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ON THE FINITE TIME BLOW-UP OF THE EULER-POISSON EQUATIONS IN $\mathbb{R}^N$*

DONGHAO CHAE† AND EITAN TADMOR‡

Abstract. We prove the finite time blow-up for C^1 solutions of the attractive Euler-Poisson equations in $\mathbb{R}^n$, $n \geq 1$, with and without background state, for a large set of 'generic' initial data. We characterize this supercritical set by tracing the spectral dynamics of the deformation and vorticity tensors.

Key words. Euler-Poisson equations, finite time blow-up

AMS subject classifications. 35Q35, 35B30

1. The Euler-Poisson equations

We are concerned with the pressureless Euler-Poisson equations in $\mathbb{R}^n$, $n \geq 1$,

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{1.1a}$$

$$\partial_t(\rho v) + \mathrm{div}(\rho u \otimes u) = -k\rho\nabla\phi, \quad -\Delta\phi = \rho - c, \tag{1.1b}$$

$$\begin{cases} v(x,0) = v_0(x), \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{1.1c}$$

The equations involve the unknown velocity field, $v = (v^1, \ldots, v^n) = v(x,t)$, local density $\rho = \rho(x,t) \geq 0$, potential function $\phi = \phi(x,t)$, and the two constants, c and k. Here, $c \geq 0$ is the constant "background" state; typical cases include the case of zero background, $c = 0$, or the case of a nonzero background given by the average mass, $c = \bar{\rho}$, where

$$\bar{\rho} := \int \rho(x,t)dx = \int \rho_0(x)dx.$$

Finally, k is a scaled physical constant which signifies whether the underlying forcing is *attractive*, when $k < 0$, or *repulsive*, when $k > 0$.

The hyperbolic-elliptic system (1.1) appears in a variety of different applications, from small scale models in charge transport and plasma collision, e.g., [18, 8], to large scale dynamics of (clusters of) stars in cosmological waves, and expansion of the cold ions, e.g., [1, 7].

For the questions of local regularity and global existence of *weak* solutions, we refer to [13, 14, 5] for local existence in the small H^s-neighborhood of a steady state, and to [16, 9] for the relaxation limit of the weak entropy solution in the isentropic and isothermal cases. Global existence with a "sufficient" amount of damping relaxation

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†Department of Mathematics, University of Chicago, Chicago, IL 60637, USA and Sungkyunkwan University, Suwon 440-746, Korea (chae@uchicago.edu).

‡Department of Mathematics, Center for Scientific Computation and Mathematical Modeling (CSCAMM) and Institute for Physical Science and Technology (IPST), University of Maryland, College Park, MD 20742 (tadmor@cscamm.umd.edu). http://www.cscamm.umd.edu/~tadmor.

can be found in [22, 23, 12]. For the model without damping relaxation, global existence was obtained by Guo [6], assuming the flow is irrotational and the data is in the small H^2-neighborhood of a constant state.

We focus our attention on the questions of *global* regularity versus finite-time breakdown of solutions for (1.1). In this case, the precise configuration of the initial data and the type of attractive vs. repulsive forcing play a decisive role. On the one hand, there are various non-existence results. A finite time breakdown result in the case of attractive forces, $k<0$, was proved by Makino and Perthame ([15]) for the spherically symmetric Euler-Poisson equations with pressure, subject to compactly supported $\rho(x,t)$ in $\mathbb{R}^3$. For the repulsive case, with similar geometry, the blow-up was deduced in [20] provided that the initial data is *sufficiently large*. The study of singularity formation in the model with diffusion and relaxation can be found in [24]. Local conditions for the finite-time loss of smoothness in the one-dimensional case with and without pressure were given in [3, 4]. On the other hand, there are various results on the long time existence of strong solutions. Global regularity results for a large class of initial data near a steady state is obtained in [6]. In [19] the stability type of result was obtained with inclusion of the pressure.

All these results leave open the question of global regularity of solutions to (1.1) subject to more general settings of initial configurations, which are not necessarily confined to a "sufficiently small" neighborhood of any prefered state (including infinity). It is in this sense that we are concerned here with the global regularity vs. finite-time breakdown of solutions to (1.1). The main difficulty lies with the nonlocal nature of the forcing term, $-k\nabla\phi$, which resembles the notorious pressure term in the 3D incompressible Euler equations. This feature was emphasized in [11], and was the main motivation for studying the so-called 'restricted Euler-Poisson' model, where the nonlocal forcing term is replaced by a local one. It is shown that in the *repulsive* case, the restricted model admits global smooth solutions for a large set of initial configurations, so called sub-critical conditions which are not necessarily confined to any prefered small neighborhood. This type of 'critical threshold phenomena' was studied in [3, 10, 11, 2] via spectral dynamics, and we will use it below to derive finite time breakdown in the attractive case.

Our aim in this paper is to show that finite time blow-up is generic for the *attractive* Euler-Poisson equations under suitable conditions for a large class of initial data. In section 2 we show how to bypass the difficulty of nonlocal forcing by tracing the spectral dynamics of the symmetric part of the velocity gradient matrix (deformation tensor) and observing that the vanishing property of the skew-symmetric part (vorticity tensor) is preserved along the particle trajectories. In contrast to this generic finite time blow-up in the attractive case, one expects global regularity for large sets of initial configurations in the *repulsive* Euler-Poisson equations. These are the sub-critical initial data identified in the one-dimensional case [3, 21] and the two-dimensional case of the restricted Euler-Poisson equations [11]. The regularity of the non-restricted repulsive Euler-Poisson equations in $n>1$ dimensions remains an outstanding open problem.

2. Finite time blow-up in the attractive case

We focus on the case $k<0$, which represents the case of *attractive* Poisson forcing. Let us define the rescaled vorticity matrix $\Omega=(\Omega_{ij})$ for the n dimensional vector field $v=(v^1,\cdots,v^n)$ as $\Omega_{ij}:=\frac{1}{2}(\partial_i v^j-\partial_j v^i)$, $i,j=1,\cdots,n$ (the extra $\frac{1}{2}$ factor will simplify our formulae below). In the 1-D case we set $\Omega\equiv 0$.

Theorem 2.1. *Consider the n-dimensional Euler-Poisson equations (1.1). Assume that the initial data* (ρ_0, v_0) *satisfies,*

$$\mathcal{S} := \left\{ a \in \mathbb{R}^n \,|\, \rho_0(a) > 0, \Omega_0(a) = 0, \operatorname{div} v_0(a) + \sqrt{-nkc} < 0 \right\} \neq \emptyset.$$

Then the C^1-*regularity of local classical solution with initial data* v_0, ρ_0 *cannot persist for arbitrarily long times, namely, there exists* $t_c < \infty$ *such that* $\operatorname{div} v(\cdot,t) \downarrow -\infty$ *as* $t \uparrow t_c$.

Remark 2.1. We refer to $\mathcal{S}$ as the set of *supercritical* configurations. We observe that in the zero-background case, $c = 0$, Theorem 2.1 implies a finite time breakdown for generic non-vacuum, supercritical initial configurations where $\Omega_0(a)=0$ and $\operatorname{div} v_0(a) < 0$. In particular, in the 1-D case and in the n-D spherically symmetric case, the requirement for vanishing vorticity, $\Omega_0(a) = 0$, is redundant.

Remark 2.2. *The finite time blow-up is a local phenomenon and does not depend on the specific domain; the same results hold for a bounded domain with a smooth boundary or a periodic domain.*

Remark 2.3. *The finite-time breakdown will occur at a critical time,* $t_c < \infty$*, which satisfies*

$$t_c \le \inf_{a \in \mathcal{S}} t_c(a) < \infty, \qquad t_c(a) := \left\{ \frac{1}{2\sqrt{-nkc}} \ln \frac{\operatorname{div} v_0(a) - \sqrt{-nkc}}{\operatorname{div} v_0(a) + \sqrt{-nkc}} \right\}.$$

Proof. Away from vacuum, where $\rho > 0$, the momentum equation (1.1b) can be rewritten solely in terms of the velocity field,

$$\partial_t v + (v \cdot \nabla) v = -k \nabla \phi, \qquad \left\{ (x,t) \in \mathbb{R}^n \times [0,\infty) \,\middle|\, \rho(x,t) > 0 \right\}. \tag{2.1}$$

Taking partial derivatives of (2.1) we obtain the Ricatti matrix equation,

$$\partial_t V + (v \cdot \nabla) V + V^2 = -k \Phi, \tag{2.2}$$

where $V := (\partial_i v^j)$ is the stress tensor and $\Phi := (\partial_i \partial_j \phi)$ is the Hessian of ϕ. The symmetric and the skew-symmetric parts of (2.2) satisfy

$$\frac{D}{Dt} \mathcal{D} = -\mathcal{D}^2 - \Omega^2 - k\Phi, \tag{2.3a}$$

and

$$\frac{D}{Dt} \Omega = -\mathcal{D}\Omega - \Omega\mathcal{D}. \tag{2.3b}$$

Here, $\mathcal{D} := \frac{1}{2}(V + V^\top)$, $\Omega := \frac{1}{2}(V - V^\top)$, and $\frac{D}{Dt} = \partial_t + (v \cdot \nabla)$ amounts to path differentiation

$$\frac{D}{Dt}[\cdot](x,t) = \frac{d}{dt}[\cdot](X(a,t),t),$$

along particle trajectories $\{X(a,t) \,|\, X_t(a,t) = v(X(a,t),t),\ X(a,0) = a\}$, where $v(x,t)$ is a classical solution of the system (1.1).

Consider a particle trajectory starting at $a \in \mathcal{S}$ and let λ be an eigenvalue of $\mathcal{D}$ associated with a normalized eigenvector r_λ. We are interested in the dynamics of the eigenvalues along such particle trajectories, $\lambda = \lambda(X(a,t),t)$. Now, the initial state at $a \in \mathcal{S}$ is assumed to be a non-vacuum state. Hence, as long as $\rho(\cdot,t)$ remains positive, (2.3a) applies, and the *spectral dynamics* of λ is governed by, e.g., [11, Sec. 3]

$$\frac{D}{Dt}\lambda = -\lambda^2 - \langle \Omega^2 r_\lambda, r_\lambda\rangle - k\langle \Phi r_\lambda, r_\lambda\rangle. \tag{2.4}$$

Since the initial vorticity vanishes at $\mathcal{S}$, equation (2.3b) tells us that Ω remains zero along the corresponding trajectory, $\Omega_0(a) = 0 \mapsto \Omega(X(a,t),t) = 0$. It follows from (2.4) that as long as the particle trajectory does not cross into a vacuum state, then

$$\frac{D}{Dt}\sum_\lambda \lambda = -\sum_\lambda \lambda^2 - k\sum_\lambda \langle \Phi r_\lambda, r_\lambda\rangle, \quad \lambda \equiv \lambda(\mathcal{D}). \tag{2.5}$$

We observe that $\sum \lambda = \mathrm{tr}(\mathcal{D}) = \mathrm{div}\, v$ and hence

$$\sum \lambda^2 \geq \frac{1}{n}\Big(\sum \lambda\Big)^2 = \frac{1}{n}(\mathrm{div}\, v)^2;$$

also,

$$\sum_\lambda \langle \Phi r_\lambda, r_\lambda\rangle = \mathrm{tr}\,\Phi = \Delta\phi = -(\rho - c).$$

Since $k\rho \leq 0$, we end up with the Ricatti-type inequality

$$\frac{D}{Dt}\mathrm{div}\, v \leq -\frac{1}{n}(\mathrm{div}\, v)^2 - kc. \tag{2.6}$$

The last inequality implies that the divergence is non-increasing along $\{X(a,t), a \in \mathcal{S}\}$[1]

$$\mathrm{div}\, v(\cdot,t) \leq M_0 := \sup_{a\in\mathcal{S}}\{\mathrm{div}\, v_0(a)\} \leq -\sqrt{-nkc},$$

and hence, by (1.1a), a particle path initiated in $a \in \mathcal{S}$ remains non-vacuum for all time

$$\frac{D}{Dt}\rho = -\mathrm{div}\, v \cdot \rho \quad \mapsto \quad \rho(X(a,t),t) \geq e^{-M_0 t}\rho_0(a) > 0.$$

Moreover, if we wait long enough, solutions of (2.6) subject to $\mathrm{div}\, v_0(a) < -\sqrt{-nkc}$ will blow up as $\lim_{t\uparrow t_c(a)} \mathrm{div}\, v = -\infty$. Thus, there is a finite time breakdown on or before $t_c(a)$ whenever $a \in \mathcal{S}$. ∎

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[1] In fact, $\mathrm{div}\, v$ remains uniformly upper-bounded for *all* non-vacuum paths, $\mathrm{div}\, v(\cdot,t) \leq \sup_{\{a:\rho_0(a)>0\}}\{\mathrm{div}\, v_0(a), \sqrt{-nkc}\}$.

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