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NEARLY LIPSCHITZEAN DIVERGENCE FREE TRANSPORT PROPAGATES NEITHER CONTINUITY NOR BV REGULARITY*

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Abstract. We give examples of divergence free vector fields

$$\mathbf{a}(x,y) \ \in \ \cap_{1\le p<\infty} W^{1,p}(\mathbf{R}^2)\,.$$

For such fields the Cauchy problem for the linear transport equation

$$\frac{\partial u}{\partial t} + \mathbf{a}_1(x,y)\frac{\partial u}{\partial x} + \mathbf{a}_2(x,y)\frac{\partial u}{\partial y} \ = \ 0\,, \quad \operatorname{div}\mathbf{a} := \frac{\partial \mathbf{a}_1}{\partial x} + \frac{\partial \mathbf{a}_2}{\partial y} = 0\,, \tag{0.1}$$

has unique bounded solutions for $u_0 \in L^\infty(\mathbf{R}^2)$. The first example has nonuniqueness in the Cauchy problem for the ordinary differential equation defining characteristics. In addition, there are smooth initial data $u_0 \in C_0^\infty(\mathbf{R}^2)$ so that the unique bounded solution is not continuous on any neighborhood of the origin.

The second example is a field of similar regularity and initial data in $W^{1,1} \subset BV$ so that for no $t > 0$ is it true that $u(t,\cdot)$ is of bounded variation.

1. Introduction

Suppose that $\mathbf{a} \in (L^\infty \cap W^{1,1})(\mathbf{R}^d)$ is a bounded divergence free vector field on d dimensional Euclidean space. For arbitrary bounded initial data $u_0 \in L^\infty(\mathbf{R}^d)$ there is a unique solution $u \in L^\infty([0,\infty[\times\mathbf{R}^d)$ of the initial value problem

$$\partial_t u + \mathbf{a}\cdot\nabla\,\mathbf{u} = \mathbf{0}\,, \qquad \mathbf{u(0,\cdot) = u_0}(\cdot),$$

(see e.g. [13]). In the planar case which we discuss the weaker hypothesis $\mathbf{a} \in L^\infty(\mathbf{R}^2)$ yields the same conclusion [10, 6]. In general dimension, a sequence of papers ([13, 14], [6], [7], [8], [3]) end with Ambrosio's recent proof of uniqueness when $\mathbf{a}$ is of bounded variation. For $d > 2$ (and $d = 2$ in the nonautonomous case) there are examples of nonuniqueness for nearly BV fields ([1], [9], [11, 12]).

If $\mathbf{a} \in W^{1,\infty}(\mathbf{R}^d) = \mathrm{Lip}(\mathbf{R}^d)$, then the transport propagates all Hölder regularity in the following sense. For $0 < \alpha < 1$, $W^{\alpha,\infty)}(\mathbf{R}^d)$ is the set of uniformly Hölder continuous functions. If $\alpha \in [0,1]$ and $u_0 := u|_{\{t=0\}} \in W^{\alpha,\infty}(\mathbf{R}^d)$, then for all $T > 0$, the solution u is also Hölder, $u \in W^{\alpha,\infty}([0,T]\times\mathbf{R}^d)$.

A formal interpolation between propagation of $W^{0,\infty}(\mathbf{R}^2)$ when $\mathbf{a} \in W^{0,\infty}(\mathbf{R}^2)$ and propagation of $W^{1,\infty}(\mathbf{R}^2)$ when $\mathbf{a} \in W^{1,\infty}(\mathbf{R}^2)$ suggests that if $\mathbf{a} \in W^{\alpha,\infty}(\mathbf{R}^2)$ and $u_0 \in W^{\alpha,\infty}(\mathbf{R}^2)$ then the solution belongs to $W^{\alpha,\infty}$. Nothing of the sort is true. In section 2, we present an example of a field in all the Hölder spaces and an initial datum which is smooth and of compact support so that the solution is not continuous on any neighborhood of the origin. In section 3, we present an example with $u_0 \in W^{1,1} \subset BV$ for which $u(t) \notin BV$ for $t > 0$.

The fields have the property that the characteristics, defined by solving the ordinary differential equations

$$\frac{dx}{dt} = \mathbf{a}_1(x(t),y(t))\,, \qquad \frac{dy}{dt} = \mathbf{a}_2(x(t),y(t))\,, \tag{1.1}$$

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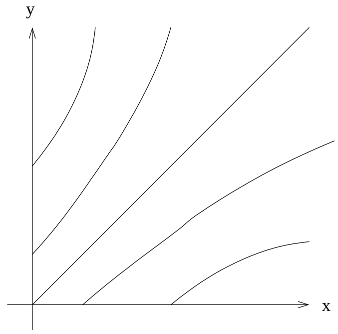

FIG. 2.1. *Positive octant phase portrait*

have nonunique solutions. The divergence free hypothesis shows that the flow preserves volumes. Nonuniqueness is an extreme form of length distortion, an interval of length zero is distorted to an interval of finite length. This is consistent with volume preservation. The length distortion explains the lack of propagation of regularity. Formally, if Φ_t is the flow generated by $\mathbf{a}$ then one thinks of the solution as the composition $u_0(\Phi_{-t}(x))$. In order for this to propagate Hölder regularity one needs Φ_{-t} to be Lipschitzean. This is guaranteed when $\mathbf{a}$ is Lipschitzean.

In the one dimensional case, $d = 1$, length distortion and volume distortion are equivalent and are often controlled by one sided inequalities on the derivative $\partial a(t,x)/\partial x$. There is an extensive literature going back at least to Oleinik's uniqueness proof [15] showing that compression is good for uniqueness while rarefaction is good for existence while bad for uniqueness ([4], [5], [16], [17]). In the Oleinik proof the entropy condition controls the possible stretching of lengths. Our examples preserve area while stretching lengths unboundedly.

2. C^∞ propagates to discontinuous

A simple explicit C^α field exhibiting nonuniqueness of characteristics and therefore infinite length distortion is the following.

Example. In the positive quadrant $\{x > 0 \text{ and } y > 0\}$ consider the divergence free double shear

$$-y^\alpha\,\partial_x \;-\; x^\alpha\,\partial_y\,. \tag{2.1}$$

Characteristics satisfy

$$\frac{dx}{dt} = -y^\alpha\,, \qquad \frac{dy}{dt} = -x^\alpha\,, \qquad \frac{d}{dt}\Big(x^{1+\alpha} - y^{1+\alpha}\Big) \;=\; 0\,.$$

The phase portrait is sketched in Figure 1.

The line $x = y$ is invariant. The solution with initial value $(x(0), y(0)) = (b, b)$ with $b > 0$ is given by

$$x(t) = y(t) = \Big(b^{1-\alpha} - (1-\alpha)\,t\Big)^{\frac{1}{1-\alpha}}\,.$$

This curve reaches the origin at the time

$$t_*(b) := \frac{b^{1-\alpha}}{1-\alpha}.$$

Through the point $(t_*(b), 0, 0)$ pass this backward characteristic and also the characteristic $x = y = 0$. If these backward paths hit $t = 0$ at points where u_0 takes distinct values, the requirement that u be constant on characteristics yields incompatible values. This is the heart of the following construction.

In the next definition note that $s(\log s)^2$ is strictly increasing for $0 < s < e^{-2}$ and converges to zero as s decreases to 0.

Definition. *Suppose that $0 \le f \in C(\mathbf{R})$ vanishes for $s \le 0$, is nondecreasing and uniformly bounded, and for $0 < s \le e^{-2}/2$*

$$f(s) = s\,(\log s)^2\,.$$

Define the bounded divergence free field

$$\mathbf{a}_1(x,y)\,\partial_x + \mathbf{a}_2(x,y)\,\partial_y \;:=\; -f(y)\,\partial_x \;-\; f(x)\,\partial_y\,. \tag{2.2}$$

Then $\mathbf{a}$ belongs to all the Hölder spaces $C^\alpha(\mathbf{R}^2)$ with $0 < \alpha < 1$, to $W^{1,p}(\mathbf{R}^2)$ for all $1 \le p < \infty$, and even more $\nabla \mathbf{a} \in BMO(\mathbf{R}^2)$.

Theorem 1. *Suppose $\mathbf{a}(x, y)$ is the vector field (2.2). Suppose that $u_0 \in C_0^\infty(\mathbf{R}^2)$ vanishes when both x and y are nonpositive, and is strictly positive when x and y are strictly positive and small. Then for any relatively open subset $\omega \subset [0,\infty[\times\mathbf{R}^2$ with $(0, 0, 0) \in \omega$, the unique solution $u \in L^\infty([0,\infty[\times\mathbf{R}^2)$ of the transport equation with these initial data is not continuous on ω.*

Proof. Supposing that u is a solution which is continuous on a neighborhood of the origin in $[0,\infty[\times\mathbf{R}^2$ we derive a contradiction.

The characteristic beginning at (b, b) with $0 < b \le e^{-2}/2$ is equal to $(x(t), x(t))$ where

$$\frac{dx}{dt} = -x\,(\log x)^2\,, \qquad x(0) = b\,.$$

Then

$$\frac{d}{dt}\,\frac{1}{\log x} = \frac{-1}{(\log x)^2}\,\frac{1}{x}\,\frac{dx}{dt} = 1\,.$$

Thus

$$\frac{1}{\log x(t)} = t + \frac{1}{\log b}\,, \qquad \log x(t) = \frac{\log b}{t\,\log b + 1}\,.$$

The path arrives at the origin at the finite time

$$t^*(b) := \frac{-1}{\log b}\,.$$

The method of characteristics in the form of the next lemma is needed. The proof is standard.

LEMMA 2. *Suppose that* $\gamma(t) = (t, x(t))$ *with* $x : [0, c] \to \mathbf{R}^2$ *is an integral curve of* $\partial_t + \mathbf{a}_1\partial_x + \mathbf{a}_2\partial_y$ *with the property that* $\mathbf{a}$ *is uniformly Lipschitzean on a neighborhood of* $x([0, c])$. *If* u *is a continuous solution of (0.1) on a neighborhood of* $\gamma([0, c])$, *then* u *is constant on* $\gamma([0, c])$.

This lemma is applied to the characteristic beginning at (b, b) near the origin in the positive quadrant. The characteristic arrives at the origin at the small time $t^*(b)$. The Lemma with $c = t^*(b) - \epsilon$ implies that

$$u\Big(t^*(b) - \epsilon\,,\, x(t^*(b) - \epsilon)\,,\, x(t^*(b) - \epsilon)\Big) = u_0(b, b)\,.$$

Passing to the limit $\epsilon \to 0$ using the continuity of u yields for b small

$$u(t^*(b), 0, 0) = u_0(b, b) > 0\,. \tag{2.3}$$

On the other hand, the field $\mathbf{a}$ vanishes in the quadrant where both x and y are negative. The Lemma implies that u is independent of time in that quadrant and therefore that $u(t, x, y)$ vanishes when both x and y are negative. Since u is continuous near the origin it follows that for t small

$$u(t, 0, 0) = 0\,.$$

For b small this contradicts the conclusion (2.3) and the proof is complete. □

Remark: The solution u is continuous at the point $(0, 0, 0)$ with $u(0, 0, 0) = 0$. In fact, the values of u in a small neighborhood of $(0, 0, 0)$ are determined by the values of u_0 on a small neighborhood of $(0, 0)$. By continuity of u_0 these values differ little from 0.

3. Bounded variation is not propagated

We give a simple example for which BV regularity is not propagated. Suppose that $g(s) \in C_0^0(\mathbf{R})$ with

$$g(s) \;=\; -s\,\log|s| + s$$

on a neighborhood of $s = 0$. Then near the origin $g' = -\log|s|$. Define the divergence free bounded field

$$\mathbf{b} \;:=\; g(y)\,\partial_x\,.$$

The flow of this field and its inverse are shears given explicitly by

$$\Phi_t(x, y) \;=\; (x + tg(y), y)\,, \qquad \Phi_{-t}(x, y) \;=\; (x - tg(y), y)\,.$$

The solution of the associated linear transport equation with initial value u_0 is given by

$$u(t, x, y) \;=\; u_0(\Phi_{-t}(x, y)) \;=\; u_0(x - tg(y), y)\,.$$

Then

$$\partial_y u \;=\; -tg'(y)\,\frac{\partial u_0(x - tg(y), y)}{\partial x} \;+\; \frac{\partial u_0(x - tg(y), y)}{\partial y}\,. \tag{3.1}$$

For $u_0 \in W^{1,1}$,

$$\left\| \frac{\partial u_0(x - tg(y), y)}{\partial x} \right\|_{L^1(\mathbf{R}^2_{x,y})} \quad \text{and} \quad \left\| \frac{\partial u_0(x - tg(y), y)}{\partial y} \right\|_{L^1(\mathbf{R}^2_{x,y})}$$

are independent of t. The strategy for small y is to take advantage of the large factor $tg'(y) \sim t\,|\log|y|\,|$ in the first summand.

Theorem 3. *Let* $r := \sqrt{x^2+y^2}$ *and suppose that* $u_0(x,y) \in (L^\infty \cap W^{1,1})(\mathbf{R}^2)$ *satisfies* $u_0(x,y) = \cos\left(r^{-1}(\log r)^{-2}\right)$ *on a neighborhood of* $(0,0)$*. Then for all* $t > 0$ *the unique bounded solution of the initial value problem*

$$u_t + g(y)\,u_x = 0\,, \qquad u(0,x,y) = u_0(x,y)$$

satisfies $u(t,\cdot) \notin BV(\mathbf{R}^2_{x,y})$.

Proof. Near the origin

$$\frac{\partial u_0}{\partial r} \sim \frac{1}{r^2\,(\log r)^2}$$

is just barely L^1. Since

$$\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial r}\frac{\partial r}{\partial x}$$

and the second factor is bounded away from zero when the argument of (x,y) is bounded away from $\pm\pi/2$ it follows that $\partial_x u_0$ is also borderline L^1.

Fix $t > 0$ and introduce the change of coordinates

$$(\underline{x}, \underline{y}) \;:=\; (x - tg(y), y)\,,$$

with associated polar coordinates $(\underline{r}, \underline{\theta})$. This change preserves area and $\underline{y} = y$.

The expression (3.1) is valid in $y \neq 0$. Therefore if $u(t,\cdot)$ belongs to $BV(\mathbf{R}^2_{x,y})$ it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\partial_y u(t,x,y)\right|\,dx\,dy \;<\; \infty\,.$$

The second summand in (3.1) belongs to $L^1(\mathbf{R}^2_{x,y})$ for all t with norm independent of t. To complete the proof it suffices to show that for $\phi \in]0,\pi/2[$ and $0 < \epsilon << 1$

$$\infty \;=\; \int_0^{\phi}\int_0^{\epsilon} \left|\,\log|\underline{y}|\;\frac{1}{\underline{r}^2\,(\log\underline{r})^2}\,\right|\,\underline{r}\;d\underline{r}\,d\underline{\theta}\,,$$

since the L^1 norm of the first summand in (3.1) is at least as large as a positive multiple of the right hand side.

On the circle of radius $\underline{r}$ in the $(\underline{x}, \underline{y})$ plane one has $|\underline{y}| \leq \underline{r}$, so $|\log|\underline{y}|\,| \geq |\log|\underline{r}|\,|$. Therefore the integrand is bounded below by $\frac{1}{\underline{r}\,\log\underline{r}}$ which is not integrable. $\square$

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