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TRANSPORT IN VISCOUS ROTATING FLUIDS *

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Abstract. We consider a uniformly rotating viscous incompressible fluid and estimate particle transport in the vertical direction (parallel to the rotation axis). We prove that for short time and regular initial data, strong rotation suppresses the vertical gradient of flow maps. The proof uses a diffusive Lagrangian formalism, and the suppression of the vertical gradient is a natural and direct byproduct of the formalism.

Key words. diffusive Lagrangian, viscous rotating fluid, Rossby number

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1. Introduction

We consider a container filled with a viscous fluid of kinematic viscosity ν, and rotating with a constant angular velocity Ω. The Navier-Stokes equations written in a frame rotating with the container are ([4])

$$\partial_t u + (u\cdot\nabla)u - \nu\triangle u + \nabla\pi + 2\Omega e_3\times u = 0 \tag{1.1}$$

$$\nabla\cdot u = 0 \tag{1.2}$$

where e_3 is the vertical unit vector, $\pi = \frac{p}{\varrho} - \frac{1}{2}|\Omega\times r|^2$, p is the pressure and ϱ the density.

In the absence of viscosity and inertia, the Taylor-Proudman theorem [4] implies that all steady slow motions in a rotating fluid are necessarily two dimensional. In classical experiments, G. I. Taylor verified this by injecting ink droplets in a uniformly rotating tank of water. He observed that if the motion was slow and steady, the ink droplets were drawn into thin sheets which remained parallel to each other and mutually perpendicular to the axis of rotation.

A well developed mathematical approach to study the effect of strong rotation on fluids is based on the averaging method. The Coriolis force introduces a linear, anti-symmetric perturbation to the nonlinear Navier-Stokes equations, and in the limit of very strong rotation only the resonant terms survive. One class of resonant terms represent two dimensional motion (see [3]). In the context of fluids, this strategy has been used by many authors ([1], [2], [3], [5], [11], [15]). The averaging method has the benefit that it leads to effective equations describing all fully three-dimensional resonant interactions. Global in time regularity of the resonant equations was proved in [3].

In this paper we employ a different approach, based on diffusive Lagrangian transformations [9]. For short time and regular initial data ($u_0\in H^{\frac{7}{2}+}$), the suppression of vertical transport in the presence of rotation is a natural and direct byproduct of the formalism.

We consider maps representing the particle motion. We show that for short time, and regular initial data the gradient in the direction of the rotation axis (vertical gradient) of the flow map and of the diffusive flow map are suppressed when the rotation is strong. More precisely, let $\bar{X}(a,t)$ be the position at time t of a particle

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in the fluid starting from a at time 0. If the initial velocity is smooth, then standard theory shows the existence of a time interval $[0,T]$ (independent of the forcing rotation Ω) where the flow map $\bar{X}$ is smooth. We show that for all $t\in[0,T]$

$$\left|\frac{\partial}{\partial a_3}\left(\bar{X}(a,t)-a\right)\right|\leqslant K\left(\rho+\nu t\|C\|^2_{L^\infty(I\times[0,T])}+\nu t\|\nabla C\|_{L^\infty(I\times[0,T])}\right)$$

where C are the commutator coefficients (defined in (2.4)), $\frac{\partial}{\partial a_3}=e_3\cdot\nabla_a$ is derivative in the vertical direction, K is a non-dimensional constant and ρ is the maximal local Rossby number. The maximal local Rossby number is defined by

$$\rho=\sup_{t\in[0,T]}\frac{\|\omega(\cdot,t)\|_{L^\infty}}{\Omega} \tag{1.3}$$

where ω is the relative vorticity, $\omega=\nabla\times u$. We also remark (in Section 5) that for given (regular) initial data, the commutator coefficients C can be bounded independently of Ω. Finally, if X is the diffusive flow map of the fluid (defined below) we show

$$\left|\frac{\partial}{\partial a_3}\left(X(a,t)-a\right)\right|\leqslant K\left(\rho+\nu t\|C\|^2_\infty\right).$$

As $\rho\to 0$ and $\nu\to 0$ one obtains a complete suppression of vertical displacement in both the traditional and diffusive flow maps. (The traditional and the diffusive maps coincide when $\nu=0$.) One should also note that, for smooth flows, the viscous correction is smaller than $O(\sqrt{\nu t})$, the correction one would get by superposing Brownian vertical drift to horizontal rigid body rotation [14].

The Eulerian-Lagrangian formulation of the Euler and Navier-Stokes equations developed in [8], [9] was extended to rotating fluids in [10]. The suppression of vertical transport was proved there only for $\nu=0$. We provide a brief introduction to the Euler-Lagrangian formulation below, and then proceed to prove suppression of the vertical gradient in the viscous case.

2. Introduction to the Eulerian-Lagrangian formulation

We first define the diffusive evolution operator $\mathcal{D}_{\nu,u}$ by

$$\mathcal{D}_{\nu,u}=\partial_t+(u\cdot\nabla)-\nu\triangle$$

where $\nu\geqslant 0$ and u is any divergence-free function. In our context, we always think of ν as the viscosity of the fluid, and u as the velocity. Where there is no ambiguity about ν and u, we will drop the subscript. Notice that when there is no viscosity, $\mathcal{D}_0$ is the convective derivative.

In [10] it was shown that the equations (1.1) – (1.2) are equivalent to the system of equations:

$$\mathcal{D}_{\nu,u}A=0 \tag{2.1}$$

$$u=\mathbb{P}((\nabla A)^{\mathrm{t}}v) \tag{2.2}$$

$$\mathcal{D}_{\nu,u}v_\beta=2\nu C^i_{j,\beta}\partial_j v_i+2\Omega\nu\left(e_3,\partial_j A,C^{\cdot}_{j,\beta}\right) \tag{2.3}$$

$$C^\alpha_{j,i}=(\nabla A)^{-1}_{ki}\partial_k\partial_j A_\alpha \tag{2.4}$$

with initial data

$$A(x,0)=x \tag{2.5}$$

$$v(x,0)=u_0(x). \tag{2.6}$$

In the absence of viscosity, A is the inverse of the flow map of the fluid – the 'back to labels map' or inverse Lagrangian. That is, if $X(a,t)$ is the position of a particle starting at a at time t, then $A(X(a,t),t)=a$ for all time. With viscosity, A is a diffusive analogue. The traditional particle paths are characteristics of the underlying hyperbolic system. In the presence of viscosity these characteristics no longer carry enough information. The diffusive maps do.

The velocity u is recovered from the diffusive inverse Lagrangian by means of the Webber formula (2.2). The $\mathbb{P}$ is the Leray-Hodge projector onto divergence free vector fields. $\mathbb{P}$ can be expressed as a combination of Riesz transforms (see for instance [6] and [16]).

In the absence of viscosity, the evolution equation of v becomes

$$\partial_t v + (u\cdot\nabla)v = 0.$$

Thus v is passively advected by the fluid, and hence

$$v = u_0 \circ A. \tag{2.7}$$

In the presence of viscosity however, v is no longer passively advected, and the evolution of v is governed by (2.3) where we use the notation (u,v,w) to denote the determinant of the 3×3 matrix with columns u, v, and w. We call v the *virtual velocity.*

One natural question to ask is about the invertability of $A(\cdot,t)$. In the absence of viscosity, A is the inverse of the flow map of the fluid: it is invertible. In the presence of viscosity, it is known [9] that

$$\mathcal{D}\ln\det(\nabla A) = \nu C^i_{k;s} C^s_{k;i} \tag{2.8}$$

showing that for small time t, the determinant of ∇A is non zero, and hence A is locally invertible.

Global invertability is a more topological question. If for short time T the velocity u is differentiable enough, then parabolic regularity [12] will ensure A is continuous as a function of space and time. In this case the map $A(\cdot,t)$ provides a continuous homotopy between the identity map and $A(\cdot,T)$ showing that the topological degree [13] of A is 1. For short time the determinant of ∇A is non zero (hence positive), so A is always orientation preserving. Combined with the fact that the degree of A is 1, A must be bijective. Thus for short time T, one can invert A spatially. Long times would require resettings; this is not in the scope of the present work.

As A can be inverted, we consider the coordinate frame given by the map A. We define the *Lagrangian gradient* ∂^A to be the gradient in the coordinates given by A. The commutator coefficients $C^k_{j,i}$ arise from the commutator relation

$$\partial^A_i \partial_j - \partial_j \partial^A_i = [\partial^A_i, \partial_j] = C^k_{j,i} \partial^A_k \tag{2.9}$$

between the Eulerian gradient ∂ and the Lagrangian gradient ∂^A. In [9] the commutator relation between the Lagrangian gradient ∂^A and the diffusive evolution operator $\mathcal{D}_\nu$ was shown to be

$$[\mathcal{D}_{\nu,u}, \partial^A_i] = 2\nu C^k_{j,i} \partial_j \partial^A_k. \tag{2.10}$$

As only the evolution equation of A is used in proving (2.10), this relation remains unchanged in the rotating frame.

We define the diffusive Lagrangian X by the equation

$$X(A(x,t),t)=x$$

and the diffusive Lagrangian displacements by

$$\ell(x,t)=A(x,t)-x$$
$$\lambda(a,t)=X(a,t)-a.$$

We study the equations (2.1) – (2.6) with spatially periodic boundary conditions (and period 1) along coordinate directions. For this we demand that the diffusive Lagrangian displacement ℓ and the initial data u_0 to be periodic. We establish some notation: we let $I=[0,1]^3$ be the unit cube and, by convention, all L^p norms will be in space only (not in space and time), unless explicitly indicated.

In this formulation we show that $\|\partial_3\lambda\|_\infty \leqslant O(\rho+\nu t\|C\|^2_{L^\infty(I\times[0,T])})$. Further, in the limit $\Omega\to\infty$, $\|\partial_3\lambda\|_\infty \leqslant O(\nu t\|C\|^2_{L^\infty(I\times[0,T])})$. We then compare the diffusive Lagrangian paths and Lagrangian paths, and show that they differ (in C^1) by $O(\nu t[\|C\|^2_{L^\infty(I\times[0,T])}+\|\nabla C\|_{L^\infty(I\times[0,T])}])$. Combining the two results we obtain that vertical gradient of the Lagrangian displacement is $O(\rho+\nu t[\|C\|^2_{L^\infty(I\times[0,T])}+\|\nabla C\|_{L^\infty(I\times[0,T])}])$.

3. Evolution and bounds for the virtual vorticity

We begin by obtaining bounds for the virtual vorticity ζ (defined below) in terms of the original vorticity ω_0 and the forcing rotation Ω.

DEFINITION 3.1. *We define the virtual vorticity ζ to be the Lagrangian curl of the virtual velocity (i.e. $\zeta=\partial^A\times v$).*

PROPOSITION 3.2. *The evolution equation for the virtual vorticity is given by*

$$\mathcal{D}\zeta_\gamma = 2\nu C^i_{j,i}\partial_j\zeta_\gamma - 2\nu C^\gamma_{j,k}\partial_j\zeta_k + \nu\epsilon_{\alpha\beta\gamma}\epsilon_{kim}C^i_{j,\beta}C^k_{j,\alpha}\zeta_m + 2\Omega\nu\epsilon_{\alpha\beta\gamma}\big(e_3, C^{\cdot}_{j,\alpha}, C^{\cdot}_{j,\beta}\big). \quad (3.1)$$

Proof. Recall the formula

$$\nabla\times v = \epsilon_{ijk}\partial_i v_j e_k$$

where ϵ_{ijk} is the signature. We apply $\epsilon_{\alpha\beta\gamma}\partial^A_\alpha$ to both sides of (2.3) and use the commutator relation (2.10) to obtain

$$\mathcal{D}\zeta_\gamma = 2\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha C^i_{j,\beta}\partial_j v_i + 2\nu\epsilon_{\alpha\beta\gamma}C^i_{j,\beta}\partial^A_\alpha\partial_j v_i + 2\nu\epsilon_{\alpha\beta\gamma}C^i_{j,\alpha}\partial_j\partial^A_i v_\beta + 2\Omega\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha\big(e_3, \partial_j A, C^{\cdot}_{j,\beta}\big). \quad (3.2)$$

Using equation (2.4) we see

$$2\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha C^i_{j,\beta}\partial_j v_i = 2\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha\partial^A_\beta\partial_j A_i\partial_j v_i.$$

As an interchange of α and β produces a sign change, the term on the right is 0, and hence

$$2\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha C^i_{j,\beta}\partial_j v_i = 0. \tag{3.3}$$

We now deal with the second and third terms on the right hand side of (3.2). Interchanging α and β in the third term, and using the commutator relation (2.9) we have

$$\begin{aligned}
&2\nu\epsilon_{\alpha\beta\gamma}C^i_{j,\beta}\partial^A_\alpha\partial_j v_i + 2\nu\epsilon_{\beta\alpha\gamma}C^i_{j,\beta}\partial_j\partial^A_i v_\alpha \\
&\quad = 2\nu\epsilon_{\alpha\beta\gamma}\left(C^i_{j,\beta}\partial_j\left[\partial^A_\alpha v_i - \partial^A_i v_\alpha\right]\right) + 2\nu\epsilon_{\alpha\beta\gamma}C^i_{j,\beta}\left[\partial^A_\alpha\partial_j - \partial_j\partial^A_\alpha\right]v_i \\
&\quad = 2\nu\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha ik}C^i_{j,\beta}\partial_j\zeta_k + 2\nu\epsilon_{\alpha\beta\gamma}C^i_{j,\beta}C^k_{j,\alpha}\partial^A_k v_i \\
&\quad = 2\nu\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha ik}C^i_{j,\beta}\partial_j\zeta_k + \nu\epsilon_{\alpha\beta\gamma}C^i_{j,\beta}C^k_{j,\alpha}(\partial^A_k v_i - \partial^A_i v_k) \\
&\quad = 2\nu C^i_{j,i}\partial_j\zeta_\gamma - 2\nu C^\gamma_{j,k}\partial_j\zeta_k + \nu\epsilon_{\alpha\beta\gamma}\epsilon_{kim}C^i_{j,\beta}C^k_{j,\alpha}\zeta_m.
\end{aligned} \tag{3.4}$$

We finally deal with the last term of (3.2). Using equation (2.9) we have

$$\begin{aligned}
\epsilon_{\alpha\beta\gamma}\partial^A_\alpha(e_3,\partial_j A, C^\cdot_{j,\beta}) &= \epsilon_{\alpha\beta\gamma}(e_3,\partial^A_\alpha\partial_j A, C^\cdot_{j,\beta}) + \epsilon_{\alpha\beta\gamma}(e_3,\partial_j A,\partial^A_\alpha C^\cdot_{j,\beta}) \\
&= \epsilon_{\alpha\beta\gamma}(e_3, C^\cdot_{j,\alpha}, C^\cdot_{j,\beta}) + \epsilon_{\alpha\beta\gamma}(e_3,\partial_j A,\partial^A_\alpha\partial^A_\beta\partial_j A).
\end{aligned}$$

Again, note that interchanging α and β, the last term in (3.5) changes sign, and hence must be 0. Thus

$$2\Omega\nu\epsilon_{\alpha\beta\gamma}\partial^A_\alpha(e_3,\partial_j A, C^\cdot_{j,\beta}) = 2\Omega\nu\epsilon_{\alpha\beta\gamma}(e_3, C^\cdot_{j,\alpha}, C^\cdot_{j,\beta}). \tag{3.5}$$

Combining equations (3.3), (3.4) and (3.5) we are done. ∎

We now prove bounds for the virtual vorticity.

Proposition 3.3. *There exists an absolute constant K such that*

$$\|\zeta\|_{L^\infty(I\times[0,T])} \leqslant \|\omega_0\|_\infty e^{\int_0^T K\nu\|C\|^2_\infty} + \Omega\left(e^{\int_0^T K\nu\|C\|^2_\infty} - 1\right)$$

where $\omega_0 = \nabla\times u_0$ is the initial vorticity. The constant K arises from counting the number of terms in the expression $C^k_{j,i}$ and can be computed explicitly.

Proof. We let $|\nabla\zeta|$ denote the Euclidean norm of $\nabla\zeta$. Let K be a constant which changes from line to line. Starting with the product rule for $\mathcal{D}$ and using equation (3.1) we have

$$\begin{aligned}
&\mathcal{D}\zeta^2_\gamma = 2\zeta_\gamma\mathcal{D}\zeta_\gamma - 2\nu|\nabla\zeta_\gamma|^2 \\
\implies\quad &\mathcal{D}|\zeta|^2 + 2\nu|\nabla\zeta|^2 \leqslant \nu K\|C\|_\infty|\zeta||\nabla\zeta| + K\nu\|C\|^2_\infty|\zeta|^2 + K\Omega\nu\|C\|^2_\infty|\zeta| \\
&\qquad\leqslant \nu|\nabla\zeta|^2 + K\nu\|C\|^2_\infty\left(|\zeta|^2 + \Omega|\zeta|\right) \\
\implies\quad &\mathcal{D}|\zeta|^2 + \nu|\nabla\zeta|^2 \leqslant K\nu\|C\|^2_\infty\left(|\zeta|^2 + \Omega|\zeta|\right).
\end{aligned}$$

Multiplying by $|\zeta|^{p-2}$ and integrating over I gives

$$\begin{aligned}
2\int_I|\zeta|^{p-1}\partial_t|\zeta| + 2\int_I|\zeta|^{p-1}(u\cdot\nabla)|\zeta| - \nu\int_I|\zeta|^{p-2}\triangle|\zeta|^2 \\
\leqslant K\nu\|C\|^2_\infty\left[\int_I\Omega|\zeta|^{p-1} + \int_I|\zeta|^p\right].
\end{aligned} \tag{3.6}$$

Integrating by parts we find

$$\begin{aligned}
\int_I |\zeta|^{p-1}(u.\nabla)|\zeta| &= -\int_I |\zeta|(u.\nabla)|\zeta|^{p-1} \\
&= -(p-1)\int_I |\zeta|^{p-1}(u.\nabla)|\zeta| \\
\implies \quad \int_I |\zeta|^{p-1}(u.\nabla)|\zeta| &= 0. \qquad (3.7)
\end{aligned}$$

Further, when $p > 2$,

$$\int_I |\zeta|^{p-2}\triangle|\zeta|^2 = -2(p-2)\int_I |\zeta|^{p-3}\left|\nabla|\zeta|^2\right|^2 \leqslant 0. \qquad (3.8)$$

Substituting (3.7) and (3.8) in (3.6) we have

$$\begin{aligned}
\frac{2}{p}\int_I \partial_t|\zeta|^p &\leqslant K\nu\|C\|_\infty^2\left[\Omega\|\zeta\|_{p-1}^{p-1} + \|\zeta\|_p^p\right] \\
\implies \quad 2\|\zeta\|_p^{p-1}\partial_t\|\zeta\|_p &\leqslant K\nu\|C\|_\infty^2\left[\Omega\|\zeta\|_p^{p-1} + \|\zeta\|_p^p\right] \qquad (3.9) \\
\implies \quad \partial_t\|\zeta\|_p &\leqslant K\nu\|C\|_\infty^2\left[\Omega + \|\zeta\|_p\right].
\end{aligned}$$

In (3.9) we used $\|\zeta\|_{p-1} \leqslant \|\zeta\|_p$, which is true because I has measure 1. Finally, integrating out the above ODE and solving we obtain

$$\|\zeta\|_p \leqslant \|\zeta_0\|_p e^{\int_0^T K\nu\|C\|_\infty^2} + \Omega\left(e^{\int_0^T K\nu\|C\|_\infty^2} - 1\right).$$

At time 0 the Lagrangian coordinates and Eulerian coordinates are the same, so we have

$$\zeta_0 = \partial^A \times v_0 = \nabla \times u_0 = \omega_0.$$

Finally, as the domain has measure 1,

$$\|\zeta\|_\infty = \lim_{p\to\infty}\|\zeta\|_p$$

and the proposition follows. □

4. Bounds for $\partial_3\ell$ and $\partial_3\lambda$

Theorem 4.1. *Consider a short time T such that*

$$\|\nabla\ell\|_{L^\infty(I\times[0,T])} = \sup_{1\leqslant i,j\leqslant 3}\|\partial_j\ell_i\|_{L^\infty(I\times[0,T])} \leqslant g. \qquad (4.1)$$

There exists a constant G depending only on g, and an absolute constant K (which arises as in Proposition 3.3) such that

$$\|\partial_3\lambda\|_{L^\infty(I\times[0,T])} \leqslant G\left[e^{\int_0^T K\nu\|C\|_\infty^2}\rho + e^{\int_0^T K\nu\|C\|_\infty^2} - 1\right] \qquad (4.2)$$

$$\text{and} \quad \|\partial_3\ell\|_{L^\infty(I\times[0,T])} \leqslant G\left[e^{\int_0^T K\nu\|C\|_\infty^2}\rho + e^{\int_0^T K\nu\|C\|_\infty^2} - 1\right]. \qquad (4.3)$$

Recall that ρ is the Rossby number is defined in (1.3).

REMARK 4.1. *Notice the right hand sides of* (4.2) *and* (4.3) *are* $O(\rho+\nu T\|C\|^2_{L^\infty(I\times[0,T])})$.

Before beginning the proof, we recall a few preliminaries.

DEFINITION 4.2. *If M is a 3×3 matrix, and $v\in\mathbb{R}^3$ we define $\mathcal{C}(M,v)$ by*

$$\mathcal{C}(M,v)=\frac{1}{2}\epsilon_{ijk}(M_{\cdot j},M_{\cdot k},v)e_i.$$

Here $M_{\cdot i}$ denotes the i^{th} column of the matrix M, and the notation (u,v,w) refers to the determinant of the 3×3 matrix with columns u, v and w.

LEMMA 4.3. *If M is invertible, then $\mathcal{C}(M,v)=\det(M)M^{-1}v$.*

Proof. It is enough to show that $M\mathcal{C}(M,v)=\det(M)v$. Notice that

$$\begin{aligned}M\mathcal{C}(M,v)&=M_{ni}\mathcal{C}(M,v)_ie_n\\&=\frac{1}{2}\epsilon_{ijk}\epsilon_{pqr}M_{ni}M_{pj}M_{qk}v_re_n\\&=\frac{1}{2}\epsilon_{pqr}\det(M)\delta_{nr}v_re_n\\&=\det(M)v.\end{aligned}$$

□

We finally recall the Cauchy formula [10], which states

$$\omega+2\Omega e_3=\mathcal{C}(\nabla A,\zeta+2\Omega e_3) \tag{4.4}$$

where $\omega=\nabla\times u$. The Cauchy formula can be proved by direct computation starting from (2.2). We do not reproduce it here, and refer the reader to [10]. We also remark that in the absence of rotation the Cauchy formula [9] is

$$\omega=\mathcal{C}(\nabla A,\zeta)$$

confirming the understanding [4] that in the presence of rotation, $\omega+2\Omega e_3$ plays the same role as ω in the absence of rotation. We are now ready for the proof of Theorem 4.1.

Proof. [Proof of Theorem 4.1]   From Cauchy formula (4.4) we have

$$\begin{aligned}\omega+2\Omega e_3&=\mathcal{C}(\nabla A,\zeta+2\Omega e_3)\\&=\mathcal{C}(\nabla A,\zeta)+2\Omega\cdot\mathcal{C}(\nabla A,e_3)\\&=\mathcal{C}(\nabla A,\zeta)+2\Omega\det(\nabla A)\cdot(\nabla X e_3),\\\implies\quad\frac{1}{2\Omega}(\omega-\mathcal{C}(\nabla A,\zeta))&=(\det(\nabla A)\nabla X-I)e_3\\&=\det(\nabla A)(\nabla X-I)e_3+(\det(\nabla A)I-I)e_3\\&=\det(\nabla A)\partial_3\lambda+(\det(\nabla A)-1)e_3,\\\implies\qquad\partial_3\lambda&=\frac{\det(\nabla X)}{2\Omega}(\omega-\mathcal{C}(\nabla A,\zeta))+(\det(\nabla X)-1)e_3,\\\implies\qquad|\partial_3\lambda|&\leqslant\frac{\det(\nabla X)}{2\Omega}|\omega-\mathcal{C}(\nabla A,\zeta)|+\det(\nabla X)-1.\end{aligned} \tag{4.5}$$

Using equation (2.8) and the maximum principle for $\mathcal{D}$ (or an argument similar to the proof of Proposition 3.3) and $\det(\nabla A(\cdot,0))=1=\det(\nabla X(\cdot,0))$ we obtain

$$e^{-\int_0^T K\nu\|C\|_\infty^2} \leqslant \det(\nabla A) \leqslant e^{\int_0^T K\nu\|C\|_\infty^2} \tag{4.6}$$

$$\text{and} \quad e^{-\int_0^T K\nu\|C\|_\infty^2} \leqslant \det(\nabla X) \leqslant e^{\int_0^T K\nu\|C\|_\infty^2}. \tag{4.7}$$

Also, counting terms in the expression $\mathcal{C}(\nabla A,\zeta)$ we see that

$$|\mathcal{C}(\nabla A,\zeta)| \leqslant 3(1+4g+6g^2)\|\zeta\|_\infty. \tag{4.8}$$

Using Proposition 3.3 and equations (4.7), (4.8) in (4.5) we obtain equation (4.2).

The bounds for $\partial_3\ell$ follow from the bounds for $\partial_3\lambda$ quickly. Note that

$$\begin{aligned} \partial_3\ell &= (\nabla A - I)e_3 \\ &= \nabla A(I-\nabla X)e_3 \\ &= -\nabla A(\partial_3\lambda) \\ \implies \|\partial_3\ell\|_\infty &\leqslant (1+3g)\|\partial_3\lambda\|_\infty. \end{aligned} \tag{4.9}$$

and using (4.2), equation (4.3) follows. ∎

We finally remark that we can deduce the inviscid case from our proof above. In [10] it was shown that in the absence of viscosity $|\partial_3\lambda| \leqslant O(\rho)$. We deduce this as follows.

Taking the Lagrangian curl of equation (2.7) immediately gives $\zeta=\omega_0\circ A$, hence we conclude

$$\|\zeta\|_\infty = \|\omega_0\|_\infty. \tag{4.10}$$

In the absence of viscosity, X is the flow map of an incompressible fluid, and so $\det(\nabla X)=1=\det(\nabla A)$. Using this, (4.8) and (4.10) in (4.5) we obtain

$$|\partial_3\lambda| \leqslant G\rho \tag{4.11}$$

where the constant G depends only on g. Thus $|\partial_3\lambda| \leqslant O(\rho)$.

5. Asymptotic behaviour under fast rotation

We now wish to investigate the behaviour of the fluid as $\Omega\to\infty$. We begin by showing that our assumption in equation (4.1) is satisfied for regular initial data.

THEOREM 5.1. *Let $u_0\in H^s$ be periodic and divergence free. There exists a time $T>0$ depending only on $\|u_0\|_{H^s}$ (but independent of Ω) such that for any $\Omega>0$, the Navier-Stokes equations in a rotating frame* (1.1) – (1.2) *with periodic boundary conditions and initial data u_0 have a solution in $C(0,T;H^s)$. Further on the interval $[0,T]$, we can bound $\|u\|_{H^s}$ independent of Ω.*

We do not provide the proof of this theorem here, however we remark that the proof follows from the fact that $e_3\times u$ is orthogonal to u, and hence if we take the inner product of equation (1.1) with u and integrate, we no longer have any Ω dependence. The same holds for higher derivatives of u, and we obtain local existence through Galerkin approximations [7] in exactly the same manner as we did in the absence of rotation.

Now, if we start with initial data $u_0\in H^s$, with $s>\frac{7}{2}$, then the Sobolev embedding theorem guarantees $u\in C^2$. Given a fixed $u\in C^2$, the evolution equation for A is a linear parabolic equation with C^2 coefficients, and standard parabolic regularity shows two derivatives of A are continuous and bounded (independently of Ω) showing equation (4.1) is satisfied.

Along these lines we note that if $u_0 \in H^s$ with $s > \frac{7}{2}$, then $\|\omega\|_\infty$ is bounded uniformly in Ω. Hence as $\Omega \to \infty$, $\rho \to 0$. We now exhibit the existence of a limit as $\Omega \to \infty$.

THEOREM 5.2. *Let $u_0 \in H^s$ with $s > \frac{7}{2}$ be a fixed divergence free vector field. Let $(\Omega_n) \to \infty$, and A_n be the diffusive Lagrangian map that solves equations* (2.1) – (2.6) *on the time interval $[0,T]$ with spatially periodic boundary conditions. Then there exists a subsequence (denoted by n for convenience), and a map A such that $A_n \to A$ strongly in $C(0,T;C^2(I))$. Further if C_n are the commutator coefficients associated to A_n, and we define*

$$g = \sup_{n\in\mathbb{N}} \|\nabla \ell_n\|_{L^\infty(I\times[0,T])}$$

$$\text{and} \quad c = \sup_{n\in\mathbb{N}} \|C_n\|_{L^\infty(I\times[0,T])}.$$

Then both c and g are finite, and there exists a constant G depending only on g, and an absolute constant K which arises as in Theorem 4.1 such that

$$\|\partial_3 A - e_3\|_{L^\infty(I\times[0,T])} \leqslant G\left(e^{K\nu T c^2} - 1\right). \tag{5.1}$$

REMARK 5.1. *Observe that the right hand side of* (5.1) *is $O(\nu T)$. As remarked earlier, this is smaller than $O(\sqrt{\nu T})$ which is the viscous correction one obtains by superposing a Brownian vertical drift to a laminar flow.*

Proof. Let u_n be the velocity associated to the map A_n. It is known ([10]) that u_n solves the Navier-Stokes equations (1.1) – (1.2) with periodic boundary conditions. From Theorem 5.1 it follows that $\|u_n\|_{H^s}$ can be bounded independently of n, and hence (by the argument preceding the theorem) $\|A_n\|_{H^s}$ is bounded independently of n on the interval $[0,T]$. Now because equation (2.1) does not involve Ω we can also bound $\|\partial_t A_n\|_{H^{s-2}}$ independently of n on the interval $[0,T]$. Thus there exists a subsequence (denoted by n) such that A_n converges weakly to A in $C(0,T,H^s)$. The Sobolev embedding and Rellich theorems show that $A_n \to A$ strongly in $C(0,T;C^2(I))$.

This immediately shows g is finite. We now let C_n be the commutator coefficients associated to A_n. Note that, by reducing T if necessary, we can ensure $\|\nabla A_n - I\|_{L^\infty(I\times[0,T])} < \frac{1}{2}$, and hence equation (2.4) guarantees c is finite. We conclude by applying Theorem 4.1 to A_n with g as above. We know $\rho \to 0$ as $\Omega \to \infty$ and $A_n \to A$ in C^2, so taking limits on both sides of (4.3) we are done. □

6. Diffusive Lagrangian paths and Lagrangian paths

We conclude by studying the difference between the Lagrangian paths and the diffusive Lagrangian paths.

PROPOSITION 6.1. *Let $\bar{X}$ be the flow map of the fluid, and $\bar{A}$ the inverse of $\bar{X}$. If $\bar{X}$ is C^3 and for short time T equation* (4.1) *holds, then there exists a constant G depending only on g such that*

$$\|A - \bar{A}\|_\infty \leqslant \int_0^T G\nu \|C\|_\infty, \tag{6.1}$$

$$\text{and} \quad \|X - \bar{X}\|_\infty \leqslant \int_0^T G\nu \|C\|_\infty. \tag{6.2}$$

Proof. Differentiating the identity $\bar{A}(\bar{X}(a,t),t)=a$ with respect to time, we have

$$\begin{aligned}\partial_t\bar{A}+(u\cdot\nabla)\bar{A}&=0,\\ \bar{A}(x,0)&=x,\end{aligned}$$

We set $\delta=A-\bar{A}$. Clearly

$$\partial_t\delta+(u\cdot\nabla)\delta=\nu\triangle A, \tag{6.3}$$

$$\delta(x,0)=0. \tag{6.4}$$

From equation (2.4), the definition of C we see that

$$\partial_i A_k C^{\alpha}_{j,k}=\partial_i\partial_j A_\alpha, \tag{6.5}$$

$$\implies \quad |\triangle A|\leqslant G\|C\|_\infty,$$

and hence, integrating (6.3) along particle paths we obtain equation (6.1).

Equation (6.2) follows quickly from this. Take any a in our domain, and let $\bar{x}=\bar{X}(a,t)$ and $\bar{a}=A(\bar{x},t)$. Then

$$\begin{aligned}&|A(\bar{x},t)-\bar{A}(\bar{x},t)|\leqslant\int_0^T G\nu\|C\|_\infty,\\ \implies\quad &|\bar{a}-a|\leqslant\int_0^T G\nu\|C\|_\infty,\\ \implies\quad &|X(\bar{a},t)-X(a,t)|\leqslant|\nabla X|\int_0^T G\nu\|C\|_\infty,\\ \implies\quad &|\bar{x}-X(a,t)|\leqslant|\nabla X|\int_0^T G\nu\|C\|_\infty,\\ \implies\quad &|\bar{X}(a,t)-X(a,t)|\leqslant|\nabla X|\int_0^T G\nu\|C\|_\infty.\end{aligned}$$

As $|\nabla X|$ is clearly controlled by g, we are done. ◻

We now investigate the difference in gradients of X and $\bar{X}$. We first require a slight variant of Gronwall's lemma.

LEMMA 6.2. *If $a,b>0$ are two constants, and $y:[0,T]\to\mathbb{R}$ is a differentiable function satisfying*

$$y'(t)\leqslant a+by(t)\quad\forall t\in[0,T], \tag{6.6}$$

then for all $t\in[0,T]$ we have

$$y(t)\leqslant(y(0)+at)e^{bt}.$$

Proof. Differentiating $y(t)e^{-bt}$ and using (6.6) immediately yields the Lemma. The details are elementary and are left to the reader. ◻

PROPOSITION 6.3. *We define $\boldsymbol{C}$ and $\boldsymbol{F}$ by*

$$\boldsymbol{C} = \|C\|^2_{L^\infty(I\times[0,T])} + \|\nabla C\|_{L^\infty(I\times[0,T])} = \sup_{0\leqslant t\leqslant T} \|C\|^2_\infty + \|\nabla C\|_\infty,$$

$$\text{and} \quad \boldsymbol{F} = \|\nabla u\|_{L^\infty(I\times[0,T])} = \sup_{0\leqslant t\leqslant T} \|\nabla u\|_\infty.$$

Then with the same notation and assumptions of Proposition 6.1, there exists a constant G depending only on g such that

$$\|\nabla A - \nabla \bar{A}\|_\infty \leqslant G\nu t \boldsymbol{C} e^{\boldsymbol{F}t}, \tag{6.7}$$

$$\|\nabla X - \nabla \bar{X}\|_\infty \leqslant G\nu t \boldsymbol{C} e^{\boldsymbol{F}t}. \tag{6.8}$$

Proof. Let $\delta = A - \bar{A}$ as before. Differentiating (6.3) we obtain

$$\partial_t \nabla\delta + (u\cdot\nabla)\nabla\delta = \nu\triangle\nabla A - (\nabla u)^t\nabla\delta. \tag{6.9}$$

Differentiating equation (6.5) we see

$$\begin{aligned} & \partial_p\partial_i\partial_j A_\alpha = (\partial_p\partial_i A_k) C^\alpha_{j,k} + \partial_i A_k \partial_p C^\alpha_{j,k}, \\ \Longrightarrow \quad & |\triangle\nabla A| \leqslant G(\|C\|^2_\infty + \|\nabla C\|_\infty) \\ & \qquad \leqslant G\boldsymbol{C}. \end{aligned}$$

Further, at time 0, $A = \bar{A}$ and hence $\nabla\delta = 0$. Integrating along particle paths and using Lemma 6.2 we immediately obtain (6.7).

For the proof of (6.8), pick any $a \in \mathbb{R}^3$. Let $x = X(a,t)$ and $\bar{x} = \bar{X}(a,t)$. We let G denote a constant (which depends only on g) that changes from line to line.

$$\begin{aligned} |\nabla X(a,t) - \nabla\bar{X}(a,t)| &= \left|(\nabla A)^{-1}(x,t) - (\nabla\bar{A})^{-1}(\bar{x},t)\right| \\ &= \big|(\nabla A)^{-1}(x,t) - (\nabla A)^{-1}(\bar{x},t) + \\ &\qquad (\nabla A)^{-1}(\bar{x},t) - (\nabla\bar{A})^{-1}(\bar{x},t)\big| \\ &\leqslant G\|C\|_\infty|x-\bar{x}| + \\ &\qquad \left|(\nabla A)^{-1}(\bar{x},t)\left[\nabla\bar{A}(\bar{x},t) - \nabla A(\bar{x},t)\right](\nabla\bar{A})^{-1}(\bar{x},t)\right| \\ &\leqslant G[\nu\boldsymbol{C}t + \nu\boldsymbol{C}te^{\boldsymbol{F}t}] \\ &\leqslant G\nu\boldsymbol{C}te^{\boldsymbol{F}t} \end{aligned}$$

□

COROLLARY 6.4. *With the same assumptions as Theorem 4.1, and with $\boldsymbol{C}$ and $\boldsymbol{F}$ as in Proposition 6.3, there exists a constant G depending only on g, and an absolute constant K (which arises as in Proposition 3.3) such that*

$$\|\partial_3\bar{\ell}\|_\infty \leqslant G\left[e^{\int_0^T K\nu\|C\|^2_\infty}\rho + e^{\int_0^T K\nu\|C\|^2_\infty} - 1 + \nu T\boldsymbol{C}e^{\boldsymbol{F}T}\right], \tag{6.10}$$

$$\text{and} \quad \|\partial_3\bar{\lambda}\|_\infty \leqslant G\left[e^{\int_0^T K\nu\|C\|^2_\infty}\rho + e^{\int_0^T K\nu\|C\|^2_\infty} - 1 + \nu T\boldsymbol{C}e^{\boldsymbol{F}T}\right], \tag{6.11}$$

where $\bar{\lambda}$ and $\bar{\ell}$ are the traditional Eulerian and Lagrangian displacements respectively.

Proof. The corollary follows directly from Theorem 4.1 and Proposition 6.3. □

REMARK 6.1. *From Theorem 5.1 we see that if the initial data* $u_0 \in H^s$ *with* $s > \frac{7}{2}$ *then* $\mathbf{F}$ *can be bounded independently of* Ω*, and hence the right hand sides of* (6.10) *and* (6.11) *are both* $O(\rho + \nu T\boldsymbol{C}) = O(\rho + \nu T\|C\|^2_{L^\infty(I\times[0,T])} + \nu T\|\nabla C\|_{L^\infty(I\times[0,T])})$.

7. Conclusion

We showed that the vertical gradient of the diffusive Lagrangian displacement is bounded uniformly $O(\rho + \nu t\|C\|^2_{L^\infty(I\times[0,T])})$ where ρ is the maximal local Rossby number, and C are the commutator coefficients. As the magnitude of the imposed rotation goes to infinity, the Rossby number tends to zero, and we showed that in the limit (of a subsequence) the vertical gradient of the diffusive Lagrangian is $O(\nu t\|C\|_{L^\infty(I\times[0,T])})$.

We compared the Lagrangian and diffusive Lagrangian maps. For fixed initial data (but arbitrary forcing rotation) they differ by $O(\nu t\|C\|^2_\infty)$ in the L^∞ spatial norm. In the C^1 (spatial) norm however, they differ by $O(\nu t[\|C\|^2_{L^\infty(I\times[0,T])} + \|\nabla C\|_{L^\infty(I\times[0,T])}])$. Hence we see that vertical gradient of the flow map is $O(\rho + \nu t[\|C\|^2_{L^\infty(I\times[0,T])} + \|\nabla C\|_{L^\infty(I\times[0,T])}])$.

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