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REMARKS ON KEEL-SMITH-SOGGE ESTIMATES AND SOME APPLICATIONS TO NONLINEAR HIGHER-ORDER WAVE EQUATIONS

Makoto Nakamura
Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

(Submitted by: Tohru Ozawa)

Abstract. The Keel-Smith-Sogge estimates are considered in a general frame in terms of the harmonic analysis. The estimates are applied to the Cauchy problem for nonlinear higher-order wave equations, and the existence of global and almost-global solutions is shown.

1. Introduction

Keel, Smith, and Sogge have shown the so-called "Keel-Smith-Sogge estimate,"

$$\begin{aligned}&(\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}u'\|_{L^2((0,T)\times\mathbb{R}^3)} \\ &\lesssim \|u'(0,\cdot)\|_{L^2(\mathbb{R}^3)}+\int_0^T\|(\partial_t^2-\Delta)u(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds\end{aligned} \tag{1.1}$$

in [22, Proposition 2.1], and have applied it to the Cauchy problem for the nonlinear wave equation,

$$\begin{cases}(\partial_t^2-\Delta)u(t,x)=Q(u') & \text{for}\quad (t,x)\in[0,\infty)\times\mathbb{R}^3\\ u(0,x)=f(x),\quad \partial_t u(0,x)=g(x) & \text{for}\quad x\in\mathbb{R}^3,\end{cases} \tag{1.2}$$

where Δ denotes the Laplacian, u' denotes the first derivatives of u by t and x, and $Q(u')$ denotes the quadratic term of u' such as $(\partial_t u)^2$ or $(\partial_t u)^2-|\nabla u|^2$. Also, they have shown the almost-global solutions to (1.2) when the initial data are sufficiently small. Here, the "almost-global" solutions mean that the lifespan T of the solution u is as long as the exponential order; namely, T can be taken as

$$T=C\exp\big(\frac{c}{\varepsilon}\big) \tag{1.3}$$

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for some positive constants C and c, where

$$\varepsilon := \sum_{|\alpha|\le 10} \|Z^\alpha f\|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha|\le 9} \|Z^\alpha g\|_{L^2(\mathbb{R}^3)}, \tag{1.4}$$

and Z denotes the standard derivatives $\{\partial_t, \nabla_x\}$ and the angular derivatives $x_i\partial_j - x_j\partial_i$, $1 \le i \ne j \le 3$. They have also considered the boundary-value problems for nonlinear wave equations. And now the estimate has become one of the useful estimates for the study of nonlinear wave equations (see for example [23, 35, 36, 38, 39, 40, 41, 42] and [53, Notes on page 94]).

Let us briefly review the almost-global solutions in the whole space $[0,\infty)\times\mathbb{R}^3$. It is known that the solution of (1.2) blows up in finite time for any nontrivial initial data when $Q(u') = (\partial_t u)^2$ (see [15] and [50]), while the solution exists globally in time for sufficiently small initial data when $Q(u') = (\partial_t u)^2 - |\nabla u|^2$ (see for example [52] and see [4], [28], [9, Theorem 6.6.2], [53, Theorem 5.1] for the null condition). The result for almost-global solutions has already been shown in [26], and the lifespan $C\exp(c/\varepsilon)$ is known to be sharp in general for the blowing-up solutions (see [17], [18], [48], [49], and also [50]). The estimate (1.1) gives a simpler proof for the almost-global solutions, and (1.1) is proved in a simple way by the use of the classical energy estimates and the Huygens principle.

It is natural and useful to generalize the Keel-Smith-Sogge estimates to other dimensions and moreover to find other formulations. Metcalfe has constructed the Keel-Smith-Sogge estimates in more than four spatial dimensions in [33] by the method of harmonic analysis. In [54, Appendix (A.10)], Sterbenz and Rodnianski have given an alternative proof for the Keel-Smith-Sogge estimates in more than three spatial dimensions based on the modified energy estimates without the use of the Huygens principle, and Metcalfe and Sogge have generalized them in [39]. In [13, Proposition B.2] (see also [14, Lemma 2.5 and its Remark]), Hidano and Yokoyama have considered the estimates in general dimensions based on the method in [33].

We should refer to the preceding results of the related weighted L^2 estimates for wave equations by Kenig, Ponce, and Vega [25, Proposition 2.6]; Mochizuki [43, §27]; Matsuyama [32]; Morawetz [44], [45]; Smith and Sogge [51]; and Strauss [55] (see also the description in [6, two lines after Lemma 2.2], [13, Remarks after Proposition B.1], [53, page 94], [1], and [2]).

Since the vector fields method gives us the estimate

$$(1+t+|x|)^{(n-1)/2}(1+||t|-|x||)^{1/2}|u(t,x)|$$

$$\lesssim \sum_{|\alpha|\leq (n+2)/2} \|\Gamma^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \tag{1.5}$$

for any function u on $(t-1,t+1)\times\mathbb{R}^n$ with suitable differentiability (see [9, p. 118, Proposition 6.5.1]), where Γ denotes the vector fields

$$\partial_t,\ \partial_j,\ t\partial_j + x_j\partial_t,\ x_j\partial_k - x_k\partial_j,\quad 1\leq j\neq k\leq n,\quad t\partial_t + \sum_{k=1}^n x_k\partial_k \tag{1.6}$$

and α denotes the multiple indices, we know by the classical energy estimates that the Cauchy problem

$$\begin{cases} (\partial_t^2-\Delta)u(t,x) = Q(u') & \text{for}\quad (t,x)\in[0,\infty)\times\mathbb{R}^n \\ u(0,x)=f(x),\quad \partial_t u(0,x) = g(x) & \text{for}\quad x\in\mathbb{R}^n \end{cases} \tag{1.7}$$

has global solutions in cases $Q(u') = O(|u'|^2)$ with $n\geq 4$, $Q(u')=O(|u'|^3)$ with $n=3$, and $Q(u')=O(|u'|^4)$ with $n=2$, and has almost-global solutions in the cases $Q(u')=O(|u'|^2)$ with $n=3$ and $Q(u')=O(|u'|^3)$ with $n=2$. While the vector-fields method uses the operator $t\partial_j + x_j\partial_t$, the Keel-Smith-Sogge estimates use only $\partial_{t,x}$ and $x_j\partial_k - x_k\partial_j$ which are commutative with the c speed D'Alembertian $\partial_t^2 - c^2\Delta$ which is not commutative with $t\partial_j + x_j\partial_t$ if $c\neq 1$. So, the estimates are also useful when we consider multiple speed wave systems.

The almost-global solutions for nonlinear wave equations have been studied in [15, 19, 21, 26, 27, 29, 31, 50] in three spatial dimensions. The case of two spatial dimensions is considered in [3, 30]. See also [9, Theorems 6.5.3, 6.5.7, 6.5.9, p. 129–p. 130, Remarks 1,2,3] or [10, Theorems 2.2.1, 2.4.4] for more references on the almost-global solutions of nonlinear wave equations. In recent years, the almost-global solutions and global solutions when $n\geq 3$ are considered based on the Keel-Smith-Sogge estimates in [11, 13, 14, 23, 33, 39].

We raise some results on the almost-global solutions for other partial differential equations. See [5, 46] for Klein-Gordon equations. See [8] for Schrödinger equations. See [17, 20, 34, 42, 56, 57] for elastic waves. See [37] for nonlinear wave equations with local dissipation in exterior domains. See [47] for fourth-order wave equations. See also the references therein.

In this paper, we consider higher-order wave equations

$$\left(\partial_t^2 + (-\Delta)^m\right)u(t,x) = F(t,x)\quad \text{for}\quad t\geq 0,\quad x\in\mathbb{R}^n,\quad n\geq 1,\quad m>0 \tag{1.8}$$

and give a generalization of the Keel-Smith-Sogge estimates for $m>0$ in Corollary 3 based on the local smoothing estimates by Kenig, Ponce, and

Vega in [24, Theorem 4.1]. Here, we put

$$(-\Delta)^m := \mathcal{F}^{-1}|\xi|^{2m}\mathcal{F}, \quad \omega := (-\Delta)^{m/2} = \mathcal{F}^{-1}|\xi|^m\mathcal{F} \tag{1.9}$$

and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. We also apply it to Cauchy problems for nonlinear higher-order wave equations, and we show the existence of global and almost-global solutions in Theorem 9 and Theorem 12.

Here, we note that the wave operator $W_m := \partial_t^2 + (-\Delta)^m$ satisfies the commutative property

$$\partial_t W_m = W_m\partial_t,\ \partial_j W_m = W_m\partial_j,\ (x_j\partial_k - x_k\partial_j)W_m = W_m(x_j\partial_k - x_k\partial_j) \tag{1.10}$$

for $1 \le j \ne k \le n$ from the direct computation of the definition of $(-\Delta)^m$ in (1.9). But W_m is not commutative with $t\partial_j + x_j\partial_t$ and $t\partial_t + \sum_{k=1}^n x_k\partial_k$ when $m \ne 1$.

We use the following notation. The Lebesgue space of order p is denoted by $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. The homogeneous Sobolev space of order s is denoted by $\dot{H}^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. For any interval I and any Banach space X, the space $L^p(I, X)$ denotes X-valued L^p functions on I. Let $\langle x\rangle := \sqrt{1+x^2}$. The notation $a \lesssim b$ denotes the inequality $a \le Cb$ for a positive constant C which is not essential for our arguments.

2. Keel-Smith-Sogge-type estimates

Let us first consider a generalization of Keel-Smith-Sogge estimates.

Lemma 1. *Let $n \ge 1$. Let $m > 0$ be a real number. Let $\phi(\xi) = |\xi|^m$ for $\xi \in \mathbb{R}^n$. Then the operator $e^{\pm it\phi(i\nabla)}$, which is defined by*

$$e^{\pm it\phi(i\nabla)} = e^{\pm it\omega} = e^{\pm it(-\Delta)^{m/2}} := \mathcal{F}^{-1}e^{\pm it\phi(\xi)}\mathcal{F},$$

satisfies the following estimates.

(1) *If $m = 1$, then*

$$(\log(e+T))^{-1/2}\|\langle x\rangle^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{2.1}$$

for any real number $T > 0$.

(2) *Let $h \in L^\infty((0,\infty),\mathbb{R})$ be a function which satisfies*

$$H := \Big(\|h\|^2_{L^\infty(0<r<1)} + \sum_{j=0}^{\infty}\|h\|^2_{L^\infty(2^j<r<2^{j+1})}\Big)^{1/2} < \infty.$$

Then

$$\|\langle x\rangle^{-1/2}h(|x|)e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)}\lesssim m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}. \tag{2.2}$$

Remark 2. For an example of h in (2) of the above lemma, we can take

$$h(r)=(1+r)^{-\gamma}\quad\text{or}\quad h(r)=(\log(e+r))^{-1/2-\gamma}$$

for any $\gamma>0$. In applications, we use the former example to show the existence of global solutions for Cauchy problems of nonlinear higher-order wave equations.

Proof of Lemma 1. For a function f, we consider the integral

$$F(t,x):=\int_{\mathbb{R}^n}e^{ix\xi\pm it\phi(\xi)}f(\xi)d\xi. \tag{2.3}$$

We start from the local smoothing estimates (see [24, p. 54, Theorem 4.1]; see also [51, Lemma 2.2] for the case $m=1$) for ϕ,

$$\int_{\mathbb{R}}\int_{\{x\in\mathbb{R}^n\,:\,|x|<R\}}|F(t,x)|^2dx\,dt\lesssim R\int_{\mathbb{R}^n}|\nabla\phi|^{-1}|f(x)|^2dx, \tag{2.4}$$

which leads to the inequality

$$\|e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}(|x|<R)}\lesssim R^{1/2}m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}. \tag{2.5}$$

In particular, we have

$$\||x|^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}(R/2<|x|<R)}\lesssim m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}. \tag{2.6}$$

When $0<T<1$, by (2.5) we have

$$\|\langle x\rangle^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\{|x|<T\})}\lesssim m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}. \tag{2.7}$$

When $T\geq 1$, we use the dyadic decomposition

$$\{|x|<T\}\subset\{|x|<1\}\cup\cup_{j=0}^{\log_2 T}\{2^j\leq|x|<2^{j+1}\}$$

and (2.6) to obtain

$$\begin{aligned}&\|\langle x\rangle^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\{|x|<T\})}\\&\lesssim(\log(e+T))^{1/2}m^{-1/2}\|f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)}.\end{aligned} \tag{2.8}$$

(1) Since the operator $e^{\pm it\phi(i\nabla)}$ is isometric, we have

$$\begin{aligned}&\||x|^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\{|x|>T\})}\\&\lesssim T^{-1/2}\|e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)}\lesssim\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{2.9}$$

So that, when $m = 1$, we obtain by (2.8)

$$(\log(e+T))^{-1/2}\|\langle x\rangle^{-1/2}e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.10}$$

(2) By (2.6) and the dyadic decomposition

$$\mathbb{R}^n = \{|x| < 1\} \cup \cup_{j=0}^{\infty}\{2^j \le |x| < 2^{j+1}\},$$

we have

$$\|\langle x\rangle^{-1/2}h(|x|)e^{\pm it\phi(i\nabla)}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)} \lesssim Hm^{-1/2}\|f\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)}. \tag{2.11}$$

Therefore, we have obtained the required results. □

Corollary 3. *Let $n \ge 1$ and $m > 0$. For any solution u which satisfies*

$$\begin{cases} (\partial_t^2 + (-\Delta)^m)u(t,x) = F(t,x) \quad \textit{for} \quad t \ge 0, \quad x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot) \\ \partial_t u(0,\cdot) = g(\cdot), \end{cases} \tag{2.12}$$

the following estimates hold.

(1) $$\|u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-m}(\mathbb{R}^n)} + \|F\|_{L^1((0,\infty),\dot{H}^{-m}(\mathbb{R}^n))}. \tag{2.13}$$

(2) *For any fixed $\gamma > 0$,*

$$\begin{aligned}\|\langle x\rangle^{-1/2-\gamma}u\|_{L^2((0,\infty)\times\mathbb{R}^n)} \lesssim m^{-1/2}\Big\{&\|f\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)} \\ &+ \|g\|_{\dot{H}^{-(m-1)/2-m}(\mathbb{R}^n)} + \|F\|_{L^1((0,\infty),\dot{H}^{-(m-1)/2-m}(\mathbb{R}^n))}\Big\}.\end{aligned} \tag{2.14}$$

(3) *If $m = 1$, then*

$$\begin{aligned}&(\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}u\|_{L^2((0,T)\times\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1}(\mathbb{R}^n)} + \|F\|_{L^1((0,T),\dot{H}^{-1}(\mathbb{R}^n))}.\end{aligned} \tag{2.15}$$

Proof. The solution u can be written by

$$u(t,\cdot) = \cos t\omega f + \frac{\sin t\omega}{\omega}g + \int_0^t \frac{\sin(t-s)\omega}{\omega}F(s)ds. \tag{2.16}$$

Then the required estimates follow from Lemma 1, where $h(|x|) = \langle x\rangle^{-\gamma}$. □

Corollary 4. *Let $n \ge 1$ and $m > 0$. For any solution u which satisfies*

$$\begin{cases} (\partial_t^2 + (-\Delta)^m)u(t,x) = F(t,x) \quad \textit{for} \quad t \ge 0, \quad x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot), \end{cases} \tag{2.17}$$

the following estimates hold.

(1) $$(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1((0,t),L^2(\mathbb{R}^n))} \quad \text{for any} \quad t \geq 0. \tag{2.18}$$

(2) $$\begin{aligned}&\|\partial_t u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} + \|\omega u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))}\\ &\lesssim \|f\|_{\dot{H}^m(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1((0,\infty),L^2(\mathbb{R}^n))}.\end{aligned} \tag{2.19}$$

(3) *For any fixed* $\gamma > 0$,

$$\begin{aligned}&\|\langle x\rangle^{-1/2-\gamma}\partial_t u\|_{L^2((0,\infty)\times\mathbb{R}^n)} + \|\langle x\rangle^{-1/2-\gamma}\omega u\|_{L^2((0,\infty)\times\mathbb{R}^n)}\\ &\lesssim m^{-1/2}\Big\{\|f\|_{\dot{H}^{(m+1)/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)} + \|F\|_{L^1((0,\infty),\dot{H}^{-(m-1)/2}(\mathbb{R}^n))}\Big\}.\end{aligned} \tag{2.20}$$

(4) *If* $m = 1$, *then*

$$\begin{aligned}&(\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\partial_t u\|_{L^2((0,T)\times\mathbb{R}^n)}\\ &\quad + (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\omega u\|_{L^2((0,T)\times\mathbb{R}^n)}\\ &\lesssim \|f\|_{\dot{H}^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1((0,T),L^2(\mathbb{R}^n))}.\end{aligned} \tag{2.21}$$

Proof. The solution u can be written as

$$u(t,\cdot) = \cos t\omega f + \frac{\sin t\omega}{\omega} g + \int_0^t \frac{\sin(t-s)\omega}{\omega} F(s)ds. \tag{2.22}$$

(1) By the L^2 invariance of the Fourier transform and the bound

$$|\sin t|\xi|^m/|\xi|^m| \lesssim t,$$

we have

$$\|u(t,\cdot)\|_{L^2} \lesssim \|f\|_{L^2} + t\|g\|_{L^2} + t\|F\|_{L^1L^2}. \tag{2.23}$$

Multiplying by $(1+t)^{-1}$, we obtain the result.

(2) Since we have

$$\begin{aligned}\omega u(t,\cdot) &= \cos t\omega(\omega f) + \sin t\omega g + \int_0^t \sin(t-s)\omega F(s,\cdot)ds\\ \partial_t u(t,\cdot) &= -\sin t\omega(\omega f) + \cos t\omega g + \int_0^t \cos(t-s)\omega F(s,\cdot)ds,\end{aligned} \tag{2.24}$$

the result follows from the L^2 invariance of the Fourier transform.

The results of (3) and (4) follow from (2.24) and Lemma 1. □

When $m = 1$, we have the following corollary.

Corollary 5. *Let $n \geq 1$. Let $\delta \geq 0$ be a real number. Let $m = 1$. We recall that $\phi(i\nabla) = \omega = \sqrt{-\Delta}$. Then we have the estimate*

$$\begin{aligned}
&\sup_{\delta>1/2} \min\left\{1, |1-2\delta|^{1/2}\right\} \|\langle x\rangle^{-\delta} e^{\pm it\omega} f\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&\qquad + \sup_{0\leq\delta<1/2} |1-2\delta|^{1/2} \langle T\rangle^{\delta-1/2} \|\langle x\rangle^{-\delta} e^{\pm it\omega} f\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&\quad + \{\log(e+T)\}^{-1/2} \|\langle x\rangle^{-1/2} e^{\pm it\omega} f\|_{L^2((0,T)\times\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \qquad (2.25)
\end{aligned}$$

where the positive constant C is independent of δ.

In particular, for any solution u which satisfies

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = F(t,x) & \text{for } t \geq 0, \quad x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot), \end{cases} \qquad (2.26)$$

the following estimate holds:

$$\begin{aligned}
&\sup_{\delta>1/2} \min\left\{1, |1-2\delta|^{1/2}\right\} \|\langle x\rangle^{-\delta} u\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&\qquad + \sup_{0\leq\delta<1/2} |1-2\delta|^{1/2} \langle T\rangle^{\delta-1/2} \|\langle x\rangle^{-\delta} u\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&\qquad\quad + \{\log(e+T)\}^{-1/2} \|\langle x\rangle^{-1/2} u\|_{L^2((0,T)\times\mathbb{R}^n)} \\
&\qquad \leq C \cdot \left\{ \|f\|_{L^2(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1}(\mathbb{R}^n)} + \|F\|_{L^1((0,T),\dot{H}^{-1}(\mathbb{R}^n))} \right\}, \qquad (2.27)
\end{aligned}$$

where the positive constant C is independent of δ.

Remark 6. Du, Metcalfe, Sogge, and Zhou have shown the related estimate of (2.27) in [6, Theorem 2.3] and [7, Theorem 3.3]. We note that the $\dot{H}^{-1}$ norm can be bounded by the $L^{2n/(n+2)}(|x| < 1)$ norm and the weighted $L^1_{|x|}((1,\infty), L^2(\mathbb{S}^{n-1}))$ norm by the use of Lemma 2.2 in [6] when $n \geq 3$. Hidano has given a similar estimate for the second term on the left-hand side of (2.27) in [12, Theorem 3.1].

Proof of Corollary 5. For $T > 0$, we put the weight function $W(T)$ as

$$W(T) := \begin{cases} \max\left\{1, \frac{1}{|1-2\delta|^{1/2}}\right\} & \text{if } \delta > \frac{1}{2} \\ \frac{1}{|1-2\delta|^{1/2}} \langle T\rangle^{1/2-\delta} & \text{if } \delta < \frac{1}{2} \\ (\log(e+T))^{1/2} & \text{if } \delta = \frac{1}{2}. \end{cases} \qquad (2.28)$$

First, we have

$$\|\langle x\rangle^{-\delta} e^{\pm it\omega} f\|_{L^2_{t,x}((0,T)\times\{|x|<1\})} \lesssim \|f\|_{L^2} \qquad (2.29)$$

by (2.5). We show

$$\|\langle x\rangle^{-\delta}e^{\pm it\omega}f\|_{L^2_{t,x}((0,T)\times\{1<|x|<T\})}\lesssim\|f\|_{L^2}\cdot W(T) \tag{2.30}$$

for any $T>1$. Indeed, we use the decomposition

$$\{1<|x|<T\}\subset\cup_{j=0}^{\log_2 T}\{2^j\le|x|<2^{j+1}\} \tag{2.31}$$

and the estimate

$$\|e^{\pm it\omega}f\|_{L^2_{t,x}((0,T)\times\{2^j<|x|<2^{j+1}\})}\lesssim 2^{j/2}\|f\|_{L^2} \tag{2.32}$$

by (2.5), to obtain

$$\|\langle x\rangle^{-\delta}e^{\pm it\omega}f\|^2_{L^2_{t,x}((0,T)\times\{1<|x|<T\})}\lesssim\sum_{j=0}^{\log_2 T}2^{(1-2\delta)j}\|f\|^2_{L^2}. \tag{2.33}$$

The required estimate follows since $\sum_{j=0}^{\log_2 T}2^{(1-2\delta)j}$ is bounded by $W(T)^2$.

Combining the estimates (2.29) and (2.30) and the simple estimate

$$\|\langle x\rangle^{-\delta}e^{\pm it\omega}f\|_{L^2_{t,x}((0,T)\times\{T<|x|\})}\lesssim\langle T\rangle^{1/2-\delta}\|f\|_{L^2}, \tag{2.34}$$

we obtain the estimate

$$\|\langle x\rangle^{-\delta}e^{\pm it\omega}f\|_{L^2_{t,x}((0,T)\times\mathbb{R}^n)}\lesssim\|f\|_{L^2}\cdot W(T), \tag{2.35}$$

which enables us to derive (2.25), and (2.27) by (2.16). □

3. Applications to nonlinear higher-order wave equations

In this section, we give two applications of Corollary 3 and Corollary 4. We prepare Sobolev-type estimates and estimates for nonlinear terms.

3.1. **Sobolev-type estimates.** Let us denote the derivatives $\partial_t,\partial_1,\dots,\partial_n$, $x_j\partial_k-x_k\partial_j$, $1\le j\ne k\le n$, by Z. For any nonnegative real number N, we denote all derivatives $Z^\alpha u$ by $|\alpha|\le N$ by $\overline{Z}^N u$, where α denotes a multi-index. We use Sobolev-type estimates to obtain some decay for x variables. The following estimate is well-known, but we add a brief sketch of the proof for the convenience of readers.

Lemma 7. *Let $n\ge1$. Then the inequality*

$$\|h(x)\|_{L^\infty(R<|x|<2R)}\lesssim R^{-(n-1)/2}\sum_{|\alpha|\le n/2+1}\|Z^\alpha h(x)\|_{L^2(R/2<|x|<4R)} \tag{3.1}$$

holds for any function h and any $R\ge1$.

Proof. Let $\chi \in C^\infty([0,\infty))$ be a bump function with $\chi(r) = 1$ for $1 \le r \le 2$ and $\chi(r) = 0$ for $r \le 1/2$ or $r \ge 4$. Then by the Sobolev estimates for polar coordinates, we have

$$\begin{aligned} &\|h(x)\|_{L^\infty(R<|x|<2R)} \le \|\chi(r/R)h(r\omega)\|_{L^\infty_{r,\omega}((0,\infty)\times\mathbb{S}^{n-1})} \\ &\lesssim \sum_{|\alpha|\le n/2+1} \|\partial^\alpha_{r,\omega}\left(\chi(r/R)h(r\omega)\right)\|_{L^2_{r,\omega}((0,\infty)\times\mathbb{S}^{n-1})} \\ &\lesssim \sum_{|\alpha|\le n/2+1} \|\partial^\alpha_{r,\omega} h(r\omega)\|_{L^2_{r,\omega}(R/2<r<4R,\omega\in\mathbb{S}^{n-1})}, \end{aligned} \tag{3.2}$$

where we note that the $\partial_{r,\omega}(\chi(r/R))$ are always bounded independent of $R \ge 1$. To bound the last term, for example, we decompose the sphere $\mathbb{S}^{n-1}$ adequately and use

$$\begin{aligned} &\int_{R/2}^{4R}\int_{|(\omega_1,\cdots,\omega_{n-1})|\le 1/2} |\partial^\alpha_{r,\omega} h(r\omega)|^2 d\omega_1\cdots d\omega_{n-1}dr \\ &\lesssim R^{-n+1}\int_{R/2<|x|<4R} |\partial^\alpha_{r,\omega}h(x)|^2dx, \end{aligned} \tag{3.3}$$

where R^{-n+1} appears from the Jacobian. We obtain the required result since $\partial_{r,\omega}$ can be written by the combination of Z. □

3.2. **Estimates for nonlinear terms.**

Lemma 8. *Let $n \ge 2$ and $m > 0$. Let p be an integer and γ a real number which satisfy*

$$p \ge \max\{2, 1+\frac{2}{n-1}\}, \qquad 0 \le \gamma \le \frac{n-1}{4}(p-1-\frac{2}{n-1}).$$

Then for any $M \ge 0$, the following estimates hold:

(1)
$$\begin{aligned} \sum_{|\alpha|\le M} \|Z^\alpha u^p\|_{L^1((0,\infty),L^2(\mathbb{R}^n))} &\lesssim \sum_{|\alpha|\le M/2+n/2+1} \|Z^\alpha u\|^{p-2}_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} \\ &\times \sum_{|\alpha|\le\max\{M/2+n/2+1,M\}} \|\langle x\rangle^{-1/2-\gamma} Z^\alpha u\|^2_{L^2((0,\infty)\times\mathbb{R}^n)}. \end{aligned} \tag{3.4}$$

(2)
$$\begin{aligned} &\sum_{|\alpha|\le M} \|Z^\alpha(u^p - v^p)\|_{L^1((0,\infty),L^2(\mathbb{R}^n))} \\ &\lesssim \Big(\max_{w=u,v} \sum_{|\alpha|\le M/2+n/2+1} \|Z^\alpha w\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))}\Big)^{p-2} \end{aligned}$$

$$\times \max_{w=u,v} \sum_{|\alpha|\le \max\{M/2+n/2+1,M\}} \|\langle x\rangle^{-1/2-\gamma} Z^\alpha w\|_{L^2((0,\infty)\times\mathbb{R}^n)}$$
$$\times \sum_{|\alpha|\le \max\{M/2+n/2+1,M\}} \|\langle x\rangle^{-1/2-\gamma} Z^\alpha (u-v)\|_{L^2((0,\infty)\times\mathbb{R}^n)}. \quad (3.5)$$

Proof. (1) First, we have

$$|\overline{Z}^M u^p(t,x)| \lesssim |\overline{Z}^{M/2} u(t,x)|^{p-1} |\overline{Z}^M u(t,x)|,$$

and we use the Sobolev estimates in Lemma 7 to obtain

$$|\overline{Z}^{M/2} u(t,x)| \lesssim \langle x\rangle^{-(n-1)/2} \|\overline{Z}^{M/2+n/2+1} u\|_{L^2_y(|x|/2<|y|<2|x|)}.$$

Taking the norm in L^2_x, we obtain the required estimates under the assumption of p and γ.

The proof of (2) also follows similarly by the use of the inequality

$$\begin{aligned} &|\overline{Z}^M (u^p - v^p)(t,x)| \qquad (3.6)\\ &\lesssim \max_{w=u,v} |\overline{Z}^M w(t,x)| \max_{w=u,v} |\overline{Z}^{M/2} w(t,x)|^{p-2} |\overline{Z}^{M/2}(u-v)(t,x)| \\ &\quad + \max_{w=u,v} |\overline{Z}^{M/2} w(t,x)|^{p-1} |\overline{Z}^M (u-v)(t,x)|. \end{aligned}$$ □

3.3. **Application to the Cauchy problem: Part 1.** In this section, we show the following application of Corollary 3.

Theorem 9. *Let $m \ge 1$ be any real number. Let $n \ge 2$, $M \ge n+m+1$, and $N := M/2 + n/2 + 1$; let p be an integer which satisfies*

$$p \ge \max\left\{1 + \tfrac{2}{n-1}, 2\right\}, \quad (3.7)$$

and let

$$\gamma := \tfrac{n-1}{4}\left(p - 1 - \tfrac{2}{n-1}\right). \quad (3.8)$$

Let λ be any real number. We consider the Cauchy problem

$$(P1) \quad \begin{cases} (\partial_t^2 + (-\Delta)^m) u(t,x) = \lambda \partial_x^{(m-1)/2+m} (u^p(t,x)) & \text{for } t \ge 0,\ x \in \mathbb{R}^n \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot), \end{cases} \quad (3.9)$$

where ∂_x^l denotes any derivative ∂_x^α with $|\alpha| = l$ if l is an integer, and the fractional derivative $\mathcal{F}^{-1}|\xi|^l \mathcal{F}$ if l is not an integer. We put

$$\varepsilon := \|\overline{Z}^M f\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M g\|_{\dot H^{-(m-1)/2-m}(\mathbb{R}^n)}. \quad (3.10)$$

If ε is sufficiently small, then the following results hold.

(1) *If* $p > 1 + 2/(n-1)$, *then* (P1) *has a unique global solution in*

$$\Big\{u : \|u\|_A := \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M u\|_{L^2((0,\infty)\times\mathbb{R}^n)} + \|\overline{Z}^N u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} < \infty\Big\}. \tag{3.11}$$

(2) *If* $p = 1 + 2/(n-1)$ *with* $m = 1$, *then* (P1) *has a unique almost-global solution in*

$$\Big\{u : \|u\|_A := (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M u\|_{L^2((0,T)\times\mathbb{R}^n))} + \|\overline{Z}^N u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} < \infty\Big\}, \tag{3.12}$$

where $T := C\exp(c/\varepsilon^{p-1})$ *for some positive constants* C *and* c.

(3) *Let* $T = \infty$ *for* (1). *In* (1) *and* (2), *the solution* u *satisfies*

$$Z^\alpha u \in C([0,T), L^2(\mathbb{R}^n)) \tag{3.13}$$

for any α *with* $|\alpha| \le N$, *and if* $\{f_j\}_{j=1}^\infty$ *and* $\{g_j\}_{j=1}^\infty$ *satisfy*

$$\lim_{j\to\infty}\Big\{\|\overline{Z}^M(f-f_j)\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M(g-g_j)\|_{\dot H^{-(m-1)/2-m}(\mathbb{R}^n)}\Big\} = 0, \tag{3.14}$$

then the solution u_j *for the Cauchy data* f_j *and* g_j *satisfies*

$$\lim_{j\to\infty}\|u-u_j\|_A = 0. \tag{3.15}$$

Remark 10. When we take $m = 1$ in the above theorem, the results show that the Cauchy problem of

$$(\partial_t^2 - \Delta)u = \lambda\partial_x(u^p) \tag{3.16}$$

for sufficiently small data has global solutions if $p \ge 2$ with $n \ge 4$, or $p \ge 3$ with $n = 3$, or $p \ge 4$ with $n = 2$. Also, it has almost-global solutions if $p = 2$ with $n = 3$ or $p = 3$ with $n = 2$ on the time interval $[0, C\exp(c/\varepsilon^{p-1})]$ for some positive constants C and c. The result of almost-global solutions for $p = 2$ with $n = 3$ has been shown by John and Klainerman [21, Main Theorem] and Lindblad [31, Theorem 2.1]. The result of global solutions for $p > 3$ with $n = 2$, and almost-global solutions for $p = 3$ with $n = 2$, has been shown by Kovalyov [30, Theorem 3 and Remark 1]. Here, ∂_x denotes any derivative of $\partial/\partial x_1, \dots, \partial/\partial x_n$.

Remark 11. In Theorem 9, we have considered the single nonlinear term in (3.9). We are able to consider the nonlinear terms

$$\sum_{j=1}^{J}\lambda_j\partial_x^{(m-1)/2+m}(u^{p_j}(t,x)), \tag{3.17}$$

where $J \geq 1$, λ_j is any real number, and p_j is any integer with

$$\max\{1 + \frac{2}{n-1}, 2\} \leq p_1 \leq p_2 \leq \cdots \leq p_J. \tag{3.18}$$

We can obtain global solutions if $p_1 > 1 + 2/(n-1)$, and almost-global solutions for $T = C\exp(c/\varepsilon^{p_1-1})$ for some positive constants C and c if $p_1 = 1 + 2/(n-1)$ with $m = 1$.

Proof of Theorem 9. We define two norms

$$\|u\|_X := \begin{cases} \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M u\|_{L^2((0,\infty)\times\mathbb{R}^n)} & \text{for } (1) \\ (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M u\|_{L^2((0,T)\times\mathbb{R}^n)} & \text{for } (2) \end{cases} \tag{3.19}$$

$$\|u\|_Y := \begin{cases} \|\overline{Z}^N u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} & \text{for } (1) \\ \|\overline{Z}^N u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} & \text{for } (2), \end{cases} \tag{3.20}$$

where $T > 0$ is determined later.

Let $F(u) := \lambda\partial_x^{(m-1)/2+m}(u^p(t,x))$. We show that the operator

$$\Phi(u)(t) := \cos t\omega f + \frac{\sin t\omega}{\omega}g + \int_0^t \frac{\sin(t-s)\omega}{\omega}F(u(s))ds \tag{3.21}$$

is a contraction on a ball in (3.11) or (3.12). By Corollary 3, we have

$$\|\Phi(u)\|_X \lesssim \|\overline{Z}^M f\|_{\dot{H}^{-\frac{m-1}{2}}} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2-m}} + \|\overline{Z}^M F(u)\|_{L^1\dot{H}^{-(m-1)/2-m}}, \tag{3.22}$$

where the last term is bounded by

$$\|\overline{Z}^M(u^p)\|_{L^1L^2}. \tag{3.23}$$

Similarly, we also have

$$\begin{aligned} \|\Phi(u)\|_Y &\lesssim \|\overline{Z}^N f\|_{L^2} + \|\overline{Z}^N g\|_{\dot{H}^{-m}} + \|\overline{Z}^N F(u)\|_{L^1\dot{H}^{-m}} \\ &\lesssim \|\overline{Z}^M f\|_{\dot{H}^{-(m-1)/2}} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2-m}} + \|\overline{Z}^M(u^p)\|_{L^1L^2}, \end{aligned} \tag{3.24}$$

where we have used $N + (m-1)/2 \leq M$. By Lemma 8, we have

$$\|\overline{Z}^M(u^p)\|_{L^1L^2} \lesssim \|u\|_Y^{p-2}\|u\|_X^2\log(e+T), \tag{3.25}$$

where $\log(e+T)$ is removed for (1).

So, we have obtained

$$\|\Phi(u)\|_{X\cap Y} \lesssim \|\overline{Z}^M f\|_{\dot{H}^{-(m-1)/2}} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2-m}} + \|u\|_{X\cap Y}^p\log(e+T), \tag{3.26}$$

where $\log(e+T)$ is removed for (1).

On the other hand, since

$$\Phi(u)-\Phi(v)=\int_0^t \frac{\sin(t-s)\omega}{\omega}(F(u(s))-F(v(s)))ds, \tag{3.27}$$

by Corollary 3, we have

$$\|\Phi(u)-\Phi(v)\|_{X\cap Y}\lesssim \|\overline{Z}^M(u^p-v^p)\|_{L^1L^2}, \tag{3.28}$$

where we have used $N+(m-1)/2\le M$. By Lemma 8, the right-hand side is bounded by

$$\max_{w=u,v}\|w\|_Y^{p-2}\max_{w=u,v}\|w\|_X\|u-v\|_X\log(e+T), \tag{3.29}$$

where $\log(e+T)$ is removed for (1). So, we have obtained

$$\|\Phi(u)-\Phi(v)\|_{X\cap Y}\lesssim \max_{w=u,v}\|w\|_{X\cap Y}^{p-1}\|u-v\|_X\log(e+T), \tag{3.30}$$

where $\log(e+T)$ is removed for (1).

Therefore, the Banach fixed-point theorem shows the unique existence of the fixed point of Φ in a ball of (3.11) or (3.12) if the data are sufficiently small.

(3) First, we show (3.13). Since we have

$$\|\overline{Z}^N(\cos(t+\varepsilon)\omega-\cos t\omega)f\|_{L^2}\lesssim \|\overline{Z}^M f\|_{\dot H^{-(m-1)/2}} \tag{3.31}$$

by the L^2 invariance of the Fourier transform, we obtain

$$\lim_{\varepsilon\to 0}\|\overline{Z}^N(\cos(t+\varepsilon)\omega-\cos t\omega)f\|_{L^2}=0 \tag{3.32}$$

by Lebesgue's convergence theorem. Similarly, we obtain

$$\lim_{\varepsilon\to 0}\|\overline{Z}^N(\frac{\sin(t+\varepsilon)\omega}{\omega}-\frac{\sin t\omega}{\omega})g\|_{L^2}=0. \tag{3.33}$$

For the inhomogeneous term, we put

$$\begin{aligned}&\int_0^{t+\varepsilon}\frac{\sin(t+\varepsilon-s)\omega}{\omega}F(u(s))ds-\int_0^t\frac{\sin(t-s)\omega}{\omega}F(u(s))ds \qquad (3.34)\\ &=\int_t^{t+\varepsilon}\frac{\sin(t+\varepsilon-s)\omega}{\omega}F(u(s))ds\\ &+\int_0^t\frac{\sin(t+\varepsilon-s)\omega-\sin(t-s)\omega}{\omega}F(u(s))ds=:I+II.\end{aligned}$$

By the L^2 invariance of the Fourier transform, we have

$$\|\overline{Z}^N I\|_{L^2}\lesssim \|\overline{Z}^M u^p\|_{L^1((t,t+\varepsilon),L^2)}\longrightarrow 0 \quad \text{as } \varepsilon\to 0. \tag{3.35}$$

Similarly to (3.32), we have

$$\lim_{\varepsilon\to 0} \|\overline{Z}^N II\|_{L^2} = 0. \tag{3.36}$$

Next, we show (3.15). Since

$$\begin{aligned}(u-u_j)(t) &= \cos t\omega(f-f_j) + \frac{\sin t\omega}{\omega}(g-g_j)\\ &\quad + \int_0^t \frac{\sin(t-s)\omega}{\omega}(F(u(s)-F(u_j(s)))ds,\end{aligned} \tag{3.37}$$

we have by Corollary 3

$$\begin{aligned}\|u-u_j\|_{X\cap Y} &\lesssim \|\overline{Z}^M(f-f_j)\|_{\dot H^{-(m-1)/2}}\\ &\quad + \|\overline{Z}^M(g-g_j)\|_{\dot H^{-(m-1)/2-m}} + \|\overline{Z}^M(u^p-u_j^p)\|_{L^1L^2}.\end{aligned} \tag{3.38}$$

By Lemma 8, the last term is bounded by

$$\max_{w=u,u_j} \|w\|_{X\cap Y}^{p-1}\|u-u_j\|_X \log(e+T), \tag{3.39}$$

where $\log(e+T)$ is removed for (1), which implies

$$\|u-u_j\|_{X\cap Y} \longrightarrow 0 \quad \text{as} \quad j\to\infty. \tag{3.40}$$

3.4. **Application to the Cauchy problem: Part 2.** We give an application of Corollary 4. We recall that $\omega = (-\Delta)^{m/2}$.

Theorem 12. *Let $m\ge 1$ be any real number. Let $n\ge 2$, $M\ge n+m+3$, and $N := M-(m-1)/2$. Also, let p be an integer which satisfies*

$$p \ge \max\left\{1+\tfrac{2}{n-1}, 2\right\}, \qquad \textit{and let} \qquad \gamma := \tfrac{n-1}{4}\left(p-1-\tfrac{2}{n-1}\right). \tag{3.41}$$

Let $P(\cdot,\cdot)$ be any homogeneous polynomial of p-order for two variables. We consider the Cauchy problem

$$(P2) \qquad \begin{cases} (\partial_t^2 + (-\Delta)^m)u(t,x) = \partial_x^{(m-1)/2}(P(\partial_t u(t,x), \omega u(t,x))) \\ \qquad\qquad\qquad\qquad \textit{for} \ \ t\ge 0, \quad x\in\mathbb{R}^n \\ u(0,\cdot) = f(\cdot), \quad \partial_t u(0,\cdot) = g(\cdot), \end{cases} \tag{3.42}$$

where ∂_x^l denotes any derivative ∂_x^α with $|\alpha| = l$ if l is an integer, and the fractional derivative $\mathcal{F}^{-1}|\xi|^l\mathcal{F}$ if l is not an integer. We put

$$\varepsilon := \|\overline{Z}^M f\|_{H^{(m+1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M g\|_{\dot H^{-(m-1)/2}(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}. \tag{3.43}$$

If ε is sufficiently small, then the following results hold.

(1) *If* $p > 1 + 2/(n-1)$, *then* (P2) *has a unique global solution in*

$$\Big\{u : \|u\|_A := \|\langle x\rangle^{-\frac{1}{2}-\gamma}\overline{Z}^M\partial_t u\|_{L^2((0,\infty)\times\mathbb{R}^n)} + \|\langle x\rangle^{-\frac{1}{2}-\gamma}\overline{Z}^M\omega u\|_{L^2((0,\infty)\times\mathbb{R}^n)}$$
$$+ \|\overline{Z}^N\partial_t u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} + \|\overline{Z}^N\omega u\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))}$$
$$+ \sup_{0<t<\infty}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty\Big\}. \tag{3.44}$$

(2) *If* $p = 1 + 2/(n-1)$ *with* $m = 1$, *then* (P2) *has a unique almost-global solution in*

$$\Big\{u : \|u\|_A := (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M\partial_t u\|_{L^2((0,T)\times\mathbb{R}^n)}$$
$$+ (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M\omega u\|_{L^2((0,T)\times\mathbb{R}^n)}$$
$$+ \|\overline{Z}^N\partial_t u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} + \|\overline{Z}^N\omega u\|_{L^\infty((0,T),L^2(\mathbb{R}^n))}$$
$$+ \sup_{0<t<T}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty\Big\}, \tag{3.45}$$

where $T := C\exp(c/\varepsilon^{p-1})$ *for some positive constants* C *and* c.

(3) *Let* $T = \infty$ *for* (1). *In* (1) *and* (2), *the solution* u *satisfies*

$$Z^\alpha u \in C([0,T), L^2(\mathbb{R}^n)) \tag{3.46}$$

for any α *with* $|\alpha| \le N$, *and if* $\{f_j\}_{j=1}^\infty$ *and* $\{g_j\}_{j=1}^\infty$ *satisfy*

$$\lim_{j\to\infty}\Big\{\|\overline{Z}^M(f-f_j)\|_{H^{(m+1)/2}(\mathbb{R}^n)} + \|\overline{Z}^M(g-g_j)\|_{\dot{H}^{-(m-1)/2}(\mathbb{R}^n)}$$
$$+ \|g-g_j\|_{L^2(\mathbb{R}^n)}\Big\}, \tag{3.47}$$

then the solution u_j *for the Cauchy data* f_j *and* g_j *satisfies*

$$\lim_{j\to\infty}\|u-u_j\|_A = 0. \tag{3.48}$$

Remark 13. When we take $m = 1$ in the above theorem, the results show that the Cauchy problem of

$$(\partial_t^2 - \Delta)u = P(\partial_t u, \nabla_x u) \tag{3.49}$$

for sufficiently small data has global solutions if $p \ge 2$ with $n \ge 4$ or $p \ge 3$ with $n = 3$ or $p \ge 4$ with $n = 2$. Also, it has almost-global solutions if $p = 2$ with $n = 3$ or $p = 3$ with $n = 2$ on the time interval $[0, C\exp(c/\varepsilon^{p-1})]$ for some positive constants C and c. Here, $P(\cdot,\cdot)$ denotes any homogeneous polynomials of order p. The result of global solutions for $p > 3$ with $n = 2$, and almost-global solutions for $p = 3$ with $n = 2$, has been shown by Kovalyov [30, Theorem 3]. The result of almost-global solutions for $p = 2$ with

$n = 3$ has been shown by Klainerman [26, Theorem 1] for spherical symmetric solutions and John and Klainerman [21, Main Theorem] for nonspherical symmetric solutions.

Remark 14. In Theorem 12, the single nonlinear term in (3.42) can be replaced with

$$\sum_{j=1}^{J} \partial_x^{\alpha_j} P_j(\partial_t u(t,x), \omega u(t,x)), \tag{3.50}$$

where $J \geq 1$, α_j is any multi-index with $|\alpha_j| = (m-1)/2$, p_j is any homogeneous polynomial of p_j order, and p_j is any integer with

$$\max\left\{1 + \tfrac{2}{n-1}, 2\right\} \leq p_1 \leq p_2 \leq \cdots \leq p_J. \tag{3.51}$$

We can obtain global solutions if $p_1 > 1 + 2/(n-1)$, and almost-global solutions for $T = C\exp(c/\varepsilon^{p_1-1})$ for some positive constants C and c if $p_1 = 1 + 2/(n-1)$ with $m = 1$.

Proof of Theorem 12. We denote $\partial_t u$ and ωu by u'. We define three notations.

$$\|u\|_X := \begin{cases} \|\langle x\rangle^{-1/2-\gamma}\overline{Z}^M u'\|_{L^2((0,\infty)\times\mathbb{R}^n)} & \text{for } (1) \\ (\log(e+T))^{-1/2}\,\|\langle x\rangle^{-1/2}\overline{Z}^M u'\|_{L^2((0,T)\times\mathbb{R}^n)} & \text{for } (2) \end{cases} \tag{3.52}$$

$$\|u\|_Y := \begin{cases} \|\overline{Z}^N u'\|_{L^\infty((0,\infty),L^2(\mathbb{R}^n))} & \text{for } (1) \\ \|\overline{Z}^N u'\|_{L^\infty((0,T),L^2(\mathbb{R}^n))} & \text{for } (2) \end{cases} \tag{3.53}$$

$$\|u\|_Z := \begin{cases} \sup_{0<t<\infty}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} & \text{for } (1) \\ \sup_{0<t<T}(1+t)^{-1}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} & \text{for } (2), \end{cases} \tag{3.54}$$

where $T > 0$ is determined later.

Let $F(u') := \partial_x^{(m-1)/2}(P(u'))$. We show that the operator

$$\Phi(u)(t) := \cos t\omega f + \frac{\sin t\omega}{\omega} g + \int_0^t \frac{\sin(t-s)\omega}{\omega} F(u'(s))ds \tag{3.55}$$

is a contraction on a ball in (3.44) or (3.45). By Corollary 4, we have

$$\|\Phi(u)\|_X \lesssim \|\overline{Z}^M f\|_{\dot H^{(m+1)/2}} + \|\overline{Z}^M g\|_{\dot H^{-(m-1)/2}} + \|\overline{Z}^M F(u')\|_{L^1\dot H^{-(m-1)/2}}, \tag{3.56}$$

where the last term is bounded by

$$\|\overline{Z}^M (P(u'))\|_{L^1L^2}. \tag{3.57}$$

Similarly, we also have

$$\begin{aligned}\|\Phi(u)\|_Y &\lesssim \|\overline{Z}^N f\|_{\dot{H}^{m/2}} + \|\overline{Z}^N g\|_{L^2} + \|\overline{Z}^N F(u')\|_{L^1L^2}\\ &\lesssim \|\overline{Z}^M f\|_{\dot{H}^{(m+1)/2}} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2}} + \|\overline{Z}^M (P(u'))\|_{L^1L^2}, \quad (3.58)\end{aligned}$$

$$\|\Phi(u)\|_Z \lesssim \|f\|_{L^2} + \|g\|_{L^2} + \|\overline{Z}^M (P(u'))\|_{L^1L^2}, \quad (3.59)$$

where we have used $N + (m-1)/2 = M$. By Lemma 8, we have

$$\|\overline{Z}^M (P(u'))\|_{L^1L^2} \lesssim \|u\|_Y^{p-2} \|u\|_X^2 \log(e+T), \quad (3.60)$$

where $\log(e+T)$ is removed for (1), and we have used the condition $M \geq m+n+3$.

So, we have obtained

$$\begin{aligned}&\|\Phi(u)\|_{X\cap Y\cap Z} \quad (3.61)\\ &\lesssim \|\overline{Z}^M f\|_{H^{(m+1)/2}} + \|\overline{Z}^M g\|_{\dot{H}^{-(m-1)/2}} + \|g\|_{L^2} + \|u\|_{X\cap Y}^p \log(e+T),\end{aligned}$$

where $\log(e+T)$ is removed for (1).

On the other hand, since

$$\Phi(u) - \Phi(v) = \int_0^t \frac{\sin(t-s)\omega}{\omega} (F(u'(s)) - F(v'(s)))ds, \quad (3.62)$$

by Corollary 4, we have

$$\|\Phi(u) - \Phi(v)\|_{X\cap Y\cap Z} \lesssim \|\overline{Z}^M (P(u') - P(v'))\|_{L^1L^2}. \quad (3.63)$$

By Lemma 8, the right-hand side is bounded by

$$\max_{w=u,v} \|w\|_Y^{p-2} \max_{w=u,v} \|w\|_X \|u-v\|_X \log(e+T), \quad (3.64)$$

where $\log(e+T)$ is removed for (1). So, we have obtained

$$\|\Phi(u) - \Phi(v)\|_{X\cap Y\cap Z} \lesssim \max_{w=u,v} \|w\|_{X\cap Y}^{p-1} \|u-v\|_X \log(e+T), \quad (3.65)$$

where $\log(e+T)$ is removed for (1).

Therefore, the Banach fixed-point theorem shows the unique existence of the fixed point of Φ if the data are sufficiently small. We note that the closed ball in X becomes a complete metric space since we are considering the norm $\|\cdot\|_Z$. Since the proof of (3) is quite analogous to that of Theorem 9, we omit it. □

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References

[1] Serge Alinhac, *Remarks on energy inequalities for wave and Maxwell equations on a curved background*, Math. Ann., 329 (2004), 707–722.

[2] Serge Alinhac, *On the Morawetz–Keel-Smith-Sogge inequality for the wave equation on a curved background*, Publ. Res. Inst. Math. Sci., 42 (2006), 705–720.

[3] Yun Mei Chen, *The life span of the solutions to Cauchy problems for nonlinear wave equations*, Nonlinear Anal., 13 (1989), 101–124.

[4] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math., 39 (1986), 267–282.

[5] Jean-Marc Delort and Daoyuan Fang, *Almost global existence for solutions of semilinear Klein-Gordon equations with small weakly decaying Cauchy data*, Comm. Partial Differential Equations, 25 (2000), 2119–2169.

[6] Yi Du, Jason Metcalfe, Christopher D. Sogge, and Yi Zhou, *Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in 4-dimensions*, Comm. Partial Differential Equations, 33 (2008), 1487–1506.

[7] Yi Du and Yi Zhou, *The lifespan for nonlinear wave equation outside of star-shaped obstacle in three space dimensions*, Comm. Partial Differential Equations, 33 (2008), 1455–1486.

[8] J. Ginibre and N. Hayashi, *Almost global existence of small solutions to quadratic nonlinear Schrödinger equations in three space dimensions*, Math. Z., 219 (1995), 119–140.

[9] Lars Hörmander, "Lectures on Nonlinear Hyperbolic Differential Equations," Mathematiques and Applications (Berlin), 26, Springer-Verlag, Berlin, 1997, viii+289 pp.

[10] Lars Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Pseudodifferential operators (Oberwolfach, 1986), 214–280, Lecture Notes in Math., 1256, Springer, Berlin, 1987.

[11] Kunio Hidano, *An elementary proof of global or almost global existence for quasi-linear wave equations*, Tohoku Math. J., 56 (2004), 271–287.

[12] Kunio Hidano, *Morawetz-Strichartz estimates for spherically symmetric solutions to wave equations and applications to semilinear Cauchy problems*, Differential Integral Equations, 20 (2007), 735–754.

[13] Kunio Hidano and Kazuyoshi Yokoyama, *A remark on the almost global existence theorems of Keel, Smith and Sogge*, Funkcial. Ekvac., 48 (2005), 1–34.

[14] Kunio Hidano and Kazuyoshi Yokoyama, *Space-time L^2-estimates and life span of the Klainerman-Machedon radial solutions to some semi-linear wave equations*, Differential Integral Equations, 19 (2006), 961–980.

[15] Fritz John, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math., 34 (1981), 29–51.

[16] Fritz John, *Lower bounds for the life span of solutions of nonlinear wave equations in three dimensions*, Comm. Pure Appl. Math., 36 (1983), 1–35.

[17] Fritz John, *Formation of singularities in elastic waves*, Trends and Applications of Pure Mathematics to Mechanics (Palaiseau, 1983), 194–210, Lecture Notes in Phys., 195, Springer, Berlin, 1984.

[18] Fritz John, *Blow-up of radial solutions of $u_{tt} = c^2(u_t)\Delta u$ in three space dimensions*, Mat. Apl. Comput., 4 (1985), 3–18.

[19] Fritz John, *Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small initial data*, Comm. Pure Appl. Math., 40 (1987), 79–109.

[20] Fritz John, *Almost global existence of elastic waves of finite amplitude arising from small initial disturbances*, Comm. Pure Appl. Math., 41 (1988), 615–666.

[21] F. John and S. Klainerman, *Almost global existence to nonlinear wave equations in three space dimensions*, Comm. Pure Appl. Math., 37 (1984), 443–455.

[22] Markus Keel, Hart F. Smith, and Christopher D. Sogge, *Almost global existence for some semilinear wave equations*, Dedicated to the memory of Thomas H. Wolff, J. Anal. Math., 87 (2002), 265–279.

[23] Markus Keel, Hart F. Smith, and Christopher D. Sogge, *Almost global existence for quasilinear wave equations in three space dimensions*, J. Amer. Math. Soc., 17 (2004), 109–153.

[24] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., 40 (1991), 33–69.

[25] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal., 127 (1995), 204–234.

[26] Sergiu Klainerman, *On "almost global" solutions to quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math., 36 (1983), 325–344.

[27] Sergiu Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math., 38 (1985), 321–332.

[28] Sergiu Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1 (Santa Fe, N.M., 1984), 293–326, Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[29] Sergiu Klainerman and Thomas C. Sideris, *On almost global existence for nonrelativistic wave equations in* 3*D*, Comm. Pure Appl. Math., 49 (1996), 307–321.

[30] Mikhail Kovalyov, *Long-time behaviour of solutions of a system of nonlinear wave equations*, Comm. Partial Differential Equations, 12 (1987), 471–501.

[31] Hans Lindblad, *On the lifespan of solutions of nonlinear wave equations with small initial data*, Comm. Pure Appl. Math., 43 (1990), 445–472.

[32] Tokio Matsuyama, *Asymptotics for the nonlinear dissipative wave equation*, Trans. Amer. Math. Soc., 355 (2003), 865–899.

[33] Jason L. Metcalfe, *Global existence for semilinear wave equations exterior to nontrapping obstacles*, Houston J. Math., 30 (2004), 259–281.

[34] Jason Metcalfe, *Elastic waves in exterior domains. I. Almost global existence*, Int. Math. Res. Not., 2006, Art. ID 69826, 41 pp.

[35] Jason Metcalfe, Makoto Nakamura, and Christopher D. Sogge, *Global existence of solutions to multiple speed systems of quasilinear wave equations in exterior domains*, Forum Math., 17 (2005), 133–168.

[36] Jason Metcalfe, Makoto Nakamura, and Christopher D. Sogge, *Global existence of quasilinear, nonrelativistic wave equations satisfying the null condition*, Japan. J. Math. (N.S.), 31 (2005), 391–472.

[37] Jason Metcalfe and Makoto Nakamura, *General quasilinear wave equations with localized dissipation in exterior domains*, J. Differential Equations, 233 (2007), 313–344.

[38] Jason Metcalfe and Christopher D. Sogge, *Hyperbolic trapped rays and global existence of quasilinear wave equations*, Invent. Math., 159 (2005), 75–117.
[39] Jason Metcalfe and Christopher D. Sogge, *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal., 38 (2006), 188–209.
[40] Jason Metcalfe and Christopher D. Sogge, *Global existence for Dirichlet-wave equations with quadratic nonlinearities in high dimensions*, Math. Ann., 336 (2006), 391–420.
[41] Jason Metcalfe and Christopher D. Sogge, *Global existence of null-form wave equations in exterior domains*, Math. Z., 256 (2007), 521–549.
[42] Jason Metcalfe and Becca Thomases, *Elastic waves in exterior domains. II. Global existence with a null structure*, Int. Math. Res. Not., IMRN 2007, Art. ID rnm034, 43 pp.
[43] Kiyoshi Mochizuki, *Scattering theory for wave equations*, Kinokuniya, Tokyo, 1984 (in Japanese).
[44] Cathleen S. Morawetz, *Time decay for the nonlinear Klein-Gordon equations*, Proc. Roy. Soc. Ser. A, 306 (1968), 291–296.
[45] Cathleen S. Morawetz, *Decay for solutions of the exterior problem for the wave equation*, Comm. Pure Appl. Math., 28 (1975), 229–264.
[46] Kazunori Moriyama, Satoshi Tonegawa, and Yoshio Tsutsumi, *Almost global existence of solutions for the quadratic semilinear Klein-Gordon equation in one space dimension*, Funkcial. Ekvac., 40 (1997), 313–333.
[47] M. Gabriella Paoli, *Almost global existence for solutions of a nonlinear fourth order hyperbolic equation*, NoDEA Nonlinear Differential Equations Appl., 4 (1997), 249–270.
[48] M.A. Rammaha, *Upper bounds for the life span of solutions to systems of nonlinear wave equations in two and three space dimensions*, Nonlinear Anal., 25 (1995), 639–654.
[49] M.A. Rammaha, *A note on a nonlinear wave equation in two and three space dimensions*, Comm. Partial Differential Equations, 22 (1997), 799–810.
[50] Thomas C. Sideris, *Global behavior of solutions to nonlinear wave equations in three dimensions*, Comm. Partial Differential Equations, 8 (1983), 1291–1323.
[51] Hart F. Smith and Christopher D. Sogge, *Global Strichartz estimates for nontrapping perturbations of the Laplacian*, Comm. Partial Differential Equations, 25 (2000), 2171–2183.
[52] Christopher D. Sogge, *Global existence for nonlinear wave equations with multiple speeds*, Harmonic Analysis at Mount Holyoke (South Hadley, MA, 2001), 353–366, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.
[53] Christopher D. Sogge, "Lectures on Non-Linear Wave Equations," Second edition, International Press, Boston, MA, 2008. x+205 pp.
[54] Jacob Sterbenz, "Angular Regularity and Strichartz Estimates for the Wave Equation, with an Appendix by Igor Rodnianski," Int. Math. Res. Not., 2005, 187–231.
[55] Walter A. Strauss, *Dispersal of waves vanishing on the boundary of an exterior domain*, Comm. Pure Appl. Math., 28 (1975), 265–278.

[56] Jie Xin and Tiehu Qin, *Almost global existence of Dirichlet initial-boundary value problem for nonlinear elastodynamic system outside a star-shaped domain*, Chinese Ann. Math. Ser. B, 25 (2004), 433–454.

[57] Xin Jie and Qin Tiehu, *Almost global existence for the initial value problem of nonlinear elastodynamic system*, J. Math. Anal. Appl., 339 (2008), 517–529.