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NONPERTURBATIVE CONTRIBUTIONS IN QUANTUM-MECHANICAL MODELS: THE INSTANTONIC APPROACH*

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Abstract. We review the euclidean path-integral formalism in connection with the one-dimensional non-relativistic particle. The configurations which allow construction of a semiclassical approximation classify themselves into either topological (instantons) and non-topological (bounces) solutions. The quantum amplitudes consist of an exponential associated with the classical contribution multiplied by the fluctuation factor which is given by a functional determinant. The eigenfunctions as well as the energy eigenvalues of the quadratic operators at issue can be written in closed form due to the shape-invariance property. Accordingly, we resort to the zeta-function method to compute the functional determinants in a systematic way. The effect of the multi-instantons configurations is also carefully considered. To illustrate the instanton calculus in a relevant model, we go to the double-well potential. The second popular case is the periodic-potential where the initial levels split into bands. The quantum decay rate of the metastable states in a cubic model is evaluated by means of the bounce-like solution.

1. Introduction.

The tunneling through classically forbidden regions represents one of the most striking phenomenon in quantum theory and therefore plays a central role in many areas of modern physics. On the other hand, together with the operator formalism of quantum mechanics we have an equivalent description by means of path-integrals. In such a case, the Schrodinger's equation is substituted by a global approach where the quantum mechanical time evolution is analysed in terms of a functional integration. Qualitatively speaking, the path-integral representation corresponds to a sum over all histories allowed to the physical system we are dealing with. To be precise, we need to take into account an imaginary exponential of the classical action associated with every path which fulfills the appropiate initial and end point conditions. Of course, a quantum amplitude so built is difficult to handle due to the oscillating character of the exponential at issue. To avoid problems of this sort, we carry out the change $t \to -i\tau$ (known in the literature as the Wick rotation). In doing so, we can take advantage of the euclidean version of the path-integral, which represents by itself a new tool for describing relevant aspects of the quantum theory.

Almost from the work on the subject, began a semiclassical treatment of the tunneling phenomena (ranging from periodic-potentials in quantum mechanics to Yang-Mills models in field theories) was performed by means of the so-called instantons. Going to more physical terms, the instantons represent localised finite-action solutions of the euclidean equation of motion. To be specific, the euclidean equation is the same as the usual one for our particle in real time except that the potential is turned upside down. Although more massive than the perturbative excitations, the instantons themselves become stable since an infinite barrier separates them from the ordinary sector of the model. The stability is reinforced by the existence of a topologically conserved charge which does not arise by Noether's theorem in terms of a well-behaved symmetry, but characterizes the global behaviour of the system when the imaginary time is large enough. Accordingly, it comes as no surprise that these

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classical solutions have been considered in the literature under the name of topological configurations. Once the appropiate classical solution is well-known, we make an expansion around the topological background to evaluate the quadratic fluctuations which arise in terms of the functional determinant of a second order differential operator. The integration is solved within the gaussian scheme except for the zero-modes which appear due to the invariances of the model. To deal with these excitations we introduce collective coordinates so that ultimately the gaussian integration is carried out only along the directions orthogonal to the zero-modes.

As a functional determinant includes an infinite product of eigenvalues, the result should be in principle a highly divergent expression. Fortunately we can regularize the fluctuation factors by means of the ratio of determinants. The eigenfunctions as well as the energy eigenvalues of the quadratic operators which appear in this article are obtained in closed form by virtue of the shape-invariance property just derived in one-dimensional supersymmetric quantum mechanics. Once we reach this point, the evaluation of the quotient of determinants is borne out by the zeta-function method.

On the other hand, the use of metastable states, defined as long-lived but eventually decaying states, is a fact of crucial relevance in the most diverse branches of physics. The classically stable configurations which allow the system to decay through quantum tunneling into the true vacuum receive in the literature the name of false vacuum.

For instance, particular interest has been devoted in recent years to the study of cosmological models where the metastability properties are of great importance to explain the different steps of the evolution of the universe as a whole. As regards the very early stages of our world, it is customarily assumed the existence of an inflationary era in which the energy density is highly dominated by that of the Higgs field trapped precisely in a false vacuum. As expected, the end of the inflation period occurs once the metastable state decays to the true ground-state of the model.

Restricting ourselves to quantum mechanics, let us consider a particle moving under the action of a potential which exhibits a metastable well around the origin. In such a case, the particle will stay there forever as far as classical mechanics is concerned. Because of tunneling, however, the potential allows the escape of the particle from the single well to infinity. In other words, the state situated around the origin is quantum mechanically unstable and the probability current outside the barrier is non-vanishing. Taking into account this fact, the energy eigenvalues become complex so that the imaginary part itself represents a measure of the decay width.

In order to compute the lifetime of the particle trapped within the single trough of the potential, different methods have been proposed. Again, the semiclassical expansion based on the euclidean version of the path-integral represents the most suitable approach in this context. When constructing a semiclassical theory for the decay of metastable states, the euclidean non-topological configuration which leads the process is often referred to as a bounce solution. To be precise, the spectrum of small fluctuations around the bounce itself contains a negative eigenvalue which becomes ultimately responsible for the metastability of the system as a whole. In summary, both topological (instantons) and non-topological (bounces) euclidean configurations represent probably the most adequate tools to study tunneling phenomena.

It remains to identify the general class of potentials for which the functional determinant can be computed according to the method based on the zeta-function. Our description takes over the shape-invariance properties of the one-dimensional Schrodinger equation. Among other things, the shape-invariance represents a sufficient condition for the solvability of the Schrodinger equation at issue [1]. In such a case, the energy eigenvalues as well as the eigenfunctions can be obtained in closed form so that the explicit evaluation of the associated zeta-function is feasible. The curious reader can find in [1] a complete list of solvable potentials giving rise to shape-invariance properties. To be more specific, the cases considered in this article belong to a general class of Posch-Teller potentials labeled by ℓ ($\ell = 1, 2, ...$). For the sake of brevity, only the first members of the series are discussed in detail, although the potentials associated with higher values of ℓ are relevant from a physical point of view [2].

On the other hand, it may be interesting at this point to compare the results obtained according to the one-loop approximation of the path integral and the ones derived from the standard WKB method. Historically, the discussions have been based on the double-well potential and the periodic sine-Gordon model. However, the exact spectrum of the Lamé potential has been found recently [3]. Although it is not apparent a priori how to handle more complicated situations, we can conclude that the aforementioned case represents an excellent benchmark to test the standard WKB procedure. In order not to clutter this paper, we refer to the article of Müller-Kirsten et al. [4] where the interested reader can find a careful discussion. For background we simply point out that only the so-called WKB-related method of matched asymptotic expansions yields the exact instanton results. In other words, we need to go beyond the well-known linear connection relations to reach the correct values. To sum up, it may be worth spelling out how the one-loop approximation based on the instantonic approach provides a relevant framework for evaluating physical magnitudes in a more reliable context than the simple WKB method.

The arrangement of the article is as follows. First of all, we review in detail the main features concerning the instantonic approach in one-dimensional particle mechanics. To illustrate the method, we resort to the double-well potential. In addition, we also study the harmonic oscillator since it represents the reference for all the models discussed along our analysis. The quantum effects of the one-instanton as well as the multi-instantons configurations are included in this chapter.

The following section is devoted to the periodic-potential. Among other things, the lowest band arising from the splitting of the initial ground-state is explained by means of the multi-instantons contribution. We conclude with the consideration of a cubic model to stick out the existence of metastable states. Different mathematical results concerning one-dimensional supersymmetric quantum mechanics, shape-invariance properties, zeta-function regularization procedures and spectral densities can be consulted in the appendices.

2. The instantonic approach.

To start from scratch let us describe in detail the instanton calculus for the one-dimensional spinless particle as can be found, for instance, in [5]. The interested reader can find there a comprehensive description of the whole subject. In order to be specific, we assume that the particle moves under the action of a confining potential $V(x)$ which yields at quantum level a pure discrete spectrum of energy eigenvalues. Unless otherwise noted, we choose the origin of the energy so that the minima of the potential satisfy $V(x) = 0$. If we set the mass of the particle equal to unity for notational simplicity, the lagrangian L which governs the behaviour of the model is given by

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) \tag{2.1}$$

Now we can start to quantize the theory. If the particle is located at the initial time $t_i = -T/2$ at the point x_i while one finds it when $t_f = T/2$ at the point x_f, the functional version of the non-relativistic quantum mechanics allows us to express the transition amplitude in terms of a sum over all paths joining the world points with coordinates $(-T/2, x_i)$ and $(T/2, x_f)$. At this point, it proves convenient to write the action S starting from the lagrangian L, i.e.

$$S = \int_{-T/2}^{T/2} L(x, \dot{x})\ dt \tag{2.2}$$

so that the contribution of a path has a phase proportional to the action S itself. Therefore, we reduce our problem to the study of a transition amplitude expressed as

$$< x_f | \exp(-iHT) | x_i >= N(T) \int [dx] \exp iS[x(t)] \tag{2.3}$$

It may be worth spelling out that H represents the hamiltonian of the model at issue while the symbol $[dx]$ indicates the integration over all functions which fulfill the adequate boundary conditions. The factor $N(T)$ will be chosen to normalize the amplitude conveniently when we discuss more mathematically the meaning of (2.3). As the hamiltonian H gives rise to a pure discrete spectrum of energy eigenvalues, namely

$$H|n >= E_n |n > \tag{2.4}$$

we can write that

$$< x_f | \exp(-iHT) | x_i >= \sum_n \exp(-iE_n T) < x_f | n >< n | x_i > \tag{2.5}$$

As we are mainly interested in the first eigenfunctions of H, it proves convenient to transform the exponentials of (2.3) into decreasing exponentials. For such a purpose, we make the change $t \to -i\tau$, known in the literature as the Wick rotation, so that

$$iS[x(t)] \to \int_{-T/2}^{T/2} \left[-\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 - V(x) \right]\ d\tau \tag{2.6}$$

In the following we resort systematically to the euclidean action S_e, i.e.

$$S_e = \int_{-T/2}^{T/2} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right]\ d\tau \tag{2.7}$$

To sum up, the euclidean formulation of the path-integral corresponds to

$$< x_f | \exp(-HT) | x_i >= N(T) \int [dx]\ \exp -S_e[x(\tau)] \tag{2.8}$$

To expose the main features of the semiclassical approximation, we start by considering a particular trajectory $x_c(\tau)$ which satisfies the boundary conditions at issue. Next, we perform the expansion of a generic $x(\tau)$ with identical Dirichlet conditions as $x_c(\tau)$ according to

$$x(\tau) = x_c(\tau) + \sum_j c_j\ x_j(\tau) \tag{2.9}$$

where $x_j(\tau)$ stands for a complete set of orthonormal functions, i.e.

$$\int_{-T/2}^{T/2} x_j(\tau)\ x_k(\tau)\ d\tau = \delta_{jk} \tag{2.10}$$

vanishing at our boundary

$$x_j(\pm T/2) = 0 \tag{2.11}$$

We carry things further and make the choice

$$[dx] = \prod_j \frac{dc_j}{\sqrt{2\pi}} \tag{2.12}$$

so that as a matter of fact, each path is completely characterized by the c_j themselves. It seems plausible to explain the functional formalism in terms of the integration over the Fourier coefficients c_j. Now we need to identify the quadratic differential operator which gives rise to the basis $x_j(\tau)$.

The quasiclassical approximation (or steepest descent method in more mathematical language) takes for granted that $x_c(\tau)$ represents a stationary point of the euclidean action. As corresponds to an extremal path, $x_c(\tau)$ verifies the equation

$$\frac{d^2x}{d\tau^2} = V'(x) \tag{2.13}$$

where the prime denotes as usual the derivative with respect to the spatial coordinate. Notice that (2.13) corresponds to the euclidean equation of motion for the particle once the potential has been turned upside down. To take into account the quantum fluctuations, we perform a functional expansion about $x_c(\tau)$. As expected, the crucial feature is the analysis of the second variational derivative, since the linear term is absent due to $x_c(\tau)$ representing an extremal path. We find that

$$S_e[x_c(\tau) + \delta x(\tau)] = S_{eo} + \int_{-T/2}^{T/2} \delta x \left[-\frac{1}{2}\frac{d^2}{d\tau^2}\delta x + \frac{1}{2}V''[x_c(\tau)]\delta x \right]\ d\tau \tag{2.14}$$

S_{eo} begin the classical action associated with the configuration $x_c(\tau)$. The conventional form of the semiclassical approximation takes over a complete set of eigenfunctions (eigenvalues) of the so-called stability equation, i.e.

$$-\frac{d^2}{d\tau^2}v_j(\tau) + V''[x_c(\tau)]v_j(\tau) = \epsilon_j v_j(\tau) \tag{2.15}$$

Now the expansion of (2.14) becomes diagonal, so that

$$S_e = S_{eo} + \frac{1}{2}\sum_j c_j^2 \epsilon_j \tag{2.16}$$

and therefore, the euclidean transition amplitude reduces itself to

$$< x_f|\exp(-HT)|x_i >= N(T)\exp(-S_{eo})\ \prod_j \epsilon_j^{-1/2} \tag{2.17}$$

Historically the product of eigenvalues has been written as

$$\prod_j \epsilon_j^{-1/2} = \left\{ Det \left[-\frac{d^2}{d\tau^2} + V''[x_c(\tau)] \right] \right\}^{-1/2} \tag{2.18}$$

according to a notation which obviously has its origin in the finite-dimensional case. In principle we assume that (2.18) represents a formal expression where all the eigenvalues ϵ_j are positive. The difficulties associated with the existence of zero-modes can be avoided by introducing collective coordinates, not to mention the negative eigenvalue which yields the metastability phenomenon (more on this later).

Now we review the properties of the euclidean transition amplitude for the well-grounded problem of the harmonic oscillator, since this model is the reference to deal with the quotient of functional determinants when we go to more relevant systems. If we are dealing with

$$V(x) = \frac{\nu^2}{2}\, x^2 \tag{2.19}$$

the amplitude between $x_i = 0$ and $x_f = 0$ is given by

$$< x_f = 0 | \exp(-H_{ho}T) | x_i = 0 >= N(T) \left\{ Det \left[-\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} \tag{2.20}$$

The interested reader can find in [6] the final form of the amplitude, namely

$$< x_f = 0 | \exp(-H_{ho}T) | x_i = 0 >= \left(\frac{\nu}{\pi}\right)^{1/2} \, (2 \sinh \nu T)^{-1/2} \tag{2.21}$$

Next, we must carry things further in order to discuss in detail models which give rise to relevant instantonic configurations far beyond the trivial cases where $x_c(\tau)$ reduces itself to a constant. (A complete description of the subject can be found in the work of Rajaraman [7]). To illustrate these ideas in a simple context, we again resort to the double-well potential given by

$$V(x) = \frac{\omega^2}{8}(x^2 - 1)^2 \tag{2.22}$$

The model is particularly well adapted for such a purpose since at the classical level, we find two degenerate minima located at $x_- = -1$ and $x_+ = 1$, as expected when considering a potential which enjoys the discrete symmetry $x \to -x$.

The question we wish to address now is the explicit description of the tunneling phenomenon in the euclidean version of the path-integral. To make clear the method, we turn the potential upside down as required by the Wick rotation. From a physical point of view it seems reasonable to evaluate the transition amplitude between the points $x_- = -1$ and $x_+ = 1$. As regards the topological configuration, which represents the basis for a quasiclassical approximation, we look for a trajectory with initial $x_i = -1$ at $t_i = -T/2$ while $x_f = 1$ when $t_f = T/2$. In other words, we need to know in closed form the instanton just interpolating between the two minima of the potential $V(x)$. It is customarily assumed that $T \to \infty$ mainly because the explicit solution of the problem is much more complicated for finite T (more on this point later).

With the appropiate boundary conditions, we must integrate (2.13) as far as the double-well is concerned. The $x_c(\tau)$ corresponds to the motion of a particle whose conserved euclidean energy $\tilde{E}$ is given by

$$\tilde{E} = \frac{1}{2}\left(\frac{dx_c}{d\tau}\right)^2 - V(x_c) \tag{2.23}$$

According to the Dirichlet conditions required by the form of our transition amplitude, we conclude that $\tilde{E} = 0$. As a matter of fact, the particle leaves the point $x_i = -1$ and takes an infinite time to climb up to the top of the mountain $x_f = 1$. Once the journey starts, the particle cannot return to the initial point or explore unlimited motions towards plus or minus infinity. The only dynamical possibility is to reach $x_f = 1$ asymptotically as $t_f \to \infty$.

The aforementioned mechanical analogy allows us to solve the problem just by integration of a first-order differential equation instead of resorting to the conventional euclidean equation of motion written in (2.13). Therefore, we have

$$\frac{dx_c}{d\tau} = \pm\sqrt{2V(x_c)} \tag{2.24}$$

where we recognize the mechanical version of the well-grounded Bogomol'nyi condition first introduced in the mid-seventies for the analysis of field theories [8]. Now we explicitly solve (2.24) by a simple quadrature so that

$$x_c(\tau) = \tanh\frac{\omega(\tau - \tau_c)}{2} \tag{2.25}$$

where the parameter τ_c indicates the point at which the instanton makes the jump. As expected the antiinstanton is obtained by means of the transformation $\tau \to -\tau$ and allows the connection between $x_i = 1$ and $x_f = -1$. Now we must evaluate the action associated with $x_c(\tau)$ according to (2.7) to get

$$S_{eo} = \frac{2\omega}{3} \tag{2.26}$$

It should be emphasized that in principle, we need classical configurations for which $x = \pm 1$ at large but finite values $\tau = \pm T/2$. However, the instantons that appear in the literature correspond to solutions where the particle takes itself an infinite time to complete the trajectory. Fortunately, the difference is so small that it can be ignored, mainly because we are precisely interested in the asymptotic limit $T \to \infty$. As corresponds to a semiclassical approximation, we now evaluate the quadratic corrections over the background provided by the classical configuration. In other words, we take into account the functional determinant at issue while the cubic and higher terms in the quantum fluctuations are neglected. First of all, we need an operator of reference for the explicit evaluation of the determinant. The most natural choice is of course the harmonic oscillator located at either of the two minima of $V(x)$, namely

$$V''(x = \pm 1) = \omega^2 \tag{2.27}$$

so that our description takes over

$$< x_f = 1|\exp(-HT)|x_i = -1 >= N(T)\left\{Det\left[-\frac{d^2}{d\tau^2} + \omega^2\right]\right\}^{-1/2}$$

$$\left\{ \frac{Det\left[-(d^2/d\tau^2) + V''[x_c(\tau)]\right]}{Det\left[-(d^2/d\tau^2) + \omega^2\right]} \right\}^{-1/2} \exp(-S_{eo}) \tag{2.28}$$

where we have multiplied and divided by the determinant of the aforementioned harmonic oscillator. Incorporating the explicit form of $x_c(\tau)$, the stability equation is given by

$$-\frac{d^2}{d\tau^2} x_j(\tau) + \left[\omega^2 - \frac{3\omega^2}{2\cosh^2(\omega\tau/2)}\right] x_j(\tau) = \epsilon_j x_j(\tau) \tag{2.29}$$

and corresponds to a Schrodinger's equation with an exactly solvable Posch-Teller potential. A complete description about the eigenfunctions and the eigenvalues of (2.29) is possible in terms of the shape-invariance properties which appear in supersymmetric quantum mechanics [1]. This modern framework is particularly well adapted to incorporate the zeta-function method [9]. (The technical details of the method can be found in appendix A).

Among other things, one finds a zero-mode $x_o(\tau)$ which could jeopardize the physical meaning of the determinant. However this eigenvalue $\epsilon_o = 0$ comes as no surprise since it reflects the translational invariance of the system. To be more specific, there is one direction in the functional space of the second variations which is incapable of changing the action. One could just as well discover the existence of a zero-mode starting from (2.13). If we perform an additional derivative with respect to τ, then

$$x_o(\tau) = \frac{1}{\sqrt{S_{eo}}} \frac{dx_c}{d\tau} \tag{2.30}$$

is just the solution of (2.15) with $\epsilon_o = 0$. Notice the normalization of $x_o(\tau)$ which is due precisely to the zero euclidean energy condition for $x_c(\tau)$. Going back to the functional arguments, we can see how the integration over the Fourier coefficient c_o becomes tantamount to the integration which takes care of the center of the instanton τ_c. To fix the jacobian of the transformation we consider a first change like

$$\Delta x(\tau) = x_o(\tau)\ \Delta c_o \tag{2.31}$$

According to the general expression written in (2.9), we observe that under a shift $\Delta\tau_c$, the effect corresponds to

$$\Delta x(\tau) = -\sqrt{S_{eo}}\ x_o(\tau) \Delta\tau_c \tag{2.32}$$

Next, the identification between (2.31) and (2.32) yields

$$dc_o = \sqrt{S_{eo}} d\tau_c \tag{2.33}$$

where the minus sign disappears, since what matters is the modulus of the jacobian at issue. In doing so we find that

$$\left\{ \frac{Det\left[-(d^2/d\tau^2) + V''[x_c(\tau)]\right]}{Det\left[-(d^2/d\tau^2) + \omega^2\right]} \right\}^{-1/2} =$$

$$\left\{ \frac{Det'\left[-(d^2/d\tau^2) + V''[x_c(\tau)]\right]}{Det\left[-(d^2/d\tau^2) + \omega^2\right]} \right\}^{-1/2} \sqrt{\frac{S_{eo}}{2\pi}}\ d\tau_c \tag{2.34}$$

where Det' stands for the so-called reduced determinant once the zero-mode has been explicitly removed.

Different procedures have been used in the literature to evaluate the typical ratio of determinants which appears in (2.34). The reasoning become much more accesible once the system is enclosed in a box of length T, thus avoiding the subleties associated with the continuous spectrum. If one imposes periodic boundary conditions, the most relevant physical information derives from the analysis of the phase shifts [5]. In the end, the limit $T \to \infty$ is necessary to achieve a meaningful result. However, we prefer from the very beginning to work in open space and compute the aforementioned ratio of determinants by resorting to the zeta-function regularization procedure [10]. Going back to (2.29) and (2.34), we perform the change of variable

$$z = \frac{\omega\tau}{2} \tag{2.35}$$

to recognize the presence of the ratio of determinants

$$Q_2 = \frac{Det'\ O_2}{Det\ P_2} \tag{2.36}$$

together with a factor β given by

$$\beta = \frac{\omega^2}{4} \tag{2.37}$$

As regards the non-zero spectrum of the operator O_2, we find a discrete level with $E_1 = 3$, while the energy of the scattering states corresponds to

$$E_k = k^2 + 4 \tag{2.38}$$

for eigenfunctions derived from the plane waves according to

$$\phi_{2,k}(z) = \frac{A_2^\dagger(z)}{\sqrt{k^2+4}}\ \frac{A_1^\dagger(z)}{\sqrt{k^2+1}}\ \left[\frac{\exp(ikz)}{\sqrt{2\pi}}\right] \tag{2.39}$$

If we bear in mind the form of the operators $A_2^\dagger(z)$ and $A_1^\dagger(z)$, namely

$$A_1^\dagger = -\frac{d}{dz} + \tanh z \tag{2.40}$$

$$A_2^\dagger = -\frac{d}{dz} + 2\tanh z \tag{2.41}$$

we can write that

$$\phi_{2,k}(z) = \left(-k^2 + 2 - \frac{3}{\cosh^2 z} - 3ik\tanh z\right)\ \frac{\exp(ikz)}{\sqrt{2\pi}\ \sqrt{k^2+4}\ \sqrt{k^2+1}} \tag{2.42}$$

As a straightforward consequence of (2.42), the regularized spectral density $\rho_r(k)$ of our problem reads

$$\rho_r(k) = -\frac{3(k^2+2)}{\pi(k^2+4)(k^2+1)} \tag{2.43}$$

In this scheme, the suitable zeta-function $\zeta_r(s)$ to compute the ratio of determinants of (2.34) can be expressed as

$$\zeta_r(s) = \zeta_{O_2}(s) - \zeta_{P_2}(s) \tag{2.44}$$

Armed with this information, we write

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \left[\exp(-3\mu) - \frac{3}{\pi} \int_{-\infty}^{\infty} \frac{(k^2+2)\exp[-(k^2+4)\mu]}{(k^2+4)(k^2+1)} \, dk \right] \tag{2.45}$$

which, in turn, gives rise to [11]:

$$\zeta_r(s) = \frac{1}{3^s} - \frac{3}{\pi} \int_{-\infty}^{\infty} \frac{(k^2+2)}{(k^2+1)(k^2+4)^{s+1}} \, dk \tag{2.46}$$

Next, it suffices to break the integral of (2.46) into more simple components to obtain $\zeta_r(s)$ in terms of Gamma and Hypergeometric Functions, namely [11]

$$\zeta_r(s) = \frac{1}{3^s} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+1}} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+3}} \frac{\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(s+2)} \, F\left(1, s+\frac{3}{2}, s+2, \frac{3}{4}\right) \tag{2.47}$$

If we incorporate [12]:

$$F\left(1, \frac{3}{2}, 2, \frac{3}{4}\right) = \frac{8}{3} \tag{2.48}$$

the evaluation of $\zeta_r(s)$ at $s = 0$ yields

$$\zeta_r(0) = -1 \tag{2.49}$$

The interested reader can find in the appendix B the fundamental steps which allow ultimately the computation of $\zeta_r'(0)$. Next, if we simply collect the partial results obtained above, the ratio of determinants R which appears in (2.34), i.e.

$$R = \frac{Det' \left[-(d^2/d\tau^2) + V''[x_c(\tau)]\right]}{Det\left[-(d^2/d\tau^2) + \omega^2\right]} \tag{2.50}$$

reduces itself to

$$R = \frac{1}{12\omega^2} \tag{2.51}$$

Combining now these pieces of information, we obtain the transition amplitude anticipated in (2.28), namely

$$< x_f = 1 | \exp(-HT) | x_i = -1 >=$$

$$\left(\frac{\omega}{\pi}\right)^{1/2} \; (2\sinh\omega T)^{-1/2} \; \sqrt{S_{eo}} \; \sqrt{\frac{6}{\pi}} \; \exp(-S_{eo}) \; \omega \; d\tau_c \tag{2.52}$$

Apart from the first factor, which represents the contribution of the harmonic oscillator of reference, we get a transition amplitude just depending on the point τ_c at which the instanton precisely makes the jump. According to the values of the interval T, the result seems plausible whenever

$$\sqrt{S_{eo}}\,\sqrt{\frac{6}{\pi}}\;\exp(-S_{eo})\;\omega\;T \ll 1 \tag{2.53}$$

a nonsense condition if T is large enough. However, in this regime, we can accommodate configurations constructed of instantons and antiinstantons which mimic the behaviour of a trajectory strictly derived from the euclidean equation of motion. In doing so, we get an additional bonus since the integration over the centers of the string of instantons and antiinstantons is carried out in a systematic way. As a matter of fact, the well-known formula for the level splitting of the double-well potential appears when considering the effect of the multi-instantons.

As all the above calculations were carried out over a single instanton, it remains to identify the contributions which take into account the effect of a string of widely separated instantons and antiinstantons along the τ axis. It is customarily assumed that these combinations of topological solutions represent no strong deviations of the trajectories just derived from the euclidean equation of motion without any kind of approximation. We shall compute the functional integral by summing over all such configurations, with j instantons and antiinstantons centered at points $\tau_1, ..., \tau_j$ whenever

$$-\frac{T}{2} < \tau_1 < ... < \tau_j < \frac{T}{2} \tag{2.54}$$

With the regions where the instantons (antiinstantons) make the jump narrow enough, the action of the proposed path is almost extremal. We can carry things further and assume that the action of the string of instantons and antiinstantons is given by the sum of the j individual actions. This scheme is well-known in the literature, where it appears with the name of dilute gas approximation [13].

Now we can evaluate transition amplitudes with closed paths, with $x_i = -1 = x_f$ for instance, so that the action at issue S_t will be an even multiple of the single instanton action, i.e.

$$S_t = 2j\;S_{eo} \tag{2.55}$$

As expected, the amplitude between $x_i = -1$ and $x_i = 1$ incorporates a contribution given by

$$S_t = (2j+1)\;S_{eo} \tag{2.56}$$

In addition, the translational degrees of freedom of the separated j instantons and antiinstantons yield an integral of the form

$$\int_{-T/2}^{T/2} \omega d\tau_j \;\int_{-T/2}^{\tau_j} \omega d\tau_{j-1} ... \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^j}{j!} \tag{2.57}$$

As regards the quadratic fluctuations around the j topological solutions, we find now that the single ratio of determinants transforms into [13]

$$\left(\frac{\omega}{\pi}\right)^{1/2}\;(2\sinh\omega T)^{-1/2}\;\left\{\frac{Det'\left[-(d^2/d\tau^2)+V''[x_c(\tau)]\right]}{Det\left[-(d^2/d\tau^2)+\omega^2\right]}\right\}^{-1/2}\;\longrightarrow$$

$$\left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \quad \left[\left\{ \frac{Det' \left[-(d^2/d\tau^2) + V''[x_c(\tau)] \right]}{Det \left[-(d^2/d\tau^2) + \omega^2 \right]} \right\}^{-1/2} \right]^j \tag{2.58}$$

according to the limit of the factor associated with the harmonic oscillator when T is large. Next, we can write the complete transition amplitudes for the double-well potential so that

$$< x_f = 1 | \exp(-HT) | x_i = -1 >= \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sum_{j=1}^{\infty} \frac{(\omega T d)^{2j+1}}{(2j+1)!} \tag{2.59}$$

where d stands for the so-called instanton density, i.e.

$$d = \sqrt{\frac{6}{\pi}} \sqrt{S_{eo}} \; \exp(-S_{eo}) \tag{2.60}$$

In summary,

$$< x_f = 1 | \exp(-HT) | x_i = -1 >= \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \; \sinh(\omega T d) \tag{2.61}$$

Similarly,

$$< x_f = 1 | \exp(-HT) | x_i = 1 >= \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sum_{j=0}^{\infty} \frac{(\omega T d)^{2j}}{(2j)!} \tag{2.62}$$

and therefore,

$$< x_f = 1 | \exp(-HT) | x_i = 1 >= \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \; \cosh(\omega T d) \tag{2.63}$$

Resorting to the limit $T \to \infty$ in (2.61), we obtain the energy eigenvalues E_0 and E_1 of the first two levels of the double-well potential:

$$E_0 = \frac{\omega}{2} - 2\omega \sqrt{\frac{\omega}{\pi}} \exp(-2\omega/3) \tag{2.64}$$

$$E_1 = \frac{\omega}{2} + 2\omega \sqrt{\frac{\omega}{\pi}} \exp(-2\omega/3) \tag{2.65}$$

As expected, the quantum mechanical tunneling transfers the wave function from one well to the other, thus lifting the degeneracy of the classical vacua. To close, notice the way in which the energy eigenvalues depend on the barrier-penetration factor, i.e. the exponential of minus the classical action of the instanton at issue. Once we have understood the main properties of the instanton calculus for the one-dimensional particle, we proceed to extend the method to arently differing models like a periodic-potential based on the sine-Gordon theory or the cubic system, which represents by itself an excellent benchmark to discuss the existence of metastable states.

3. The periodic-potential.

In this section, we consider the particle under the effects of a periodic-potential $V(x)$. At first, one can ignore the barrier-penetration so that the energy eigenkets are infinitely degenerate. Going to more physical terms, we construct a set of harmonic oscillators centered at the bottom of each well. However, the quantum-mechanical tunneling dramatically changes this naive picture. As a matter of fact, the single energy eigenvalue transforms into a continuous band while the eigenstates are given in terms of Bloch waves. Our objective is to describe how the instanton method serves to explain these results, which can be obtained of course by means of conventional procedures in any solid-state physics course. More specific all, the periodic-potential is the most commonly used model for the study of electrons in a one-dimensional lattice.

Once we have carried out the discussion of the one-instanton amplitude, the dilute-gas approximation takes over a set of instantons and antiinstantons freely distributed along the τ-axis. As expected, each topological configuration (instanton or antiinstanton) starts just at the point where its predecessor ends. In addition, the number of instantons minus the number of antiinstantons represents the observed change in the coordinate x between the initial and final points of the transition amplitude at issue. For reference, we choose a concrete periodic-potential $V(x)$ given by

$$V(x) = \omega^4 \left[1 - \cos\left(\frac{x}{\omega}\right)\right] \tag{3.1}$$

so that the minima of $V(x)$ lie at

$$x = 2\pi j\omega, \qquad j \in Z \tag{3.2}$$

Notice that as usual, the minima satisfy $V(x) = 0$. This periodic-potential can be understood as the quantum-mechanical version of the well-grounded sine-Gordon model, which consists of a real scalar field $\phi(x,t)$ in $d = 1+1$ dimensions governed by a potential function similar to (3.1). As a matter of fact, the instanton we are interested in represents the soliton in the corresponding bidimensional field theory.

From a qualitative perspective, the starting point should be the so-called *tight-binding* approximation where the tunneling of the low-energy eigenkets from one well to the next one is irrelevant. In doing so, we have an infinitely degenerate ground-state with energy given by

$$E_o = \frac{V''(2\pi j\omega)}{2} \tag{3.3}$$

as corresponds to a set of harmonic oscillators distributed along the one-dimensional lattice. When considering the tunneling effects, the single level E_o yields a complete band, i.e.

$$E_\theta = E_o - \alpha\ \cos\theta \tag{3.4}$$

α being a constant to determine, while θ serves to denote the different states along the band itself [14].

Next, we explain the way in which the instanton calculus provides an accurate description of this Bloch wave. Starting, from scratch, we take into account the transition amplitude with $x_i = 0$ at $t_i = -T/2$ and $x_f = 2\pi\omega$ when $t_f = T/2$. In the process, the functional integral is dominated by the instanton which interpolates between the two adjoining minima of the potential $V(x)$. The explicit form of the topological configuration derives once more from the Bogomol'nyi condition. Solving (2.24) by means of a quadrature, we find

$$x_c(\tau) = 4\omega \arctan\left[\exp \omega(\tau - \tau_c)\right] \tag{3.5}$$

so that the action associated with $x_c(\tau)$ reads

$$S_{eo} = 8\omega^3 \tag{3.6}$$

Next, we must write the stability equation over the instanton, i.e.

$$-\frac{d^2}{d\tau^2} x_j(\tau) + \left[\omega^2 - \frac{2\omega^2}{cosh^2 \omega\tau}\right] x_j(\tau) = \epsilon_j x_j(\tau) \tag{3.7}$$

to obtain again a Posch-Teller potential. In any case, we conclude that

$$< x_f = 2\pi\omega | \exp(-HT) | x_i = 0 >= \left(\frac{\omega}{\pi}\right)^{1/2} \; (2 \sinh \omega T)^{-1/2}$$

$$\left\{ \frac{Det' \left[-(d^2/d\tau^2) + V''[x_c(\tau)]\right]}{Det \left[-(d^2/d\tau^2) + \omega^2\right]} \right\}^{-1/2} \; \sqrt{\frac{S_{eo}}{2\pi}} \; d\tau_c \tag{3.8}$$

since $V''(2\pi j\omega) = \omega^2$ while τ_c represents as usual the collective coordinate of the problem. Now we perform the change of variable

$$z = \omega\tau \tag{3.9}$$

so that the relevant quotient of determinants corresponds to

$$Q_1 = \frac{Det' \; O_1}{Det \; P_1} \tag{3.10}$$

where the global factor is now $\beta = \omega^2$. The discrete spectrum reduces itself to the zero-mode, while the scattering states have energies of the form

$$E_k = k^2 + 1 \tag{3.11}$$

for eigenkets like

$$\phi_{1,k}(z) = \frac{A_1^\dagger(z)}{\sqrt{k^2+1}} \; \left[\frac{\exp(ikz)}{\sqrt{2\pi}}\right] \tag{3.12}$$

In other words,

$$\phi_{1,k}(z) = (-ik + \tanh z) \; \frac{\exp(ikz)}{\sqrt{2\pi} \; \sqrt{k^2+1}} \tag{3.13}$$

As regards the expression of the regularized spectral density $\rho_r(k)$, we find that

$$\rho_r(k) = -\frac{1}{\pi(k^2+1)} \tag{3.14}$$

The structure that is at work corresponds therefore to the zeta-function $\zeta_r(s)$ given by

$$\zeta_r(s) = \zeta_{O_1}(s) - \zeta_{P_1}(s) \tag{3.15}$$

To be more specific,

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \left[-\frac{1}{\pi} \int_{-\infty}^\infty \frac{\exp[-(k^2+1)\mu]}{(k^2+1)} \, dk \right] \tag{3.16}$$

In accordance with (3.16), we find that

$$\zeta_r(s) = -\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \tag{3.17}$$

Taking advantage of the well-known properties of the Gamma Function, we obtain

$$\zeta_r(0) = -1 \tag{3.18}$$

together with (see appendix B)

$$\zeta_r'(0) = 2 \ln 2 \tag{3.19}$$

This approach provides the conventional ratio of determinants, i.e.

$$R = \frac{1}{4\omega^2} \tag{3.20}$$

which in turn gives rise to the transition amplitude we are looking for. To sum up

$$< x_f = 2\pi\omega | \exp(-HT) | x_i = 0 >=$$

$$\left(\frac{\omega}{\pi}\right)^{1/2} \; (2 \sinh \omega T)^{-1/2} \; \sqrt{S_{eo}} \, \sqrt{\frac{2}{\pi}} \; \exp(-S_{eo}) \; \omega \; d\tau_c \tag{3.21}$$

Now, when going to the dilute-gas approximation we must consider a set of instantons and antiinstantons so that each topological configuration starts where its predecessor ends. In terms of the transition amplitude (3.21), we require that the number of instantons (n) minus the number of antiinstantons ($\bar{n}$) fulfill the condition $n - \bar{n} = 1$ as corresponds to the jump between $x_i = 0$ and $x_f = 2\pi\omega$. Because of indistinguishability, the dilute-gas approximation includes a combinatorial factor F in order to eliminate the over-counting of configurations which correspond to the exchange of center-positions. For the amplitude (3.21), we find that

$$F = \frac{(n+\bar{n})!}{n! \; \bar{n}!} \tag{3.22}$$

Notice the difference with the double-well potential where the instantons strictly alternate with the antiinstantons, since the problem has only two minima so that the combinatorial factor is not necessary at all.

By integrating over the translational degrees of freedom, we obtain now a term I of the form

$$I = \frac{(\omega T)^{n+\bar{n}}}{(n+\bar{n})!} \tag{3.23}$$

Armed with this information, we can write the transition amplitude in the limit $T \to \infty$, i.e.

$$< x_f = 2\pi\omega | \exp(-HT) | x_i = 0 >= \left(\frac{\omega}{\pi}\right)^{1/2} \exp(-\omega T/2) \sum_{n,\bar{n}}^{\infty} \frac{(\omega T d)^{n+\bar{n}}}{n!\ \bar{n}!} \delta_{n-\bar{n},1} \tag{3.24}$$

where $\delta_{n-\bar{n},1}$ is the Kronecker symbol, while d stands as usual for the so-called instanton density, namely

$$d = \sqrt{\frac{2}{\pi}}\ \sqrt{S_{eo}}\ \exp(-S_{eo}) \tag{3.25}$$

with S_{eo} as given in (3.6). Now the sum over n and $\bar{n}$ decouples by resorting to the integral representation of the Kronecker symbol, i.e.

$$\delta_{n-\bar{n},1} = \int_0^{2\pi} \frac{d\theta}{2\pi}\ \exp[-i\theta(n - \bar{n} - 1)] \tag{3.26}$$

so that ultimately we have two independent exponential series. Consequently, our main result should be

$$< x_f = 2\pi\omega | \exp(-HT) | x_i = 0 >=$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \exp(i\theta) \left(\frac{\omega}{\pi}\right)^{1/2} \exp(2\omega d T \cos\theta - \omega T/2) \tag{3.27}$$

so that the left-hand side of (3.27) is dominated when $T \to \infty$ by a continuous band parametrised in terms of θ with $0 \leq \theta \leq 2\pi$. Therefore, the energy of the states contained in this low-lying band should be

$$E_\theta = \frac{\omega}{2} - 2\omega d \cos\theta \tag{3.28}$$

As expected, the result is consistent with the expression written in (3.4), so that we have described by means of the instanton calculus the behaviour of the lowest band of a periodic-potential.

4. Tunneling and decay.

Next, we take into account the decay of metastable states by tunneling processes. In principle, the states with a finite lifetime are analytically described by means of energy eigenvalues which appear in the lower half of a complex plane due to a negative imaginary part. According to the temporal evolution of a standard wave function, it is the case that such imaginary part represents a measure of the decay width Γ through the relation [5]

$$\Gamma = -2Im\ E \tag{4.1}$$

As a matter of fact, the negative sign of this imaginary contribution for the energy eigenvalues serves to prevent the existence of unphysical states with exponentially growing norm. Our purpose in this section should be the computation of the magnitude Γ for the lowest state of a concrete problem. From a physical point of view, we assume as usual a high barrier-potential so that the lifetime at issue is long. In other words, the state is approximately stationary so that a quasiclassical analysis represents the most suitable tool to discuss the main characteristics of the system.

More specifically, let us consider the point particle in the presence of a cubic potential like

$$V(x) = \frac{\omega^2}{2}x^2 - \frac{1}{6}x^3 \tag{4.2}$$

Clearly, the potential has a minimum located at $x = 0$. According to classical mechanics, once the particle is trapped in the single trough of $V(x)$ (with negligible kinetic energy) the stability is strictly guaranteed. However, the situation is dramatically different in the quantum regime because of tunneling. As a matter of fact, the particle can go towards $+\infty$ with a non-vanishing probability current there. Consequently, non-selfadjoint boundary conditions must be added to the hamiltonian so that the energy eigenvalues become complex. The question we wish to address now is the computation of Γ for the lowest state of (4.2). Qualitatively speaking, the classical configuration which dominates the tunneling process is called a bounce. As expected, the metastability properties derive from the behaviour of the fluctuations built over the classical configuration. When considering non-topological bounces, the spectrum of the stability equation has one negative eigenvalue, which is therefore responsible for the decay phenomenon. It may be interesting at this point to revisit the case of the double-well potential in this context.

A particle released from rest at the top of the left-hump of $-V(x)$ when $t_i = -\infty$ arrives at the other hump with velocity zero at time $t_f = \infty$. In other words, the velocity increases from zero to the corresponding maximum and decreases to recover asymptotically the zero again. However, this velocity represents the zero-mode of the stability equation. Simply put, such zero-mode is a nodeless wave function, so that the second variational derivative of the euclidean action is positive semidefinite.

In the semiclassical approximation, we shall compute the transition matrix element between $x_i = 0$ and $x_f = 0$. As usual, the development relies on the solution obtained by solving the imaginary-time equation of motion, which is equivalent to a conventional trajectory in the reversed potential $-V(x)$. The particle starts at $x_i = 0$, goes through the minimum of $-V(x)$ and then returns to the initial point. Because of this return motion the non-topological solution receives the name of bounce.

Next, we proceed in the same manner as in the previous sections. First of all, we resort again to the Bogomol'nyi condition to get the form of the bounce-like configuration, i.e.

$$x_c(\tau) = \frac{3\omega^2}{\cosh^2[\omega(\tau - \tau_c)/2]} \tag{4.3}$$

so that the euclidean action of (4.3) should be

$$S_{eo} = \frac{24\omega^5}{5} \tag{4.4}$$

The Schrodinger equation which governs the behaviour of the fluctuations over the bounce reads

$$-\frac{d^2}{d\tau^2}x_j(\tau) + \left[\omega^2 - \frac{3\omega^2}{cosh^2\omega\tau/2}\right]x_j(\tau) = \epsilon_j x_j(\tau) \tag{4.5}$$

In the case of this cubic potential, the velocity of the particle at issue has a zero located precisely at the turning point. To sum up, the second variational derivative of the euclidean action is expected to possess one lower even eigenket with negative energy eigenvalue. This is the point at which the path integral formalism introduces the metastability. However, the explicit derivation of $Im\ E_0$ requires a careful procedure.

First of all, we need to perform the integration over the eigenstate $x_{-1}(\tau)$ with eigenvalue $\epsilon_{-1} < 0$. For such a purpose, one takes into account the analytic continuation procedure through a continuous sequence of paths drawn in the functional space at issue. In order not to clutter the paper, we omit the mathematical details of the computation to emphasize the physical consequences of the result. The interested reader can find in [5] a complete description of the analytic continuation method as tailored to this specific integral. Therefore, we find that

$$\int \exp(-\epsilon_{-1} c_{-1}^2/2)\ \frac{dc_{-1}}{\sqrt{2\pi}} = \frac{i}{2} \frac{1}{\sqrt{|\epsilon_{-1}|}} \tag{4.6}$$

so that the tunneling rate formula will include the presence of the additional factor 1/2 together with the modulus of the quotient of determinants. According to these arguments, the transition amplitude over the bounce reads

$$< x_f = 0 | \exp(-HT) | x_i = 0 >= \left(\frac{\omega}{\pi}\right)^{1/2} \ (2 \sinh \omega T)^{-1/2}$$

$$\frac{i}{2} \left| \frac{Det' \left[-(d^2/d\tau^2) + V''[x_c(\tau)] \right]}{Det \left[-(d^2/d\tau^2) + \omega^2 \right]} \right|^{-1/2} \ \sqrt{\frac{S_{eo}}{2\pi}}\ d\tau_c \tag{4.7}$$

because $V''(0) = \omega^2$. Once we incorporate the change of variable (2.35), the ratio of determinants now corresponds to

$$Q_3 = \left| \frac{Det'\ (O_3 - 5)}{Det\ P_2} \right| \tag{4.8}$$

with the factor β given in (2.37). As regards the relevant discrete spectrum of $O_3 - 5$, we have $E_{-1} = -5$ and $E_1 = 3$ while the energy of the scattering states corresponds again to (2.38). As a matter of fact, these eigenkets derive from the plane waves, i.e.

$$\phi_{3,k}(z) = \frac{A_3^\dagger(z)}{\sqrt{k^2+9}} \frac{A_2^\dagger(z)}{\sqrt{k^2+4}} \ \frac{A_1^\dagger(z)}{\sqrt{k^2+1}} \ \left[\frac{\exp(ikz)}{\sqrt{2\pi}} \right] \tag{4.9}$$

to give

$$\phi_{3,k}(z) = \left[-\frac{15 \sinh z}{\cosh^3 z} - 6(k^2 - 1) \tanh z + ik \left(\frac{15}{\cosh^2 z} + k^2 - 11 \right) \right]$$

$$\frac{\exp(ikz)}{\sqrt{2\pi}\ \sqrt{k^2+9}\ \sqrt{k^2+4}\ \sqrt{k^2+1}} \tag{4.10}$$

Consequently, the regularized spectral density $\rho_r(k)$ is of the form

$$\rho_r(k) = -\frac{6(k^4 + 8k^2 + 11)}{\pi(k^2+9)(k^2+4)(k^2+1)} \tag{4.11}$$

As expected, the zeta-function $\zeta_r(s)$ reads

$$\zeta_r(s) = \zeta_{O_{3-5}}(s) - \zeta_{P_2}(s) \tag{4.12}$$

so that

$$\zeta_r(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu$$

$$\left[\exp(-3\mu) + \exp(-5\mu) - \frac{6}{\pi} \int_{-\infty}^{\infty} \frac{(k^4 + 8k^2 + 11) \exp[-(k^2 + 4)\mu]}{(k^2 + 9)(k^2 + 4)(k^2 + 1)} \; dk\right] \tag{4.13}$$

In other words,

$$\zeta_r(s) = \frac{1}{3^s} + \frac{1}{5^s} - \frac{6}{\pi} \int_{-\infty}^{\infty} \frac{(k^4 + 8k^2 + 11)}{(k^2 + 9)(k^2 + 1)(k^2 + 4)^{s+1}} \; dk \tag{4.14}$$

Breaking (4.14) into more simple integrals as usual we find

$$\zeta_r(s) = \frac{1}{3^s} + \frac{1}{5^s} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s}} \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma(s+1)} - \frac{3}{\sqrt{\pi}} \frac{1}{2^{2s+3}} \frac{\Gamma\left(s + \frac{3}{2}\right)}{\Gamma(s+2)} \; F\left(s + \frac{3}{2}, 1, s+2, \frac{3}{4}\right) +$$

$$+ \frac{5}{\sqrt{\pi}} \frac{1}{3^{2s+2}} \frac{\Gamma\left(s + \frac{3}{2}\right)}{\Gamma(s+2)} \; F\left(s + \frac{3}{2}, s+1, s+2, \frac{5}{9}\right) \tag{4.15}$$

If we bear in mind that

$$F\left(\frac{3}{2}, 1, 2, \frac{5}{9}\right) = \frac{9}{5} \tag{4.16}$$

we get again that

$$\zeta_r(0) = -1 \tag{4.17}$$

Taking advantage of the explicit results collected in appendix B the transition amplitude at issue reduces itself to

$$< x_f = 0 | \exp(-HT) | x_i = 0 >= \left(\frac{\omega}{\pi}\right)^{1/2} \; (2 \sinh \omega T)^{-1/2}$$

$$\frac{i}{2} \sqrt{S_{eo}} \; \sqrt{\frac{30}{\pi}} \; \exp(-S_{eo}) \; \omega \; d\tau_c \tag{4.18}$$

To close the section, we evaluate the transition amplitude by summing over configurations which contain a set of separated bounces. In this case, we have no restriction to an even or odd number of bounces, so that the complete exponential series arises. To sum up,

$$< x_f = 0 | \exp(-HT) | x_i = 0 >= \left(\frac{\omega}{\pi}\right)^{1/2} \; \exp(-\omega T/2) \; \exp(id\omega/2) \tag{4.19}$$

for

$$d = \sqrt{\frac{30}{\pi}}\ \sqrt{S_{eo}}\ \exp(-S_{eo}) \tag{4.20}$$

In doing so we obtain the final expression of the decay width, i.e.

$$\Gamma = \omega\sqrt{\frac{30}{\pi}}\ \sqrt{S_{eo}}\ \exp(-S_{eo}) \tag{4.21}$$

where the prefactor in (4.21) is known as the *quantum attempt frequency*, while the second term is interpreted as the Boltzman weight for the appearance of the bounce, which ultimately leads the tunneling phenomenon.

5. Appendix A.

In this article, we focus the attention on the set of hamiltonians O_ℓ given by

$$O_\ell = -\frac{d^2}{dz^2} - \frac{\ell(\ell+1)}{\cosh^2 z} + \ell^2 \tag{5.1}$$

where $\ell = 1, 2, ...$, although only the first members of the series lead to relevant models in physics. The operator O_ℓ can be factorized in terms of a superpotential $W(z,\ell)$ like [1]

$$W(z,\ell) = \ell \tanh z \tag{5.2}$$

so that $O_\ell = Q_\ell^\dagger Q_\ell$ for

$$Q_\ell = \frac{d}{dz} + \ell \tanh z \tag{5.3}$$

$$Q_\ell^\dagger = -\frac{d}{dz} + \ell \tanh z \tag{5.4}$$

The shape-invariance condition appears once we write the partner hamiltonian $\tilde{O}_\ell$, so that in our case the mapping between the old parameter ℓ and the new one $\tilde{\ell}$ reduces to

$$\tilde{\ell} = \ell - 1 \tag{5.5}$$

By iteration of the procedure, we construct a family of well-behaved superpotentials which allow us to solve O_ℓ. The discrete spectrum includes as usual a normalizable zero-energy mode $\phi_{\ell,o}(z)$ of the form

$$\phi_{\ell,o}(z) = \frac{\sqrt{2(2\ell-1)!}}{2^\ell(\ell-1)!}\ \frac{1}{\cosh^\ell z} \tag{5.6}$$

together with the set of states $\phi_{\ell,m}(z)$ given by

$$\phi_{\ell,m}(z) = \frac{\sqrt{2(2\ell-2m-1)!}}{2^{\ell-m}(\ell-m-1)!}\ \frac{1}{\sqrt{\prod_{j=0}^{m-1}(E_m-E_j)}}\ Q_\ell^\dagger(z)\ \ldots\ Q_{\ell-m+1}^\dagger(z)\ \left[\frac{1}{\cosh^{\ell-m} z}\right] \tag{5.7}$$

with energies

$$E_m = \ell^2 - (\ell - m)^2 \tag{5.8}$$

for $m = 1, ..., \ell - 1$. The continuous spectrum corresponds to

$$\phi_{\ell,k}(z) = \frac{Q_\ell^\dagger(z)}{\sqrt{k^2+\ell^2}} \; \frac{Q_{\ell-1}^\dagger(z)}{\sqrt{k^2+(\ell-1)^2}} \cdots \frac{Q_1^\dagger(z)}{\sqrt{k^2+1}} \left[\frac{\exp(ikz)}{\sqrt{2\pi}}\right] \tag{5.9}$$

with energy eigenvalues

$$E_k = k^2 + \ell^2 \tag{5.10}$$

and normalization as

$$\int_{-\infty}^{\infty} \phi_{\ell,k}^*(z) \; \phi_{\ell,k'}(z) \; dz = \delta(k - k') \tag{5.11}$$

Let us now sketch the way in which we can compute the ratio of the determinants associated with O_ℓ and P_ℓ, which is given by

$$P_\ell = -\frac{d^2}{dz^2} + \ell^2 \tag{5.12}$$

As a matter of fact, P_ℓ itself represents a hamiltonian of comparison for O_ℓ. In such a case, we need to deal with

$$Q = \frac{Det' \; O_\ell}{Det \; P_\ell} \tag{5.13}$$

where $Det' \; O_\ell$ denotes as usual the reduced determinant once the zero-mode has been explicitly removed. To take into account the continuous spectrum, we turn to the density matrix $\Xi_{O_\ell}(k, z, w)$ written as [15]

$$\Xi_{O_\ell}(k, z, w) = \phi_{\ell,k}^*(z) \; \phi_{\ell,k}(w) \tag{5.14}$$

When going to P_ℓ we get

$$\Upsilon_{P_\ell}(k, z, w) = \frac{\exp -ik(z-w)}{2\pi} \tag{5.15}$$

as expected for a free-particle. The conventional subtraction procedure yields the regularized spectral density $\rho_r(k)$ expressed as

$$\rho_r(k) = \int_{-\infty}^{\infty} \left[\Xi_{O_\ell}(k, z, z) - \Upsilon_{P_\ell}(k, z, z)\right] \; dz \tag{5.16}$$

In this scheme, the zeta-function for the analysis of (5.13) would be [9]

$$\zeta_r(s) = \zeta_{O_\ell}(s) - \zeta_{P_\ell}(s) \tag{5.17}$$

and therefore,

$$\begin{aligned}\zeta_r(s) =& \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} d\mu \left\{ \exp\left[-\sum_{m=1}^{\ell-1} [\ell^2 - (\ell-m)^2]\mu\right] \right. \\ & \left. + \int_{-\infty}^{\infty} \rho_r(k) \exp\left[-(k^2+\ell^2)\mu\right] \; dk \right\}\end{aligned} \tag{5.18}$$

6. Appendix B.

Let us come down to the concrete details which allow the evaluation of $\zeta_r'(0)$ when considering the instanton of the double-well potential. According to the expression of $\zeta_r(s)$ (see (2.47)), we find that

$$\zeta_r'(0) = -\ln 3 + 8\ln 2 - \frac{1}{2} - \frac{3}{16} F'(s+\frac{3}{2}, 1, s+2, \frac{3}{4})|_{s=0} \tag{6.1}$$

once we take into account that [12]

$$\Gamma'(\frac{1}{2}) = -\sqrt{\pi}\ (\gamma + 2\ln 2) \tag{6.2}$$

$$\Gamma'(1) = -\gamma \tag{6.3}$$

$$\Gamma'(\frac{3}{2}) = -\sqrt{\pi}\ (\frac{\gamma}{2} + \ln 2 - 1) \tag{6.4}$$

$$\Gamma'(2) = -\gamma + 1 \tag{6.5}$$

where γ is the Euler's constant. To obtain the derivative of the hypergeometric function, we turn to the integral representation, namely [12]

$$F(s+\frac{3}{2}, 1, s+2, \frac{3}{4}) = \frac{\Gamma(s+2)}{\Gamma(1)\ \Gamma(s+1)} \int_0^1 (1-t)^s \left(1 - \frac{3t}{4}\right)^{-s-\frac{3}{2}} dt \tag{6.6}$$

so that

$$F'(s+\frac{3}{2}, 1, s+2, \frac{3}{4})|_{s=0} = I_1 + I_2 + I_3 \tag{6.7}$$

for integrals of the form

$$I_1 = \int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} dt \tag{6.8}$$

$$I_2 = \int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} \ln(1-t)\ dt \tag{6.9}$$

$$I_3 = -\int_0^1 \left(1 - \frac{3t}{4}\right)^{-\frac{3}{2}} \ln\left(1 - \frac{3t}{4}\right) dt \tag{6.10}$$

If we have

$$I_1 = \frac{8}{3} \tag{6.11}$$

$$I_2 = \frac{32}{3} \ln\frac{2}{3} \tag{6.12}$$

$$I_3 = -\frac{16}{3} + \frac{32}{3}\ln 2 \tag{6.13}$$

then $\zeta_r'(0)$ is obviously

$$\zeta_r'(0) = 4\ln 2 + \ln 3 \tag{6.14}$$

In terms of the bounce solution of the cubic potential, we obtain that

$$\zeta_r'(0) = 9\ln 2 - \ln 5 + \frac{1}{2} + \frac{5}{18}F'(s+\frac{3}{2}, s+1, s+2, \frac{5}{9})|_{s=0} \tag{6.15}$$

Taking into account that

$$F(s+\frac{3}{2}, s+1, s+2, \frac{5}{9}) = \frac{\Gamma(s+2)}{\Gamma(1)\ \Gamma(s+1)} \int_0^1 t^s \left(1-\frac{5t}{9}\right)^{-s-\frac{3}{2}} dt \tag{6.16}$$

we find again a situation of the form

$$F'(s+\frac{3}{2}, s+1, s+2, \frac{3}{4})|_{s=0} = I_1 + I_2 + I_3 \tag{6.17}$$

where, in this case, we have

$$I_1 = \int_0^1 \left(1-\frac{5t}{9}\right)^{-\frac{3}{2}} dt \tag{6.18}$$

$$I_2 = \int_0^1 \left(1-\frac{5t}{9}\right)^{-\frac{3}{2}} \ln t\ dt \tag{6.19}$$

$$I_3 = -\int_0^1 \left(1-\frac{5t}{9}\right)^{-\frac{3}{2}} \ln\left(1-\frac{5t}{9}\right) dt \tag{6.20}$$

After a tedious but straightforward computation, we can write that

$$I_1 = \frac{9}{5} \tag{6.21}$$

$$I_2 = \frac{36}{5}\ln 5 - \frac{36}{5}\ln 2 - \frac{36}{5}\ln 3 \tag{6.22}$$

$$I_3 = -\frac{54}{5}\ln 2 + \frac{54}{5}\ln 3 - \frac{18}{5} \tag{6.23}$$

with $\zeta_r'(0)$ corresponding to

$$\zeta_r'(0) = 4\ln 2 + \ln 3 + \ln 5 \tag{6.24}$$

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