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INTRASEASONAL MULTI-SCALE MOIST DYNAMICS OF THE TROPICAL TROPOSPHERE*

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Abstract. We derive a multi-scale model of moist tropical dynamics which is valid on horizontal synoptic scales, zonal planetary scales, and synoptic and intraseasonal time scales. The Intraseasonal Multi-Scale Moist Dynamics (IMMD) framework builds on the IPESD framework of [A.J. Majda and R. Klein, J. Atmos. Sci., 60, 393–408, 2003]. It generalizes the latter by allowing for strong zonal winds (the Trade Winds) and the pressure and stratification variations that they generate. The framework consists of three pieces. The first, called TH, are planetary scale climatology modulation equations which govern the Trade Winds and Hadley Circulation. Self-consistency of the asymptotic theory requires that the meridional component of the Hadley Circulation is an order of magnitude weaker than the zonal component. The second piece, S, is a linear system of equations which govern synoptic scale velocity, temperature, and pressure fluctuations forced by synoptic scale heating fluctuations. Unlike the IPESD theory, these fluctuations are advected by part of the planetary scale climatology from TH. Since the meridional component of TH is an order of magnitude weaker than the zonal component, the synoptic scale fluctuations are only advected by the latter. The third, P, govern the planetary scale anomalies which, like IPESD, are driven both by planetary scale mean heating and by upscale fluxes from the synoptic scales. These planetary scale anomalies are advected both by the zonal component of the Trade Winds and by the meridional component of the Hadley Circulation and, furthermore, respond to an in-scale flux from the mean climatology. We also present an asymptotic analysis of the equations of bulk cloud thermodynamics in order to lay out a self-contained path for incorporating synoptic scale cloud models into the IMMD framework. This framework has potentially important implications for the development of models describing the Madden-Julian Oscillation (MJO) since the MJO manifests itself as planetary scale anomalies from a mean climatology which it modulates on intraseasonal time scales.

Key words. Multi-scale asymptotics, tropical atmospheric dynamics, Madden-Julian oscillation.

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1. Introduction

The dominant component of intraseasonal variability in the tropics is the 40–50 day tropical intraseasonal oscillation, often called the Madden-Julian oscillation (MJO) after its discoverers [23]. In the troposphere, the MJO is an equatorial planetary scale wave envelope of complex multi-scale convective processes which propagates across the Indian Ocean and Western Pacific at a speed of roughly $5\,\mathrm{ms}^{-1}$ [34, 11, 10, 29]. The planetary scale circulation anomalies associated with the MJO significantly affect monsoon development, intraseasonal predictability in mid-latitudes, and impact the development of the El Nino Southern Oscillation (ENSO) in the Pacific Ocean [24, 38, 40, 41]. An overview lecture by the co-discoverer of the MJO, R. Madden, describing its dynamics is provided as a webcast in [22]. This non-technical description is well suited for mathematically inclined readers who seek a more in depth, yet accessible description of the phenomenon; such a description is beyond the scope of this paper.

It has been hypothesized that organized convection on multiple scales is an essential ingredient of the Madden-Julian oscillation [33, 20, 9, 37, 39, 41]; this paper

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will specifically focus on the multi-scale nature of the phenomenon. Present day computer general circulation models (GCM) typically poorly represent the MJO [35] and it is likely that this poor performance of GCMs is due to the inadequate treatment across multiple spatial scales of the interaction of the hierarchy of organized structures which generate the MJO as their envelope. This multi-scale organization in the MJO is manifested in part as correlations between different components of the dynamical fields (i.e. velocity or temperature) which yield net upscale fluxes in those fields and drive the planetary scale organized flow.

Recently the authors have developed a multi-scale theory of the Madden-Julian oscillation (MJO) [26, 2, 3, 5] in which the structure of the planetary scale MJO is determined by upscale momentum and temperature fluxes from the synoptic scales in addition to direct heating on the planetary scale. The model begins with a prescribed phase speed for the MJO and is shown to have many features in common with the observational record [10, 11, 20, 13, 12, 14] in particular the spatial correlation of synoptic scale vertically tilted convective structures with the large scale organization of the MJO.

In this theory, the synoptic scale flows are determined by diabatic fluctuating heating due to latent heat release, resolved on the synoptic scales. The equations governing synoptic scale fluctuations are the linear equatorial equations forced by latent heating [32, 8]. Planetary scale flows are governed by quasi-linear equatorial long wave equations forced by upscale fluxes from the planetary scales and direct, though weaker heating, resolved on the planetary scale.

The multi-scale asymptotic framework which underlies this model is the Intraseasonal Planetary/Equatorial Synoptic Dynamics (IPESD) framework, first derived in [27]; for alternate derivations see [25, 3]. IPESD is, inherently, a theory of anomalies from a mean climatology. The derivation relies on the assumption that the heating due to the release of latent heat of condensation is about 10 Kelvin per day when averaged on the synoptic scales and less than about 2 Kelvin per day when averaged on the planetary scales.

In order to construct a predictive closed theory of the Madden-Julian oscillation the dynamic effects of moisture must be included in the IPESD framework and the MJO model. As a particular example, the multi-cloud moisture models of Khouider and Majda [15, 16, 17, 18, 19] (hereafter we shall refer to this body of work as KM) can be processed through the asymptotic procedure stipulated in the derivation of IPESD. Ideally this would yield a closed MJO model where synoptic scale fluctuations drive upscale fluxes of temperature and momentum while large scale flows modify the local synoptic scale environment where the fluctuations are generated. Practically, though, there is no *a priori* way to ensure that latent heating on the planetary scale remains of order 2 K/day, as is required by the IPESD asymptotics. Physically, this corresponds to the fact that the nonlinearities in any model of cloud physics have the ability to generate mean heating which exceeds a few Kelvin per day, thereby modifying the local radiative-convective equilibrium and the large scale flow in which the Madden-Julian oscillation is embedded. In the tropics, this large scale flow consists of the Hadley Circulation, and most importantly for the IMMD theory, the stronger zonal component of this flow.

Therefore, in order to include moist dynamics in a multi-scale model of the tropics, the IPESD framework must be generalized to allow for stronger mean zonal winds and thermodynamic stratification, both of which can be modulated on the zonal planetary scales and on intraseasonal time scales. Though the IPESD framework does allow for stronger heating and thereby generates a Hadley circulation, it requires that this stronger heating be exactly balanced by zonal momentum dissipation (e.g. from cumulus drag). By introducing the stronger zonal flows, this exact balance is no longer required since the lack of balance is naturally directed to the forcing of the zonal component of the Trade Winds. We call the resulting non-linear multi-scale framework the "Intraseasonal Multi-scale Moist Dynamics" (IMMD), and it naturally separates into three systems of equations for the flow which are coupled through advective nonlinearities, cross-scale fluxes, and scale averaged diabatic heating.

S: Synoptic scale fluctuating flows forced by synoptic scale fluctuating forcing (diabatic heating, drag and radiation), advected by the zonal trade winds and modulated by the potential temperature and moisture environment. The synoptic scale fluctuations also respond to the the cross-scale flux of zonal momentum and potential temperature through the divergence of fluxes which couple the synoptic scale flows to the planetary scale climatology.

P: Planetary scale mean flows which are anomalies from a mean climatology. These are allowed to vary on both synoptic and intraseasonal time scales, and their evolution on each of these time scales separate into two systems of equations. These flows are advected by the Trade Winds and, on the intraseasonal time scale, also by the Hadley circulation. As with the synoptic scale fluctuating flow, they also respond to the stronger potential temperature stratification which is modulated on the intraseasonal time scale. Like the original IPESD, they are forced by upscale fluxes from the synoptic scales. However, unlike IPESD, they are also coupled to the mean climatology through the flux of zonal momentum and potential temperature from the mean climatology to the fluctuations.

TH: Stronger planetary scale zonal flow, pressure, potential temperature, and moisture stratification which is modulated on the intraseasonal time scale and determines the Trade Winds and the Hadley circulation. The assumption that the synoptic scale fluctuating heating is order 10 K/day implies that the upscale fluxes from the synoptic scales are too weak to modulate this flow on intraseasonal timescales. Therefore, all of the upscale fluxes drive the planetary scale anomalies, P, and the Trade Winds and Hadley circulation are driven solely by planetary scale and intraseasonally averaged latent heating and dissipation.

There are nonlinearities in the systems of equations for each of the scales of the dynamics owing to advective coupling with the large scale flows, fluxes across the scales, and coupling due to latent heat release. However, the system of equations describing Hadley Circulation (TH) is coupled to the other components of the flow only through the forcing terms arising from the planetary scale mean of the latent heat release; therefore, only in this system of equations are the advective terms fully non-linear.

Before we construct the multi-scale asymptotic framework, we will present the derivation of the equations of moisture dynamics which are relevant on the synoptic (and larger) scales in section 2. This derivation follows that of [7, 16], but is included here in order to provide a fully self-contained moist asymptotic theory for synoptic and larger scales in the tropics and in order to clearly spell out the scales of validity of the asymptotic theory. In section 3 the large scale equation for water vapor derived in section 2 is coupled to the hydrostatic equatorial β-plane equations to derive the multi-scale IMMD framework. Through the course of the derivation we detail the relevant scales and magnitudes which the theory describes. Ultimately, the IMMD framework is completely independent of the particular physics associated with either a cloud model or boundary layer coupling. The derivation that we present lays out a precise procedure for taking a particular cloud/boundary layer model (specifically, the multi-cloud model of KM) and including it in a full multi-scale model for tropical dynamics on synoptic/planetary length scales and synoptic/intraseasonal time scales. We will present this more detailed application in a forthcoming paper.

2. Moisture dynamics on synoptic scales

The starting point for the derivation are the anelastic, hydrostatic Euler equations on an equatorial β-plane, which are widely accepted as the appropriate equations for large scale (synoptic or larger) phenomena in the tropics

$$\begin{aligned}
\frac{D}{Dt}u - yv &= -p_x + S_u \\
\frac{D}{Dt}v + yu &= -p_y + S_v \\
\frac{D}{Dt}\theta + w &= \quad H + S^\theta \\
p_z &= \theta \\
(\rho u)_x + (\rho v)_y + (\rho w)_z &= 0,
\end{aligned} \tag{2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{2.2}$$

is the three dimensional advective derivative. The equations have been non-dimensionalized [2, 3] so that time is measured in units of the equatorial time scale, $T_E = (c\beta)^{-1/2} \approx 8.3\,\mathrm{hrs} \approx 1/3\,\mathrm{day}$, the horizontal length scales are in units of the equatorial deformation radius, $L_E = (c/\beta)^{1/2} = 1500\,\mathrm{km}$, the vertical length scale is in units of the troposphere height divided by π, $H_T/\pi \approx 5\,\mathrm{km}$, and the temperature is measured in units of the thermal lapse rate in the atmosphere across one unit of the vertical scale, $\theta_0 \equiv 6.5\,\mathrm{K/km} \times 5\,\mathrm{km} \approx 33\,\mathrm{K}$. These parameters correspond to a Brunt-Väisälä frequency of $10^{-2}\,\mathrm{s}^{-1}$, which arises from a constant linear gradient of potential temperature, so that θ is the deviation from this linear gradient. The equatorial β-plane in the free troposphere occupies the domain $0 \le z \le \pi$, $-40/3 \le x \le 40/3$ and $-10/3 \le y \le 10/3$. Notice that the natural horizontal velocity scale is $L_E/T_E \approx 50\,\mathrm{m/s}$, which is also the dry gravity wave speed and the unit for vertical velocity is $(H_T/\pi)/T_E \approx 16\,\mathrm{cm/s}$. The density is scaled to that at the base of the free troposphere, $\rho(z=0) = 1$ (and is elsewhere less than or equal to one). Hereafter we shall use the assumption of incompressibility, $\rho(z) = 1$ everywhere, which does not change the conclusions and makes for more streamlined notation.

The sources of zonal and meridional momentum forcing, S^u, S^v are upscale fluxes from meso- and smaller scales and drag due to downscale cascade to these same scales, as would be expected in a turbulent environment. At this stage we wish to simply model the dynamics below the synoptic scales; we shall employ only a linear drag law closure to describe the momentum dissipation to sub-synoptic scales, $S^u = -du$ and $S^v = -dv$. These should only be thought of as placeholders which contain the necessary time scales of momentum dissipation as resolved on the synoptic spatial scales. No aspect of the following asymptotic analysis relies on the detailed description of drag, only its magnitude. For example, in their theory of the MJO, the authors have already discussed a more general boundary layer drag prescription for IPESD [3]. However, linear models are common in the atmospheric literature and the dissipation time associated with these models is ascribed to cumulus drag; it has been estimated to be about 5 days [21] from observations of large scale tropical flows, which yields a dissipation rate of $d = .07$ in non-dimensional units.

One non-dimensional unit of the thermal forcing corresponds to a 100 Kelvin per day heating rate; it has been divided into two pieces on the right hand side of equation (2.1). The second term, S^θ, describes radiative damping; note that this notation is slightly modified from previous work [27, 2, 3, 5]. For elucidation, we shall use a linear Newtonian cooling law, $S^\theta = -d_\theta\,\theta$ where observational estimates for the cooling time are of the order 15 days, $d_\theta \approx .023$ [21, 31].

The first term, H, describes latent heat release due to condensation in the bulk of the troposphere. Since it arises from moisture, a description of moisture dynamics is necessary at this stage in the derivation. Our basis for including moisture are the equations of bulk cloud microphysics, their fully dimensional version

$$\begin{aligned}
\frac{D\hat{\theta}}{D\hat{t}} &= \frac{L\hat{\theta}}{c_p\hat{T}}(C_d - E_r)\\
\frac{D\hat{q}_v}{D\hat{t}} &= -(C_d - E_r)\\
\frac{D\hat{q}_c}{D\hat{t}} &= C_d - A_r - C_r\\
\frac{D\hat{q}_r}{D\hat{t}} &= \frac{\partial(\hat{V}_T\,\hat{q}_r)}{\partial\hat{z}} + A_r + C_r - E_r
\end{aligned} \tag{2.3}$$

was also used as the basis for the cloud/moisture models in [7, 16]; for further details of bulk cloud microphysics see [6]. For consistency with IPESD, we use the (dimensional) potential temperature in the first microphysics equation and employ the anelastic constraint to remove the density from the advective derivative. The actual temperatue appears in the denominator on the right hand side of the potential temperature equation and can be written in terms of the potential temperature and the total tropospheric pressure,

$$\hat{T} = \hat{\theta}\left(\frac{\hat{P}}{P_0}\right)^{\frac{R}{c_p}}. \tag{2.4}$$

The new equations describe the mass fractions of water vapor, $\hat{q}_v$, water in clouds, $\hat{q}_c$, and rain water, $\hat{q}_c$. Therefore, the water conversion terms on the right hand sides of these equations are simply rates (grams per kilogram per unit time) and for our derivation we shall not describe the specific physics associated with them. It suffices to record that C_d is the condensation rate of water vapor forming cloud water, E_r is the re-evaporation rate of rain into dry air, A_r is the autoconversion rate of cloud into rain, and C_r is the rate at which falling rain collects liquid water from clouds. Rain is assumed to quickly move downward at its terminal velocity, $\hat{V}_T$. In these equations, the total liquid water, $\hat{q}_v + \hat{q}_c + \hat{q}_r$ is materially conserved, other than for its loss through the downward rain flux. The ratio of the latent heat released through the condensation of one gram of water to the specific heat of dry air sets an important temperature scale

$$\frac{L}{c_p} = \frac{2260\text{J/g}}{1.004\text{J/(g}\cdot\text{K)}} = 2250\text{K} \equiv \theta_L. \tag{2.5}$$

Hats have been used above the variables of equation (2.3) to distinguish these quantities from their non-dimensional or, in the case of the mixing ratios, their rescaled version which will be used to develop the asymptotic framework.

The rainwater conversion terms can be eliminated from this system of equations by writing an equation for the mixing ratio of the total liquid water $\hat{q}_l = \hat{q}_c + \hat{q}_r$

$$\frac{D\hat{q}_l}{D\hat{t}} = \frac{\partial(\hat{V}_T \hat{q}_r)}{\partial \hat{z}} + \hat{H}, \tag{2.6}$$

where $\hat{H} = C_d - E_r$ is the net conversion rate of liquid water, whose dimensions are inverse time. Thus we consider the simpler system

$$\begin{aligned} \frac{D\hat{\theta}}{D\hat{t}} &= \theta_L \frac{\hat{\theta}}{\hat{T}} \hat{H} \\ \frac{D\hat{q}_v}{D\hat{t}} &= -\hat{H} \end{aligned} \tag{2.7}$$

along with equation (2.6) as describing moist processes. Of course, this system is not closed since there is no explicit equation for $\hat{q}_r$, but this does not cause problems since the asymptotic theory considers synoptic and larger scale dynamics so that the specifics of rain and cloud processes must be modeled anyway. The ratio of potential temperature to actual temperature appearing on the right hand side of the first equation in (2.7) is dominated by the mean temperature and potential temperature soundings in the troposphere, equivalently, by the background pressure stratification from equation (2.4). The ratio effectively acts as a vertical weighting of the heating associated with the condensation of water. The analysis of the moisture equations and the derivation of IMMD are the same whether or not the vertical dependence of the ratio is retained. In order to simplify the equations as in [7], we fix this ratio to its value at the base of the troposphere, $\hat{\theta}/\hat{T} = 1$.

The water mixing ratios are on the order of $10\text{g/kg} \sim 10^{-2}$, so it will be convenient to rescale the vapor and liquid water mixing ratios according to

$$\hat{q}_v = q_0 q, \quad \hat{q}_l = \frac{q_0}{R} q_l, \quad \text{and} \quad \hat{q}_r = \frac{q_0}{R} q_r, \tag{2.8}$$

where q_0 is the typical water vapor mixing ratio scale and $R > 1$ is the ratio of this scale to the typical liquid water mixing ratio scale. These rescalings are substituted into the temperature and water equations with time and potential temperature non-dimensionalized as before. The conversion rate of rain is non-dimensionalized by

$$\hat{H} = H_0 H, \tag{2.9}$$

where

$$H_0 = \frac{\theta_0}{\theta_L T_E} = 4 \times 10^{-2}\,\text{day}^{-1}. \tag{2.10}$$

The water vapor scale q_0 is chosen to remove constants from the water vapor conversion equation,

$$q_0 = H_0 T_E = 1.5 \times 10^{-2}. \tag{2.11}$$

Since typical mixing water ratios at saturation are of order 10^{-2} in the lower troposphere, then $H=1$ means that a typical parcel of water could go from dry to saturation a few times per day. Using these definitions, the potential temperature and water vapor equations take the simple form

$$\begin{aligned}\frac{D\theta}{Dt} &= H\\ \frac{Dq}{Dt} &= -H.\end{aligned} \tag{2.12}$$

A cloud model constitutes a specification of H, either through a deterministic or stochastic dynamical model. In particular, the multi-cloud models considered in KM use three types of clouds: congestus, deep, and stratiform, corresponding to heating rates which project strongly onto the lower, middle and upper troposphere, respectively.

Turning our attention to the liquid water equation, we non-dimensionalize the terminal velocity by the vertical length scale, equatorial deformation time, and the ratio R,

$$\hat{V}_T = R\,\frac{H_T}{\pi T_E}\,V_T. \tag{2.13}$$

Assuming typical rain terminal velocities of about 3 m/s yields $R=20$ and the liquid water equation becomes

$$\frac{Dq_l}{Dt} = R\left[\frac{\partial}{\partial z}(V_T\,q_r) + H\right]. \tag{2.14}$$

Notice that the large value of R suggests that the water vapor of a parcel of air exceeds the liquid water in a parcel of air by a factor of R (from the rescalings in (2.8)); most of the air in the tropical troposphere exists as moist unsaturated air, not cloud or rain.

Before concluding the discussion of moisture, we mention two additional features which aid in the incorporation of the multi-cloud models of KM. The first concerns the vertical average of the liquid water advection equation (2.14). In the limit $R \longrightarrow \infty$, the sum of the terms in parentheses on the right hand side of equation (2.14) must vanish in order for the advection of q_l to remain finite. Integrating the right hand side over the height of the troposphere and defining the total precipitation, $\mathbf{P}$, as the flux of the liquid water mixing ratio at the ground, we find

$$\mathbf{P} \equiv \frac{V_T}{H_T}\,q_l\Big|_{z=0} = \frac{1}{\pi}\int_0^{\pi} H\,dz, \tag{2.15}$$

where we have used the fact that the flux of rain through the top of the troposphere is zero. As in most GCMs the cloud models of KM and [7] are built on the assumption that column integrated liquid water processes are fast compared to the equatorial synoptic time scale.

The final point regarding moisture concerns the source of water vapor in equation (2.12) and liquid water in equation (2.14). There is apparently no source of total water in these equations and only the sink due to the flux of rain through the lower boundary. Physically the moisture source of the free troposphere arises through its interaction with a dynamic moist boundary layer and the strong coupling of this boundary layer to the warm ocean. The system of equation (2.1) and the moisture equation in (2.12) constitute an almost closed system of equations, except for a model of the heating, H. Coupling to a boundary layer is readily included by considering a layer below $z=0$ and matching fluxes through this layer. The careful incorporation of boundary layer effects is a necessary further step in the asymptotic analysis which is beyond the scope of this paper. However a carefully derived system of equations which includes boundary layer effects and the conservation of column integrated moist static energy is the basis for the multi-cloud models of KM. See [36] for the derivation of non-linear fluxes in the boundary layer.

3. Derivation of IMMD

Now that the non-dimensionalized primitive equation (2.1) and the moisture equation in (2.12) are established and we have a precise understanding of the magnitudes and scales over which these equations are valid, we are in a position to derive the Intraseasonal Multi-scale Moist Dynamics equations (IMMD).

There are only two assumptions involved in the derivation of IPESD from the equatorial β-plane equations, and only one of these will be used in the derivation of IMMD. That is, though the forcing terms on the right hand side of (2.1) and (2.12) have been non-dimensionalized to 150 m/s/day, 100 K/day and a moist mass fraction conversion rate of .04 day^{-1}, the equations themselves describe motions on synoptic and larger scales and actual average rates over these scales is much smaller. The measured atmospheric heating in the tropics associated with latent heat release and resolved on synoptic scales and larger is of order 10 K/day, a tenth of the non-dimensional scale. Furthermore, the drag dissipation and Newtonian cooling rate are also small. This motivates the introduction of a small parameter

$$\epsilon = \frac{\text{Measured heating rate on synoptic scales}}{\text{Nondimensional unit of heating rate}} = 0.1 \tag{3.1}$$

from which the IPESD and IMMD asymptotics emerge. In the IPESD asymptotics, ϵ also becomes the Froude number of the flow, i.e., the ratio of horizontal flow speeds to the dry gravity wave speed, and the parameter which measures the separation of the synoptic and planetary scales and synoptic and intraseasonal scales. While ϵ still measures the separation of scales in the IMMD framework, it is no longer a measure of the strength of the winds. Since IMMD allows for zonal flows which are of the same order of magnitude as the dry gravity wave speed, the Froude number (at least when measured using the modulating zonal winds) is no longer small. However, IMMD does maintain the notion of a small Froude number for the anomalies from the climatology, which are of order ϵ times the planetary zonal wind speeds. Rescaling the latent heating, momentum, and temperature dissipation rates by ϵ the primitive equations are rewritten as

$$\begin{aligned}
\frac{D}{Dt}u - yv &= -p_x - \epsilon du \\
\frac{D}{Dt}v + yu &= -p_y - \epsilon dv \\
\frac{D}{Dt}\theta + w &= \epsilon H - \epsilon d_\theta \theta \\
\frac{D}{Dt}q - Q_0 w &= -\epsilon H \\
p_z &= \theta \\
u_x + v_y + w_z &= 0
\end{aligned} \tag{3.2}$$

where, now, $d = 0.7$, $d_\theta = 0.23$, and one unit of latent heating corresponds to 10 K/day or a moist mass fraction rate of $4\times 10^{-3}\,\mathrm{day}^{-1}$. Much like the temperature equation, we have redefined q as the deviation from a background profile which decreases monotonically with height with gradient $-Q_0(z)$ [7]. Since the moisture stratification tends to be an exponential function of height in the troposphere, it is appropriate to retain the vertical dependence of Q_0.

The machinery for establishing the multi-scale IPESD or IMMD frameworks is that of multiple scales asymptotic theory. In particular, IPESD and IMMD are derived by trying to establish the dynamics on the synoptic zonal length and time scales. The solvability condition of the resultant system then requires modulating the dynamics on longer zonal length scales and time scales; these are the planetary zonal scales and the intraseasonal time scales [27, 25]. The primary length and time scales are $L_E = 1500\,\mathrm{km}$ and $T_E = 8.3\,\mathrm{hours}$, the equatorial deformation radius and time. L_E is also referred to as the equatorial synoptic length scale and we shall employ that terminology in this paper. T_E is the time it takes for linear wave solutions of equation (3.2) to travel one synoptic length scale, and following [2, 4] we refer to this as the synoptic time scale. The planetary length scale and intraseasonal time scales are large compared to the synoptic length and time scales and it will turn out that ϵ from the definition in equation (3.1) also measures the separation of these scales. Therefore, the planetary scale is $L_P = \epsilon^{-1} L_E \approx 15000\,\mathrm{km}$ and the intraseasonal time scale is $T_I = \epsilon^{-1} T_E \approx 3.5\,\mathrm{days}$ for the parameter values we have chosen.

The first step in a multiple scales theory is the ansatz that the solutions corresponding to any of the physical variables in equation (3.2) can be approximated by functions which vary on both length scales and both time scales. Formally, this corresponds to augmenting the independent variables by new coordinates which describe the planetary zonal scale and the intraseasonal time scale and which are independent of the original synoptic zonal and time scales. For example, for the zonal velocity field, this corresponds to the replacement,

$$u(x,t) \longrightarrow u^\epsilon(x,\epsilon x,t,\epsilon t) \tag{3.3}$$

where the dependence on y and z is suppressed since they do not actively participate in the multiple scales asymptotics. For the applied reader, we would like to clearly spell out the interpretation of the functional replacement in equation (3.3). Focus on the time variables; the zonal variables will have the same interpretation. We allow that when any of the arguments of u^ϵ varies an order one amount, then u^ϵ can, at most, vary an order one amount . An order one variation in t equals 8.3 hours, which is a synoptic time unit. An order one variation in ϵt means that $\epsilon t \approx 1$, or that t varies an order of ϵ^{-1}, which is equal to 10 times 8.3 hours or 3 days, an intraseasonal time unit. Introducing the planetary zonal variable and intraseasonal time variable

$$X = \epsilon x \quad T = \epsilon t \tag{3.4}$$

amounts to assuming a multi-scale solution

$$u(x,t) = u^\epsilon(x,X,t,T) \tag{3.5}$$

where zonal derivatives and time derivatives are replaced by their multi-scale counterparts

$$\begin{aligned} \frac{\partial u}{\partial x} &\longrightarrow \frac{\partial u^\epsilon}{\partial x} + \epsilon \frac{\partial u^\epsilon}{\partial X} \\ \frac{\partial u}{\partial t} &\longrightarrow \frac{\partial u^\epsilon}{\partial t} + \epsilon \frac{\partial u^\epsilon}{\partial T}. \end{aligned} \tag{3.6}$$

This same ansatz holds for meridional and vertical velocity fields, pressure, potential temperature and moisture. Implicit in the ansatz of equation (3.5) is that if any of the independent variables x, X, t or T varies an amount which is order one then u^ϵ can also vary an amount which is order one, but it cannot vary by amounts larger than order one.

The second step in the theory is to pose an asymptotic expansion of the dynamic variables. This expansion is meant to be well ordered for all synoptic time and for planetary time scales of order ϵ^{-1} and longer. A consistent solution of the moist dynamics in equation (3.2) requires the asymptotic expansion

$$\begin{aligned}
u^\epsilon &= U(X,T) + \epsilon(u'(x,X,t,T) + \bar{u}(X,t,T)) + \epsilon^2 u_2 + \ldots \\
v^\epsilon &= \epsilon(v'(x,X,t,T) + \bar{v}(X,t,T)) + \epsilon^2 v_2 + \ldots \\
w^\epsilon &= \epsilon(w'(x,X,t,T) + \bar{w}(X,t,T)) + \epsilon^2 w_2 + \ldots \\
p^\epsilon &= P(X,T) + \epsilon(p'(x,X,t,T) + \bar{p}(X,t,T)) + \epsilon^2 p_2 + \ldots \\
\theta^\epsilon &= \Theta(X,T) + \epsilon(\theta'(x,X,t,T) + \bar{\theta}(X,t,T)) + \epsilon^2 \theta_2 + \ldots \\
q^\epsilon &= Q(X,T) + \epsilon(q'(x,X,t,T) + \bar{q}(X,t,T)) + \epsilon^2 q_2 + \ldots
\end{aligned} \tag{3.7}$$

where, again, all of the fields also depend on y and z. The overbars denote averaging with respect to the synoptic zonal scale

$$\bar{f}(X,t,T) \equiv \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(x,X,t,T)\, dx \tag{3.8}$$

and the primed variables are mean zero,

$$f(x,X,t,T) = \bar{f}(X,t,T) + f'(x,X,t,T) \quad \text{where} \quad \bar{f}' = 0. \tag{3.9}$$

For the applied scientist who may be less familiar with the intricacies of multi-scale asymptotics, we emphasize an important feature of the asymptotic expansion posed in equation (3.7). Based on the assumptions of the model — for IMMD, equation (3.1) - and considering initial conditions which are consistent with the expansion in (3.7), then the multiscale asymptotic procedure seeks solutions which are consistent with this expansion for long times; in the case of IMMD, intraseasonal time scales. Furthermore, while the theory allows uppercase variables (the climatology) to vary on intraseasonal (3.5 day time unit) or longer timescales, their being stationary over these timescales is also consistent with the asymptotics.

Establishing the theory requires that each successive order of the asymptotic expansion be solvable — in particular, u_2 is solvable. This further implies that the error of our approximation on these scales is less than $O(\epsilon^2)$, which for the zonal velocity corresponds to 0.5 m/s. There is no *a priori* reason that the asymptotics should yield the structure in equation (3.7); it ultimately reveals itself upon attempting to solve the successive orders of the theory.

Except for the moisture variable q and the lowest order terms, U, P, Θ, Q, the expansion in (3.7) is the same as that in the IPESD framework [27, 25]. The inclusion of a lowest order zonal flow, U, means that the IMMD framework can account for zonal flows of order 50 m/s and the pressure, P, which is in meridional geostrophic and hydrostatic balance with such flows. Thus, unlike IPESD, IMMD is not a low Froude number theory; it is able to describe the stronger zonal flows associated with the Trade Winds. The pressure that these trade winds generate will generate a potential temperature perturbation, Θ, from the hydrostatic constraint and the theory also accounts for a moisture stratification which is the same strength as this temperature perturbation. In the IMMD framework the Trade Winds, and their companion thermodynamic variables, modulate on the planetary zonal scale, X, and the intraseasonal time scale, T, but not on the synoptic zonal or temporal scales. In the appendix, we discuss how the loosening of the restriction on the synoptic zonal mean heating rate naturally leads to the addition of the stronger zonal flow.

The first order correction to the Trade Winds have horizontal velocities on the order of 5 m/s. We explicitly separate these flows into their synoptic scale fluctuating components, u',v',w',p',θ' and q' and their synoptic scale means, $\bar{u},\bar{v},\bar{w},\bar{p},\bar{\theta}$ and $\bar{q}$. The expansion (3.7) allows for a second order correction in the terms with a subscript $'2'$ and it will become evident below how these terms participate in the theory.

For conciseness of exposition we have written the asymptotic expansion of equation (3.2) in Appendix A, and in the following subsections we describe how to solve them at each order, thereby obtaining a closed, multi-scale asymptotic theory. We define the synoptic (or fast) time average

$$\langle f\rangle(x,X,T) \equiv \lim_{t_*\to\infty} \frac{1}{2t_*}\int_{-t_*}^{t_*} f(x,X,t,T)\,dt, \tag{3.10}$$

and, since the derivation will separately describe the fast and slow variations of a zonal synoptic mean field, we define

$$\tilde{f}(X,t,T) \equiv \bar{f}(X,t,T) - \langle \bar{f}\rangle(X,T). \tag{3.11}$$

Clearly $\left\langle \tilde{f}\right\rangle \equiv 0$ and $\tilde{f}$ describes all of the synoptic time scale variation of the zonal mean fields.

3.1. Trade winds and Hadley circulation (TH). The asymptotic expansion of equation (3.2) are written in Appendix A equations (A.1)–(A.7) and we shall often refer to these in deriving the IMMD equations. The lowest order of equations (A.1)–(A.7) is ϵ^0 and occurs in equation (A.2) and (A.6). They describe the meridional geostrophic and hydrostatic balance of the pressure and zonal wind

$$\begin{aligned} yU + P_y &= 0 \\ P_z &= \Theta. \end{aligned} \tag{3.12}$$

At order ϵ^1, we arrive at the first result of IMMD which is more general than the IPESD framework. Taking the synoptic zonal and time averages of equation (A.1), (A.3), (A.4), and (A.7) we find

$$\begin{aligned} \frac{D}{DT}U - yV + P_X &= -dU \\ \frac{D}{DT}\Theta + W &= \left\langle \bar{H}\right\rangle - d_\theta \Theta \\ \frac{D}{DT}Q - Q_0 W &= -\left\langle \bar{H}\right\rangle \\ U_X + V_y + W_z &= 0, \end{aligned} \tag{3.13}$$

where the notation

$$V \equiv \langle \bar{v}\rangle \quad \text{and} \quad W \equiv \langle \bar{w}\rangle \tag{3.14}$$

is introduced in order to describe the meridional/vertical structure of the Hadley Circulation. The advective derivative is taken with respect to the intraseasonal time scale and the full three dimensional flow described by the Hadley Circulation

$$\frac{D}{DT} \equiv \frac{\partial}{\partial T} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z}. \tag{3.15}$$

Notice that all zonal derivatives are with respect to the planetary zonal scale, X. The choice of a cloud model constitutes a prescription of H, and the coupling of the free troposphere to a boundary layer would affect the lower boundary conditions on the system of equation (3.12)–(3.13). Otherwise, this system constitutes a closed description of the Hadley Circulation, (U,V,W), and the pressure, temperature, and moisture stratification induced by this flow, P,Θ,Q. As such, it is the TH scale of the IMMD framework. We emphasize that while we have chosen to begin with incompressible dynamics in order to simplify the discussion, the anelastic constraint would have applied had we chosen to include the density at the outset [27].

The TH scale consists of three dimensional incompressible flows which are in equatorial meridional geostrophic and hydrostatic balance. The equations in (3.12) and (3.13) are nonlinear equatorial long wave equations which allow for zonal jets, barotropic waves and equatorial baroclinic Rossby and Kelvin waves while filtering out equatorial inertio-gravity and mixed Rossby gravity waves. They can describe both a mean Hadley Circulation and its zonal modulation on planetary scales over timescales no faster than the intraseasonal timescale. Therefore, linear models of the Trade Wind/Hadley circulation must result from the additional assumption that the strength of the forcing on the planetary scales is $O(\epsilon)$ weaker than that which is allowed in this IMMD derivation. Conversely, if the actual Trade Wind/Hadley circulation is observed to modulate on intraseasonal timescales with wind strengths of 50 m/s zonally and 5 m/s meridionally, then equations (3.13) imply that advective non-linearities are not negligible.

TH can couple to smaller scales through the cloud model and the boundary layer. Models for H such as KM or Betts-Miller type parameterizations [1] are strongly non-linear functions of the thermodynamic variables so spatio-temporal averages of these functions cannot be separated into different, uncoupled scales. This heating term, therefore, is the primary source of coupling between the Trade Winds/Hadley circulation and synoptic and smaller scale flows.

3.2. Zonal synoptic scale fluctuations (S). Focusing on order ϵ^1 terms in equations (A.1), (A.3), (A.2), (A.4), and (A.7) we subtract from these terms their synoptic scale zonal mean to find

$$\begin{aligned} u'_t + U u'_x + (u'U)_x + (v'U)_y + (w'U)_z - y v' + p'_x &= 0 \\ v'_t + U v'_x + y u' + p'_y &= 0 \\ \theta'_t + U\theta'_x + w' + (u'\Theta)_x + (v'\Theta)_y + (w'\Theta)_z &= H' \\ q'_t + U q'_x - Q_0 w' + (u'Q)_x + (v'Q)_y + (w'Q)_z &= -H' \\ p'_z &= \theta' \\ u'_x + v'_y + w'_z &= 0, \end{aligned} \tag{3.16}$$

which describe the synoptic scale fluctuating dynamics (S). Since the mean wind, temperature, and moisture do not depend on the zonal synoptic scale, the synoptic scale incompressibility constraint has been used to rewrite the terms corresponding to the advection of the mean state by the fluctuations explicitly as divergence terms, which is to say

$$v'U_y + w'U_z = (u'U)_x + (v'U)_y + (w'U)_z, \tag{3.17}$$

and similarly for Θ and Q.

Considered in isolation, the synoptic scale fluctuating flows are described by linear equations — the nonlinearities arise through coupling to the planetary scale or the cloud model, H'. The synoptic scale flows are advected by the zonal component of the Trade Winds but do not respond to the meridional flow of the Hadley Circulation. They are additionally coupled to the zonal component of the Hadley Circulation and the stratification through the divergence of zonal momentum and temperature *across scales*; the divergence terms describe a wave/mean flow coupling on the synoptic spatial and temporal scales. Also, these equations retain the features of the IPESD synoptic scale theory [27, 25] in that the momentum and temperature dissipation does not directly affect the synoptic scale flows, but will do so through their upscale fluxes to the planetary scales.

3.3. Planetary scale anomalies from the climatology (P). Consider the synoptic zonal mean of the order ϵ^1 terms in equations (A.1)–(A.7). Subtracting from these the synoptic time average of the same equations (which is tantamount to subtracting equation (3.13)) and using the definition in (3.11) yields

$$\begin{aligned}
\tilde{u}_t + (\tilde{v}U)_y + (\tilde{w}U)_z - y\tilde{v} &= 0\\
\tilde{v}_t + y\tilde{u} + \tilde{p}_y &= 0\\
\tilde{\theta}_t + \tilde{w} + (\tilde{v}\Theta)_y + (\tilde{w}\Theta)_z &= \tilde{H}\\
\tilde{q}_t - Q_0\tilde{w} + (\tilde{v}Q)_y + (\tilde{w}Q)_z &= -\tilde{H}\\
\tilde{p}_z &= \tilde{\theta}\\
\tilde{v}_y + \tilde{w}_z &= 0,
\end{aligned} \tag{3.18}$$

which describes the fast time scale modulation of planetary scale anomalies from the modulating climatology described by TH. As in section 3.2, we have used the incompressibility constraint to write the coupling to the planetary scales as the divergence of a flux.

Equations (3.18) describe fast moving waves whose amplitudes, by definition, have synoptic time mean equal to zero. Unlike the synoptic scale fluctuations, these waves are not advected by the zonal component of the Trade Winds; in fact, there is no zonal coupling in these equations since there are no zonal derivatives. These waves propagate only in the meridional and vertical directions. Their physical interpretation is much like that of an intertio-gravity wave, which communicates the presence of an imposed heating source and leaves an adjusted density stratification in its wake. In the presence of a heating source at the same planetary scale zonal location, the waves in (3.18) communicate the adjustment to this heat source in the meridional and vertical direction.

We now derive the equations governing the zonal planetary scale anomalies from the mean climatology which, themselves are modulated on intraseasonal time scales. To do so, we introduce one more notational simplification

$$\begin{aligned}
\langle\bar{u}\rangle &\to u, & \langle\bar{p}\rangle &\to p,\\
\langle\bar{\theta}\rangle &\to \theta, & \langle\bar{q}\rangle &\to q,\\
\langle\bar{v}_2\rangle &\to v, & \langle\bar{w}_2\rangle &\to w
\end{aligned} \tag{3.19}$$

and the equations that we seek will describe the modulation of these variables.

Consider the zonal and temporal mean of the terms of order ϵ^1 in equation (A.2), (A.6), and the terms of order ϵ^2 in equation (A.7). The constraints

$$
\begin{aligned}
yu+p_y &= -dV \\
p_z &= \theta \\
u_X+v_y+w_z &= 0
\end{aligned} \tag{3.20}
$$

arise after exploiting the fact that the synoptic time average of a synoptic time derivative must vanish

$$\langle \bar{v}_t \rangle = \langle \tilde{v}_t \rangle = \lim_{t_* \to \infty} \frac{\tilde{v}(t_*)-\tilde{v}(-t_*)}{2t_*} \longrightarrow 0. \tag{3.21}$$

Notice that the anomalies are not in strict meridional geostrophic balance since the meridional component of the Hadley circulation appears as a forcing for the geostrophic constraint. See [3] for a discussion of how such a forcing can be incorporated into a zonal momentum forcing and heating.

The order ϵ^2 terms in equation (A.1), (A.3), and (A.4) constitute an inhomogeneous linear system of PDEs for the second order corrections to the zonal momentum, temperature, and moisture, u_2,θ_2,q_2, respectively. They can be written concisely as

$$
\begin{aligned}
u_{2,t}+U u_{2,x}+p_{2,x} &= G_u \\
\theta_{2,t}+U\theta_{2,x} &= G_\theta \\
q_{2,t}+U q_{2,x} &= G_q
\end{aligned} \tag{3.22}
$$

where the pressure is given by the hydrostatic relation $p_{2,z}=\theta_2$ from equation (A.6) and G_u,G_θ, and G_q encapsulate all of the remaining terms at second order. In order for the asymptotic expansion posed in equation (3.7) to be well ordered for all time, these equations cannot describe secular growth in u_2,θ_2,q_2. Since U is independent of x,t, its variables, X,T (and y,z, which have been suppressed) appear as parameters in the inhomogeneous equation (3.22); effectively, each X,T,y are decoupled in equation (3.22), and different vertical levels couple because of the hydrostatic pressure relation. Therefore, for each X,T,y,z, a necessary and sufficient condition for non-secular growth of the synoptic scale zonal means of u_2,θ_2,q_2 is that the synoptic zonal and temporal mean of the forcing terms on each of the right hand sides of the equations in (3.22) must vanish. Taking the synoptic zonal and time average of the right hand sides results in the following system of equations for the intraseasonal dynamics of the synoptic zonal and time averaged planetary scale zonal momentum, temperature, and moisture anomaly

$$
\begin{aligned}
\frac{D}{DT}u+(uU)_X+(vU)_y+(wU)_z-yv+p_X &= F^u-du \\
\frac{D}{DT}\theta+w+(u\Theta)_X+(v\Theta)_y+(w\Theta)_z &= F^\theta+\left\langle \bar{H}_2 \right\rangle - d_\theta\theta \\
\frac{D}{DT}q-Q_0w+(uQ)_X+(vQ)_y+(wQ)_z &= F^q-\left\langle \bar{H}_2 \right\rangle,
\end{aligned} \tag{3.23}
$$

where the advective derivatives are defined in equation (3.15) and the fluxes on the right hand side are given by

$$
\begin{aligned}
F^u &= -\left\langle \overline{v'u'} \right\rangle_y - \left\langle \overline{w'u'} \right\rangle_z - \langle \tilde{v}\tilde{u} \rangle_y - \langle \tilde{w}\tilde{u} \rangle_z \\
F^\theta &= -\left\langle \overline{v'\theta'} \right\rangle_y - \left\langle \overline{w'\theta'} \right\rangle_z - \left\langle \tilde{v}\tilde{\theta} \right\rangle_y - \left\langle \tilde{w}\tilde{\theta} \right\rangle_z \\
F^q &= -\left\langle \overline{v'q'} \right\rangle_y - \left\langle \overline{w'q'} \right\rangle_z - \langle \tilde{v}\tilde{q} \rangle_y - \langle \tilde{w}\tilde{q} \rangle_z .
\end{aligned} \tag{3.24}
$$

Together, equation (3.23) and the upscale fluxes (3.24) describe the evolution of the planetary scale anomalies in the IMMD framework.

The planetary scale anomalies are advected by all components of the Hadley Circulation and evolve on the intraseasonal time scale, just like the background climatology. They further experience an in-scale flux through their interaction with the planetary scale flow. Because of the incompressibility constraint, this flux can also be thought of as the advection of the background climatology by the anomalies.

The anomalies are directly forced by the flux terms, F^u, F^θ, and F^q which are defined in equation (3.24), each of which consists of two pieces. The first two terms in each flux correspond to the spatial upscale flux from synoptic scale fluctuations to the planetary scale and are further divided into meridional and vertical components. The second two terms defining each flux correspond to fast, mean zero fluctuations of the planetary scale anomalies nonlinearly driving time averaged planetary scale anomalies; again, these have both meridional and vertical components. Due to the original, physically motivated choice that the dissipation timescales are intraseasonal the anomalies are also subject to both zonal momentum and temperature dissipation.

The last term appearing in the planetary scale theory is the synoptic zonal and time average of the second order mean heating; dimensionally, this term is measured in units of 1.5 K/day. Examining the definition in equation (A.5), we can see that separating this term out from the lowest order mean heating makes no sense *a priori* since one can always redefine $\left\langle \bar{H} \right\rangle$ to include $\left\langle \bar{H}_2 \right\rangle$ by simply redefining (equivalently, renormalizing) the small parameter, ϵ. Therefore, the presence of $\left\langle \bar{H}_2 \right\rangle$ makes sense only when $\left\langle \bar{H} \right\rangle$ vanishes, that is to say, only when we can say with certainty that the forcing from latent heating is everywhere no greater than a few Kelvin per day. Otherwise any planetary scale mean heating can always be absorbed into the term $\left\langle \bar{H} \right\rangle$. We elaborate on this point in Appendix B where IPESD and IMMD are compared.

3.4. On describing the synoptic time dynamics by a single system of equations. As it has been derived, IMMD breaks up into four systems of equations, each system being coupled to one or more of the others. We can reduce this to three systems of equations by combining the synoptic scale fluctuating fields (u',v',w',p',θ',q'), which have synoptic zonal mean equal to zero, with the planetary scale fast oscillating flows $(\tilde{u},\tilde{v},\tilde{w},\tilde{p},\tilde{\theta},\tilde{q})$ which have synoptic time average equal to zero. The corresponding fields in equations (3.16) and (3.18) are the same order in ϵ, so they can be added to one another while maintaining asymptotic consistency. Defining the sum of the synoptic fluctuations and the fast time fluctuations of the planetary scale means as

$$
f^* = f' + \tilde{f} \quad \text{so that} \quad \left\langle \bar{f^*} \right\rangle = 0, \tag{3.25}
$$

then the first order terms in the asymptotic expansion in (3.7) can be re-expressed as

$$
f = \left\langle \bar{f} \right\rangle + f^*. \tag{3.26}
$$

Therefore, neither the spatial nor temporal mean of f^* vanish, but together its spatial-temporal mean is zero. The starred variables are thus the synoptic space/time fluctuations. Adding equations (3.16) and (3.18) and substituting the definition of the starred variables, we can replace those two systems of equations with the following single system of equations for the synoptic space/time fluctuations

$$\begin{aligned} u_t^* + U u_x^* + (u^* U)_x + (v^* U)_y + (w^* U)_z - y v^* + p_x^* &= 0 \\ v_t^* + U v_x^* + y u^* + p_y^* &= 0 \\ \theta_t^* + U \theta_x^* + w^* + (u^* \Theta)_x + (v^* \Theta)_y + (w^* \Theta)_z &= H^* \\ q_t^* + U q_x^* - Q_0 w^* + (u^* Q)_x + (v^* Q)_y + (w^* Q)_z &= -H^* \\ p_z^* &= \theta^* \\ u_x^* + v_y^* + w_z^* &= 0. \end{aligned} \tag{3.27}$$

Structurally, this system is identical to that in (3.16), with the exception that the synoptic space/time average of H^* is zero — in contrast to the synoptic spatial average of H' vanishing in (3.16).

The fluxes defined in equation (3.24) which are the primary forcing for the anomalies in (3.23), are now replaced by simpler expressions in terms of the synoptic space/time fluctuations

$$\begin{aligned} F^u &= -\left\langle \overline{v^* u^*} \right\rangle_y - \left\langle \overline{w^* u^*} \right\rangle_z \\ F^\theta &= -\left\langle \overline{v^* \theta^*} \right\rangle_y - \left\langle \overline{w^* \theta^*} \right\rangle_z \\ F^q &= -\left\langle \overline{v^* q^*} \right\rangle_y - \left\langle \overline{w^* q^*} \right\rangle_z . \end{aligned} \tag{3.28}$$

This more concise representation clearly shows how the combined synoptic space/time fluctuations create the fluxes which force the anomalies from the background climatology.

4. Discussion

We have used multiscale asymptotic analysis to systematically derive the moist multi-scale dynamics of the tropical atmosphere. By making the sole assumption that the forcing of the tropical troposphere is of order 10 Kelvin/day when resolved on the synoptic scales and larger, we have constructed a multi-time and multi-space theory for the evolution of synoptic and planetary scale flows on synoptic and intraseasonal time scales. The Intraseasonal Multi-scale Moist Dynamics framework for the tropical troposphere (IMMD) is valid on horizontal synoptic scales and zonal planetary scales and describes variations on equatorial synoptic and intraseasonal time scales. There are three scales of motion in the IMMD framework.

The first, TH, governs the modulation of the Trade Winds and Hadley circulation on planetary zonal length scales (15000 km) and intraseasonal time scales (3.5 days). Zonal flows of order 50 m/s, meridional flows of order 5 m/s, and temperature variations of order 33 Kelvin are allowed by the theory. The TH system consists of equation (3.13) with the constraints (3.12) which are fully non-linear equatorial moist long wave equations forced by the synoptic zonal and time averaged diabatic heating rate and dissipated by linear dissipation. Coupling to smaller scales occurs through the non-linearities involved in averaging the heating rate.

The second set of equations governs the synoptic scale fluctuations, S. This system is written in equation (3.16). Synoptic scale fluctuations occur on the 8 hour time scale and are of order 5 m/s in the horizontal direction with temperature perturbations of order 3.3 Kelvin. The synoptic scale flows are advected only by the zonal component of the Hadley Circulation. They further couple to the modulating climatology through the advection of the climatology by the synoptic scale fluctuations. Equivalently, these terms can be interpreted as the cross-scale flux of zonal momentum, temperature, and moisture from the planetary scales to the synoptic scales. The synoptic scale fluctuating flows are directly forced by the latent heat fluctuations, but they are not damped by the weaker dissipation.

The third system of equations describes planetary scale anomalies, P, which evolve on both the synoptic and intraseasonal time scales.

The synoptic time evolution of the planetary scale anomaly is given by the equations in (3.18) where the flow in the meridional/vertical plane is two-dimensional incompressible. From the second equation in (3.18), the meridional velocity is accelerated by the portion of the zonal velocity and pressure which are not in (meridional) geostrophic balance. The vertical velocity is determined from the meridional velocity through the incompressibility constraint: the last equation in (3.18). Therefore the dynamics in equation (3.18) describe meridional and vertically propagating waves which adjust the planetary scale anomalies to a steadily (on the synoptic time scale) forced system in meridional geostrophic balance. These waves interact with the intraseasonally modulating Hadley Circulation and thermodynamic stratification, but disperse too quickly to modify them. The horizontal flow generated by these waves is of order 5 m/s and the temperature perturbation is 3.3 Kelvin.

In section 3.4 we showed how the equations governing the synoptic scale fluctuations in S and the synoptic time evolution of the planetary scale anomalies from P can be reconstituted so that all synoptic time dynamics are described by one system of equation (3.27).

The intraseasonal evolution of the planetary scale anomaly is governed by the equations (3.20) and (3.23). While zonal flow anomalies remain of order 5 m/s and temperature anomalies are of order 3.3 Kelvin, the meridional flow anomaly is smaller, of order 0.5 m/s; this is due to the anisotropic scaling in the equatorial long wave theory. The planetary scale anomalies show a rich dynamical structure and it is in these fields that the MJO envelope presents itself. In the planetary scale theory, synoptic mean zonal winds, temperature and moisture are advected on the intraseasonal time scales by the Hadley Circulation and Trade Winds. Furthermore, the flow set up by these anomalies interacts with the zonal component of the Trade Winds and the stratification which modulates the anomaly but not the mean climatology. This is expressed as an in-scale flux divergence in equation (3.23). The anomalies are further forced by two flux divergences from equation (3.24). The first is an upscale flux from the synoptic scale fluctuating flow in equation (3.16) while the second is an in-scale flux from the fast oscillating adjustment waves in equation (3.18).

The IMMD framework, the three systems of equation (3.13), (3.23), and (3.27), plus the constraints (3.12) and (3.20), provide a conceptual basis for making dynamical models and for the interpretation of observations. The climatology is only forced by the mean heating whereas the synoptic scale fluctuations are forced by the synoptic scale fluctuating heating. On the other hand, the primary driver of planetary scale flow anomalies are the upscale fluxes from the synoptic scales. The existence of a mean upscale flux from the synoptic scale is equivalent to the synoptic scale flow having some average organization, such as would arise if there are vertically or meridionally tilted heating profiles; this organization was discussed in the IPESD MJO models [2, 3, 5] and has been extracted from the observations [30]. Ultimately, mean heating sets up a climatology and synoptic organization modulates that climatology to generate a MJO.

One could ask, why not combine the equations for the climatology and the anomalies just like the equations for synoptic timescale dynamics in (3.27)? First, the amplitudes of the two flows are different orders in the asymptotic expansion, so such a combination is not asymptotically consistent. Second, the computational simplification that the flows are only advected by the mean climatology — not the anomalies — is lost if the two are combined. Third, such a combination would obscure the point that the planetary anomalies are forced by synoptic scale organization — therefore the notion that synoptic scale organization is necessary to produce the MJO. Fourth, one could envision an MJO model where the mean heating is stationary on the intraseasonal time scales, thereby generating a stationary climatology. In this scenario, only the synoptic organization develops in time, thereby driving the MJO envelope, and this would reduce the model to two systems of equations much like IPESD.

Finally, implicit in the IMMD framework is a multi-scale algorithm for the numerical integration of large scale tropical dynamics which we describe as follows. Filter initial conditions onto planetary scale climatology and synoptic scale fluctuations. Integrate the synoptic scale fluctuations (3.27) in the presence of a cloud model (for H^*) over one unit of fast, synoptic time. At every sub-step of the fast time integration (fractions of a synoptic timescale) take the synoptic zonal spatial average of the heating rate and the dynamical fields and subtract them from the result. The remaining fluctuating components are used to continue the integration of the synoptic scale fluctuations and to compute the upscale fluxes using (3.28) after one synoptic timescale is reached. Use the synoptic time/space averaged upscale fluxes to integrate the planetary scale anomaly. Use the temporally integrated synoptic averaged mean heating (which was subtracted from the previous integration) to integrate the climatology. Iterate this process. Notice that this algorithm discards the small mean flow that is generated in the process of integrating the synoptic scale fluctuation equations. However, it preserves the mean heating that is generated from the synoptic scales and uses it to integrate the TH equations. As long as the mean heating remains order one throughout the integration — which is consistent with the 10 Kelvin/day heating rate — then discarding the mean flows they generate over synoptic timescales is consistent with the asymptotics.

The IMMD framework provides a clear multi-scale framework for discussing the modulation and rectification of the Hadley Circulation as well as the Madden-Julian Oscillation. Latent heat release from synoptic scale flows directly drives synoptic scale fluctuations which are advected by the trade winds in the mean stratification. The synoptic scale means of the latent heating modulate the mean stratification and Hadley Circulation. The upscale fluxes from the synoptic scale fluctuations drive the planetary scale anomalies, which are advected by the zonal component Hadley Circulation and constitute the envelope of the Madden-Julian Oscillation. Section 2 presents a road map for adding moisture models to the IMMD dynamics and we will incorporate the multi-cloud models of KM into this multi-scale framework in a forthcoming paper.

Appendix A. Full multi-scale asymptotic expansion of the primitive equations. Substituting the multi-scale asymptotic expansion from equation (3.7) into the rescaled moist primitive equation (3.2) yields a coupled set of asymptotic PDEs. We record them below up to the order of relevance needed for the derivation of the IMMD framework and explicitly write the order of the neglected terms in each equation. The zonal velocity equation is

$$\begin{aligned}&\epsilon[U_T+u'_t+\bar{u}_t+U(U_X+u'_x)+(v'+\bar{v})U_y+(w'+\bar{w})U_z-yv'-y\bar{v}+P_X+p'_x+dU]\\&+\epsilon^2[u_{2,t}+u'_T+\bar{u}_T+U(u'_X+\bar{u}_X+u_{2,x})+(u'+\bar{u})(U_X+u'_x)+(v'+\bar{v})(u'+\bar{u})_y\\&+v_2U_y+(w'+\bar{w})(u'+\bar{u})_z+w_2U_z-yv_2+p'_X+\bar{p}_X+p_{2,x}+d(u'+\bar{u})]=O(\epsilon^3).\end{aligned}\tag{A.1}$$

The meridional velocity equation is

$$yU+P_y+\epsilon\left[v'_t+\bar{v}_t+Uv'_x+y(u'+\bar{u})+p'_y+\bar{p}_y+dV\right]=O(\epsilon^2).\tag{A.2}$$

The potential temperature,

$$\begin{aligned}&\epsilon\left[\Theta_T+\theta'_t+\bar{\theta}_t+U(\Theta_X+\theta'_x)+(v'+\bar{v})\Theta_y+(w'+\bar{w})\Theta_z-H'-\bar{H}+d_\theta\Theta\right]\\&+\epsilon^2\left[\theta_{2,t}+\theta'_T+\bar{\theta}_T+U(\theta'_X+\bar{\theta}_X+\theta_{2,x})+(u'+\bar{u})(\Theta_X+\theta'_x)+(v'+\bar{v})(\theta'+\bar{\theta})_y\right.\\&\left.+v_2\Theta_y+(w'+\bar{w})(\theta'+\bar{\theta})_z+w_2\Theta_z+d_\theta(\theta'+\bar{\theta})\right]=O(\epsilon^3),\end{aligned}\tag{A.3}$$

and moisture equations,

$$\begin{aligned}&\epsilon\left[Q_T+q'_t+\bar{q}_t+U(Q_X+q'_x)+(v'+\bar{v})Q_y+(w'+\bar{w})Q_z+H'+\bar{H}\right]\\&+\epsilon^2[q_{2,t}+q'_T+\bar{q}_T+U(q'_X+\bar{q}_X+q_{2,x})+(u'+\bar{u})(Q_X+q'_x)\\&+(v'+\bar{v})(q'+\bar{q})_y+v_2Q_y+(w'+\bar{w})(q'+\bar{q})_z+w_2Q_z]=O(\epsilon^3),\end{aligned}\tag{A.4}$$

are essentially identical and we have explicitly split the heating rate into the mean, the fluctuating components, and the possibility of heating which is weaker than either of these two

$$H=H'+\bar{H}+\epsilon H_2.\tag{A.5}$$

We shall find that H_2 is only relevant if the mean of the lower order $\bar{H}$ vanishes. Otherwise, all weaker heating can be incorporated into the either of the lower order terms. What remains are the expansions of the hydrostatic

$$P_z+\epsilon(p'_z+\bar{p}_z)+\epsilon^2p_{2,z}=\Theta+\epsilon(\theta'+\bar{\theta})+\epsilon^2\theta_2+O(\epsilon^3)\tag{A.6}$$

and incompressible constraints

$$\epsilon\left[U_X+u'_x+v'_y+\bar{v}_y+w'_z+\bar{w}_z\right]+\epsilon^2[u'_X+\bar{u}_X+v_{2,y}+w_{2,y}]=O(\epsilon^3).\tag{A.7}$$

Appendix B. Comparison of IMMD with IPESD and the MJO models. The IPESD framework [27, 25] is a dry theory concerned with synoptic scale fluctuations and planetary scale anomalies only. It therefore does not describe the zonal component of the Hadley Circulation or the modulation of the mean stratification of temperature that arises from this wind. In order to compare these two theories, we must entirely neglect the moisture equation, set $U=0$, and allow only vertical variations of the pressure and potential temperature perturbations, $\Theta(z)$, $P(z)$. Therefore the squared Brunt-Väisälä frequency is $N^2=1+d\Theta/dz$. Furthermore, IPESD allows for a general zonal momentum forcing of the order 15 m/s/day, S^u. Making these substitutions in equation (3.13) yields the system of equations

$$\begin{aligned}-yV&=\langle\bar{S^u}\rangle\\N^2W&=\langle\bar{H}\rangle\\V_y+W_z&=0,\end{aligned}\tag{B.1}$$

describing the Hadley Circulation, V,W; notice that this notation differs from [27]. Since these are three equations for the two unknowns there must be a solvability condition. Isolating V in the first equation in (B.1) and setting the divergence of the circulation equal to zero yields the constraint

$$\left(\frac{\langle \bar{S^u} \rangle}{y}\right)_y = \left(\frac{\langle \bar{H} \rangle}{N^2}\right)_z \tag{B.2}$$

which is discussed in [27, 25, 3]. This is a not unexpected requirement that a steady climatology requires a careful balance between mean heating and zonal momentum dissipation. Though such a balance is not physically implausible, requiring it a priori reduces the generality of the IPESD framework. The allowance of a mean zonal wind obviates this balance. We emphasize that, with the modification described above and the additional caveat that IMMD includes an equation for the moisture, the IMMD framework exactly reduces to the IPESD framework.

The MJO models of [26, 2, 3, 5] consider mean heating rates which are smaller than 10 K/day, whereby $\langle \bar{H} \rangle = 0$. Clearly, the Hadley Circulation is zero in this setting so that the there remain no terms corresponding to *advection by*, or *cross-scale interaction with* the background flow in the equations for the synoptic scale fluctuations, (3.27). In the equations for the planetary anomaly, the advective time derivative is replaced by the regular partial derivative with respect to the intraseasonal time scale and, again, there is no mean flow. In this instance, the weaker mean heating $\langle \bar{H}_2 \rangle$, which is of order a few Kelvin/day, plays an essential role in the MJO theory. As discussed in section 3.3, if the mean heating rate were stronger, then $\langle \bar{H}_2 \rangle$ could be incorporated into $\langle \bar{H} \rangle$ but the equations for the modulating Hadley Circulation and background stratification (3.13) would need to be included.

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