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POSITIVE SOLUTIONS FOR INFINITE SEMIPOSITONE PROBLEMS ON EXTERIOR DOMAINS

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Abstract. We study positive radial solutions to the problem

$$\begin{cases} -\Delta u = \lambda K(|x|)f(u) & x \in \Omega, \\ \quad u = 0 & \text{if } |x| = r_0, \\ \quad u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{0.1}$$

where λ is a positive parameter, $\Delta u = \operatorname{div}\big(\nabla u\big)$ is the Laplacian of u, $\Omega = \{x \in \mathbb{R}^n, n > 2 : |x| > r_0\}$ is an exterior domain and $f : (0,\infty) \to \mathbb{R}$ belongs to a class of sublinear functions at ∞ such that they are continuous and $f(0^+) = \lim_{s\to 0^+} f(s) < 0$. In particular we also study infinite semipositone problems where $\lim_{s\to 0^+} f(s) = -\infty$. Here $K : [r_0,\infty) \to (0,\infty)$ belongs to a class of continuous functions such that $\lim_{r\to\infty} K(r) = 0$. We establish various existence results for such boundary value problems and also extend our results to classes of systems. We prove our results by the method of sub-/supersolutions.

1. Introduction

Consider the boundary value problem:

$$\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega, \\ \quad u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $\Delta u = \operatorname{div}\big(\nabla u\big)$ is the Laplacian of u and Ω is a bounded domain. In the case when $f(0) > 0$ (positone problems)

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there is a very rich history in the study of positive solutions (see Amann [3], Brown [6], Grandall [13], de Figueiredo [14], Gidas [15], Joseph [17], Kazdan [18], Laetsch [19], and Rabinowitz [24]). The case when $f(0) < 0$, which is referred in the literature as a semipositone problem, is mathematically more challenging. However, when Ω is a bounded domain many existence results have been proved for semipositone problems (See [1], [2], [4], [7], [23], and [28]). Also recently, existence of positive solutions has been studied in the singular case called infinite semipositone problems (here $\lim_{s\to 0^+} f(s) = -\infty$) (see [25], [21], and [22]). One of the main tools used in these studies is the method of sub and supersolutions. By a subsolution of (1.1) we mean a function $\psi \in C^2(\Omega)\bigcap C(\bar\Omega)$ that satisfies

$$\begin{cases} -\Delta\psi \le \lambda f(\psi) & \text{in } \Omega, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and by a supersolution of (1.1) we mean a function $Z \in C^2(\Omega)\cap C(\bar\Omega)$ that satisfies

$$\begin{cases} -\Delta Z \ge \lambda f(Z) & \text{in } \Omega, \\ Z > 0 & \text{in } \Omega, \\ Z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the following lemma holds (see [3, 26, 27]).

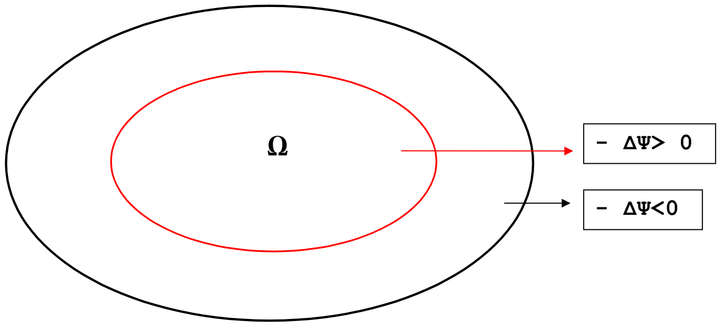
Lemma 1.1. *Let ψ be a subsolution of* (1.1) *and Z be a supersolution of* (1.1) *such that $\psi \le Z$. Then* (1.1) *has a solution u such that $\psi \le u \le Z$.*

Construction of a subsolution is challenging in the semipositone case (see [5] and [20]). Here our test functions for a positive subsolution must come from positive functions ψ such that $-\Delta\psi < 0$ near the boundary and $-\Delta\psi > 0$ in a large part of the interior.

Infinite semipositone problems are more challenging because in this case the subsolution must also satisfy $\lim\limits_{x\to\partial\Omega} -\Delta\psi = -\infty$ since $\lim\limits_{s\to 0^+} f(s) = -\infty$.

The main purpose of our paper is to establish existence results for infinite semipositone problems in **exterior domains**. We study positive radial solutions to the problem

$$\begin{cases} -\Delta u = \lambda K(|x|) f(u) & x \in \Omega, \\ u = 0 & \text{if } |x| = r_0, \\ u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$



where $\Omega = \{x \in \mathbb{R}^n, n > 2 : |x| > r_0\}$ is an exterior domain, λ is a positive parameter and $f : (0,\infty) \to \mathbb{R}$ belongs to a class of sublinear functions at ∞ such that they are continuous and $f(0^+) = \lim_{s\to 0^+} f(s) < 0$. In particular we also study the case when $\lim_{s\to 0^+} f(s) = -\infty$. Here $K : [r_0,\infty) \to (0,\infty)$ belongs to a class of continuous functions such that $\lim_{r\to\infty} K(r) = 0$. By using transformations $r = |x|$ and $s = (\frac{r}{r_0})^{2-n}$ we can reduce (see Appendix) equation (1.3) to the following boundary value problem:

$$\begin{cases} -z'' = \lambda h(s) f(z(s)) & 0 < s < 1, \\ z(0) = z(1) = 0, \end{cases} \tag{1.4}$$

where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$.

If $K(r) < \frac{1}{r^{n+\sigma}}$ for $r \gg 1$ and for some $\sigma \geq n-2$, then $h(t)$ is nonsingular at 0 and $h \in C([0,1],(0,\infty))$. In this case problem (1.4) (hence also problem (1.3)) can be studied using ideas similar to those used in [21] and [22], where the case $h(t) \equiv 1$ was considered. Hence in this paper we will focus on the challenging case when $\sigma < n-2$, which forces h to be singular at 0. Note that in this singular case $\hat{h} = \inf_{t\in(0,1)} h(t) > 0$.

We first consider positive solutions to the problem

$$\begin{cases} -\Delta u = \lambda K(|x|) \frac{g(u)}{u^\rho} & x \in \Omega, \quad 0 \leq \rho < 1, \\ \quad u = 0 & \text{if } |x| = r_0, \\ \quad u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

where Ω is as before and $g \in C([0,\infty),\mathbb{R})$ with $g(0) < 0$. By using transformations as before we can reduce equation (1.5) to the following boundary value problem:

$$\begin{cases} -z'' = \lambda h(s)\frac{g(z(s))}{z^\rho} & 0 < s < 1, \quad 0 \leq \rho < 1, \\ z(0) = z(1) = 0, \end{cases} \tag{1.6}$$

where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$.

The class of weight functions K that we will study are such that $K \in C([r_0, \infty), (0, \infty))$ and satisfies

$$K(r) < \frac{1}{r^{n+\sigma}} \text{ for } r \gg 1, \text{ and for some } \sigma \text{ such that } (n-2)\rho < \sigma < n-2. \tag{1.7}$$

Note that $h \in C((0,1], (0,\infty))$, and since K satisfies (1.7), equivalently h satisfies

$\exists\ \epsilon_1 > 0$ and a constant $d > 0$ such that

$$h(t) \leq \frac{d}{t^\beta} \text{ for all } t \in (0, \epsilon_1) \text{ for } \beta = \frac{(n-2)-\sigma}{n-2}. \tag{1.8}$$

Remark. Note that $\beta < 1 - \rho$ since $\sigma > (n-2)\rho$.

We now state some hypotheses on g which we will use to state Theorems 1.2 and 1.3.

- (H_1) $\lim_{s\to\infty} g(s) = \infty$.
- (H_2) $\lim_{s\to\infty} \frac{g(s)}{s} = 0$.
- (H_3) There exist $\delta > 0, A > 0$ such that $g(s) \geq As^\delta$ *for* $s \gg 1$.
- (H_4) There exist $\gamma > 0, B > 0$ such that $\gamma < \rho + 1$ and $g(s) \leq Bs^\gamma$ for all $s \geq 0$.

We first establish the following results for (1.6).

Theorem 1.2. *Let $\rho = 0$ and assume (H_1) and (H_2). Then* (1.6) *has a positive solution for $\lambda \gg 1$.*

Theorem 1.3. *Let $0 < \rho < 1$ and assume (H_3) and (H_4). Then* (1.6) *has a positive solution for $\lambda \gg 1$.*

Next we consider the problem

$$\begin{cases} -\Delta u = K(|x|)(au - bu^2 - \frac{c}{u^\rho}) & x \in \Omega, \quad 0 \leq \rho < 1, \\ \quad u = 0 & \text{if } |x| = r_0, \\ \quad u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.9}$$

Here Ω and K are as before and a, b, and c are positive constants. By using the same transformations as before (1.9) reduces to

$$\begin{cases} -u'' = h(t)[au - bu^2 - \frac{c}{u^\rho}] \quad 0 < t < 1, \quad 0 \le \rho < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.10}$$

Let λ_1 be the first eigenvalue of the problem $-u'' = \lambda u$, $u(0) = 0 = u(1)$. We establish the following results for (1.10).

Theorem 1.4. *Let $a > \frac{\lambda_1}{\hat{h}}$. Then $\exists\ c^* = c^*(a, b)$ such that for $c < c^*$,* (1.10) *has a positive solution.*

We next extend our results for (1.5) to systems of the form

$$\begin{cases} -\Delta u = \lambda K_1(|x|)\frac{g_1(v)}{u^\rho} & x \in \Omega, \\ -\Delta v = \lambda K_2(|x|)\frac{g_2(u)}{v^\rho} & x \in \Omega, \\ \quad u, v = 0 & \text{if } |x| = r_0, \\ \quad u, v \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.11}$$

where $g_i(0) < 0$ and $g_i's$ are continuous and nondecreasing. By using the same transformations as before (1.11) reduces to

$$\begin{cases} -u'' = \lambda h_1(t)\frac{g_1(v(t))}{u^\rho}, \quad 0 < t < 1 \\ -v'' = \lambda h_2(t)\frac{g_2(u(t))}{v^\rho}, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.12}$$

where $h_i(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K_i(r_0 s^{\frac{1}{2-n}})$, $i = 1, 2$. $K_i : [r_0, \infty) \to (0, \infty)$ are continuous functions which satisfy

$$K_i(r) < \frac{1}{r^{n+\sigma}} \ \text{ for } \ r \gg 1 \text{ and for } \sigma \text{ such that } (n-2)\rho < \sigma < n-2,\ i = 1, 2. \tag{1.13}$$

Note that $h_i \in C((0, 1], (0, \infty))$ and since K_i satisfies (1.13), equivalently h_i satisfies: $\exists \epsilon_1 > 0$ and a constant $d > 0$ such that

$$h_i(t) \le \frac{d}{t^\beta} \ \text{ for all } \ t \in (0, \epsilon_1) \ \text{ for } \ \beta = \frac{(n-2) - \sigma}{n-2},\ i = 1, 2. \tag{1.14}$$

We now state some additional hypotheses which we will use to state Theorems 1.5 and 1.6.

(H_5) $\lim_{s\to\infty} g_i(s) = \infty$, $i = 1, 2$.
(H_6) $\lim_{s\to\infty} \frac{g_1(M(g_2(s)))}{s} = 0$ for every $M > 0$.
(H_7) There exist $\delta > 0$, $A > 0$ such that $g_i(s) \ge As^\delta$ for $s \gg 1$, $i = 1, 2$.

(H_8) There exist $\gamma > 0$, $B > 0$ such that $\gamma < \rho + 1$ and $g_i(s) \leq Bs^\gamma$ for all $s \geq 0$, $i = 1, 2$.

We establish the following results for (1.12).

Theorem 1.5. *Let $\rho = 0$ and assume (H_5) and (H_6). Then* (1.12) *has a positive solution for $\lambda \gg 1$.*

Theorem 1.6. *Let $0 < \rho < 1$ and assume (H_7) and (H_8). Then* (1.12) *has a positive solution for $\lambda \gg 1$.*

Finally, we extend our results for (1.9) to systems of the form

$$\begin{cases} -\Delta u = K_1(|x|)[a_1 u - b_1 u^2 - \frac{c_1}{v^\rho}] & x \in \Omega,\ 0 < \rho < 1, \\ -\Delta v = K_2(|x|)[a_2 v - b_2 v^2 - \frac{c_2}{u^\rho}] & x \in \Omega,\ 0 < \rho < 1, \\ \quad u, v = 0 & \text{if } |x| = r_0, \\ \quad u, v \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where the K_i's and Ω are as before, and a_i, b_i, and c_i are positive constants. Again we first reduce this system to

$$\begin{cases} -u'' = h_1(t)[a_1 u - b_1 u^2 - \frac{c_1}{v^\rho}] & 0 < t < 1, \quad 0 < \rho < 1, \\ -v'' = h_2(t)[a_2 v - b_2 v^2 - \frac{c_2}{u^\rho}] & 0 < t < 1, \quad 0 < \rho < 1, \\ \quad u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.15}$$

where $h_i(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K_i(r_0 s^{\frac{1}{2-n}})$, $i = 1, 2$. Let λ_1 be as defined before and $\hat{h} = \min\{\inf_{t \in (0,1)} h_1(t), \inf_{t \in (0,1)} h_2(t)\}$. We establish the following result for (1.15).

Theorem 1.7. *Let $\min\{a_1, a_2\} > \frac{\lambda_1}{\hat{h}}$. Then $\exists\ c^* = c^*(a_i, b_i) > 0$ such that* (1.15) *has a positive solution when $\max\{c_1, c_2\} < c^*$.*

We will establish our existence results by the method of sub-supersolutions. Consider the boundary value problem

$$\begin{cases} -z'' = \lambda h(s) f(z), \quad 0 < s < 1 \\ z(0) = z(1) = 0, \end{cases} \tag{1.16}$$

where h and f are as before. Here we restate the definition of sub and supersolutions for the problem (1.16). By a subsolution of (1.16) we mean a function $\psi \in C^2(0,1) \cap C[0,1]$ that satisfies

$$\begin{cases} -\psi'' \le \lambda h(t) f(\psi) & 0 < t < 1, \\ \quad \psi > 0 & 0 < t < 1, \\ \psi(0) = 0 = \psi(1), \end{cases}$$

and by a supersolution of (1.16) we mean a function $Z \in C^2(0,1) \bigcap C[0,1]$ that satisfies

$$\begin{cases} -Z'' \ge \lambda h(t) f(Z) & 0 < t < 1, \\ \quad Z > 0 & 0 < t < 1, \\ Z(0) = 0, \quad Z(1) = 0. \end{cases}$$

Then the following lemma holds.

Lemma 1.8. ([27]) *Let ψ be a subsolution of* (1.16) *and Z be a supersolution of* (1.16) *such that $\psi \le Z$. Then* (1.16) *has a solution u such that $\psi \le u \le Z$.*

In Sections 2–7, we prove all the results which are stated above. Section 8 (Appendix) describes the transformation of an exterior domain problem to a two-point boundary value problem.

2. Proof of Theorem 1.2

Consider

$$-\phi'' = \lambda \phi(t), \qquad \phi(0) = \phi(1) = 0. \tag{2.1}$$

Let $\phi_1 \in C^2[0,1]$ be an eigenfunction corresponding to the first eigenvalue λ_1 of (2.1) such that $\phi_1 > 0$. Then there exists $d_1 > 0$ such that

$$0 < \phi_1(t) \le d_1 t(1-t) \text{ for } t \in (0,1).$$

Let $\alpha \in (1, 2-\beta)$, $\epsilon < \epsilon_1$, $m > 0$, and $\mu > 0$ be such that

$$-m > [\lambda_1 \alpha \phi_1^2 - \alpha(\alpha-1) \left|\phi_1'\right|^2] \text{ in } (0,\epsilon] \cup [1-\epsilon, 1)$$

and $\phi_1 > \mu$ in $(\epsilon, 1-\epsilon)$. This is possible since $\phi_1 = 0$ and $|\phi_1'| > 0$ at $t = 0, 1$. Define $\psi = \lambda k_0 \phi_1^\alpha$, where $-k_0 < \frac{d_1^{2-\alpha} d}{m} \min_{t \in [0,\infty)} g(t)$. Then,

$$\begin{aligned} \psi' &= \lambda k_0 \alpha (\phi_1)^{\alpha-1} \phi_1' \\ -\psi'' &= -\lambda k_0 \alpha(\alpha-1) \phi_1^{\alpha-2} \left|\phi_1'\right|^2 - \lambda k_0 \alpha \phi_1^{\alpha-1} \phi_1'' \\ &= \lambda [\lambda_1 k_0 \alpha \phi_1^\alpha - k_0 \alpha(\alpha-1) \frac{|\phi_1'|^2}{\phi_1^{2-\alpha}}]. \end{aligned}$$

For $t \in (0, \epsilon]$,

$$-\psi'' = \lambda \frac{k_0}{\phi_1^{2-\alpha}}[\lambda_1 \alpha \phi_1^2 - \alpha(\alpha-1)\left|\phi_1'\right|^2] \leq -\lambda \frac{k_0}{d_1^{2-\alpha} t^{2-\alpha}} m$$
$$\leq -\lambda \frac{k_0}{d_1^{2-\alpha} t^\beta} m \leq -\lambda \frac{k_0 h(t)}{d_1^{2-\alpha} d} m \leq \lambda h(t) \min_{t \in [0,\infty)} g(t) \leq \lambda h(t) g(\psi).$$

Since h does not have a singularity in $[1-\epsilon, 1]$ it is easier to prove $-\psi'' \leq \lambda h(t) g(\psi)$ for $t \in [1-\epsilon, 1)$. Now for $t \in (\epsilon, 1-\epsilon)$, since $\phi_1(t) \geq \mu$ and $\lim_{s\to\infty} g(s) = \infty$, $g(\lambda k_0 \phi_1^\alpha(t)) \geq \frac{1}{\hat{h}} \lambda_1 k_0 \alpha \phi_1^\alpha(t)$ for $\lambda \gg 1$. Thus for $\lambda \gg 1$,

$$-\psi'' \leq \lambda \lambda_1 k_0 \alpha \phi_1^\alpha(t) \leq \lambda \hat{h} g(\lambda k_0 \phi_1^\alpha(t)) \leq \lambda h(t) g(\psi).$$

Hence for $\lambda \gg 1$, ψ is a positive subsolution of (1.6). Next we construct a positive supersolution. Let $Z = M(\lambda) e$, where e is the solution of

$$-e'' = h(t), \quad 0 < t < 1, \quad e(0) = e(1) = 0.$$

Define $\hat{g}(x) = \max_{u \in [0,x]} g(u)$; then $\hat{g}$ satisfies (H_1) and (H_2) and is nondecreasing. Choose $M(\lambda) \gg 1$ such that

$$\frac{1}{\|e\|_\infty \lambda} \geq \frac{\hat{g}(M(\lambda)\|e\|_\infty)}{M(\lambda)\|e\|_\infty}.$$

Then

$$-Z'' = M(\lambda) h(t) \geq \lambda \hat{g}(M(\lambda)\|e\|_\infty) h(t) \geq \lambda \hat{g}(M(\lambda) e) h(t) \geq \lambda h(t) g(Z).$$

Hence Z is a positive supersolution of (1.6). Choose $M(\lambda) \gg 1$ such that $\psi \leq Z$. Thus we know that (1.6) has a positive solution $u \in [\psi, Z]$.

3. Proof of Theorem 1.3

Let ϕ_1 be as defined before, $\alpha \in (1, \frac{2-\beta}{1+\rho})$ and $r \in (\frac{1}{1+\rho}, \frac{1}{1+\rho-\delta})$. Define $\psi = \lambda^r \phi_1^\alpha$. Then

$$\psi' = \lambda^r \alpha \phi_1^{\alpha-1} \phi_1'$$
$$-\psi'' = \lambda^r [\lambda_1 \alpha \phi_1^\alpha - \alpha(\alpha-1) \frac{|\phi_1'|^2}{\phi_1^{2-\alpha}}].$$

Let $m > 0$ and $\epsilon > 0$ be such that $\alpha(\alpha-1)|\phi_1'|^2 - \lambda_1 \alpha \phi_1^2 \geq m$ in $(0, \epsilon] \cup [1-\epsilon, 1)$, where $\epsilon < \epsilon_1$ as in the previous section. Let $k > 0$ be such that $g(s) \geq -k$ for all $s \geq 0$. Then in $(0, \epsilon] \cup [1-\epsilon, 1)$, for $\lambda \gg 1$

$$\lambda_1 \alpha \phi_1^2 - \alpha(\alpha-1)|\phi_1'|^2 \leq -m \leq \frac{\lambda d d_1^\beta(-k)}{\lambda^r \lambda^{r\rho}},$$

since $1-r-r\rho<0$. Hence in $(0,\epsilon]$, for $\lambda \gg 1$

$$\begin{aligned} -\psi'' &= \lambda^r[\lambda_1\alpha\phi_1^\alpha - \alpha(\alpha-1)\frac{|\phi_1'|^2}{\phi_1^{2-\alpha}}] \le \frac{\lambda d d_1^\beta(-k)}{\lambda^{r\rho}\phi_1^{2-\alpha}} \\ &\le \frac{\lambda d d_1^\beta(-k)}{\lambda^{r\rho}\phi_1^{\beta+\alpha\rho}} \le \frac{\lambda d(-k)}{(\lambda^r\phi_1^\alpha)^\rho t^\beta} \le \frac{\lambda(-k)h(t)}{(\lambda^r\phi_1^\alpha)^\rho} \le \frac{\lambda g(\lambda^r\phi_1^\alpha)h(t)}{(\lambda^r\phi_1^\alpha)^\rho}. \end{aligned} \tag{3.1}$$

Here again we note that since h does not have any singularity near $t=1$, an easier proof will show that $-\psi'' \le \lambda\frac{h(t)g(\psi)}{\psi^\rho}$ in $[1-\epsilon,1)$.

Next in $(\epsilon,1-\epsilon)$, since there exists $\mu>0$ such that $\phi_1 \ge \mu$, from (H_3)

$$g(\lambda^r\phi_1^\alpha) \ge A(\lambda^r\phi_1^\alpha)^\delta, \text{ for } \lambda \gg 1.$$

Since $1+r(\delta-\rho)-r>0$, in $(\epsilon,1-\epsilon)$

$$-\psi'' \le \lambda^r\lambda_1\alpha\phi_1^\alpha \le \lambda\hat{h}A(\lambda^r\phi_1^\alpha)^{\delta-\rho}, \text{ for } \lambda \gg 1.$$

Hence, for $\lambda \gg 1$ we have

$$-\psi'' \le \frac{\lambda\hat{h}A(\lambda^r\phi_1^\alpha)^\delta}{(\lambda^r\phi_1^\alpha)^\rho} \le \frac{\lambda h(t)g(\lambda^r\phi_1^\alpha)}{(\lambda^r\phi_1^\alpha)^\rho}. \tag{3.2}$$

Combining (3.1) and (3.2) we see that

$$-\psi'' \le \lambda h(t)\frac{g(\psi)}{\psi^\rho} \text{ in } (0,1) \text{ for } \lambda \gg 1.$$

Thus ψ is a positive subsolution. Now we construct a supersolution $Z \ge \psi$. Note that in (H_4), without loss of generality we can choose $\rho \le \gamma < \rho+1$; hence for $m(\lambda) \gg 1$

$$(m(\lambda))^{1+\rho-\gamma} \ge \lambda B e^{\gamma-\rho},$$

where e is as before. Hence for $m(\lambda) \gg 1$

$$m(\lambda) \ge \lambda\frac{B(m(\lambda)e)^\gamma}{(m(\lambda)e)^\rho}.$$

Define $Z = m(\lambda)e$. Then

$$-Z'' = m(\lambda)h(t) \ge \lambda\frac{B(m(\lambda)e)^\gamma}{(m(\lambda)e)^\rho}h(t) \ge \lambda h(t)\frac{g(m(\lambda)e)}{(m(\lambda)e)^\rho}.$$

Thus Z is a supersolution. Further $m(\lambda)$ can be chosen large such that $Z \ge \psi$. Hence (1.6) has a positive solution for $\lambda \gg 1$ when $0<\rho<1$.

4. PROOF OF THEOREM 1.4

Consider the boundary value problem

$$-z'' - \lambda z = -1, \quad 0 < t < 1, \qquad z(0) = z(1) = 0. \tag{4.1}$$

From an anti-maximum principle (see [8]) there exists $\delta_1 > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ the solution, z_λ of (4.1) is positive in $(0,1)$ and $|z'_\lambda| > 0$ at $t = 0, 1$. Also there exist $d_2 > 0$ such that

$$0 < z_\lambda \leq d_2 t(1-t) \text{ for } t \in (0,1).$$

Let $\alpha \in (1, \min\{\frac{a\hat{h}}{\lambda_1}, \frac{2-\beta}{1+\rho}\})$ and fix $\lambda^* \in (\lambda_1, \min\{\frac{a\hat{h}}{\alpha}, \lambda_1 + \delta_1\})$. Define $\psi = k_0 z_{\lambda^*}^\alpha$, where z_{λ^*} is the solution of (4.1) for $\lambda = \lambda^*$ and

$$k_0 = \min\{\frac{\alpha}{b\|z_{\lambda^*}\|_\infty^{2\alpha-1} d_2^{2-\alpha} d}, \frac{(a - \frac{\alpha\lambda^*}{\hat{h}})}{2b\|z_{\lambda^*}\|_\infty^\alpha}\}.$$

Then

$$\begin{aligned}
\psi' &= k_0 \alpha z_{\lambda^*}^{\alpha-1} z'_{\lambda^*} \\
-\psi'' &= -k_0 \alpha(\alpha-1) z_{\lambda^*}^{\alpha-2} |z'_{\lambda^*}|^2 - k_0 \alpha z_{\lambda^*}^{\alpha-1} z''_{\lambda^*}
\end{aligned}$$

$$= k_0 \alpha z_{\lambda^*}^\alpha \lambda^* - k_0 \alpha z_{\lambda^*}^{\alpha-1} - k_0 \alpha(\alpha-1) \frac{|z'_{\lambda^*}|^2}{z_{\lambda^*}^{2-\alpha}} \tag{4.2}$$

and

$$h(t)(a\psi - b\psi^2 - \frac{c}{\psi^\rho}) = h(t)(ak_0 z_{\lambda^*}^\alpha - bk_0^2 z_{\lambda^*}^{2\alpha} - \frac{c}{(k_0 z_{\lambda^*}^\alpha)^\rho}). \tag{4.3}$$

Let $\mu > 0$ and $m > 0$ be such that $|z_{\lambda^*}| \leq 1$, and $|z'_{\lambda^*}| \geq m$ in $(0, \epsilon] \cup [1-\epsilon, 1)$ and $z_{\lambda^*} \geq \mu$ in $(\epsilon, 1-\epsilon)$, where $\epsilon < \epsilon_1$. Also let

$$c^* = \min\{\frac{k_0^{\rho+1} \alpha(\alpha-1) m^2}{d_2^\beta d}, \frac{1}{2} k_0^{\rho+1} \mu^{\alpha(\rho+1)} (a - \frac{\alpha\lambda^*}{\hat{h}})\}.$$

In $(0, \epsilon]$ we compare (4.2) and (4.3) term by term to see that for $c < c^*$

$$-\psi'' \leq h(t)(a\psi - b\psi^2 - \frac{c}{\psi^\rho}).$$

Since $\lambda^* \alpha < a\hat{h}$

$$k_0 \alpha z_{\lambda^*}^\alpha \lambda^* \leq k_0 \alpha z_{\lambda^*}^\alpha \frac{a\hat{h}}{\alpha} \leq k_0 z_{\lambda^*}^\alpha h(t) a. \tag{4.4}$$

Next, we see that

$$\begin{aligned}-k_0\alpha z_{\lambda^*}^{\alpha-1} &= \frac{-k_0\alpha z_{\lambda^*}}{z_{\lambda^*}^{2-\alpha}} \leq \frac{-k_0\alpha z_{\lambda^*}}{d_2^{2-\alpha}t^{2-\alpha}} \leq \frac{-k_0\alpha z_{\lambda^*}}{d_2^{2-\alpha}t^{\beta}} \\ &\leq -k_0\alpha z_{\lambda^*}\frac{h(t)}{d_2^{2-\alpha}d} = \frac{-k_0^2\alpha z_{\lambda^*}h(t)}{k_0 d_2^{2-\alpha}d}.\end{aligned}$$

Now from the choice of k_0, $\frac{-1}{k_0} \leq \frac{-b||z_{\lambda^*}||_\infty^{2\alpha-1}d_2^{2-\alpha}d}{\alpha}$. Hence,

$$-k_0\alpha z_{\lambda^*}^{\alpha-1} \leq -bk_0^2\|z_{\lambda^*}\|_\infty^{2\alpha-1}z_{\lambda^*}h(t) \leq -bk_0^2 z_{\lambda^*}^{2\alpha-1}z_{\lambda^*}h(t) = -bk_0^2 z_{\lambda^*}^{2\alpha}h(t).$$

Since $2-\alpha > \beta+\alpha\rho$ and $c < \frac{k_0^{\rho+1}\alpha(\alpha-1)m^2}{d_2^{\beta}d}$,

$$\begin{aligned}\frac{-k_0\alpha(\alpha-1)|z_{\lambda^*}'|^2}{z_{\lambda^*}^{2-\alpha}} &\leq \frac{-k_0\alpha(\alpha-1)m^2}{z_{\lambda^*}{}^{\beta}z_{\lambda^*}{}^{\alpha\rho}} \leq \frac{-k_0\alpha(\alpha-1)m^2h(t)}{d_2^{\beta}dz_{\lambda^*}^{\alpha\rho}} \\ &\leq \frac{-k_0^{1+\rho}\alpha(\alpha-1)m^2h(t)}{d_2^{\beta}d(k_0 z_{\lambda^*}^{\alpha})^{\rho}} \leq -\frac{ch(t)}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}}.\end{aligned}$$

Hence we get $-\psi'' \leq h(t)(a\psi - b\psi^2 - \frac{c}{\psi^\rho})$ in $(0,\epsilon]$. It is easier to prove $-\psi'' \leq h(t)(a\psi - b\psi^2 - \frac{c}{\psi^\rho})$ in $[1-\epsilon,1)$. Now in $(\epsilon,1-\epsilon)$ since $z_\lambda^* \geq \mu$ we have $c \leq \frac{1}{2}k_0^{\rho+1}(z_{\lambda^*}^{\alpha})^{\rho+1}(a-\frac{\alpha\lambda^*}{\hat{h}})$ and by our choice of k_0, $bk_0 z_{\lambda^*}^{\alpha} \leq \frac{1}{2}(a-\frac{\alpha\lambda^*}{\hat{h}})$. Hence, for t $\in(\epsilon,1-\epsilon)$,

$$\begin{aligned}-\psi'' &\leq k_0\alpha z_{\lambda^*}^{\alpha}\lambda^* = \frac{\hat{h}k_0\alpha z_{\lambda^*}^{\alpha}\lambda^*}{\hat{h}} \leq h(t)[\frac{1}{2}\frac{\alpha}{\hat{h}}\lambda^* k_0 z_{\lambda^*}^{\alpha} + \frac{1}{2}\frac{\alpha}{\hat{h}}\lambda^* k_0 z_{\lambda^*}^{\alpha}] \\ &\leq h(t)[(\frac{1}{2}k_0 z_{\lambda^*}^{\alpha}a - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}}) + k_0 z_{\lambda^*}^{\alpha}(\frac{1}{2}a - bk_0 z_{\lambda^*}^{\alpha})] \\ &\leq h(t)[\frac{1}{2}k_0 z_{\lambda^*}^{\alpha}a - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}} + \frac{1}{2}k_0 z_{\lambda^*}^{\alpha}a - bk_0^2 z_{\lambda^*}^{2\alpha}] \\ &= h(t)[ak_0 z_{\lambda^*}^{\alpha} - bk_0^2 z_{\lambda^*}^{2\alpha} - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}}].\end{aligned}$$

Hence, ψ is a positive subsolution of (1.10). Next we construct a supersolution. We know that there exist a large $\bar{M} > 0$ such that $au - bu^2 - \frac{c}{u^\rho} \leq \bar{M}$ for all $u > 0$ and $\bar{M}e \geq \psi$ in (0,1), where e is as defined before. Let $Z = \bar{M}e$. Then Z is a positive supersolution of (1.10). Thus Theorem 1.4 is proven.

5. Proof of Theorem 1.5

Let ϕ_1 be as defined before, $\alpha \in (1, 2-\beta)$ and $-k_0 < \frac{d_1^{2-\alpha} d}{m} \min\{\bar{g_1}, \bar{g_2}\}$, where $\bar{g_i} = \min_{x\in[0,\infty)} g_i(x)$, $i = 1, 2$, and d_1 and m are as in the proof of Theorem 1.2. Define $\psi_1 = \psi_2 = \lambda k_0 \phi_1^\alpha$. Following the steps in the proof of Theorem 1.2 it is now easy to show that for $\lambda \gg 1$, (ψ_1, ψ_2) is a subsolution of (1.12). Now we define

$$Z_1 = M(\lambda)e_1, \quad Z_2 = \lambda g_2(M(\lambda)\|e_1\|_\infty)e_2,$$

where e_i is solution of $-u'' = h_i(t)$, $0 < t < 1$, $u(0) = u(1) = 0$, $i = 1, 2$. Choose $M(\lambda) \gg 1$ such that

$$\frac{1}{\|e_1\|_\infty \lambda} \geq \frac{g_1(\lambda\|e_2\|_\infty g_2(M(\lambda)\|e_1\|_\infty))}{M(\lambda)\|e_1\|_\infty}.$$

Now

$$\begin{aligned}
-Z_1'' &= M(\lambda)h_1(t) \geq \lambda g_1(\lambda\|e_2\|_\infty g_2(M(\lambda)\|e_1\|_\infty))h_1(t) \\
&\geq \lambda g_1(\lambda e_2 g_2(M(\lambda)\|e_1\|_\infty))h_1(t) = \lambda g_1(Z_2)h_1(t), \\
-Z_2'' &= \lambda g_2(M(\lambda)\|e_1\|_\infty)h_2(t) \geq \lambda g_2(M(\lambda)e_1)h_2(t) = \lambda g_2(Z_1)h_2(t).
\end{aligned}$$

Hence (Z_1, Z_2) is a positive supersolution of (1.12). Choose $M(\lambda) \gg 1$ such that $\psi_1 \leq Z_1$ and $\psi_2 \leq Z_2$. Thus Theorem 1.5 is proven.

6. Proof of Theorem 1.6

Let ϕ_1 be as defined before, $\alpha \in (1, \frac{2-\beta}{1+\rho})$ and $r \in (\frac{1}{1+\rho}, \frac{1}{1+\rho-\delta})$. Define $\psi_1 = \psi_2 = \lambda^r \phi_1^\alpha$. A similar proof as in Theorem 1.3 will show that (ψ_1, ψ_2) is a subsolution of (1.12) for $\lambda \gg 1$. Now we construct a supersolution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. There exist $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$e_2 \leq \tau_1 e_1, \quad \text{and} \quad e_1 \leq \tau_2 e_2,$$

where e_i's are as in the proof of Theorem 1.5. As in Theorem 1.3 we can choose $\rho \leq \gamma < \rho + 1$; hence, for $m(\lambda) \gg 1$

$$(m(\lambda))^{1+\rho-\gamma} \geq \lambda B \tau_i^\gamma e_i^{\gamma-\rho}, \quad i = 1, 2,$$

and

$$m(\lambda) \geq \lambda \tau_1^\gamma \frac{B(m(\lambda)e_1)^\gamma}{(m(\lambda)e_1)^\rho} \geq \lambda \frac{B(m(\lambda)e_2)^\gamma}{(m(\lambda)e_1)^\rho}.$$

Similarly,

$$m(\lambda) \geq \lambda \frac{B(m(\lambda)e_1)^\gamma}{(m(\lambda)e_2)^\rho}.$$

Define $(Z_1, Z_2) = (m(\lambda)e_1, m(\lambda)e_2)$. Then

$$-Z_1'' = m(\lambda)h_1(t) \geq \lambda \frac{B(m(\lambda)e_2)^\gamma}{(m(\lambda)e_1)^\rho} h_1(t)$$
$$\geq \lambda h_1(t) \frac{g_1(m(\lambda)e_2)}{(m(\lambda)e_1)^\rho} = \lambda h_1(t) \frac{g_1(Z_2)}{(Z_1)^\rho}$$

and similarly,

$$-Z_2'' \geq \lambda h_2(t) \frac{g_2(Z_1)}{(Z_2)^\rho}.$$

Thus (Z_1, Z_2) is a supersolution. Further $m(\lambda)$ can be chosen large such that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. Hence (1.12) has a positive solution for $\lambda \gg 1$ when $0 < \rho < 1$.

7. Proof of Theorem 1.7

Let $a = \min(a_1, a_2)$ and $b = \max(b_1, b_2)$. Define $\psi_1 = \psi_2 = k_0 z_{\lambda^*}^\alpha$, where z_{λ^*} is the solution of (4.1) for $\lambda = \lambda^* \in (\lambda_1, \min(\frac{a\hat{h}}{\alpha}, \lambda_1 + \delta_1))$, $k_0 = \min(\frac{\alpha}{b||z_{\lambda^*}^{2\alpha-1}||_\infty d_2^{2-\alpha} d}, \frac{(a - \frac{\alpha\lambda^*}{\hat{h}})}{2b||z_{\lambda^*}^\alpha||_\infty})$ and $\alpha \in (1, \min(\frac{a\hat{h}}{\lambda_1}, \frac{2-\beta}{1+\rho}))$. By following the proof of Theorem 1.4 we can easily show that

$$\exists\ c^* = \min\{\frac{k_0^{\rho+1}\alpha(\alpha-1)m^2}{d_2^\beta d}, \frac{1}{2}k_0^{\rho+1}\mu^{\alpha(\rho+1)}(a - \frac{\alpha\lambda^*}{\hat{h}})\}$$

such that for $\max\{c_1, c_2\} < c^*$, (ψ_1, ψ_2) is a positive subsolution of (1.15). Define $Z_1 = \bar{M}e_1$ and $Z_2 = \bar{M}e_2$, where $\bar{M} > 0$ is such that $a_i u - b_i u^2 - \frac{c_i}{u^\rho} \leq \bar{M}$ for $i = 1, 2$, $\bar{M}e_1 > \psi_1$, and $\bar{M}e_2 > \psi_2$. It is easy to see that (Z_1, Z_2) is a supersolution of (1.15). Hence Theorem 1.7 is proven.

8. Appendix

Consider the problem

$$\begin{cases} -\Delta u &= \lambda K(|x|) f(u) \quad x \in \Omega, \\ u &= 0 \qquad \text{if} \quad |x| = r_0, \\ u &\to 0 \qquad \text{as} \quad |x| \to \infty, \end{cases} \tag{8.1}$$

where $K : [r_0, \infty) \to (0, \infty)$ is continuous. Set $r = |x|$ and $v(r) = u(x)$. Then

$$\Delta u \quad = \quad v''(r) + \frac{n-1}{r} v'(r),$$

which reduces (8.1) to the following:

$$\begin{cases} -v''(r) - \frac{n-1}{r}v'(r) & = \lambda K(r)f(v(r)) \quad r_0 < r < \infty, \\ v(r_0) = 0, \quad v(r) & \to 0 \qquad\qquad \text{as} \quad r \to \infty. \end{cases} \tag{8.2}$$

Now set $s = (\frac{r}{r_0})^{2-n}$ and $z(s) = v(r)$; then

$$-v''(r) - \frac{n-1}{r}v'(r) = -\frac{(2-n)^2}{r_0^2} s^{\frac{2(1-n)}{2-n}} z''(s),$$

which again reduces the problem (8.2) to the following boundary value problem:

$$\begin{cases} -z'' & = \lambda h(s)f(z(s)) \quad 0 < s < 1, \\ z(0) & = z(1) = 0, \end{cases}$$

where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$.

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