and Tao developed well-posedness theory for an abstract semilinear evolution equation and applied it to a quadratic nonlinear Schrödinger equation to establish the discontinuity of the data-to-solution map $S : H^s \to C([0,T] : H^s)$. In our setting, we essentially follow their argument but give a direct and concrete proof.

Proof of Theorem 1.2. We write $\mathcal{Q} = \alpha_1\mathcal{Q}_1 + \alpha_2\mathcal{Q}_2 + \alpha_3\mathcal{Q}_3$ corresponding to the three types of nonlinearity. A direct calculation shows that

$$\begin{aligned}
\mathcal{F}_x\mathcal{Q}_1(u_0)(t,\xi) &= -i\int\Big(e^{it\xi^2}\int_0^t e^{-2it'\xi_1(\xi-\xi_1)}\,dt'\Big)\widehat{u_0}(\xi_1)\widehat{u_0}(\xi-\xi_1)\,d\xi_1,\\
\mathcal{F}_x\mathcal{Q}_2(u_0)(t,\xi) &= -i\int\Big(e^{it\xi^2}\int_0^t e^{-it'(\xi^2+\xi_1^2+(\xi-\xi_1)^2)}\,dt'\Big)\\
&\qquad\qquad\qquad\qquad \times \overline{\widehat{u_0}(-\xi_1)\widehat{u_0}(-\xi+\xi_1)}\,d\xi_1,\\
\mathcal{F}_x\mathcal{Q}_3(u_0)(t,\xi) &= -i\int\Big(e^{it\xi^2}\int_0^t e^{-2it'\xi(\xi-\xi_1)}\,dt'\Big)\widehat{u_0}(\xi_1)\overline{\widehat{u_0}(-\xi+\xi_1)}\,d\xi_1.
\end{aligned}$$

Let $N \gg 1$ and $0 < \delta \ll 1$. We first put $\widehat{\phi_{N,\delta}} = \delta N^{(2a+3)/4}\chi_{[N-\frac{1}{N},N+\frac{1}{N}]}$. For this $\phi_{N,\delta}$ we have $\|\phi_{N,\delta}\|_{H^{s,a}} \sim \delta N^{(4s+2a+1)/4}$ and

$$|\mathcal{F}_x\mathcal{Q}_1(\phi_{N,\delta})(t,\xi)| \leq t\int \widehat{\phi_{N,\delta}}(\xi_1)\widehat{\phi_{N,\delta}}(\xi-\xi_1)\,d\xi_1 \lesssim t\delta^2 N^{(2a+1)/2}\chi_{[2N-\frac{2}{N},2N+\frac{2}{N}]},$$

which implies $\|\mathcal{Q}_1(\phi_{N,\delta})(t)\|_{H^{s,a}} \lesssim t\delta^2 N^{s+a}$. The same argument is applied to $\mathcal{Q}_2$ and leads to the same bound. On the other hand, if $|\xi| \leq 1/2N$, we have $|\xi(\xi-\xi_1)| \leq 1$ whenever $\widehat{\phi_{N,\delta}}(\xi_1) \neq 0$. It then follows, for sufficiently small t (say $t < 1/100$), that

$$|\mathcal{F}_x \mathcal{Q}_3(\phi_{N,\delta})(t,\xi)| \gtrsim t\chi_{[-\frac{1}{2N},\frac{1}{2N}]}(\xi) \int_{N-\frac{1}{2N}}^{N+\frac{1}{2N}} \widehat{\phi_{N,\delta}}(\xi_1)\overline{\widehat{\phi_{N,\delta}}(-\xi+\xi_1)}\, d\xi_1$$
$$\gtrsim t\delta^2 N^{(2a+1)/2}\chi_{[-\frac{1}{2N},\frac{1}{2N}]}(\xi)$$

and

$$\|\mathcal{Q}_3(\phi_{N,\delta})(t)\|_{H^{s,a}} \gtrsim t\delta^2 N^{(2a+1)/2}\Big(\int_{-1/2N}^{1/2N} |\xi|^{2a}\, d\xi\Big)^{1/2}.$$

This example is sufficient for the case $a \le -1/2$; in fact, $\|\mathcal{Q}_3(\phi_{N,\delta})(t)\|_{H^{s,a}}$ is infinite if $a \le -1/2$, and $\|\phi_{N,\delta}\|_{H^{s,a}}$, $\|\mathcal{Q}_1(\phi_{N,\delta})(t)\|_{H^{s,a}}$, $\|\mathcal{Q}_2(\phi_{N,\delta})(t)\|_{H^{s,a}}$ are all finite, so the estimate (4.8) cannot hold whenever $\alpha_3 \neq 0$.

We now assume $|a| < 1/2$ and $s < s_a = -(2a+1)/4$, which implies

$$4s + 2a + 1 < 4s_a + 2a + 1 = 0, \qquad s + a < s_a + a < 0.$$

In this case $\|\mathcal{Q}_3(\phi_{N,\delta})(t)\|_{H^{s,a}} \gtrsim t\delta^2$, and thus we have

$$\|\phi_{N,\delta}\|_{H^{s_a,a}} \sim \delta \gg \delta N^{(4s+2a+1)/4} \sim \|\phi_{N,\delta}\|_{H^{s,a}}, \tag{4.9}$$

$$\|U(t)\phi_{N,\delta}\|_{H^{s,a}} = \|\phi_{N,\delta}\|_{H^{s,a}} \lesssim \delta N^{(4s+2a+1)/4} \ll \delta, \tag{4.10}$$

$$\|\mathcal{Q}(\phi_{N,\delta})(t)\|_{H^{s,a}} \gtrsim t\delta^2. \tag{4.11}$$

Let $u_{N,\delta}(t)$ be the standard smooth solution to (1.1) with initial data $\phi_{N,\delta} \in H^\infty$ defined above, and let $v_{N,\delta}(t) := u_{N,\delta}(t) - U(t)\phi_{N,\delta} - \mathcal{Q}(\phi_{N,\delta})(t)$. We see that $v_{N,\delta}$ satisfies the integral equation

$$v_{N,\delta}(t) = -i\int_0^t U(t-t')\Big[N\big(U(t')\phi_{N,\delta} + \mathcal{Q}(\phi_{N,\delta})(t') + v_{N,\delta}(t')\big) \\ - N\big(U(t')\phi_{N,\delta}\big)\Big]\, dt'.$$

Estimating the right-hand side by Proposition 2.7, 3.1, 2.6 and (4.9), we obtain

$$V(\delta) \lesssim V(\delta)^2 + (\delta+\delta^2)V(\delta) + (\delta^3+\delta^4),$$

where $V(\delta) := \|v_{N,\delta}\|_{Z^{s_a,a}_{[0,1]}} + \|v_{N,\delta}\|_{L^\infty([0,1]:H^{s_a,a})}$. Since $V(\delta)$ is continuous and $V(0) = 0$, we have

$$\|v_{N,\delta}(t)\|_{H^{s_a,a}} \le V(\delta) \lesssim \delta^3, \tag{4.12}$$

for any $t \in [0,1]$ and sufficiently small δ. From (4.11) and (4.12) we can choose $\delta > 0$ for each small $t > 0$, so that

$$\|\mathcal{Q}(\phi_{N,\delta})(t) + v_{N,\delta}(t)\|_{H^{s,a}} \ge \|\mathcal{Q}(\phi_{N,\delta})(t)\|_{H^{s,a}} - \|v_{N,\delta}(t)\|_{H^{s_a,a}} \gtrsim t\delta^2,$$

for all $N \gg 1$. Finally, from (4.10) we have

$$\|u_{N,\delta}(t)\|_{H^{s,a}} \geq \|\mathcal{Q}(\phi_{N,\delta})(t) + v_{N,\delta}(t)\|_{H^{s,a}} - \|U(t)\phi_{N,\delta}\|_{H^{s,a}} \gtrsim t\delta^2,$$

for sufficiently large N. Since $\|\phi_{N,\delta}\|_{H^{s,a}} \to 0$ as $N \to \infty$, this shows the discontinuity of the flow map.

The above example contains an interaction between high-frequency components of the data, while for the case $a \geq 1/2$ we will investigate the low-frequency interactions. Moreover, in contrast to the other cases, we can show the same ill-posedness result for $a \geq 1/2$ unless the nonlinearity vanishes (i.e. $\alpha_1 = \alpha_2 = \alpha_3 = 0$), even if $\alpha_3 = 0$.

Define

$$\widehat{\phi_N} = \begin{cases} |\xi|^{-1}, & \text{for } 1/N \leq |\xi| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We see

$$\|\phi_N\|_{H^{s,a}}^2 \sim \begin{cases} \log N & \text{if } a = 1/2, \\ 1 & \text{if } a > 1/2, \end{cases}$$

and

$$\mathcal{F}_x \mathcal{Q}(\phi_N)(t,\xi) = \int \Big(\int_0^t \theta(t,t',\xi,\xi_1)\, dt' \Big) \widehat{\phi_N}(\xi_1)\widehat{\phi_N}(\xi - \xi_1)\, d\xi_1,$$

$$\begin{aligned} &\theta(t,t',\xi,\xi_1) \\ =&e^{it\xi^2}\left(\alpha_1 e^{-2it'\xi_1(\xi-\xi_1)} + \alpha_2 e^{-it'(\xi^2+\xi_1^2+(\xi-\xi_1)^2)} + \alpha_3 e^{-2it'\xi(\xi-\xi_1)}\right). \end{aligned}$$

If we consider $e^{i\pi/2}\phi_N$ or $e^{i\pi/4}\phi_N$ instead of ϕ_N, then $(\alpha_1,\alpha_2,\alpha_3)$ in the above expression will be replaced with $(-\alpha_1,-\alpha_2,\alpha_3)$ or $(i\alpha_1,-i\alpha_2,\alpha_3)$, respectively. Since $\alpha_1+\alpha_2+\alpha_3 = -\alpha_1-\alpha_2+\alpha_3 = i\alpha_1 - i\alpha_2 + \alpha_3 = 0$ implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we may assume $\alpha_0 := \alpha_1+\alpha_2+\alpha_3 \neq 0$ unless the nonlinearity vanishes.

We choose t small so that $0 < t < |\alpha_0|/100 \max\{|\alpha_1|,|\alpha_2|,|\alpha_3|\}$. Then, using the inequality $|1-e^{iA}| \leq |A|$ for $A \in \mathbb{R}$, we have

$$\begin{aligned} \Big|1 - \frac{\theta(t,t',\xi,\xi_1)}{\alpha_0}\Big| \leq{}& \frac{|\alpha_1|}{|\alpha_0|}|1 - e^{it\xi^2 - 2it'\xi_1(\xi-\xi_1)}| \\ &+ \frac{|\alpha_2|}{|\alpha_0|}|1 - e^{it\xi^2 - it'(\xi^2+\xi_1^2+(\xi-\xi_1)^2)}| + \frac{|\alpha_3|}{|\alpha_0|}|1 - e^{it\xi^2-2it'\xi(\xi-\xi_1)}| \leq \frac{1}{2}, \end{aligned}$$

whenever $\xi_1,\ \xi-\xi_1 \in \operatorname{supp}\phi_N$ and $0 \le t' \le t$. In this case $\Re[\theta(t,t',\xi,\xi_1)/\alpha_0] \ge 1/2$ and

$$|\alpha_0^{-1}\mathcal{F}_x\mathcal{Q}(\phi_N)(t,\xi)| \ge \frac{t}{2}\int \widehat{\phi_N}(\xi_1)\widehat{\phi_N}(\xi-\xi_1)\,d\xi_1,$$

$$\begin{aligned}
&\|\mathcal{Q}(\phi_N)(t)\|_{H^{s,a}}^2 \gtrsim t^2\int_{-2}^{2}|\xi|^{2a}\Big(\int \widehat{\phi_N}(\xi_1)\widehat{\phi_N}(\xi-\xi_1)\,d\xi_1\Big)^2 d\xi \\
&\ge t^2\int_0^{1/4}\xi^{2a}\Big(\int_{\xi+\frac{1}{N}}^{1}\frac{d\xi_1}{\xi_1(\xi_1-\xi)}\Big)^2 d\xi \\
&= t^2\int_0^{1/4}\xi^{2a-2}\log^2(1-\xi)(1+N\xi)\,d\xi \\
&\ge t^2N^{1-2a}\int_0^{N/4}\xi^{2a-2}\log^2(1+\xi/2)\,d\xi \gtrsim \begin{cases} t^2\log^3 N & \text{if } a=1/2, \\ t^2\log^2 N & \text{if } a>1/2. \end{cases}
\end{aligned}$$

At the fourth line of the above estimate we have used the fact that

$$(1-\xi)(1+N\xi) > 1+N\xi/2 > 1,$$

for $0<\xi<1/4$. Therefore, (4.8) does not hold for sufficiently large N. $\square$

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