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RIGOROUS DERIVATION OF THE X-Z SEMIGEOSTROPHIC EQUATIONS *

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Abstract. We prove that smooth solutions of the semigeostrophic equations in the incompressible $x-z$ setting can be derived from the Navier-Stokes equations with the Boussinesq approximation.

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1. Introduction

We consider the Navier-Stokes equations with the Boussinesq approximation (NSB):

$$\epsilon(\partial_t v+(v\cdot\nabla)v)+\alpha Kv+\nabla p=y,\ \nabla\cdot v=0, \tag{1.1}$$

$$\partial_t y+(v\cdot\nabla)y=G(x,y), \tag{1.2}$$

where $x\in D$, D being a smooth bounded domain in R^d $(d=2,3)$, $v=v(t,x)\in R^d$ is the velocity field, $p=p(t,x)$ is the pressure field, $y=y(t,x)\in R^d$ is a vector-valued forcing term, $G(x,y)$ is a given smooth vector-valued source term $D\times R^d\to R^d$, $\epsilon,\alpha>0$ are scaling factors and K is the linear dissipative operator $Kv=-\Delta v$. We assume that the fluid sticks to the boundary: $v=0$ along ∂D.

We now consider the formal limit of these equations obtained by dropping the inertia term and the dissipative term (i.e. setting $\varepsilon=\alpha=0$) in the NSB equations,

$$\nabla p=y,\ \nabla\cdot v=0,\ v\parallel\partial D, \tag{1.3}$$

$$\partial_t y+(v\cdot\nabla)y=G(x,y). \tag{1.4}$$

We are going to show that these equations can be justified under a strong uniform convexity assumption on the pressure field p. The situation of interest in this paper is the case when $d=2$ and the source term

$$G(x,y)=(x_2,y_1-x_1). \tag{1.5}$$

Then (1.3),(1.4) are the semigeostrophic Eady model equations in the special incompressible "$x-z$" situation. By $x-z$, we mean that D is part of a vertical section, the second coordinate x_2 of each point $x=(x_1,x_2)\in D$ being the vertical one. The source term in (1.5) represents the effect of the missing third dimension. In this identification, y represents the effects of rotation and stratification, and the relation $\nabla p=y$ in (1.3) expresses geostrophic and hydrostatic balance.

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The semigeostrophic model was considered by Hoskins [11] to model front formation in atmospheric sciences. The Eady model is defined in chapter 6 of [10], and models a quasi-periodic evolution in which fronts form and decay. There has been a lot of interest in these equations (see for instance [9, 1, 7, 6, 10]), due to their beautiful geometric structure and their deep links with the Monge-Ampère equation and optimal transport theory [5, 2, 3, 13]. The rigorous derivation of the full 3 dimensional SG equations is still a challenging problem. The present short note is just the first step toward this goal.

2. Motivation for a convexity assumption In their study of the SG equations, Cullen and Purser have introduced a convexity assumption on the pressure field p, based on a combination of physical and mathematical arguments. Convexity is also natural in the case of the general equations (1.3)–(1.4), independently of the choice of the source term G, for the following reasons. At first glance, these equations look strange since there is no evolution equation for v. However, y is constrained to be a gradient. Therefore, v can be seen as a kind of Lagrange multiplier for this constraint. (Vaguely speaking, due to the presence of a source term, in order to stay a gradient, the field y needs to be continuously rearranged in a volume-preserving fashion under the action of a time-dependent divergence-free vector field v.) As a matter of fact, it is (formally) very easy to get an equation for v, once $y=\nabla p$ is known. To do that, let us start with the 2 dimensional case and write

$$y(t,x)=(\partial_1 p,\partial_2 p)(t,x_1,x_2),\ \ v(t,x)=(-\partial_2\psi,\partial_1\psi)(t,x_1,x_2)$$

(at least locally), where ψ is a "stream-function". Then, let us "curl" equation (1.4) and obtain

$$-\partial_{11}^2 p\,\partial_{22}^2\psi+2\partial_{12}^2 p\,\partial_{12}^2\psi-\partial_{22}^2 p\,\partial_{11}^2\psi=\partial_1(G_2(x,\nabla p))-\partial_2(G_1(x,\nabla p)). \tag{2.1}$$

This is a linear second order elliptic equation in ψ, whenever p is a given strictly uniformly convex (or concave) function of x, i.e., when $D_x^2 p>0$, in the sense of symmetric matrices, (or <0). In three space dimensions, we get some "magnetostatic" version of equation (2.1). Indeed, since v is divergence-free, we can (at least locally) write $v=\nabla\times A$ for some "potential vector" $A=A(t,x)\in R^3$, that we may assume to be itself divergence-free. Then, by curling equation (1.4), we get a linear system for A when p is convex, namely:

$$\nabla\times(M(t,x)\nabla\times A)=\nabla\times(G(x,\nabla p)). \tag{2.2}$$

This system is elliptic whenever the symmetric matrix $M=D_x^2 p(t,x)$ is uniformly positive and bounded, which means that p is convex in a strong sense. In higher dimensions, v should be viewed as a $d-1$ form and p as a zero form. The divergence free condition (locally) means that $v=dA$, where A is a $d-2$ form. Then, again taking the curl of equation (1.4), we get the multidimensional generalization of system (2.1): $d(M(t,x)*dA)=d(G(x,dp)$ (where $*$ denotes Hodge duality and $M=D^2p$) which, again, is an elliptic system in A when $D_x^2 p$ is uniformly bounded and positive.

Thus we see that requiring p to be convex is a natural solvability condition for equations (1.3)–(1.4).

3. Rigorous derivation from the Navier-Stokes equations

The generalized Cullen-Purser convexity condition plays a crucial role in the rigorous derivation of equations (1.3)–(1.4) from the NSB equations.

THEOREM 3.1. *Let D be a smooth bounded convex domain. Assume G to be smooth with bounded derivatives up to second order. Let $(y^\varepsilon, v^\varepsilon, p^\varepsilon)$ be a Leray-type solution to the NSB equations (1.1),(1.2), where $K=-\Delta$, with $\alpha=O(\varepsilon)$. Let $(y=\nabla p, v)$ be a smooth solution to equations (1.3),(1.4) on a given finite time interval $[0,T]$. We assume $p(t,x)$ to have a smooth convex extension for all $x\in R^d$ so that its Legendre transform*

$$p^*(t,y)=\sup_{x\in R^d} x\cdot y - p(t,x) \tag{3.1}$$

is also smooth for $y\in R^d$ with Hessian $D_y^2 p^*(t,y)$ bounded away from zero and $+\infty$.

Then, the L^2 distance between y^ε and y stays uniformly of order $\sqrt{\varepsilon}$ as ε goes to zero, uniformly in $t\in[0,T]$, provided it does at $t=0$ and the initial velocity $v^\varepsilon(t=0,x)$ stays uniformly bounded in L^2.

Notice that the theorem is meaningful, since the local existence of smooth solutions has been proven by Loeper [12] (in the SG case) at least for periodic boundary conditions, provided that that $y(0,x)-x$ is not too large in some appropriate sense.

Proof. For the convergence, we use a relative entropy trick quite similar to the one used by the author for the hydrostatic limit of the 2D Euler equations in a thin domain [4]. We introduce the so-called Bregman function (or relative entropy) attached to p^*

$$\eta_{p^*}(t,z,z')=p^*(t,z')-p^*(t,z)-(\nabla p^*)(t,z)\cdot(z'-z)\sim|z'-z|^2 \tag{3.2}$$

and the related functional

$$e(t)=\int_D(\epsilon\frac{|v^\varepsilon(t,x)-v(t,x)|^2}{2}+\eta_{p^*}(t,y(t,x),y^\varepsilon(t,x)))\,dx. \tag{3.3}$$

Given a weak solution $(y^\varepsilon, v^\varepsilon)$ to the NSB equations (1.1),(1.2) and a solution y of (1.3),(1.4), we want to get an estimate of the form:

$$\frac{d}{dt}(e(t)+O(\epsilon))\leq(e(t)+O(\epsilon))c, \tag{3.4}$$

where c depends only on the limit solution (y,v) on a fixed finite time interval $[0,T]$ on which (y,v) is smooth. From this estimate (3.4), we immediately get that $y-y^\varepsilon$ is of order $O(\sqrt{\epsilon})$ in $L^\infty([0,T],L^2(D))$. So, we are left with proving (3.4). To save time, we do calculations just as if the Leray solutions were smooth solutions. Let us compute

$$I=I_1+I_2+I_3+I_4,$$

$$I_1=\frac{d}{dt}\int_D p^*(t,y^\varepsilon(t,x))\,dx,$$

$$I_2=-\frac{d}{dt}\int_D p^*(t,y(t,x))\,dx,$$

$$I_3=-\frac{d}{dt}\int_D(\nabla p^*(t,y(t,x)))\cdot y^\varepsilon(t,x)\,dx,$$

$$I_4 = \frac{d}{dt}\int_D (\nabla p^*(t,y(t,x)))\cdot y(t,x)\,dx.$$

We first obtain

$$I_1 = \int_D [(\partial_t p^*)(t,y^\varepsilon(t,x)) + (\nabla p^*)(t,y^\varepsilon(t,x))\cdot G(x,y^\varepsilon(t,x))]\,dx,$$

(using that v^ε is divergence free and parallel to ∂D). Similarly,

$$I_2 = -\int_D [(\partial_t p^*)(t,y(t,x)) + (\nabla p^*)(t,y(t,x))\cdot G(x,y(t,x))]\,dx.$$

Next,

$$I_3 = -\int_D [(\partial_t \nabla p^*)(t,y(t,x))\cdot y^\varepsilon(t,x) + (D_y^2 p^*)(t,y(t,x))(\partial_t y(t,x),y^\varepsilon(t,x))]\,dx$$

$$-\int_D (\nabla p^*)(t,y(t,x))\cdot G(x,y^\varepsilon(t,x))\,dx + I_5,$$

where

$$I_5 = \int_D x\cdot(v^\varepsilon(t,x)\cdot\nabla)y^\varepsilon(t,x)\,dx$$

$$= -\int_D v^\varepsilon(t,x)\cdot y^\varepsilon(t,x)\,dx,$$

(where we have used, for the two last lines, that $(\nabla p^*)(t,y(t,x)) = x$, which follows from Legendre duality). Since v^ε solves the NSB equations, we find

$$I_5 = -\int_D [\varepsilon(\partial_t + v^\varepsilon\cdot\nabla)v^\varepsilon + \nabla p^\varepsilon + \alpha K v^\varepsilon]\cdot v^\varepsilon\,dx$$

$$= -\frac{\varepsilon d}{2dt}\int_D |v^\varepsilon|^2\,dx - \int v^\varepsilon\cdot\alpha K v^\varepsilon\,dx.$$

Similarly

$$I_4 = \int_D [(\partial_t \nabla p^*)(t,y(t,x))\cdot y(t,x) + (D_y^2 p^*)(t,y(t,x))(\partial_t y(t,x),y(t,x))]\,dx +$$

$$+\int_D \nabla p^*(t,y(t,x))\cdot G(x,y(t,x))\,dx.$$

Collecting all terms, we obtain

$$I = I_5 + I_6 + I_7 + I_8 + I_9,$$

where

$$I_6 = \int_D \eta_{\partial_t p^*}(t,y(t,x),y^\varepsilon(t,x))\,dx,$$

(which involves the Bregman functional associated with $\partial_t p^*$ and therefore is bounded by $e(t)c$ where c is a constant depending only on the limit solutions $y=\nabla p$),

$$I_7 = -\int_D [(\nabla p^*)(t,y) - (\nabla p^*)(t,y^\varepsilon)] \cdot G(x,y^\varepsilon)\, dx,$$

$$I_8 = \int (D_y^2 p^*)(t,y)(G(x,y), y-y^\varepsilon)\, dx,$$

$$I_9 = \int (D_y^2 p^*)(t,y)(\partial_t y - G(x,y), y-y^\varepsilon)\, dx,$$

$$= \int (D_y^2 p^*)(t,y)((v\cdot\nabla)y, y^\varepsilon - y)\, dx.$$

We easily see that

$$I_7 + I_8 = \int_D \eta_{\nabla p^*}(t, y(t,x), y^\varepsilon(t,x)) \cdot G(x,y)\, dx$$

$$+ \int_D [(\nabla p^*)(t,y) - (\nabla p^*)(t,y^\varepsilon)] \cdot (G(x,y) - G(x,y^\varepsilon))\, dx,$$

(which is again bounded by $e(t)c$ where c is a constant depending only on the limit solutions $y=\nabla p$). Let us finally consider the most delicate term I_9. We can write I_9 in index notation as

$$I_9 = \int \sum_{ijk} \partial_{ij}^2 p^*(t,y) v_k \partial_k y_i (y^\varepsilon - y)_j,$$

$$= \int \sum_{ijk} \delta_{jk} v_k (y^\varepsilon - y)_j = \int v\cdot(y^\varepsilon - y),$$

(indeed, p^* is the Legendre transform of p and $y=\nabla p$, thus $D^2p^*(y)Dy = D^2p^*(\nabla p)D^2p = Id$)

$$= \int v\cdot y^\varepsilon,$$

(since y is a gradient and v is divergence free and parallel to ∂D)

$$= \int v\cdot(\varepsilon(\partial_t + v^\varepsilon\cdot\nabla)v^\varepsilon + \alpha K v^\varepsilon),$$

(using the NSB equations)

$$= J_1 + J_2,$$

where

$$J_1 = \frac{d}{dt}\varepsilon \int v^\varepsilon \cdot v,$$

and

$$|J_2| \leq \varepsilon(\int |v^\varepsilon|^2 + 1)c \leq \varepsilon(\int |v^\varepsilon - v|^2 + 1)c \leq (e(t) + \varepsilon)c,$$

where c are constants only depending on the limit solution v. Thus, again collecting all terms, and using that

$$I_5 = -\frac{\varepsilon d}{2dt} \int_D |v^\varepsilon|^2 \, dx - \alpha \int v^\varepsilon \cdot K v^\varepsilon \, dx,$$

we have obtained

$$I + \frac{d}{dt}(\frac{\varepsilon}{2} \int |v^\varepsilon - v|^2 + O(\varepsilon)) + \alpha \int v^\varepsilon \cdot K v^\varepsilon \, dx \leq (e(t) + O(\varepsilon))c,$$

which leads to the desired inequality (3.4) and completes the proof.

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