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THREE-FACTOR INTEREST RATE MODELS*

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Abstract. A three-factor interest rate model defined on a finite domain has been provided. All the functions in the model can be obtained from the real markets. It has been proven that a final-value problem of the corresponding partial differential equation on a finite domain has a unique solution. Because the formulation of the problem is on a finite domain and correct, it is not difficult to design efficient numerical methods for the problem. Therefore interest rate derivatives can be evaluated without any difficulty and the results can readily be used in practice.

1. Introduction

There are a number of papers devoted to interest rate models. Many of them discuss one-factor models (see Vasicek (1977), Cox, Ingersoll & Ross (1985), Ho & Lee (1986), Hull & White (1990), Black, Derman & Toy (1990) and Black & Karasinski (1991)). Since the information on interests is given by a random curve, for example, by a zero-coupon bond curve, even the best one-factor model is hard to describe the major features of the curve. However, it has been pointed out that the main feature of a random curve related to interest rates can be described by three or four random variables (see Jarrow (1996), Frye (1997), Hull (2000), James & Webber (2000) and Wilmott (2000)). Therefore it is possible that a multi-factor model can describe the major random features of such a curve. There exist several multi-factor interest rate models (see Brennan & Schwartz (1982), Heath, Jarrow & Morton (1992) and James & Webber (2000)). However, there are still some problems when those models are used in practice. In this paper a three-factor interest rate model has been suggested. In this model, the price of an interest rate derivative is evaluated by solving a final-value problem of a degenerate parabolic partial differential equation on a finite domain. For such a problem there exist efficient numerical methods (see Zhu & Li (2003)). Hence the model can be used to price interest rate derivatives quite quickly.

Using the same idea, n-factor models with $n > 3$ can be established. The reason why we discuss three-factor models is that usually three-factor models might be good enough and three-dimensional parabolic problem can still be solved quite fast on today's computer, so that the model can be readily adopted in practice.

The rest of the paper is organized as follows. In Sections 1 and 2, we discuss how to reduce a random zero-coupon bond curve to three or four random variables with a small error. In Section 3 the partial differential equation for interest rate derivatives is derived. In Sections 4 and 5, the major results are given. First the uniqueness theorem for final-value problems of degenerate parabolic partial differential equations on finite domains is proven. Then based on the theorem, we show that when our model is used, the price of interest rate derivatives can be evaluated by solving a final-value problem of degenerate parabolic partial differential equations on finite domains. Since the formulation of the problem is correct, it is not difficult to design efficient numerical methods to compute solutions for this problem.

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2. Approximation to zero-coupon bond curves

Let $Z(t,t+T)$ denote the T-year zero-coupon bond price at time t and we use the notation $Z_i(t) = Z(t,t+T_i)$ for T_i, $i = 0,1,\cdots,N$. Here we also assume $T_i < T_{i+1}$, $i = 0,1,\cdots,N-1$, $T_0 = 0$ and $Z_0(t) = 1$, which means that the face value of bonds is one. According to $Z_i(t), i = 0,1,\cdots,N$, we can have an interpolation function $\bar{Z}(t,t+T)$ for $T \in [0,T_N]$ by requiring $\bar{Z}(t,t+T)$ to be a continuous function with continuous first and second derivatives in the form:

$$\bar{Z}(t,t+T) = \begin{cases} a_{0,1} + a_{1,1}T + a_{2,1}T^2, & 0 \le T \le T_1, \\ a_{0,i} + a_{1,i}T + a_{2,i}T^2 + a_{3,i}T^3, & T_{i-1} \le T \le T_i, \\ & i = 2,\cdots,N-1, \\ a_{0,N} + a_{1,N}T + a_{2,N}T^2, & T_{N-1} \le T \le T_N. \end{cases} \tag{2.1}$$

In this function, there are $4(N-2)+6 = 4N-2$ coefficients. Since we have $N+1$ conditions on the value of the function

$$\bar{Z}(t,t+T_i) = Z_i(t), \quad i = 0,1,\cdots,N$$

and $3(N-1)$ continuity conditions on the function, first and second derivatives at $T_1, T_2, \cdots, T_{N-1}$, the total number of conditions is also $4N-2$. Therefore, those coefficients in (2.1) can be determined by these conditions uniquely. This method is called a cubic spline interpolation. A zero-coupon bond curve is a monotone function with respect to T. If the interpolation approximation (2.1) for a set of $Z_i(t)$, $i = 0,1,\cdots,N$ does not possess this property, the approximation needs to be modified so that the monotone is guaranteed.

We assume that $\bar{Z}(t,t+T)$ is a very good approximation to the zero-coupon bond curve $Z(t,t+T)$, i.e., from $Z_i(t)$, $i = 0,1,\cdots,N$, we can determine a very good approximation to the zero-coupon bond curve. In this way, a random curve is reduced to N random variables with a small error.

3. Reducing the number of random variables

In the last section a random curve has been reduced to N random variables. In this section the number of random variables will be reduced to K from N by the principal component analysis.

Suppose that we have N random variables

$$S_i(t), \quad i = 1,2,\cdots,N$$

and the covariance between S_i and S_j is

$$\mathrm{Cov}[S_i S_j] = b_i b_j \rho_{i,j}, \quad i,j = 1,2,\cdots,N,$$

where $-1 \le \rho_{i,j} = \rho_{j,i} \le 1$ and $\rho_{i,i} = 1$. Let

$$c_i^2 \quad \text{and} \quad \mathbf{a}_i = \begin{pmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,N} \end{pmatrix}, \quad i = 1,2,\cdots,N,$$

be the eigenvalues and unit eigenvectors of the covariance matrix

$$\mathbf{B} = \begin{pmatrix} b_1^2 & b_1 b_2 \rho_{1,2} & \cdots & b_1 b_N \rho_{1,N} \\ b_2 b_1 \rho_{2,1} & b_2^2 & \cdots & b_2 b_N \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_N b_1 \rho_{N,1} & b_N b_2 \rho_{N,2} & \cdots & b_N^2 \end{pmatrix}.$$

That is, there is the following relation:

$$\mathbf{B}\mathbf{A}^T = \mathbf{A}^T\mathbf{C},$$

where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$ and

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c_1^2 & 0 & \cdots & 0 \\ 0 & c_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_N^2 \end{pmatrix}.$$

Let $\bar{S}_1, \bar{S}_2, \cdots, \bar{S}_N$ be N other random variables defined by

$$\begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_N \end{pmatrix} = \mathbf{A} \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{pmatrix}.$$

For simplicity, this relation can be written as

$$\bar{\mathbf{S}} = \mathbf{A}\mathbf{S},$$

where

$$\bar{\mathbf{S}} = \begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_N \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{pmatrix}.$$

Then

$$\begin{aligned} \text{Cov}\ \left[\bar{S}_i \bar{S}_j\right] &= \text{E}\left[\left(\bar{S}_i - \text{E}\left[\bar{S}_i\right]\right)\left(\bar{S}_j - \text{E}\left[\bar{S}_j\right]\right)\right] \\ &= \text{E}\left[\left(\sum_{k=1}^{N} a_{ik}\left(S_k - \text{E}\left[S_k\right]\right)\right)\left(\sum_{l=1}^{N} a_{jl}\left(S_l - \text{E}\left[S_l\right]\right)\right)\right] \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} a_{ik}a_{jl}\ \text{Cov}\left[S_k S_l\right] \\ &= \begin{cases} 0, & i \neq j, \\ c_i^2, & i = j. \end{cases} \end{aligned}$$

That is, $\mathbf{C}$ is the covariance matrix of the random vector $\bar{\mathbf{S}}$. We furthermore suppose that $c_i \geq c_j$ for $i < j$ and $c_i \ll c_K, i = K+1, \cdots, N$. Assume that on some day

$$\mathbf{S} = \begin{pmatrix} S_1^* \\ S_2^* \\ \vdots \\ S_N^* \end{pmatrix}$$

and

$$\bar{\mathbf{S}} = \begin{pmatrix} \bar{S}_1^* \\ \bar{S}_2^* \\ \vdots \\ \bar{S}_N^* \end{pmatrix} = \mathbf{A} \begin{pmatrix} S_1^* \\ S_2^* \\ \vdots \\ S_N^* \end{pmatrix}.$$

Since $c_i, i = K+1, \cdots, N$ are very small, for a period starting from that day, we neglect the uncertainty caused by the last $N-K$ components of $\bar{\mathbf{S}}$. That is, we assume in that period $\bar{\mathbf{S}}$ has the following form:

$$\bar{\mathbf{S}} = \begin{pmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \\ \bar{S}_{K+1}^* \\ \vdots \\ \bar{S}_N^* \end{pmatrix},$$

where $\bar{S}_1, \cdots, \bar{S}_K$ can take all possible values. In this case

$$\mathbf{S} = \mathbf{A}^T \begin{pmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \\ \bar{S}_{K+1}^* \\ \vdots \\ \bar{S}_N^* \end{pmatrix}. \tag{3.1}$$

Under this assumption, among $S_1, S_2, \cdots, S_N$, only K components are independent. Suppose that

$$\begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{K,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{K,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,K} & a_{2,K} & \cdots & a_{K,K} \end{vmatrix} \neq 0.$$

Then we can choose $S_1, S_2, \cdots, S_K$ as independent components. Rewrite (3.1) as

$$\begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix} = \mathbf{A}_1^T \begin{pmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \end{pmatrix} + \mathbf{A}_2^T \begin{pmatrix} \bar{S}_{K+1}^* \\ \vdots \\ \bar{S}_N^* \end{pmatrix},$$

$$\begin{pmatrix} S_{K+1} \\ \vdots \\ S_N \end{pmatrix} = \mathbf{A}_3^T \begin{pmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \end{pmatrix} + \mathbf{A}_4^T \begin{pmatrix} \bar{S}_{K+1}^* \\ \vdots \\ \bar{S}_N^* \end{pmatrix},$$

where

$$\mathbf{A}_1^T = \begin{pmatrix} a_{1,1} & \cdots & a_{K,1} \\ \vdots & \ddots & \vdots \\ a_{1,K} & \cdots & a_{K,K} \end{pmatrix}, \quad \mathbf{A}_2^T = \begin{pmatrix} a_{K+1,1} & \cdots & a_{N,1} \\ \vdots & \ddots & \vdots \\ a_{K+1,K} & \cdots & a_{N,K} \end{pmatrix},$$

$$\mathbf{A}_3^T = \begin{pmatrix} a_{1,K+1} & \cdots & a_{K,K+1} \\ \vdots & \ddots & \vdots \\ a_{1,N} & \cdots & a_{K,N} \end{pmatrix}, \quad \mathbf{A}_4^T = \begin{pmatrix} a_{K+1,K+1} & \cdots & a_{N,K+1} \\ \vdots & \ddots & \vdots \\ a_{K+1,N} & \cdots & a_{N,N} \end{pmatrix}.$$

Then, for $S_{K+1}, \cdots, S_N$, we have

$$\begin{pmatrix} S_{K+1} \\ \vdots \\ S_N \end{pmatrix} = \mathbf{A}_3^T \left(\mathbf{A}_1^T\right)^{-1} \left[\begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix} - \mathbf{A}_2^T \begin{pmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{pmatrix}\right] + \mathbf{A}_4^T \begin{pmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{pmatrix}. \quad (3.2)$$

Therefore for given $S_1, \cdots, S_K$, using (3.2) we can get all other components of a vector $\mathbf{S}$. Consequently (3.2) defines a class of vectors with K parameters. Therefore by (3.2), we actually determine a class of $\mathbf{S}$, where only $S_1, \cdots, S_K$ are independent. Here we take $S_1, \cdots, S_K$ as independent components. However, it is also possible to choose other K components as independent components.

Letting $S_i = Z_i/T_i$, $i = 1, 2, \cdots, N$, by the principal component analysis described above we can find a class of vectors $(Z_1/T_1, \cdots, Z_N/T_N)^T$ with K parameters[1] and using the method given in the last section, we can further determine the curve $\bar{Z}(t, t+T)$ for $T \in [0, T_N]$. From the books by Jarrow (1996), Hull (2000), James & Webber (2000) and Wilmott (2000), we know that K usually is equal to three or four for the random curves related to interest rates. Thus all the curves determined by (3.2) form a class of curves with three or four parameters. The zero-coupon bond curve at that day is one of such curves and the projections of any vector $\mathbf{S}$ determined by (3.2) on the eigenvectors corresponding to the eigenvalues $c_{K+1}, \cdots, c_N$ are the same as those of $\mathbf{S}^*$. Those projections are different for different $\mathbf{S}^*$, so this is a feature belonging to $\mathbf{S}^*$. It is clear that the class of curves with such a feature needs to be considered most for derivative-pricing problems. Hence when $K = 3$ or 4, the class contains all possible and need-to-be-considered-most zero-coupon bond curves. As soon as we have a zero-coupon bond curve, we can determine various interest rates at t, including the spot interest rate at time t: $-\left.\dfrac{\partial \bar{Z}(t, t+T)}{\partial T}\right|_{T=0}$. In what follows, such a function is denoted by $r(Z_1, \cdots, Z_K, t)$, or simply, by r.

4. Three-factor interest rate model

Suppose that Z_1, Z_2, Z_3 are prices of zero-coupon bonds with maturities T_1, T_2, T_3 respectively. Assume $T_1 < T_2 < T_3$, which implies the relations $1 \ge Z_1 \ge Z_2 \ge Z_3$. Furthermore we assume $Z_1 \ge Z_{1,l}$, $Z_2 \ge Z_{2,l}$ and $Z_3 \ge Z_{3,l}$, where $Z_{1,l} \ge Z_{2,l} \ge Z_{3,l} \ge 0$. Z_1, Z_2, Z_3 are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i\left(Z_1, Z_2, Z_3, t\right) dt + \sigma_i\left(Z_1, Z_2, Z_3, t\right) dX_i, \quad i = 1, 2, 3$$

[1] If the conditions $Z_i \ge Z_{i+1}$, $i = 0, 1, \cdots, N-1$ are not satisfied, then some modification needs to be done in order to guarantee the monotonicity.

on the domain Ω: $\{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$. dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij} dt$ with $-1 \le \rho_{ij} \le 1$. The coefficients μ_i, σ_i and their first and second order derivatives are assumed to be bounded on the domain Ω.

Let $V(Z_1, Z_2, Z_3, t)$ be any function of Z_1, Z_2, Z_3, t. Consider the portfolio

$$\Pi = V - \sum_{i=1}^{3} \Delta_i Z_i.$$

We have

$$d\Pi = dV - \sum_{i=1}^{3} \Delta_i dZ_i.$$

According to Itô's lemma

$$dV = \frac{\partial V}{\partial t} dt + \sum_{i=1}^{3} \frac{\partial V}{\partial Z_i} dZ_i + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \sigma_i \sigma_j \rho_{i,j} dt,$$

we have

$$d\Pi = \frac{\partial V}{\partial t} dt + \sum_{i=1}^{3} \left(\frac{\partial V}{\partial Z_i} - \Delta_i \right) dZ_i + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \sigma_i \sigma_j \rho_{i,j} dt.$$

Choosing $\Delta_i = \dfrac{\partial V}{\partial Z_i}$, we obtain

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \sigma_i \sigma_j \rho_{i,j} dt.$$

In this case there is no risk. Any short time investment with no risk should have a return rate of the spot interest rate $r(Z_1, Z_2, Z_3, t)$. Therefore we also have

$$d\Pi = r\Pi dt = r \left(V - \sum_{i=1}^{3} \frac{\partial V}{\partial Z_i} Z_i \right) dt.$$

Consequently we finally arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2 V}{\partial Z_i \partial Z_j} \sigma_i \sigma_j \rho_{i,j} + r \sum_{i=1}^{3} \frac{\partial V}{\partial Z_i} Z_i - rV = 0.$$

Let

$$\mathbf{L}_{3z} = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial^2}{\partial Z_i \partial Z_j} \sigma_i \sigma_j \rho_{i,j} + r \sum_{i=1}^{3} Z_i \frac{\partial}{\partial Z_i} - r. \tag{4.1}$$

The equation above can be written as

$$\frac{\partial V}{\partial t} + \mathbf{L}_{3z} V = 0.$$

This is the equation any derivative $V(Z_1, Z_2, Z_3, t)$ should satisfy. For a derivative security, at the maturity date T, its price should be equal to the payoff $V_T(Z_1, Z_2, Z_3)$. Therefore any European interest rate derivatives under this model should be solutions of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_{3z} V = 0 \quad \text{ on } \Omega \times [0, T], \\ V(Z_1, Z_2, Z_3, T) = V_T(Z_1, Z_2, Z_3) \quad \text{ on } \Omega. \end{cases} \tag{4.2}$$

Introduce the following transformation:

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\ \xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}}. \end{cases} \tag{4.3}$$

Through this transformation, the domain Ω in the (Z_1, Z_2, Z_3)-space is transformed into the domain $\tilde{\Omega}$: $[0, 1] \times [0, 1] \times [0, 1]$ in the (ξ_1, ξ_2, ξ_3)-space. Since

$$\begin{aligned} &\frac{\partial \xi_1}{\partial Z_1} = \frac{1}{1 - Z_{1,l}}, \\ &\frac{\partial \xi_2}{\partial Z_1} = \frac{-\xi_2}{Z_1 - Z_{2,l}}, \quad \frac{\partial \xi_2}{\partial Z_2} = \frac{1}{Z_1 - Z_{2,l}}, \\ &\frac{\partial \xi_3}{\partial Z_2} = \frac{-\xi_3}{Z_2 - Z_{3,l}}, \quad \frac{\partial \xi_3}{\partial Z_3} = \frac{1}{Z_2 - Z_{3,l}}, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial V}{\partial Z_1} &= \frac{1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1} - \frac{\xi_2}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2}, \\ \frac{\partial V}{\partial Z_2} &= \frac{1}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2} - \frac{\xi_3}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3}, \\ \frac{\partial V}{\partial Z_3} &= \frac{1}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3}, \\ \frac{\partial^2 V}{\partial Z_1^2} &= \frac{1}{(1 - Z_{1,l})^2} \frac{\partial^2 V}{\partial \xi_1^2} - \frac{2\xi_2}{(1 - Z_{1,l})(Z_1 - Z_{2,l})} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} \\ &\quad + \frac{\xi_2^2}{(Z_1 - Z_{2,l})^2} \frac{\partial^2 V}{\partial \xi_2^2} + \frac{2\xi_2}{(Z_1 - Z_{2,l})^2} \frac{\partial V}{\partial \xi_2}, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 V}{\partial Z_2^2} &= \frac{1}{\left(Z_1 - Z_{2,l}\right)^2}\frac{\partial^2 V}{\partial \xi_2^2} - \frac{2\xi_3}{(Z_1 - Z_{2,l})(Z_2 - Z_{3,l})}\frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} \\
&\quad + \frac{\xi_3^2}{\left(Z_2 - Z_{3,l}\right)^2}\frac{\partial^2 V}{\partial \xi_3^2} + \frac{2\xi_3}{(Z_2 - Z_{3,l})^2}\frac{\partial V}{\partial \xi_3}, \\
\frac{\partial^2 V}{\partial Z_3^2} &= \frac{1}{\left(Z_2 - Z_{3,l}\right)^2}\frac{\partial^2 V}{\partial \xi_3^2}, \\
\frac{\partial^2 V}{\partial Z_1 \partial Z_2} &= \frac{-1}{(Z_1 - Z_{2,l})^2}\frac{\partial V}{\partial \xi_2} + \frac{1}{Z_1 - Z_{2,l}}\left(\frac{1}{1 - Z_{1,l}}\frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} - \frac{\xi_2}{Z_1 - Z_{2,l}}\frac{\partial^2 V}{\partial \xi_2^2}\right) \\
&\quad - \frac{\xi_3}{Z_2 - Z_{3,l}}\left(\frac{1}{1 - Z_{1,l}}\frac{\partial^2 V}{\partial \xi_1 \partial \xi_3} - \frac{\xi_2}{Z_1 - Z_{2,l}}\frac{\partial^2 V}{\partial \xi_2 \partial \xi_3}\right), \\
\frac{\partial^2 V}{\partial Z_1 \partial Z_3} &= \frac{1}{Z_2 - Z_{3,l}}\left(\frac{1}{1 - Z_{1,l}}\frac{\partial^2 V}{\partial \xi_1 \partial \xi_3} - \frac{\xi_2}{Z_1 - Z_{2,l}}\frac{\partial^2 V}{\partial \xi_2 \partial \xi_3}\right), \\
\frac{\partial^2 V}{\partial Z_2 \partial Z_3} &= \frac{-1}{(Z_2 - Z_{3,l})^2}\frac{\partial V}{\partial \xi_3} + \frac{1}{Z_2 - Z_{3,l}}\left(\frac{1}{Z_1 - Z_{2,l}}\frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} - \frac{\xi_3}{Z_2 - Z_{3,l}}\frac{\partial^2 V}{\partial \xi_3^2}\right).
\end{aligned}
$$

Therefore the operator $\mathbf{L}_{3Z}$ defined by (4.1) can be rewritten as

$$
\begin{aligned}
\mathbf{L}_{3\xi} = {} & \frac{1}{2}\tilde{\sigma}_1^2\frac{\partial^2}{\partial \xi_1^2} + \frac{1}{2}\tilde{\sigma}_2^2\frac{\partial^2}{\partial \xi_2^2} + \frac{1}{2}\tilde{\sigma}_3^2\frac{\partial^2}{\partial \xi_3^2} \\
& + \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\frac{\partial^2}{\partial \xi_1 \partial \xi_2} + \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\frac{\partial^2}{\partial \xi_1 \partial \xi_3} + \tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}\frac{\partial^2}{\partial \xi_2 \partial \xi_3} \\
& + b_1\frac{\partial}{\partial \xi_1} + b_2\frac{\partial}{\partial \xi_2} + b_3\frac{\partial}{\partial \xi_3} - r, \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{2}\tilde{\sigma}_1^2 &= \frac{\frac{1}{2}\sigma_1^2}{\left(1 - Z_{1,l}\right)^2}, \\
\frac{1}{2}\tilde{\sigma}_2^2 &= \frac{\frac{1}{2}\left(\sigma_1^2\xi_2^2 - 2\sigma_1\sigma_2\xi_2\rho_{1,2} + \sigma_2^2\right)}{\left(Z_1 - Z_{2,l}\right)^2}, \\
\frac{1}{2}\tilde{\sigma}_3^2 &= \frac{\frac{1}{2}\left(\sigma_2^2\xi_3^2 - 2\sigma_2\sigma_3\xi_3\rho_{2,3} + \sigma_3^2\right)}{\left(Z_2 - Z_{3,l}\right)^2}, \\
\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2} &= \frac{\sigma_1\left(\sigma_2\rho_{1,2} - \sigma_1\xi_2\right)}{\left(1 - Z_{1,l}\right)\left(Z_1 - Z_{2,l}\right)}, \\
\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3} &= \frac{\sigma_1\left(\sigma_3\rho_{1,3} - \sigma_2\xi_3\rho_{1,2}\right)}{\left(1 - Z_{1,l}\right)\left(Z_2 - Z_{3,l}\right)}, \\
\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3} &= \frac{-\sigma_2^2\xi_3 + \sigma_1\sigma_2\rho_{1,2}\xi_2\xi_3 - \sigma_1\sigma_3\rho_{1,3}\xi_2 + \sigma_2\sigma_3\rho_{2,3}}{\left(Z_1 - Z_{2,l}\right)\left(Z_2 - Z_{3,l}\right)}, \\
b_1 &= \frac{rZ_1}{1 - Z_{1,l}}, \\
b_2 &= \frac{r\left(Z_2 - Z_1\xi_2\right)}{Z_1 - Z_{2,l}} + \frac{\sigma_1\left(\sigma_1\xi_2 - \sigma_2\rho_{1,2}\right)}{\left(Z_1 - Z_{2,l}\right)^2}, \\
b_3 &= \frac{r\left(Z_3 - Z_2\xi_3\right)}{Z_2 - Z_{3,l}} + \frac{\sigma_2\left(\sigma_2\xi_3 - \sigma_3\rho_{2,3}\right)}{\left(Z_2 - Z_{3,l}\right)^2}.
\end{aligned}
$$

Consequently problem (4.2) can be rewritten as

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_{3\xi} V = 0 \quad \text{on } \tilde{\Omega} \times [0, T], \\ V(\xi_1, \xi_2, \xi_3, T) = \tilde{V}_T(\xi_1, \xi_2, \xi_3) \quad \text{on } \tilde{\Omega}, \end{cases} \tag{4.5}$$

where $\mathbf{L}_{3\xi}$ is defined by (4.4) and

$$\tilde{V}_T(\xi_1, \xi_2, \xi_3) = V_T(Z_{1,l} + \xi_1(1 - Z_{1,l}),\ Z_{2,l} + \xi_2(Z_1 - Z_{2,l}),\ Z_{3,l} + \xi_3(Z_2 - Z_{3,l})).$$

This is a final-value problem on a rectangular domain. It will be proven that this problem has a unique solution. In such a case this problem can be solved by numerical methods without big difficulties.

We would like to point out the relations among $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{12}$, $\tilde{\rho}_{13}$, $\tilde{\rho}_{23}$ and $d\xi_1$, $d\xi_2$, $d\xi_3$. Using Itô's lemma, from the definitions of ξ_1, ξ_2, ξ_3, we have

$$\begin{aligned} d\xi_1 &= \tilde{\mu}_1 dt + \tilde{\sigma}_1 dX_1, \\ d\xi_2 &= \tilde{\mu}_2 dt + \tilde{\sigma}_2 d\tilde{X}_2, \\ d\xi_3 &= \tilde{\mu}_3 dt + \tilde{\sigma}_3 d\tilde{X}_3, \end{aligned}$$

where $d\tilde{X}_2$ and $d\tilde{X}_3$ are two new Wiener processes. Therefore $\tilde{\sigma}_1^2$, $\tilde{\sigma}_2^2$ and $\tilde{\sigma}_3^2$ are $\text{Var}[d\xi_1]/dt$, $\text{Var}[d\xi_2]/dt$ and $\text{Var}[d\xi_3]/dt$ respectively. It can also be shown that

$$\text{Cov}[dX_1 d\tilde{X}_2]/dt = \tilde{\rho}_{12}, \quad \text{Cov}[dX_1 d\tilde{X}_3]/dt = \tilde{\rho}_{13}$$

and

$$\text{Cov}[d\tilde{X}_2 d\tilde{X}_3]/dt = \tilde{\rho}_{23}.$$

5. Uniqueness of solution

Consider the parabolic final-value problem on a finite domain Ω

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_{3S} V = 0, \quad \text{on } \Omega \times [0, T], \\ V(S_1, S_2, S_3, T) = V_T(S_1, S_2, S_3), \quad \text{on } \Omega, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} \mathbf{L}_{3S} = {} & \frac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial S_2^2} + \frac{1}{2}\sigma_3^2 \frac{\partial^2}{\partial S_3^2} \\ & + \sigma_1\sigma_2\rho_{1,2} \frac{\partial^2}{\partial S_1 \partial S_2} + \sigma_2\sigma_3\rho_{2,3} \frac{\partial^2}{\partial S_2 \partial S_3} + \sigma_1\sigma_3\rho_{1,3} \frac{\partial^2}{\partial S_1 \partial S_3} \\ & + \mu_1 \frac{\partial}{\partial S_1} + \mu_2 \frac{\partial}{\partial S_2} + \mu_3 \frac{\partial}{\partial S_3} - r, \\ & \sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{2,3}, \rho_{1,3}, \mu_1, \mu_2, \mu_3, r \text{ being given function of } S_1, S_2, S_3, t \end{aligned}$$

and the matrix

$$P = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix}$$

is a symmetric non-negative definite matrix. It is clear that the operators in problems (4.2) and (4.5) have such a form.

The operator $\mathbf{L}_{3S}$ can be rewritten as

$$\begin{aligned}\mathbf{L}_{3S} = &\frac{1}{2}\frac{\partial}{\partial S_1}\left(\sigma_1^2\frac{\partial}{\partial S_1}+\sigma_1\sigma_2\rho_{1,2}\frac{\partial}{\partial S_2}+\sigma_1\sigma_3\rho_{1,3}\frac{\partial}{\partial S_3}\right)\\ &+\frac{1}{2}\frac{\partial}{\partial S_2}\left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial}{\partial S_1}+\sigma_2^2\frac{\partial}{\partial S_2}+\sigma_2\sigma_3\rho_{2,3}\frac{\partial}{\partial S_3}\right)\\ &+\frac{1}{2}\frac{\partial}{\partial S_3}\left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial}{\partial S_1}+\sigma_3\sigma_2\rho_{3,2}\frac{\partial}{\partial S_2}+\sigma_3^2\frac{\partial}{\partial S_3}\right)\\ &+a_1\frac{\partial}{\partial S_1}+a_2\frac{\partial}{\partial S_2}+a_3\frac{\partial}{\partial S_3}-r,\end{aligned}$$

where

$$\begin{aligned}a_1 &= \mu_1-\frac{1}{2}\frac{\partial\sigma_1^2}{\partial S_1}-\frac{1}{2}\frac{\partial\left(\sigma_2\sigma_1\rho_{2,1}\right)}{\partial S_2}-\frac{1}{2}\frac{\partial\left(\sigma_3\sigma_1\rho_{3,1}\right)}{\partial S_3},\\ a_2 &= \mu_2-\frac{1}{2}\frac{\partial\left(\sigma_1\sigma_2\rho_{1,2}\right)}{\partial S_1}-\frac{1}{2}\frac{\partial\sigma_2^2}{\partial S_2}-\frac{1}{2}\frac{\partial\left(\sigma_3\sigma_2\rho_{3,2}\right)}{\partial S_3},\\ a_3 &= \mu_3-\frac{1}{2}\frac{\partial\left(\sigma_1\sigma_3\rho_{1,3}\right)}{\partial S_1}-\frac{1}{2}\frac{\partial\left(\sigma_2\sigma_3\rho_{2,3}\right)}{\partial S_2}-\frac{1}{2}\frac{\partial\sigma_3^2}{\partial S_3}\end{aligned}$$

and we define $\rho_{2,1}=\rho_{1,2}$, $\rho_{3,1}=\rho_{1,3}$ and $\rho_{3,2}=\rho_{2,3}$. In what follows, $\overline{\Omega}$ denotes the boundary of the domain Ω and $\mathbf{N}=(n_1,n_2,n_3)$ represents the outer unit normal vector of $\overline{\Omega}$.

For problem (5.1), we have the following theorem.

THEOREM 5.1. *Suppose that*

i) on $\overline{\Omega}$

$$\begin{cases} n_1\sigma_1+n_2\sigma_2\rho_{2,1}+n_3\sigma_3\rho_{3,1}=0,\\ n_1\sigma_1\rho_{1,2}+n_2\sigma_2+n_3\sigma_3\rho_{3,2}=0,\\ n_1\sigma_1\rho_{1,3}+n_2\sigma_2\rho_{2,3}+n_3\sigma_3=0;\end{cases} \tag{5.2}$$

ii) on $\overline{\Omega}$

$$n_1a_1+n_2a_2+n_3a_3\le 0; \tag{5.3}$$

iii)

$$\max_{\Omega}\left|\frac{\partial a_1}{\partial S_1}+\frac{\partial a_2}{\partial S_2}+\frac{\partial a_3}{\partial S_3}+2r\right|\le c_1. \tag{5.4}$$

In this case the solution of (5.1) is unique.

Proof. Define

$$\tau = T-t \quad\text{and}\quad U(\tau)=\iiint_{\Omega}V^2(S_1,S_2,S_3,T-\tau)d\Omega.$$

Then from the partial differential equation, we have

$$
\begin{aligned}
\frac{1}{2}\frac{dU}{d\tau} &= \frac{1}{2}\frac{d}{d\tau}\iiint_\Omega V^2 d\Omega = \iiint_\Omega V\frac{\partial V}{\partial \tau}d\Omega = \iiint_\Omega V\mathbf{L}V\,d\Omega \\
&= \frac{1}{2}\iiint_\Omega \left[\frac{\partial}{\partial S_1}\left(\sigma_1^2\frac{\partial V}{\partial S_1} + \sigma_1\sigma_2\rho_{1,2}\frac{\partial V}{\partial S_2} + \sigma_1\sigma_3\rho_{1,3}\frac{\partial V}{\partial S_3}\right)\right. \\
&\qquad + \frac{\partial}{\partial S_2}\left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial V}{\partial S_1} + \sigma_2^2\frac{\partial V}{\partial S_2} + \sigma_2\sigma_3\rho_{2,3}\frac{\partial V}{\partial S_3}\right) \\
&\qquad \left. + \frac{\partial}{\partial S_3}\left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial V}{\partial S_1} + \sigma_3\sigma_2\rho_{3,2}\frac{\partial V}{\partial S_2} + \sigma_3^2\frac{\partial V}{\partial S_3}\right)\right] V\,d\Omega \\
&\quad + \iiint_\Omega \left(a_1\frac{\partial V}{\partial S_1} + a_2\frac{\partial V}{\partial S_2} + a_3\frac{\partial V}{\partial S_3}\right) V\,d\Omega - \iiint_\Omega rV^2 d\Omega.
\end{aligned}
$$

Let us look at the first two integrals on the right hand side of the equation above. For the first integral, we have

$$
\begin{aligned}
&\frac{1}{2}\iiint_\Omega \left[\frac{\partial}{\partial S_1}\left(\sigma_1^2\frac{\partial V}{\partial S_1} + \sigma_1\sigma_2\rho_{1,2}\frac{\partial V}{\partial S_2} + \sigma_1\sigma_3\rho_{1,3}\frac{\partial V}{\partial S_3}\right)\right. \\
&\qquad + \frac{\partial}{\partial S_2}\left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial V}{\partial S_1} + \sigma_2^2\frac{\partial V}{\partial S_2} + \sigma_2\sigma_3\rho_{2,3}\frac{\partial V}{\partial S_3}\right) \\
&\qquad \left. + \frac{\partial}{\partial S_3}\left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial V}{\partial S_1} + \sigma_3\sigma_2\rho_{3,2}\frac{\partial V}{\partial S_2} + \sigma_3^2\frac{\partial V}{\partial S_3}\right)\right] V\,d\Omega \\
&= \frac{1}{2}\iiint_\Omega \left\{\frac{\partial}{\partial S_1}\left[\left(\sigma_1^2\frac{\partial V}{\partial S_1} + \sigma_1\sigma_2\rho_{1,2}\frac{\partial V}{\partial S_2} + \sigma_1\sigma_3\rho_{1,3}\frac{\partial V}{\partial S_3}\right)V\right]\right. \\
&\qquad + \frac{\partial}{\partial S_2}\left[\left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial V}{\partial S_1} + \sigma_2^2\frac{\partial V}{\partial S_2} + \sigma_2\sigma_3\rho_{2,3}\frac{\partial V}{\partial S_3}\right)V\right] \\
&\qquad \left. + \frac{\partial}{\partial S_3}\left[\left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial V}{\partial S_1} + \sigma_3\sigma_2\rho_{3,2}\frac{\partial V}{\partial S_2} + \sigma_3^2\frac{\partial V}{\partial S_3}\right)V\right]\right\} d\Omega \\
&\quad -\frac{1}{2}\iiint_\Omega \left\{\left(\sigma_1^2\frac{\partial V}{\partial S_1} + \sigma_1\sigma_2\rho_{1,2}\frac{\partial V}{\partial S_2} + \sigma_1\sigma_3\rho_{1,3}\frac{\partial V}{\partial S_3}\right)\frac{\partial V}{\partial S_1}\right. \\
&\qquad + \left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial V}{\partial S_1} + \sigma_2^2\frac{\partial V}{\partial S_2} + \sigma_2\sigma_3\rho_{2,3}\frac{\partial V}{\partial S_3}\right)\frac{\partial V}{\partial S_2} \\
&\qquad \left. + \left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial V}{\partial S_1} + \sigma_3\sigma_2\rho_{3,2}\frac{\partial V}{\partial S_2} + \sigma_3^2\frac{\partial V}{\partial S_3}\right)\frac{\partial V}{\partial S_3}\right\} d\Omega \\
&\le \frac{1}{2}\iint_{\overline{\Omega}} \left\{n_1\left(\sigma_1^2\frac{\partial V}{\partial S_1} + \sigma_1\sigma_2\rho_{1,2}\frac{\partial V}{\partial S_2} + \sigma_1\sigma_3\rho_{1,3}\frac{\partial V}{\partial S_3}\right)V\right. \\
&\qquad + n_2\left(\sigma_2\sigma_1\rho_{2,1}\frac{\partial V}{\partial S_1} + \sigma_2^2\frac{\partial V}{\partial S_2} + \sigma_2\sigma_3\rho_{2,3}\frac{\partial V}{\partial S_3}\right)V \\
&\qquad \left. + n_3\left(\sigma_3\sigma_1\rho_{3,1}\frac{\partial V}{\partial S_1} + \sigma_3\sigma_2\rho_{3,2}\frac{\partial V}{\partial S_2} + \sigma_3^2\frac{\partial V}{\partial S_3}\right)V\right\} d\overline{\Omega}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\iint_{\overline{\Omega}} \left[\sigma_1\left(n_1\sigma_1 + n_2\sigma_2\rho_{2,1} + n_3\sigma_3\rho_{3,1}\right)\frac{\partial V^2}{\partial S_1}\right.\\
&\qquad +\sigma_2\left(n_1\sigma_1\rho_{1,2} + n_2\sigma_2 + n_3\sigma_3\rho_{3,2}\right)\frac{\partial V^2}{\partial S_2}\\
&\qquad \left. +\sigma_3\left(n_1\sigma_1\rho_{1,3} + n_2\sigma_2\rho_{2,3} + n_3\sigma_3\right)\frac{\partial V^2}{\partial S_3}\right] d\overline{\Omega}\\
&= 0.
\end{aligned}
$$

Here we have used Gauss's divergence theorem, condition (5.2) and the fact that P is a nonnegative definite matrix.

For the second integral, the following is true:

$$
\begin{aligned}
&\iiint_{\Omega}\left(a_1\frac{\partial V}{\partial S_1} + a_2\frac{\partial V}{\partial S_2} + a_3\frac{\partial V}{\partial S_3}\right) V d\Omega\\
&= \frac{1}{2}\iiint_{\Omega}\left(\frac{\partial(a_1V^2)}{\partial S_1} + \frac{\partial(a_2V^2)}{\partial S_2} + \frac{\partial(a_3V^2)}{\partial S_3}\right) d\Omega\\
&\quad -\frac{1}{2}\iiint_{\Omega}\left(\frac{\partial a_1}{\partial S_1} + \frac{\partial a_2}{\partial S_2} + \frac{\partial a_3}{\partial S_3}\right) V^2 d\Omega\\
&= \frac{1}{2}\iint_{\overline{\Omega}}\left(n_1a_1 + n_2a_2 + n_3a_3\right) V^2 d\overline{\Omega}\\
&\quad -\frac{1}{2}\iiint_{\Omega}\left(\frac{\partial a_1}{\partial S_1} + \frac{\partial a_2}{\partial S_2} + \frac{\partial a_3}{\partial S_3}\right) V^2 d\Omega\\
&\le -\frac{1}{2}\iiint_{\Omega}\left(\frac{\partial a_1}{\partial S_1} + \frac{\partial a_2}{\partial S_2} + \frac{\partial a_3}{\partial S_3}\right) V^2 d\Omega.
\end{aligned}
$$

Here we have used Gauss's divergence theorem and condition (5.3). Consequently, noticing condition (5.4), we have

$$
\begin{aligned}
\frac{1}{2}\frac{dU}{d\tau} &\le -\frac{1}{2}\iiint_{\Omega}\left(\frac{\partial a_1}{\partial S_1} + \frac{\partial a_2}{\partial S_2} + \frac{\partial a_3}{\partial S_3} + 2r\right) V^2 d\Omega\\
&\le \frac{1}{2}c_1 U.
\end{aligned}
$$

From this relation we further have

$$U(\tau) \le U(0)e^{c_1\tau}.$$

Therefore if $U(0) = 0$, then $U(\tau) = 0$ for any $\tau \in [0, T]$, which means that problem (5.1) has a unique solution. □

Here we give a remark. When (5.2) holds on $\overline{\Omega}$, we say that the parabolic partial differential equation is degenerate on $\overline{\Omega}$. For a parabolic partial differential equation to degenerate on $\overline{\Omega}$, whether or not a boundary condition is needed depends upon the value of $n_1a_1 + n_2a_2 + n_3a_3$. When condition (5.3) holds, no boundary condition is needed. When a piece of boundary is a part of a plane with $\mathbf{N} = (-1, 0, 0)$, then conditions (5.2) and (5.3) become $\sigma_1 = 0$ and $a_1 \ge 0$. Similarly if $\mathbf{N} = (1, 0, 0)$, then $\sigma_1 = 0$ and $a_1 \le 0$. From the paper by Zhu and Li (2003), we know that these are the reversion conditions on the left and right boundaries respectively if the domain is rectangular. Therefore conditions (5.2) and (5.3) are called the reversion condition on a general domain. If on a piece of the boundary $n_1a_1 + n_2a_2 + n_3a_3 > 0$, then on that piece a boundary condition might be required in order to have a unique solution.

6. Application of the uniqueness theorem to three-factor models

Using Theorem 5.1, we can prove that under certain conditions the solutions of both problem (4.2) and problem (4.5) are unique. The conclusion that problem (4.2) has a unique solution is equivalent to the conclusion that the solution of problem (4.5) is unique. Thus we need to prove the uniqueness only for one case. Here we choose to prove the result for problem (4.5).

Theorem 6.1. *Suppose all the coefficients and their first and second order derivatives in problem (4.5) are bounded on the boundary of the domain $\tilde{\Omega}$, the following conditions are satisfied:*

i) on surface I: $\{\xi_1 = 0, \quad 0 \le \xi_2 \le 1, \quad 0 \le \xi_3 \le 1\}$,

$$\sigma_1 = 0 \ \ (\textit{or equivalently } \tilde{\sigma}_1 = 0) \ \ \textit{and} \ \ rZ_1 \ge 0; \tag{6.1}$$

ii) on surface II: $\{\xi_1 = 1, \quad 0 \le \xi_2 \le 1, \quad 0 \le \xi_3 \le 1\}$,

$$\sigma_1 = 0 \ \ (\textit{or equivalently } \tilde{\sigma}_1 = 0) \ \ \textit{and} \ \ rZ_1 \le 0; \tag{6.2}$$

iii) on surface III: $\{0 \le \xi_1 \le 1, \quad \xi_2 = 0, \quad 0 \le \xi_3 \le 1\}$,

$$\sigma_2 = 0 \ \ (\textit{or equivalently } \tilde{\sigma}_2 = 0) \ \ \textit{and} \ \ rZ_2 \ge 0; \tag{6.3}$$

iv) on surface IV: $\{0 \le \xi_1 \le 1, \quad \xi_2 = 1, \quad 0 \le \xi_3 \le 1\}$,

$$\sigma_1 = \sigma_2, \quad \rho_{1,2} = 1 \ \textit{and} \ -r(Z_1 - Z_2) \le 0; \tag{6.4}$$

v) on surface V: $\{0 \le \xi_1 \le 1, \quad 0 \le \xi_2 \le 1, \quad \xi_3 = 0\}$,

$$\sigma_3 = 0 \ \ (\textit{or equivalently } \tilde{\sigma}_3 = 0) \ \ \textit{and} \ \ rZ_3 \ge 0; \tag{6.5}$$

vi) on surface VI: $\{0 \le \xi_1 \le 1, \quad 0 \le \xi_2 \le 1, \quad \xi_3 = 1\}$,

$$\sigma_2 = \sigma_3, \quad \rho_{2,3} = 1 \ \textit{and} \ -r(Z_2 - Z_3) \le 0. \tag{6.6}$$

Then problem (4.5) has a unique solution.

Before proving this theorem, we give three remarks:

- The first two parts of condition (6.4) are equivalent to

$$\tilde{\sigma}_2\big|_{\xi_2=1} = \frac{\sqrt{\sigma_1^2 - 2\sigma_1\sigma_2\rho_{1,2} + \sigma_2^2}}{Z_1 - Z_{2,l}} = 0. \tag{6.7}$$

It is clear that from (6.4) we can have (6.7). Now suppose (6.7) holds, i.e.,

$$\sigma_1^2 - 2\sigma_1\sigma_2\rho_{1,2} + \sigma_2^2 = 0$$

or

$$\sigma_1^2 + \sigma_2^2 = 2\sigma_1\sigma_2\rho_{1,2}.$$

Since

$$\sigma_1^2 + \sigma_2^2 \ge 2\sigma_1\sigma_2$$

and $-1 \le \rho_{1,2} \le 1$, $\sigma_1 \ge 0$, $\sigma_2 \ge 0$, in order for the equality above to be true, there must be

$$\sigma_1 = \sigma_2 \text{ and } \rho_{1,2} = 1.$$

Here when $\sigma_1 = \sigma_2 = 0$, we define $\rho_{1,2} = 1$. Thus we have our conclusion. Similarly the first two parts of condition (6.6) are equivalent to

$$\tilde{\sigma}_3\big|_{\xi_3=1} = \frac{\sqrt{\sigma_2^2 - 2\sigma_2\sigma_3\rho_{2,3} + \sigma_3^2}}{Z_2 - Z_{3,l}} = 0. \tag{6.8}$$

- The last parts of conditions (6.1)-(6.6) hold automatically for this case. Because r, Z_1, Z_2 and Z_3 are nonnegative, the last parts of (6.1), (6.3) and (6.5) are fulfilled. On surface II, $\xi_1 = 1$, so $Z_1 = 1$ and the value of a zero-coupon bond curve must be equal to one identically for $T \in [0, T_1]$ because the zero-coupon bond curve is a non-increasing curve, which implies $r(1, Z_2, Z_3, t) = 0$. Thus the last part of (6.2) is fulfilled. The last parts of (6.4) and (6.6) are also satisfied because $Z_1 = Z_2$ on surface IV due to $\xi_2 = 1$ and $Z_2 = Z_3$ on surface VI due to $\xi_3 = 1$. Thus for proving this theorem we can take these conditions away. The main reason why we list them is that the conditions in this theorem are actually not only sufficient but also necessary to conditions (5.2) and (5.3) in Theorem 5.1. That is, problem (4.5) has a unique solution if and only if conditions (6.1)-(6.6) hold.
- According to the Feynman-Kac formula, the partial differential equation in (4.2) is related to the stochastic differential equations (see Karatzas & Shreve (1988))

$$dZ_i = rZ_i dt + \sigma_i dX_i, \quad i = 1, 2, 3. \tag{6.9}$$

Therefore we may think that dZ_i satisfy the stochastic differential equations (6.9) when we consider problem (4.2). Intuitively, conditions (6.1)-(6.6) guarantee that a random point (Z_1, Z_2, Z_3) satisfying the stochastic differential equations (6.9) will not move from inside of the domain Ω to its outside. For example, on surface I, the corresponding outer unit normal vector in the (Z_1, Z_2, Z_3)-space is $(-1, 0, 0)$. The conditions $\sigma_1 = 0$ and $rZ_1 \geq 0$ cause

$$-dZ_1 = -rZ_1 dt - \sigma_1 dX_i = -rZ_1 dt \leq 0,$$

so a random point in Ω will not move outside the domain Ω from that part of the boundary. On the surface IV, $\sigma_1 = \sigma_2$, $\rho_{1,2} = 1$ and $-r(Z_1 - Z_2) \leq 0$. In this case $dX_1 = dX_2$, and

$$\begin{aligned} -dZ_1 + dZ_2 &= -rZ_1 dt + rZ_2 dt - \sigma_1 dX_1 + \sigma_2 dX_2 \\ &= -r(Z_1 - Z_2)dt - \sigma_1 dX_1 + \sigma_1 dX_1 \\ &= -r(Z_1 - Z_2)dt \leq 0. \end{aligned}$$

On that surface the corresponding outer unit normal vector in the (Z_1, Z_2, Z_3) -space is $\left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right)$. Therefore $-dZ_1 + dZ_2 \leq 0$ also means that a random point will not move outside Ω from that part of the boundary.

Proof. Since all the coefficients and their first and second order derivatives are bounded on the domain $\tilde{\Omega}$, condition (5.4) will hold. Thus if we can show that conditions (5.2) and (5.3) hold on the entire boundary of $\tilde{\Omega}$, then we have the conclusion we need.

First let us check conditions (5.2) and (5.3) on surface I. There $\mathbf{N} = (-1, 0, 0)$. Thus conditions (5.2) and (5.3) are in the form

$$\begin{cases} -\tilde{\sigma}_1 = 0, \\ -\tilde{\sigma}_1 \tilde{\rho}_{1,2} = 0, \\ -\tilde{\sigma}_1 \tilde{\rho}_{1,3} = 0 \end{cases}$$

and

$$-\left(b_1 - \frac{1}{2}\frac{\partial \tilde{\sigma}_1^2}{\partial \xi_1} - \frac{1}{2}\frac{\partial\left(\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\rho}_{2,1}\right)}{\partial \xi_2} - \frac{1}{2}\frac{\partial\left(\tilde{\sigma}_3\tilde{\sigma}_1\tilde{\rho}_{3,1}\right)}{\partial \xi_3}\right) \leq 0$$

respectively. Therefore when (6.1) is fulfilled, i.e., $\tilde{\sigma_1} = 0$, then (5.2) holds. It is obvious that $\partial\tilde{\sigma}_1^2/\partial\xi_1 = 0$ on surface I. The partial derivative with respect to ξ_2 or ξ_3 is a derivative on surface $\xi_1 = 0$ and $\tilde{\sigma}_1 = 0$ on the entire surface I, so $\partial(\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\rho}_{2,1})/\partial\xi_2 = 0$ and $\partial(\tilde{\sigma}_3\tilde{\sigma}_1\tilde{\rho}_{3,1})/\partial\xi_3 = 0$ on surface I. Thus condition (5.3) becomes

$$b_1 = \frac{rZ_1}{1 - Z_{1,l}} \geq 0,$$

which is always true because both r and Z_1 are non-negative. Therefore when (6.1) is satisfied, (5.2) and (5.3) hold on surface I. When (6.3) holds on surfaces III, the way to show (5.2) being satisfied and (5.3) being reduced to $b_2 \geq 0$ is the same. In this case $\sigma_2 = 0$ and $\xi_2 = 0$, so (5.3) can be further reduced to

$$\frac{rZ_2}{Z_1 - Z_{2,l}} \geq 0.$$

It is clear that this is true, so (5.2) and (5.3) are fulfilled when (6.3) holds. On surface V the situation is similar. That is, in the same way it can be shown that (5.2) and (5.3) are fulfilled when (6.5) holds.

On surface II, if (6.2) is fulfilled, then (5.2) holds and (5.3) can be reduced to $rZ_1 \leq 0$. Recalling that $r = 0$ when $Z_1 = 1$, we know (5.3) also holds.

On surface IV, $\mathbf{N} = (0, -1, 0)$, condition (5.2) becomes

$$\begin{cases} -\tilde{\sigma}_2 \tilde{\rho}_{2,1} = 0, \\ -\tilde{\sigma}_2 = 0, \\ -\tilde{\sigma}_2 \tilde{\rho}_{2,3} = 0. \end{cases}$$

As demonstrated earlier, the first two parts of condition (6.4), $\sigma_1 = \sigma_2$ and $\rho_{1,2} = \rho_{2,1} = 1$, are equivalent to $\tilde{\sigma}_2 = 0$, so condition (5.2) is satisfied. In this case condition (5.3) becomes $b_2 \leq 0$ and furthermore is reduced to

$$\frac{r(Z_2 - Z_1)}{Z_1 - Z_{2,l}} \leq 0$$

due to $\xi_2 = 1$, which holds when $Z_1 = Z_2$ on surface IV. Therefore if condition (6.4) is satisfied, then (5.3) is fulfilled. Similarly, (5.2) and (5.3) are satisfied on surface VI when (6.6) holds. Therefore we have proved that when (6.1)-(6.6) hold, (5.2) and

(5.3) are satisfied on the entire boundary of $\tilde{\Omega}$. Consequently, from Theorem 1 we conclude that problem (4.5) has a unique solution. □

Finally we will say a few words about how to use this model to evaluate interest rate derivatives. First we need to choose Z_1, Z_2, Z_3 and find σ_1, σ_2, σ_3, $\rho_{1,2}$, $\rho_{1,3}$ and $\rho_{2,3}$ satisfying conditions (6.1)-(6.6), or find $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$ and $\tilde{\rho}_{2,3}$ satisfying conditions (6.1)-(6.3), (6.5) and (6.7)-(6.8). Finding these functions can be done from the historic and present data on markets by statistics, including the forecast technique on time series. After that, problem (4.5) needs to be solved. Suppose that $t = 0$ today and the derivative is European style. On the maturity date T, for each point (ξ_1, ξ_2, ξ_3) in $\tilde{\Omega}$, we can have Z_1, Z_2, Z_3 by (4.3) and then determine a zero-coupon bond curve by using the methods given in Sections 1 and 2. On having such a curve, the value of payoff for the point can be obtained. For example, suppose we want to price a half year option on 5-year swaps with exercise swap rate r_{se}. From what described in Sections 1 and 2, for a set of three zero coupon bond values Z_1, Z_2 and Z_3, we can have a zero-coupon bond curve $\bar{Z}(t, t+T^*; Z_1, Z_2, Z_3)$. Here we use $\bar{Z}(t, t+T^*; Z_1, Z_2, Z_3)$ instead of $\bar{Z}(t, t+T^*)$ in order to explain the dependence of the curve on Z_1, Z_2, Z_3 explicitly. When we have this curve, the value of the option at the maturity date T can be obtained by (see Zhu, Wu & Chern, 2002)

$$Q\left[1 - \frac{r_{se}}{2}\sum_{k=1}^{2N} \bar{Z}\left(T, T+\frac{k}{2}; Z_1, Z_2, Z_3\right) - \bar{Z}\left(T, T+N; Z_1, Z_2, Z_3\right)\right],$$

where $N = 5$ because we consider 5-year swaps. This can be done for all points (ξ_1, ξ_2, ξ_3) in the domain $\tilde{\Omega}$ for $t = T$. When we have the final value, we can solve the final-value problem (4.5) from $t = T$ to $t = 0$ and get the value of the derivative at $t = 0$ for all the points in $\tilde{\Omega}$. Since from Theorem 6.1 we know that the price of an interest rate derivative can be determined uniquely by the final condition (the payoff) given on a rectangular domain $\tilde{\Omega}$, without requiring any boundary condition. Therefore it is not difficult to design a numerical method to price such a problem. In the paper by Zhu & Li (2003), the details of numerical methods for such a two-dimensional problem are given. For such a three-dimensional problem, the idea is the same and the details will be given in another paper in the near future. For American style derivatives, the situation is similar. The only difference is that the value of derivative must be greater than the constraint. Since the value of the constraint can be obtained by the zero-coupon bond curve at all points in $\tilde{\Omega} \times [0, T]$, the value of a derivative can be determined without any difficulty. However, in this case usually a free boundary will appear.

7. Summary

A three-factor interest rate model and the corresponding parabolic partial differential equation for derivative securities have been provided. This model has the following features.

- The state variables are prices of three zero-coupon bonds with different maturities which can be traded on the markets, so the coefficients of the first derivatives with respect to the bond prices Z_i in the partial differential equation simply are rZ_i.
- The volatilities of these zero-coupon bonds and their correlation coefficients can be found directly from the real markets by statistics, so the model will have the real major feature of the markets.
- All the zero-coupon bond curves having appeared in the real market can be produced quite accurately. This is the base of a model giving correct results.

If taking three random variables is not good enough, four-factor models can be adopted. Generalizing three-factor models to four-factor models is straightforward.

- In other models, the partial differential equation is defined on an infinite domain. For this model the corresponding partial differential equation is defined on a finite domain and it has been proved that no boundary condition is needed for its final-value problem to have a unique solution. Thus it is not difficult to design correct and efficient numerical methods to price interest rate derivatives.

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