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On the Number of Negative Eigenvalues for a Schrödinger Operator with Magnetic Field

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Abstract: We consider the Schrödinger operator with magnetic field

$$H = \sum_{j=1}^{n} \left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j\right)^2 + V \quad \text{in } \mathbb{R}^n .$$

Under certain conditions on the magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{a}$, we generalize the Fefferman–Phong estimates (Bull. A. M. S. **9**, 129–206 (1983)) on the number of negative eigenvalues for $-\Delta + V$ to the operator H. Upper and lower bounds are established. Our estimates incorporate the contribution from the magnetic field. The conditions on $\mathbf{B}$ in particular are satisfied if the magnetic potentials $a_j(x)$ are polynomials.

Introduction

This paper concerns the Schrödinger operator with magnetic field:

$$H = H(\mathbf{a}, V) = \sum_{j=1}^{n} \left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j\right)^2 + V \quad \text{in } \mathbb{R}^n, \quad n \geqq 3 , \tag{0.1}$$

where $i = \sqrt{-1}$, $V : \mathbb{R}^n \to \mathbb{R}$ is the electric potential and $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic potential.

Let $N(\lambda, H)$ denote the number of eigenvalues (counting multiplicity) of H smaller than λ (or in general the dimension of the spectral projection for H corresponding to the interval $(-\infty, \lambda)$). In the case $\mathbf{a}(x) \equiv \mathbf{0}$, i.e., $H = H(\mathbf{0}, V) = -\Delta + V$, a basic theorem of Cwickel, Lieb and Rosenblum states that

$$N(\lambda, -\Delta + V) \leqq c_n |\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi|^2 + V(x) < \lambda\}| . \tag{0.2}$$

See [Si2, p. 95]. Using a sharper form of the uncertainty principle, C. Fefferman and D.H. Phong were able to refine the classical estimate (0.2). Indeed, it was shown

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in [F] that, for $p>1$ and $\lambda \leqq 0$, $N(\lambda, -\Delta+V)$ is bounded by $C \cdot N_0$, where N_0 is the number of minimal (disjoint) dyadic cubes which satisfy

$$l(Q)^2\left(\frac{1}{|Q|}\int_Q |V|^p\,dx\right)^{1/p} \geqq c>0, \quad l(Q)<\frac{1}{\sqrt{|\lambda|}}, \tag{0.3}$$

$l(Q)$ denotes the side length of cube Q and C, c depend only on n and p. A lower bound was also established. See [F, p. 145, Theorem 6]. The results of Fefferman and Phong are particularly useful in the cases when the right side of (0.2) becomes infinite [R, Si3, F].

Concerning the case where the magnetic potential $\mathbf{a}(x)$ is present, it is known that estimate (0.2) still holds for H by the diamagnetic inequality, although one can not expect $N(\lambda, H(\mathbf{a}, V)) \leqq N(\lambda, H(\mathbf{0}, V))$. See [A-H-S]. A obvious problem is that this estimate does not involve the magnetic field.

The purpose of this paper is to generalize the Fefferman–Phong estimate to the magnetic Schrödinger operators under certain conditions on the magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{a}$. We establish upper and lower bounds of $N(\lambda, H)$ for $\lambda \leqq 0$. The conditions on $\mathbf{B}$ in particular are satisfied if the magnetic potentials $a_j(x)$, $j=1,2,\ldots,n$ are polynomials. More importantly, our estimates incorporate the contribution from the magnetic field in an effective way.

To state the main results, we need to introduce an auxiliary function.

Definition 0.4. *For a nonnegative function W, the function $m(x, W)$ is defined by*

$$\frac{1}{m(x,W)} = \sup\left\{r>0 : \frac{r^2}{|Q(x,r)|}\int_{Q(x,r)} W(y)\,dy \leqq 1\right\},$$

where $Q(x,r)$ denotes the cube centered at x with side length r.

The function $m(x,|\mathbf{B}|)$, which behaves like $|\mathbf{B}|^{1/2}$ in scale, plays a crucial role in this paper. What makes $m(x,|\mathbf{B}|)$ so important is the fact that, under suitable conditions, we can bound the operator $H(\mathbf{a},0)$ from below by $c\{m(x,|\mathbf{B}|)\}^2$, i.e.,

$$(H(\mathbf{a},0)f,f) \geqq c(m(\,\cdot\,,|\mathbf{B}|)f, m(\,\cdot\,,|\mathbf{B}|)f)\,. \tag{0.5}$$

See Theorem 4.1. One may consider the above estimate, which is proved in [Sh3, Theorem 2.7] for a more general case, as a form of the uncertainty principle.

We also need to introduce a class of functions which satisfy the reverse Hölder inequality. This class has been studied extensively in harmonic analysis. See [St2].

Definition 0.6. *Suppose $W \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ $(1<p\leqq\infty)$ and $W \geqq 0$ a.e. on $\mathbb{R}^n$. We say $W \in (RH)_p$ if there exists $C_0 \geqq 1$ such that*

$$\left(\frac{1}{|Q|}\int_Q W^p(x)\,dx\right)^{1/p} \leqq C_0 \cdot \frac{1}{|Q|}\int_Q W(x)\,dx \tag{0.7}$$

for every cube Q in $\mathbb{R}^n$.

We remark that, if $W = |P(x)|^\alpha$, where $P(x)$ is a polynomial of degree k and $\alpha > 0$, then $W \in (RH)_\infty$ and

$$m(x, W) \approx \sum_{|\beta| \leqq k} |\partial_x^\beta P(x)|^{\frac{\alpha}{\alpha|\beta|+2}} . \tag{0.8}$$

See [Sh2].

Let

$$\mathbf{B}(x) = \operatorname{curl} \mathbf{a}(x) = (b_{jk}(x))_{1 \leqq j,k \leqq n} \tag{0.9}$$

be the magnetic field generated by $\mathbf{a}(x)$, where

$$b_{jk}(x) = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j} . \tag{0.10}$$

We are now in a position to state the main results of the paper.

Theorem 0.11. *Let $n \geqq 3$. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p > 1$. Also assume that $|\mathbf{B}| \in (RH)_{n/2}$ and*

$$|\nabla \mathbf{B}(x)| \leqq C_1 \{m(x, |\mathbf{B}|)\}^3 , \tag{0.12}$$

where $|\mathbf{B}| = |\mathbf{B}(x)| = \sum_{j,k} |b_{jk}(x)|$. Then, there exist $C = C(n) > 0$ and $c = c(C_0, C_1, n, p) > 0$, such that, for $\lambda \leqq 0$, $N(\lambda, H)$ is bounded by $C \cdot N_0$, where N_0 is the number of minimal (disjoint) dyadic cubes which satisfy

$$l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V|^p \, dx \right)^{1/p} \geqq c, \qquad l(Q) < \frac{1}{\sqrt{|\lambda|}} \tag{0.13}$$

and

$$l(Q)^2 \left(\frac{1}{|Q|} \int_Q |\mathbf{B}|^2 dx \right)^{1/2} \leqq 1 . \tag{0.14}$$

Remark 0.15. In Theorem 0.11, we have implicitly assumed that H has a self-adjoint realization on $L^2(\mathbb{R}^n)$. Under the assumption that $\mathbf{a} \in C^2(\mathbb{R}^n)$ and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we may define the quadratic form

$$q[f, g] = \sum_{j=1}^n \int_{\mathbb{R}^n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) f \overline{\left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g} \, dx + \int_{\mathbb{R}^n} V f \overline{g} \, dx \tag{0.16}$$

for f, $g \in C_0^\infty(\mathbb{R}^n)$. The key estimate (see (0.29) and (0.30) below) of this paper implies that if $M_p < \gamma(n, p, C_0, C_1)$ for $|\lambda|$ sufficiently large, then $q[f, g]$ is semi-bounded from below and closable. In this case, H can be extended to the unique self-adjoint operator associated with the quadratic form (0.16).

Remark 0.17. Note that the conditions $|\mathbf{B}| \in (RH)_{n/2}$ and $|\nabla \mathbf{B}(x)| \leqq C\{m(x, |\mathbf{B}|)\}^3$ in Theorem 0.11 are dilation invariant. Roughly speaking, these two conditions mean that the values of $|\mathbf{B}|$ do not fluctuate too much on the average and $|\nabla \mathbf{B}|$ is uniformly bounded in the scale $\{m(x, |\mathbf{B}|)\}^{-1}$. Although some condition on $\nabla \mathbf{B}$ seems to be necessary if $n \geqq 3$, the assumption in Theorem 0.11 is more restrictive than one would hope. Nevertheless, these conditions are satisfied if the magnetic potentials

$a_j(x)$ are polynomials. This follows easily from the estimate (0.8). Moreover, in this case, the constants C_0, C_1 depend only on n and the degrees of polynomials.

Remark 0.18. In Theorem 0.11, the condition (0.14) may be replaced by

$$l(Q)^2\left(\frac{1}{|Q|}\int_Q |\mathbf{B}|^q\,dx\right)^{1/q} \leqq 1 \tag{0.19}$$

for any $0 < q \leqq \infty$. Indeed, in the proof of the theorem, we will show that $N(\lambda, H) \leqq C_n \cdot \tilde{N}_0$, where $\tilde{N}_0$ is the number of minimal dyadic cubes which satisfy (0.13) and

$$l(Q) < \inf_{x\in Q} \frac{\alpha}{m(x, |\mathbf{B}|)}\,. \tag{0.20}$$

Since (0.12) implies that $|\mathbf{B}(x)| \leqq C\{m(x, |\mathbf{B}|)\}^2$ [Sh4, Remark 1.8], we conclude that

$$l(Q)^2\left(\frac{1}{|Q|}\int_Q |\mathbf{B}|^q\,dx\right)^{1/q} \leqq l(Q)^2\cdot C\cdot \sup_{x\in Q}\{m(x,|\mathbf{B}|)\}^2 \leqq \alpha^2 C \leqq 1$$

if α is small.

Note that, if Q is a cube satisfying (0.13) and (0.20) with $\lambda = 0$ and $p \geqq n/2$, then

$$\begin{aligned} c &\leqq l(Q)^{2-\frac{n}{p}}\left(\int_Q |V|^p\,dx\right)^{1/p} \\ &\leqq \left\{\inf_{x\in Q}\frac{1}{m(x,|\mathbf{B}|)}\right\}^{2-\frac{n}{p}}\left\{\int_Q |V|^p\,dx\right\}^{1/p} \\ &\leqq \left\{\int_Q \frac{|V(x)|^p}{\{m(x,|\mathbf{B}|)\}^{2p-n}}\,dx\right\}^{1/p}. \end{aligned}$$

Thus,

$$\int_Q \frac{|V(x)|^p}{\{m(x,|\mathbf{B}|)\}^{2p-n}}\,dx \geqq c\,.$$

Summing over all minimum cubes which satisfy (0.13) and (0.20), and using Remark 0.18, we obtain the following.

Corollary 0.21. *Under the same assumption as in Theorem* 0.11, *we have*

$$N(0,H) \leqq C\int_{\mathbb{R}^n} \frac{|V(x)|^p}{\{m(x,|\mathbf{B}|)\}^{2p-n}}\,dx \tag{0.22}$$

for $p \geqq n/2$, *where* C *depends on* n, p, C_0 *and* C_1.

Clearly, we may replace the domain $\mathbb{R}^n$ in (0.22) by the set $\{x \in \mathbb{R}^n : V(x) < 0\}$. In the case $p = n/2$, this is the classical Cwickel–Lieb–Rosenblum estimate. We remark that Corollary 0.21 may be deduced directly from estimates (0.5) and the diamagnetic inequality.

Corollary 0.23. *Let λ_j, $j = 1, \ldots,$ denote the negative eigenvalues of H. Under the same assumption as in Theorem* 0.11, *we have*

$$\sum_j |\lambda_j| \leqq C \int_{\{x \in R^n : V(x) < 0\}} \frac{|V(x)|^{p+1}}{\{m(x, |\mathbf{B}|)\}^{2p-n}} \, dx \tag{0.24}$$

for $p \geqq n/2$, where C depends on n, p, C_0, and C_1.

Corollary (0.23) follows from Corollary (0.21) by a simple integration argument as in the classical case.

Example 0.25. Let $\mathbf{B}(x)$ be a constant magnetic field in $\mathbb{R}^3$. Using Corollary (0.21), we get

$$N(0,H) \leqq \frac{C_p}{|\mathbf{B}|^{p-\frac{3}{2}}} \int_{\{x \in R^3 : V(x) < 0\}} |V|^p \, dx \quad p \geqq 3/2 \, .$$

The following lower bound estimate suggests that the upper bound in Theorem 0.11 is almost optimal.

Theorem 0.26. *Suppose $\mathbf{a} \in C^1(\mathbb{R}^n)$, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \leqq 0$ a.e. on $\mathbb{R}^n$. Then, there exists $C_2 > 0$ depending only on n, such that, if there exists a collection of cubes $\{Q_k, k = 1, 2, \ldots, N_0\}$, whose doubles are pointwise disjoint, with the properties*

$$l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V| dx \right) \geqq C_2, \quad l(Q) < \frac{1}{\sqrt{|\lambda|}} \tag{0.27}$$

and

$$l(Q)^2 \left(\frac{1}{|Q|} \int_{2Q} |\mathbf{B}|^2 dx \right)^{1/2} \leqq 1 \, , \tag{0.28}$$

then

$$N(\lambda, H) \geqq N_0 \, .$$

The paper is organized as follows. In Sect. 1 we give the proof of Theorem 0.26. This will be done by constructing a certain subspace of $L^2(\mathbb{R}^n)$ and using the minimax principle. To prove Theorem 0.11, we follow the approach of Fefferman and Phong [F]. Also see [K-Sa]. The key step, which requires the systematic control over the magnetic field $\mathbf{B}$, is to establish the following trace inequality:

$$\int_{\mathbb{R}^n} |V| \, |g|^2 \, dx \leqq C \cdot M_p \left\{ \sum_{j=1}^n \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 \, dx \right\}, \tag{0.29}$$

where

$$M_p = \sup_Q l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V|^p \, dx \right)^{1/p} \tag{0.30}$$

and the supremum is over all dyadic cubes satisfying

$$l(Q) < \inf_{x\in Q} \frac{\alpha}{m(x, |\mathbf{B}| + |\lambda|)} .$$

Note that (0.29) is equivalent to

$$\int_{\mathbb{R}^n} |V| \, |(H(\mathbf{a},0) + |\lambda|)^{-1/2} f|^2 \, dx \leqq C \cdot M_p \int_{\mathbb{R}^n} |f|^2 \, dx . \tag{0.31}$$

Let $K_\lambda(x,y)$ denote the kernel function of the operator $(H(\mathbf{a},0)+|\lambda|)^{-1/2}$. In Sect. 2 we will show that, for any integer $k>0$,

$$|K_\lambda(x,y)| \leqq \frac{C_k}{\{1+|x-y| m(x, |\mathbf{B}|+|\lambda|)\}^k} \cdot \frac{1}{|x-y|^{n-1}} . \tag{0.32}$$

This decay estimate is closely related the lower bound (0.5) for $H(\mathbf{a},0)$.

In Sect. 3 we establish the trace inequality (0.29) by using (0.32) and techniques from harmonic analysis. The proof of Theorem 0.11, which adapts the argument in [F, K-Sa], is given in Sect. 4.

In recent years there has been a great deal of interest in the magnetic Schrödinger operator H. For references on the spectral theory of H, we refer the reader to a survey paper by Mohamed and Raikov [M-R]. We remark that in [Sh4], under certain conditions similar to that in Theorem 0.11, we study the eigenvalue asymptotics of $H(\mathbf{a}, V)$ with nonnegative potential V. In particular, we show that $H(\mathbf{a}, V)$ has a discrete spectrum if and only if $\lim_{|x|\to\infty} m(x, |\mathbf{B}|+V) = \infty$.

We fix some notation. By dyadic cubes, we mean cubes in $\mathbb{R}^n$ whose side have length 2^k, and whose vertices are members of the lattice of points of the form $(m_1 2^k, \ldots, m_n 2^k)$ with k, m_j being arbitrary integers. Throughout this paper, unless otherwise indicated, we will use C and c to denote positive constants, which are not necessarily the same at each occurrence, which depend at most on C_0, C_1, n and p.

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1. The Lower Bound

In this section we will give the proof of Theorem 0.26 stated in the Introduction.

Proof of Theorem 0.26. Suppose that there exists a collection of cubes $\{Q_l = Q(x_l, r_l) : l = 1, 2, \ldots, N_0\}$ such that, $2Q_{l_1} \cap 2Q_{l_2} = \emptyset$ for $l_1 \neq l_2$,

$$\frac{r_l^2}{|Q_l|} \int_{Q_l} |V| \, dx \geqq C_2, \quad r_l < \frac{1}{\sqrt{|\lambda|}} \tag{1.1}$$

and

$$l(Q_l)^2 \left(\frac{1}{|Q_l|} \int_{2Q_l} |\mathbf{B}|^2 \, dx \right)^{1/2} \leqq 1 . \tag{1.2}$$

To prove $N(\lambda, H) \geqq N_0$, by the minimax principle, it suffices to show that, there exists a subspace $\mathscr{H}$ such that $\dim \mathscr{H} = N_0$ and for any $g \in \mathscr{H}$,

$$\sum_{j=1}^n \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + \int_{\mathbb{R}^n} V |g|^2 \, dx \leqq \lambda \int_{\mathbb{R}^n} |g|^2 \, dx ,$$

i.e.,

$$\sum_{j=1}^{n} \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 \, dx \leqq \int_{\mathbb{R}^n} |V| \, |g|^2 \, dx \,, \tag{1.3}$$

since $\lambda \leqq 0$ and $V \leqq 0$.

To this end, we let, for $x \in \tilde{Q}_l = 2Q_l$,

$$h_j^l(x) = \frac{1}{|\tilde{Q}_l|} \int_{\tilde{Q}_l} \left\{ \sum_{k=1}^{n} (x_k - y_k) \int_0^1 b_{jk}(y + t(x-y)) \, t \, dt \right\} dy \tag{1.4}$$

and

$$\Phi^l(x) = \frac{1}{|\tilde{Q}_l|} \int_{\tilde{Q}_l} \left\{ \sum_{k=1}^{n} (x_k - y_k) \int_0^1 a_k(y + t(x-y)) \, dt \right\} dy \,, \tag{1.5}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

A computation shows that

$$a_j(x) = h_j^l(x) + \frac{\partial \Phi^l}{\partial x_j}(x), \quad \text{for } 1 \leqq j \leqq n \tag{1.6}$$

and

$$\int_{\tilde{Q}_l} |\mathbf{h}^l|^2 \, dx \leqq C \cdot l(Q_l)^2 \int_{\tilde{Q}_l} |\mathbf{B}|^2 \, dx \,. \tag{1.7}$$

See [I, p. 365].

Note that, by (1.6), for any $g \in C^1(\mathbb{R}^n)$,

$$\left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) (g e^{i\Phi^l}) = e^{i\Phi^l} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - h_j^l \right) g \,. \tag{1.8}$$

Now, let $\psi_l \in C_0^\infty(2Q_l)$ such that $\psi \equiv 1$ on Q_l, $0 \leqq \psi_l \leqq 1$ and $|\nabla \psi_l| \leqq c_n / r_l$. Let

$$\mathscr{H} = \operatorname{Span}\{ e^{i\Phi^l} \psi_l, \ l = 1, 2, \ldots, N_0 \} \,. \tag{1.9}$$

If $g = e^{i\Phi^l} \psi_l$, then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 \, dx \\
&= \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - h_j \right) \psi_l \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |\psi_l|^2 \, dx \\
&\leqq 2 \int_{\mathbb{R}^n} |\nabla \psi_l|^2 \, dx + 2 \int_{\mathbb{R}^n} |\mathbf{h}^l|^2 |\psi_l|^2 \, dx + |\lambda| \int_{\mathbb{R}^n} |\psi_l|^2 \, dx \\
&\leqq C \left\{ \frac{1}{r_l^2} |Q_l| + r_l^2 \int_{\tilde{Q}_l} |\mathbf{B}|^2 \, dx + |\lambda| \, |Q_l| \right\} \\
&\leqq C \cdot r_l^{-2} \cdot |Q_l| \,,
\end{aligned}$$

where we have used (1.1) and (1.2) in the last inequality.

Also note that, by (1.1),

$$\int_{\mathbb{R}^n} |V|\,|g|^2\,dx \geqq \int_{Q_l} |V|\,dx \geqq C_2 \cdot r_l^{-2} \cdot |Q_l|\,.$$

It follows that (1.3) holds for $g = e^{i\Phi^l}\psi_l$ if $C_2 = C_2(n) > 0$ is sufficiently large. Since ψ_l's have disjoint supports, we conclude that (1.3) holds for any $g \in \mathscr{H}$. This completes the proof of Theorem 0.26.

2. Estimates of Kernel Functions

In this section we will establish a size estimate for the kernel function of the operator $(H(\mathbf{a},0)+\lambda)^{-1/2}$ $(\lambda > 0)$. This estimate will be used in the next section to prove the desired trace inequality.

The following lemma may be found in [Sh1, Lemma 1.4, p. 519].

Lemma 2.1. *Suppose* $W \in (RH)_{n/2}$. *Then*

(a) $m(x,W) \approx m(y,W)$ *if* $|x-y| \leqq \dfrac{1}{m(x,W)}$,

(b) $m(y,W) \leqq C\{1+|x-y|m(x,W)\}^{k_0} m(x,W)$,

(c) $m(x,W) \geqq \dfrac{c\,m(y,W)}{\{1+|x-y|m(y,W)\}^{k_0/(k_0+1)}}$,

(d) $1+|x-y|m(x,W) \leqq C\{1+|x-y|m(y,W)\}^{k_0+1}$

for some $k_0 > 0$.

Since

$$(H(\mathbf{a},0)+\lambda)^{-1/2} = \frac{1}{\pi}\int_0^\infty \alpha^{-1/2}\,(H(\mathbf{a},0)+\lambda+\alpha)^{-1}\,d\alpha\,, \tag{2.2}$$

we shall first estimate the kernel function $\Gamma_\lambda(x,y)$ of $(H(\mathbf{a},0)+\lambda)^{-1}$.

The following theorem is a special case of Theorem 1.13 in [Sh4].

Theorem 2.3. *Let* $\mathbf{a} \in C^2(\mathbb{R}^n)$ *and* $\lambda \geqq 1$. *Assume that*

$$\left(\frac{1}{|Q|}\int_Q |\mathbf{B}|^{n/2}\,dx\right)^{2/n} \leqq C_0\left(\lambda + \frac{1}{|Q|}\int_Q |\mathbf{B}|\,dx\right) \tag{2.4}$$

for any cube Q *with* $l(Q) \leqq 1$, *and*

$$|\nabla\mathbf{B}(x)| \leqq C_1\{m(x,|\mathbf{B}|+\lambda)\}^3\,. \tag{2.5}$$

Then

$$|\Gamma_\lambda(x,y)| \leqq \frac{C_k}{\{1+|x-y|m(x,|\mathbf{B}|+\lambda)\}^k}\cdot\frac{1}{|x-y|^{n-2}}$$

for $|x-y| \leqq 1$ *and any integer* $k > 0$.

Theorem 2.6. *Let* $\mathbf{a} \in C^2(\mathbb{R}^n)$. *Assume that* $|\mathbf{B}| \in (RH)_{n/2}$ *and*

$$|\nabla \mathbf{B}(x)| \leqq C_1\{m(x, |\mathbf{B}|)\}^3, \quad x \in \mathbb{R}^n . \tag{2.7}$$

Then

$$|\Gamma_\lambda(x, y)| \leqq \frac{C_k}{\{1 + |x - y| m(x, |\mathbf{B}| + \lambda)\}^k} \cdot \frac{1}{|x - y|^{n-2}}$$

for $x, \; y \in \mathbb{R}^n$, $\lambda > 0$ *and any integer* $k > 0$.

Proof. We derive this theorem from Theorem 2.3 by a rescaling argument.

For $0 < \varepsilon < \sqrt{\lambda}$, let

$$a_j^\varepsilon(x) = \frac{1}{\varepsilon} a_j\left(\frac{x}{\varepsilon}\right). \tag{2.8}$$

Then

$$b_{jk}^\varepsilon(x) = \frac{\partial a_j^\varepsilon}{\partial x_k} - \frac{\partial a_k^\varepsilon}{\partial x_j} = \frac{1}{\varepsilon^2} b_{jk}\left(\frac{x}{\varepsilon}\right). \tag{2.9}$$

Note that

$$\left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j^\varepsilon\right)(g)(x) = \frac{1}{\varepsilon}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j\right)(f)\left(\frac{x}{\varepsilon}\right), \tag{2.10}$$

where $g(x) = f(\frac{x}{\varepsilon})$. It follows that

$$\left(H(\mathbf{a}^\varepsilon, 0) + \frac{\lambda}{\varepsilon^2}\right)(g)(x) = \frac{1}{\varepsilon^2}\,(H(\mathbf{a}, 0) + \lambda)(f)\left(\frac{x}{\varepsilon}\right). \tag{2.11}$$

Hence, if $\Gamma_\lambda^\varepsilon(x, y)$ denotes the kernel function for $(H(\mathbf{a}^\varepsilon, 0) + \frac{\lambda}{\varepsilon^2})^{-1}$, we have

$$\Gamma_\lambda(x, y) = \varepsilon^{n-2}\Gamma_\lambda^\varepsilon(\varepsilon x, \varepsilon y). \tag{2.12}$$

Finally, note that, $|\mathbf{B}^\varepsilon| \in (RH)_{n/2}$ with a constant C_0 independent of $\varepsilon > 0$,

$$m\left(x, |\mathbf{B}^\varepsilon| + \frac{\lambda}{\varepsilon^2}\right) = \frac{1}{\varepsilon} m\left(\frac{x}{\varepsilon}, |\mathbf{B}| + \lambda\right), \tag{2.13}$$

where $\mathbf{B}^\varepsilon = (b_{jk}^\varepsilon)$, and

$$\begin{aligned} |\nabla \mathbf{B}^\varepsilon(x)| &= \frac{1}{\varepsilon^3}\left|\nabla \mathbf{B}\left(\frac{x}{\varepsilon}\right)\right| \leqq \frac{C_1}{\varepsilon^3}\left\{m\left(\frac{x}{\varepsilon}, |\mathbf{B}|\right)\right\}^3 \leqq \frac{C_1}{\varepsilon^3}\left\{m\left(\frac{x}{\varepsilon}, |\mathbf{B}| + \lambda\right)\right\}^3 \\ &= C_1\left\{m\left(x, |\mathbf{B}^\varepsilon| + \frac{\lambda}{\varepsilon^2}\right)\right\}^3 \end{aligned}$$

by (2.7) and (2.13). Thus, by (2.12) and Theorem 2.3, if $|x - y| \leqq \frac{1}{\varepsilon}$,

$$\begin{aligned} |\Gamma_\lambda(x, y)| &\leqq \frac{C_k \cdot \varepsilon^{n-2}}{\{1 + |\varepsilon x - \varepsilon y| m(\varepsilon x, |\mathbf{B}^\varepsilon| + \frac{\lambda}{\varepsilon^2})\}^k} \cdot \frac{1}{|\varepsilon x - \varepsilon y|^{n-2}} \\ &= \frac{C_k}{\{1 + \varepsilon|x - y| m(\varepsilon x, |\mathbf{B}^\varepsilon| + \frac{\lambda}{\varepsilon^2})\}^k} \cdot \frac{1}{|x - y|^{n-2}} \\ &= \frac{C_k}{\{1 + |x - y| m(x, |\mathbf{B}| + \lambda)\}^k} \cdot \frac{1}{|x - y|^{n-2}}, \end{aligned}$$

where we have used (2.13) in the last step.

The proof is then finished since ε can be made arbitrarily small.

Using (2.2), we may write

$$(H(\mathbf{a},0)+\lambda)^{-1/2}f(x)=\int_{\mathbb{R}^n} K_\lambda(x,y)f(y)\,dy\,, \tag{2.14}$$

where

$$K_\lambda(x,y)=\frac{1}{\pi}\int_0^\infty \alpha^{-1/2}\Gamma_{\lambda+\alpha}(x,y)\,d\alpha\,. \tag{2.15}$$

Theorem 2.16. *Under the same hypothesis as in Theorem* 2.6, *we have*

$$|K_\lambda(x,y)|\leqq \frac{C_k}{\{1+|x-y|m(x,|\mathbf{B}|+\lambda)\}^k}\cdot\frac{1}{|x-y|^{n-1}}$$

for $x,\ y\in\mathbb{R}^n$, $\lambda>0$ *and any integer* $k>0$.

Proof. It follows from (2.15) and Theorem 2.6 that

$$\begin{aligned}
|K_\lambda(x,y)| &\leqq \frac{C_k}{|x-y|^{n-2}}\int_0^\infty \frac{\alpha^{-1/2}\,d\alpha}{\{1+|x-y|m(x,|\mathbf{B}|+\lambda+\alpha)\}^k}\\
&\leqq \frac{1}{|x-y|^{n-2}}\cdot\frac{C_k}{\{1+|x-y|m(x,|\mathbf{B}|+\lambda)\}^{k-2}}\int_0^\infty\frac{\alpha^{-1/2}\,d\alpha}{\{1+|x-y|\sqrt{\alpha}\}^2}\\
&\leqq \frac{C_k}{\{1+|x-y|m(x,|\mathbf{B}|+\lambda)\}^{k-2}}\cdot\frac{1}{|x-y|^{n-1}}\,,
\end{aligned}$$

where in the second inequality we have used $m(x,|\mathbf{B}|+\lambda+\alpha)\geqq m(x,|\mathbf{B}|+\lambda)$ and $m(x,|\mathbf{B}|+\lambda+\alpha)\geqq\sqrt{\alpha}$.

The theorem then follows since $k>0$ is arbitrary.

Remark 2.17. By part (d) of Lemma 2.1, we also have

$$|K_\lambda(x,y)|\leqq \frac{C_k}{\{1+|x-y|m(y,|\mathbf{B}|+\lambda)\}^k}\cdot\frac{1}{|x-y|^{n-1}}\,. \tag{2.18}$$

3. A Trace Inequality

This section is devoted to the proof of the following trace inequality.

Theorem 3.1. *Suppose that* $\mathbf{a}\in C^2(\mathbb{R}^n)$ *and* $V\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *for some* $p>1$ *Also assume that* $|\mathbf{B}|\in(RH)_{n/2}$ *and*

$$|\nabla\mathbf{B}(x)|\leqq C_2\{m(x,|\mathbf{B}|)\}^3\quad on\ \mathbb{R}^n\,.$$

Then, for $g\in C_0^1(\mathbb{R}^n)$ *and* $\lambda\geqq 0$,

$$\int_{\mathbb{R}^n}|V|\,|g|^2\,dx\leqq C\cdot M_p\left\{\sum_{j=1}^n\int_{\mathbb{R}^n}\left|\left(\frac{1}{i}\frac{\partial}{\partial x_j}-a_j\right)g\right|^2dx+\lambda\int_{\mathbb{R}^n}|g|^2\,dx\right\}, \tag{3.2}$$

where

$$M_p = \sup_Q l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V|^p \, dx \right)^{1/p}$$

and the supremum is over all dyadic cubes Q *with the property*

$$l(Q) < \inf_{x \in Q} \frac{\alpha}{m(x, |\mathbf{B}|) + \sqrt{\lambda}}$$

and C *depends on* C_0, C_1, n, p *and* α (*to be chosen later*).

It is not hard to see that (3.2) is equivalent to

$$\int_{\mathbb{R}^n} |V| \, |(H+\lambda)^{-1/2} f|^2 \, dx \leqq C \cdot M_p \int_{\mathbb{R}^n} |f|^2 \, dx \,. \tag{3.3}$$

To show (3.3), we note that, by (2.18),

$$\begin{aligned}
|(H+\lambda)^{-1/2} f(x)| &\leqq C \int_{R^n} \frac{|f(y)| \, dy}{\{1 + m(y, |\mathbf{B}| + \lambda)|x-y|\}^k |x-y|^{n-1}} \\
&\leqq C \int_{|x-y| < \frac{c_0}{m(y, |\mathbf{B}|+\lambda)}} \frac{|f(y)| \, dy}{|x-y|^{n-1}} \\
&\quad + C \int_{|x-y| \geqq \frac{c_0}{m(y, |\mathbf{B}|+\lambda)}} \frac{|f(y)| \, dy}{\{m(y, |\mathbf{B}| + \lambda)|x-y|\}^k |x-y|^{n-1}} \\
&= C \{T_1(|f|)(x) + T_2(|f|)(x)\} \,,
\end{aligned}$$

where $c_0 > 0$ is a small constant to be determined later.

The desired estimate of $T_2(|f|)$ is fairly straightforward.

Lemma 3.4. *Under the same assumption as in Theorem* 3.1, *we have*

$$\int_{\mathbb{R}^n} |V| \, |T_2 f|^2 \, dx \leqq C \cdot M_1 \int_{\mathbb{R}^n} |f|^2 \, dx \,.$$

Proof. Note that, by part (d) of Lemma 2.1,

$$\int_{|x-y| \geqq \frac{c_0}{m(y, |\mathbf{B}|+\lambda)}} \frac{dy}{\{m(y, |\mathbf{B}| + \lambda)|x-y|\}^k |x-y|^{n-1}} \leqq \frac{C}{m(x, |\mathbf{B}| + \lambda)} \,.$$

Thus, by the Cauchy inequality, $T_2 f(x)$ is bounded by

$$\frac{C}{\{m(x, |\mathbf{B}| + \lambda)\}^{1/2}} \left\{ \int_{|x-y| \geqq \frac{c_0}{m(y, |\mathbf{B}|+\lambda)}} \frac{|f(y)|^2 \, dy}{\{m(y, |\mathbf{B}| + \lambda)|x-y|\}^k |x-y|^{n-1}} \right\}^{1/2} .$$

It follows that

$$\int_{\mathbb{R}^n} |V(x)|\,|T_2 f(x)|^2\,dx \leqq C \int_{\mathbb{R}^n} |f(y)|^2\,dy$$

$$\times \left\{ \int_{|x-y| \geqq \frac{c_0}{m(y,|\mathbf{B}|+\lambda)}} \frac{|V(x)|dx}{m(x,|\mathbf{B}|+\lambda)\{m(y,|\mathbf{B}|+\lambda)|x-y|\}^k |x-y|^{n-1}} \right\}.$$

We will show that

$$\int_{|x-y| \geqq \frac{c_0}{m(y,|\mathbf{B}|+\lambda)}} \frac{|V(x)|dx}{m(x,|\mathbf{B}|+\lambda)\{m(y,|\mathbf{B}|+\lambda)|x-y|\}^k |x-y|^{n-1}} \leqq C \cdot M_1 . \quad (3.5)$$

Clearly, this gives the estimate in the lemma.

To see (3.5), note that, if t is an integer and $2^{t-1} > c_0$,

$$\int_{|x-y| \approx \frac{2^t}{m(y,|\mathbf{B}|+\lambda)}} \frac{|V(x)|dx}{m(x,|\mathbf{B}|+\lambda)\{m(y,|\mathbf{B}|+\lambda)|x-y|\}^k |x-y|^{n-1}}$$

$$\leqq \frac{C\{m(y,|\mathbf{B}|+\lambda)\}^{n-1}}{(2^t)^{k+n-1}} \int_{|x-y| \approx \frac{2^t}{m(y,|\mathbf{B}|+\lambda)}} \frac{|V(x)|dx}{m(x,|\mathbf{B}|+\lambda)}$$

$$\leqq \frac{C\{m(y,|\mathbf{B}|+\lambda)\}^{n-2}}{(2^t)^{k+n-1-\frac{k_0}{k_0+1}}} \int_{E(y)} |V(x)|\,dx ,$$

where $E(y)$ is the ball centered at y with radius $C2^t/m(y,|\mathbf{B}|+\lambda)$ and we have used part (c) of Lemma 2.1 in the last inequality.

Now, fix y, we may cover $E(y)$ by a collection of cubes $\{Q_l = Q(x_l,r_l)\}$ with $r_l = 1/m(x_l,|\mathbf{B}|+\lambda)$ such that $x_l \in E(y)$ and $\{Q(x_l,r_l/5),\ l=1,2,\ldots\}$ are disjoint. Then

$$\int_{E(y)} |V(x)|\,dx \leqq \sum_l \int_{Q_l} |V(x)|\,dx \leqq M_1 \sum_l r_l^{n-2}$$

$$= M_1 \sum_l r_l^n \cdot \{m(x_l,|\mathbf{B}|+\lambda)\}^2$$

$$\leqq C \cdot M_1 \cdot (2^t)^{2k_0} \cdot \{m(y,|\mathbf{B}|+\lambda)\}^2 \sum_l \left| Q\left(x_l, \frac{r_l}{5}\right) \right|$$

$$\leqq C \cdot M_1 \cdot (2^t)^{2k_0} \cdot \{m(y,|\mathbf{B}|+\lambda)\}^2 \left| \left\{ x \in \mathbb{R}^n : |x-y| \leqq \frac{C2^t}{m(y,|\mathbf{B}|+\lambda)} \right\} \right|$$

$$\leqq C \cdot M_1 \cdot (2^t)^{2k_0+n} \{m(y,|\mathbf{B}|+\lambda)\}^{2-n} ,$$

where we have used the fact that, for $x \in Q(x_l, r_l/5)$,

$$|x-y| \leqq |x-x_l| + |x_l - y| \leqq \frac{C\{2^t + (1+2^t)^{k_0/(k_0+1)}\}}{m(y, |\mathbf{B}|+\lambda)}$$
$$\leqq \frac{C2^t}{m(y, |\mathbf{B}|+\lambda)} .$$

Thus,

$$\int\limits_{|x-y|\approx \frac{2^t}{m(y,|\mathbf{B}|+\lambda)}} \frac{|V(x)|dx}{m(x,|\mathbf{B}|+\lambda)\{m(y,|\mathbf{B}|+\lambda)|x-y|\}^k |x-y|^{n-1}} \leqq \frac{C \cdot M_1}{(2^t)^{k-\frac{k_0}{k_0+1}-2k_0-1}} .$$

Equation (3.5) then follows easily if we choose k sufficiently large and sum up the above estimate over t. This completes the proof of Lemma 3.4.

It remains to show that

$$\int\limits_{\mathbb{R}^n} |T_1 f|^2 |V| \, dx \leqq C \cdot M_p \int\limits_{\mathbb{R}^n} |f|^2 \, dx , \tag{3.6}$$

where

$$T_1 f(x) = \int\limits_{|x-y|<\frac{c_0}{m(y,|\mathbf{B}|+\lambda)}} \frac{f(y)\,dy}{|x-y|^{n-1}} . \tag{3.7}$$

Our approach to $T_1 f$ will be similar to that in the case of $-\Delta + V$, where the corresponding operator is

$$(-\Delta)^{-1/2} f(x) = c_n \int\limits_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-1}} .$$

See [F, K-Sa].

Let

$$V^+(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int\limits_Q |V|^p dx \right)^{1/p} , \tag{3.8}$$

where the supremum is over all dyadic cubes such that $x \in Q$ and

$$l(Q) < \inf_{y \in Q} \frac{\alpha}{m(y,|\mathbf{B}|)+\sqrt{\lambda}} . \tag{3.9}$$

Clearly, $|V(x)| \leqq V^+(x)$. We will show that

$$\int\limits_{\mathbb{R}^n} |T_1 f|^2 V^+ dx \leqq C \cdot M_p \int\limits_{\mathbb{R}^n} |f|^2 \, dx . \tag{3.10}$$

Note that

$$\int\limits_{\mathbb{R}^n} T_1 f(x) g(x) V^+(x) \, dx = \int\limits_{\mathbb{R}^n} f(y) \left\{ \int\limits_{|x-y|<\frac{c_0}{m(y,|\mathbf{B}|+\lambda)}} \frac{g(x)V^+(x)\,dx}{|x-y|^{n-1}} \right\} dy .$$

By duality, it suffices to show that

$$\int\limits_{\mathbb{R}^n} |T(gV^+)|^2 dx \leqq C \cdot M_p \int\limits_{\mathbb{R}^n} |g|^2 V^+ dx , \tag{3.11}$$

where

$$Tg(x) = \int\limits_{|y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)}} \frac{g(y)\,dy}{|x-y|^{n-1}} \,. \tag{3.12}$$

Lemma 3.13. *Let*

$$Mf(x) = \sup_{Q \ni x} \frac{1}{l(Q)^{n-1}} \int\limits_Q |f(y)|\,dy \,,$$

where the supremum is over all cubes Q such that $x \in Q$ and

$$l(Q) < \inf_{y \in Q} \frac{C_4\, c_0}{m(y, |\mathbf{B}| + \lambda)} \,.$$

Then, for $0 < q < \infty$,

$$\int\limits_{\mathbb{R}^n} |Tf|^q\,dx \leqq C_q \int\limits_{\mathbb{R}^n} |Mf|^q\,dx \,.$$

Proof. Without loss of generality, we may assume that $f \geqq 0$ a.e.

Let $\beta > 0$ and

$$E = \{x \in \mathbb{R}^n : |Tf(x)| > \beta\} \,.$$

By the Whitney decomposition [St1, p. 167], there exists a collection of disjoint dyadic cubes $\{Q_l,\ l = 1,2,\dots\}$ such that $E = \bigcup_l Q_l$ and $5Q_l \cap (\mathbb{R}^n \backslash E) \neq \emptyset$.

We will show that, for $\gamma > 0$ small,

$$|\{x \in Q_l : |Tf(x)| > 2\beta,\ Mf(x) \leqq \gamma\beta\}| \leqq C\gamma^{\frac{n}{n-1}} |Q_l| \,. \tag{3.14}$$

This implies the estimate in the lemma. Indeed, summing over all cubes Q_l, we obtain

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tf(x)| > 2\beta\}| &\leqq |\{x \in \mathbb{R}^n : Mf(x) > \gamma\beta\}| \\ &+ C\gamma^{\frac{n}{n-1}} |\{x \in \mathbb{R}^n : |Tf(x)| > \beta\}| \,. \end{aligned}$$

We then multiply both sides of this inequality by $q\beta^{q-1}$ and integrate in β over $(0,\infty)$. The estimate in the lemma follows by choosing γ small and bringing the second integral in the right side to the left.

To show (3.14), we fix $Q = Q_l = Q(x_0, r_0)$. We need to consider two cases.

We begin with the case when

$$r_0 < \frac{\varepsilon\, c_0}{m(x_0, |\mathbf{B}| + \lambda)} \,. \tag{3.15}$$

Let $f = f_1 + f_2$, where $f_1 = f\chi_{6Q}$. Let $z \in 5Q \cap (\mathbb{R}^n \setminus E)$. For any $x \in Q$, we write

$$
\begin{aligned}
&T(f_2)(x) - T(f_2)(z) \\
&= \int\limits_{\substack{|y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)} \\ |y-z| \geqq \frac{c_0}{m(z,|\mathbf{B}|+\lambda)}}} \frac{f_2(y)\,dy}{|y-x|^{n-1}} - \int\limits_{\substack{|y-z| < \frac{c_0}{m(z,|\mathbf{B}|+\lambda)} \\ |y-x| \geqq \frac{c_0}{m(x,|\mathbf{B}|+\lambda)}}} \frac{f_2(y)\,dy}{|y-z|^{n-1}} \\
&\quad + \int\limits_{\substack{|y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)} \\ |y-z| < \frac{c_0}{m(z,|\mathbf{B}|+\lambda)}}} f_2(y) \left\{ \frac{1}{|y-x|^{n-1}} - \frac{1}{|y-z|^{n-1}} \right\} dy \\
&= I_1 + I_2 + I_3 .
\end{aligned}
$$

Note that, by (3.15) and part (a) of Lemma 2.1, $m(w, |\mathbf{B}| + \lambda) \approx m(x_0, |\mathbf{B}| + \lambda)$ if $w \in 5Q$. Thus, for I_1,

$$
\begin{aligned}
|y-x| \geqq |y-z| - |z-x| &\geqq \frac{c_0}{m(z, |\mathbf{B}| + \lambda)} - c_n l(Q) \\
&\geqq \frac{c_0}{m(z, |\mathbf{B}| + \lambda)} - \frac{\varepsilon\, c_n\, c_0}{m(x_0, |\mathbf{B}| + \lambda)} \geqq \frac{c}{m(x, |\mathbf{B}| + \lambda)}
\end{aligned}
$$

if ε is small.

It follows that

$$
\begin{aligned}
|I_1| &\leqq C \cdot \{m(x, |\mathbf{B}| + \lambda)\}^{n-1} \int\limits_{|y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)}} f(y)\,dy \\
&\leqq C M f(x) .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\leqq C \cdot \{m(z, |\mathbf{B}| + \lambda)\}^{n-1} \int\limits_{|y-z| < \frac{c_0}{m(z,|\mathbf{B}|+\lambda)}} f(y)\,dy \\
&\leqq C \cdot \{m(x, |\mathbf{B}| + \lambda)\}^{n-1} \int\limits_{|y-x| < \frac{C c_0}{m(x,|\mathbf{B}|+\lambda)}} f(y)\,dy \\
&\leqq C M f(x) .
\end{aligned}
$$

To estimate I_3, note that

$$
\begin{aligned}
|I_3| &\leqq C \int\limits_{|y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)}} f_2(y) \cdot \frac{|x-z|}{|y-x|^n}\,dy \\
&\leqq C \sum_{j=0}^{\infty} \frac{1}{[2^j l(Q)]^{n-1}} \int\limits_{\substack{|y-x| \sim 2^j l(Q) \\ |y-x| < \frac{c_0}{m(x,|\mathbf{B}|+\lambda)}}} f(y)\,dy \cdot \frac{1}{2^j} \\
&\leqq C M f(x) .
\end{aligned}
$$

This, together with estimates for I_1 and I_2, shows that

$$\begin{aligned} |Tf_2(x)| &\leqq |T(f_2)(z)| + CMf(x) \leqq |T(f)(z)| + CMf(x) \\ &\leqq \beta + C\gamma\beta \leqq 3\beta/2 \end{aligned}$$

if $Mf(x) \leqq \gamma\beta$, and γ is small.

Thus,

$$\begin{aligned} &|\{x \in Q : |Tf(x)| > 2\beta,\ Mf(x) \leqq \gamma\beta\}| \\ &\leqq |\{x \in Q : |T(f_1)(x)| > \beta/2,\ Mf(x) \leqq \gamma\beta\}| \\ &\leqq |\{x \in \mathbb{R}^n : |(-\Delta)^{-1/2}(f_1)(x)| > c_n\beta\}| \\ &\leqq \left(\frac{C_n}{\beta}\int_{\mathbb{R}^n}|f_1|dx\right)^{\frac{n}{n-1}} \leqq \frac{C_n}{\beta^{\frac{n}{n-1}}}\left(\int_{6Q}|f|dx\right)^{\frac{n}{n-1}} \\ &\leqq \frac{C_n}{\beta^{\frac{n}{n-1}}}\cdot|Q|\cdot\left(\frac{1}{l(6Q)^{n-1}}\int_{6Q}|f|dx\right)^{\frac{n}{n-1}} \\ &\leqq \frac{C_n}{\beta^{\frac{n}{n-1}}}\cdot|Q|\cdot(\gamma\beta)^{\frac{n}{n-1}} = C_n\gamma^{\frac{n}{n-1}}|Q|, \end{aligned}$$

where we have used the weak-type $(1, \frac{n}{n-1})$ estimate for the fractional integral $(-\Delta)^{-1/2}$ (see [St1, p. 119–120]) in the third inequality.

Next, we consider the case when

$$r_0 \geqq \frac{\varepsilon c_0}{m(x_0, |\mathbf{B}| + \lambda)}.$$

In this case, we bisect Q repeatedly, stopping at $I_k = Q(x_k, r_k)$ if

$$r_k < \frac{\varepsilon c_0}{m(x_k, |\mathbf{B}| + \lambda)}. \tag{3.16}$$

We then obtain $Q = \bigcup_k I_k$. Since I_k is a maximal element among subcubes which satisfy (3.16), by part (a) of Lemma 2.1,

$$r_k \geqq \frac{c_0}{C\, m(x_k, |\mathbf{B}| + \lambda)}. \tag{3.17}$$

Note that, if $x \in I_k$ and $|y - x| < c_0/m(x, |\mathbf{B}| + \lambda)$,

$$|y - x_k| \leqq |y - x| + |x - x_k| \leqq \frac{c_0}{m(x, |\mathbf{B}| + \lambda)} + \sqrt{n}\, r_k \leqq \frac{C c_0}{m(x_k, |\mathbf{B}| + \lambda)}. \tag{3.18}$$

Now let $\widetilde{I_k} = Q(x_k, Cc_0/m(x_k, |\mathbf{B}| + \lambda))$. By (3.16), (3.17) and (3.18), $|I_k| \approx |\widetilde{I_k}|$,

$$l(\widetilde{I_k}) = \frac{Cc_0}{m(x_k, |\mathbf{B}| + \lambda)} \leqq \inf_{y \in \widetilde{I_k}} \frac{C_4 c_0}{m(y, |\mathbf{B}| + \lambda)},$$

$$Tf(x) = T(f\chi_{\widetilde{I_k}})(x) \quad \text{for} \quad x \in I_k.$$

It then follows from the weak type $(1, \frac{n}{n-1})$ estimates of the fractional integral $(-\Delta)^{-1/2}$ [St1, p. 119] that

$$
\begin{aligned}
&|\{x \in Q : |T(f)(x)| > 2\beta,\ Mf(x) \leqq \gamma\beta\}| \\
&\leqq \sum_k |\{x \in I_k : (-\Delta)^{-1/2}(f\chi_{\widetilde{I}_k})(x) > c\beta,\ Mf(x) \leqq \gamma\beta\}| \\
&\leqq \frac{C}{\beta^{\frac{n}{n-1}}} \sum_k \left(\int_{\widetilde{I}_k} |f(x)|dx \right)^{n/(n-1)} \\
&\leqq \frac{C}{\beta^{\frac{n}{n-1}}} \sum_k |\widetilde{I}_k| \cdot (\gamma\beta)^{n/(n-1)} \leqq C\gamma^{\frac{n}{n-1}} |Q|,
\end{aligned}
$$

where we have assumed that $\{x \in I_k : Mf(x) \leqq \gamma\beta\} \neq \emptyset$.

The proof of Lemma 3.13 is now complete.

Let

$$
M^{dy} f(x) = \sup_{Q \ni x} \frac{1}{l(Q)^{n-1}} \int_Q |f(y)|\, dy, \tag{3.19}
$$

where the supremum is taken over all *dyadic* cubes with the property

$$
l(Q) < \inf_{y \in Q} \frac{C_4^2 c_0}{m(y, |\mathbf{B}| + \lambda)}. \tag{3.20}
$$

Lemma 3.21. *For $0 < q < \infty$, we have*

$$
\int_{\mathbb{R}^n} |Mf(x)|^q\, dx \leqq C_q \int_{\mathbb{R}^n} |M^{dy} f(x)|^q\, dx.
$$

Proof. The lemma follows from

$$
|\{x \in \mathbb{R}^n : Mf(x) > \beta\}| \leqq c_n |\{x \in \mathbb{R}^n : M^{dy} f(x) > 2^{-n}\beta\}|. \tag{3.22}
$$

We omit the details. See [K-Sa, p. 215] for a similar estimate.

Finally, we have to show

$$
\int_{\mathbb{R}^n} |M^{dy}(fV^+)|^2 dx \leqq C \cdot M_p \int_{\mathbb{R}^n} |f|^2 V^+\, dx. \tag{3.23}
$$

Lemma 3.24. *Suppose that Q is a dyadic cube which satisfies* (3.9). *Then*

$$
\frac{1}{|Q|} \int_Q V^+(x)\, dx \leqq C \sup_{Q' \supseteq Q} \left(\frac{1}{|Q'|} \int_{Q'} |V(x)|^p dx \right)^{1/p}, \tag{3.25}
$$

where the supremum is over all dyadic cubes Q' which contain Q and satisfy (3.9).

Proof. Let $c(Q)$ denote the supremum in the right side of (3.25) and

$$V_Q^+(x) = \sup_{\substack{Q' \subset Q, x \in Q' \\ Q' \text{ dyadic}}} \left(\frac{1}{|Q'|} \int_{Q'} |V(y)|^p dy \right)^{1/p} .$$

Then, for $x \in Q$,

$$V^+(x) \leqq c(Q) + V_Q^+(x) .$$

By the maximal theorem,

$$\begin{aligned} \frac{1}{|Q|} \int_Q V^+ dx &\leqq c(Q) + \frac{1}{|Q|} \int_Q V_Q^+(x)\, dx \\ &\leqq c(Q) + C \left(\frac{1}{|Q|} \int_Q |V|^p dx \right)^{1/p} \\ &\leqq C \cdot c(Q) . \end{aligned}$$

Lemma 3.24 is proved.
It follows from (3.25) that

$$\frac{1}{|Q|} \int_Q V^+(x)\, dx \leqq C \cdot \inf_{x \in Q} V^+(x) , \tag{3.26}$$

i.e., $V^+(x)$ is an A_1 weight on cubes satisfying (3.9). Thus, if $E \subset Q$ and Q satisfies (3.9),

$$\int_E V^+(x)\, dx \geqq |E| \cdot \inf_{x \in E} V^+(x) \geqq |E| \cdot \inf_{x \in Q} V^+(x) \geqq \frac{|E|}{C} \cdot \frac{1}{|Q|} \int_Q V^+(x)\, dx .$$

Hence,

$$\frac{\int_E V^+ dx}{\int_Q V^+ dx} \geqq \frac{1}{C} \cdot \frac{|E|}{|Q|} . \tag{3.27}$$

Let

$$L = \sup_Q \frac{1}{l(Q)^{n-2}} \int_Q V^+(x)\, dx , \tag{3.28}$$

where supremum is over all dyadic cubes with the property

$$l(Q) < \inf_{x \in Q} \frac{C_4^2 c_0}{m(x, |\mathbf{B}| + \lambda)} .$$

Lemma 3.29. *We have*

$$\int_{\mathbb{R}^n} |M^{dy}(f V^+)|^2\, dx \leqq C \cdot L \int_{\mathbb{R}^n} |f|^2 V^+\, dx .$$

Proof. Let j be an integer. Suppose Q_j^k's are the maximal element among all dyadic cubes which satisfy

$$\frac{1}{l(Q)^{n-1}} \int_Q |f| V^+ dx > 2^j \quad \text{and} \quad l(Q) < \inf_{x \in Q} \frac{C_4^2 c_0}{m(x, |\mathbf{B}| + \lambda)} . \tag{3.30}$$

Then Q_j^k are disjoint for a fixed j and

$$\{x \in \mathbb{R}^n : M^{dy}(fV^+)(x) > 2^j\} = \bigcup_k Q_j^k .$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^n} |M^{dy}(fV^+)|^2 dx &\leqq \sum_j 2^{2j+2} |\{x \in \mathbb{R}^n : 2^j < M^{dy}(fV^+)(x) \leqq 2^{j+1}\}| \\ &= \sum_{j,k} 2^{2j+2} |E_j^k| , \end{aligned}$$

where

$$E_j^k = Q_j^k \backslash \{x \in \mathbb{R}^n : M^{dy}(fV^+)(x) > 2^{j+1}\} . \tag{3.31}$$

Thus,

$$\int_{\mathbb{R}^n} |M^{dy}(fV^+)|^2 \, dx \leqq 4 \sum_{j,k} \left\{ \frac{1}{l(Q_j^k)^{n-1}} \int_{Q_j^k} |f| V^+ dx \right\}^2 |E_j^k|$$

$$= C \sum_{j,k} \left\{ \frac{1}{V^+(Q_j^k)} \int_{Q_j^k} |f| V^+ dx \right\}^2 \cdot V^+(E_j^k) \cdot \frac{V^+(Q_j^k)}{V^+(E_j^k)} \cdot \frac{|E_j^k|}{|Q_j^k|} \cdot \frac{V^+(Q_j^k)}{l(Q_j^k)^{n-2}}$$

$$\leqq C \cdot L \sum_{j,k} \left\{ \frac{1}{V^+(Q_j^k)} \int_{Q_j^k} |f| V^+ dx \right\}^2 \cdot V^+(E_j^k) \cdot \frac{V^+(Q_j^k)}{V^+(E_j^k)} \cdot \frac{|E_j^k|}{|Q_j^k|} ,$$

where we have used the notation

$$V^+(E) = \int_E V^+(x) \, dx .$$

Now, note that, if c_0 is small,

$$l(Q_j^k) \leqq \inf_{x \in Q_j^k} \frac{C_4^2 \, c_0}{m(x, |\mathbf{B}| + \lambda)} < \inf_{x \in Q_j^k} \frac{\alpha}{m(x, |\mathbf{B}|) + \sqrt{\lambda}} .$$

By (3.26),

$$\frac{V^+(Q_j^k)}{V^+(E_j^k)} \cdot \frac{|E_j^k|}{|Q_j^k|} \leqq C .$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |M^{dy}(fV^+)|^2 dx &\leqq C \cdot L \sum_{j,k} \left\{ \frac{1}{V^+(Q_j^k)} \int_{Q_j^k} |f| V^+ dx \right\}^2 \cdot V^+(E_j^k) \\ &\leqq C \cdot L \sum_{j,k} \int_{E_j^k} |M_{V^+}(f)|^2 V^+ dx \\ &\leqq C \cdot L \int_{\mathbb{R}^n} |M_{V^+}(f)|^2 V^+ dx ,\end{aligned}$$

where

$$M_{V^+}(f)(x) = \sup_{\substack{Q \ni x \\ Q \text{ dyadic}}} \frac{1}{V^+(Q)} \int_Q |f| V^+ dy . \tag{3.32}$$

The lemma then follows from the L^2 inequality for the dyadic Hardy–Littlewood maximal function in the space $L^2(\mathbb{R}^n, V^+ dx)$. We remark that in the dyadic case, the doubling condition on the measure is not needed.

We now are in a position to give the

Proof of Theorem 3.1. We may assume that $\lambda > 0$. By Lemma 3.4 and $M_1 \leqq M_p$,

$$\int_{\mathbb{R}^n} |V| \, |(H(\mathbf{a},0) + \lambda)^{-1/2} f|^2 dx \leqq C \cdot M_p \int_{\mathbb{R}^n} |f|^2 dx + C \int_{\mathbb{R}^n} |V| \, |T_1(|f|)|^2 \, dx .$$

By duality, the desired estimate of $T_1(|f|)$ follows from

$$\int_{\mathbb{R}^n} |T(gV^+)|^2 dx \leqq C \cdot M_p \int_{\mathbb{R}^n} |g|^2 V^+ \, dx . \tag{3.33}$$

See (3.11). By Lemma 3.13, Lemma 3.21 and Lemma 3.29,

$$\begin{aligned}\int_{\mathbb{R}^n} |T(gV^+)|^2 \, dx &\leqq C \int_{\mathbb{R}^n} |M(gV^+)|^2 \, dx \leqq C \int_{\mathbb{R}^n} |M^{dy}(gV^+)|^2 \, dx \\ &\leqq C \cdot L \int_{\mathbb{R}^n} |g|^2 V^+ \, dx .\end{aligned}$$

Finally, note that

$$\begin{aligned}L &= \sup_Q l^2(Q) \left(\frac{1}{|Q|} \int_Q V^+(x) \, dx \right) \\ &\leqq C \sup_Q l(Q)^2 \left\{ \sup_{Q' \supseteq Q} \left(\frac{1}{|Q'|} \int_{Q'} |V|^p dx \right)^{1/p} \right\} \\ &\leqq C \sup_{Q'} l(Q')^2 \left(\frac{1}{|Q'|} \int_{Q'} |V|^p dx \right)^{1/p} = C \cdot M_p ,\end{aligned}$$

where Q and Q' are dyadic cubes satisfying (3.9). The proof is complete.

4. The Proof of Theorem 0.11

In this section we give the proof of Theorem 0.11 stated in the Introduction.
We begin with a lower bound for the operator $H(\mathbf{a}, 0)$.

Theorem 4.1. *Under the same hypothesis as in Theorem* 0.11, *we have, for* $g \in C_0^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(x, |\mathbf{B}|)g(x)|^2\, dx \leqq C \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g(x) \right|^2 dx\,.$$

Theorem 4.1 follows from [Sh3, Theorem 2.7] by a rescaling argument similar to that in the proof of Theorem 2.6. We omit the details.

Definition 4.2. *Let $\mathscr{B}$ be a collection of dyadic cubes in $\mathbb{R}^n$. For Q, $Q' \in \mathscr{B}$, we say that Q' is a descendent of Q if Q' is maximal with respect to the property of being properly contained in Q. We shall say Q branches if Q has at least two descendents.*

The proof of the following lemma can be found in [F, pp. 156–157].

Lemma 4.3. *Suppose $\mathscr{B}$ is a collection of dyadic cubes in $\mathbb{R}^n$. Let $\mathscr{B}_0$ be the subset of $\mathscr{B}$ consisting of* (i) *the maximal cubes in $\mathscr{B}$,* (ii) *the branching cubes in $\mathscr{B}$,* (iii) *the descendents of branching cubes in $\mathscr{B}$. Then the number of cubes in $\mathscr{B}_0$ is bounded by $C_n \cdot N$, where N is the number of minimal cubes in $\mathscr{B}$.*

We also need a lemma which may be found in [K-Sa, p. 224].

Lemma 4.4. *Suppose $Q_1, \dots, Q_k$ are pairwise disjoint dyadic subcubes of a dyadic cube Q in $\mathbb{R}^n$. Then there are* (*not necessarily dyadic or disjoint*) *cubes $I_1, \dots, I_m$ such that $Q \backslash \bigcup_{j=1}^{k} Q_j = \bigcup_{l=1}^{m} I_l$ and $m \leqq C_n \cdot k$.*

For $\lambda \leqq 0$ and $\alpha > 0$, let $\mathscr{A} = \mathscr{A}(|\mathbf{B}|, \alpha, \lambda)$ denote the collection of dyadic cubes Q in $\mathbb{R}^n$ which satisfy

$$l(Q) < \inf_{x \in Q} \frac{\alpha}{m(x, |\mathbf{B}|) + \sqrt{|\lambda|}}\,.$$

Clearly, if $Q \in \mathscr{A}$, so does any dyadic subcube of Q.
We now give the

Proof of Theorem 0.11 Let $\mathscr{B}$ be the collection of all dyadic cubes Q in $\mathscr{A}$ with the property

$$l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V|^p dx \right)^{1/p} \geqq c_1 > 0\,. \tag{4.5}$$

Suppose that $Q_1, Q_2, \dots, Q_{\tilde{N}_0}$ are the minimal cubes in $\mathscr{B}$. We will show that there exists a subspace $\mathscr{H}$ of $L^2(\mathbb{R}^n)$ of codimension less than or equal to $C_n \tilde{N}_0$, such that, for any $g \in \mathscr{H}$,

$$\sum_{j=1}^{n} \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2\, dx \geqq \int_{\mathbb{R}^n} |V||g|^2\, dx\,. \tag{4.6}$$

By the minimax principle, (4.6) implies that

$$N(\lambda, H) \leqq C_n \tilde{N}_0 \leqq C_n N_0 \quad \text{for } \lambda \leqq 0\,,$$

where N_0 is the number of minimal (disjoint) dyadic cubes satisfying (0.13) and (0.14) in Theorem 0.11. See Remark 0.18.

As in [F], we need to introduce some additional cubes $Q_{\tilde{N}_0+1}, Q_{\tilde{N}_0+2}, \dots, Q_{M_0}$ to consist of (i) the maximal cubes in $\mathscr{B}$, (ii) the branching cubes in $\mathscr{B}$, (iii) the descendents of branching cubes in $\mathscr{B}$. It follows from Lemma 4.3 that $M_0 \leqq C_n \tilde{N}_0$.

Now, let

$$\begin{aligned} E_0 &= \mathbb{R}^n \setminus \bigcup_{j=1}^{M_0} Q_j\,, \\ E_j &= Q_j \setminus \bigcup_{\substack{k \neq j \\ Q_k \subset Q_j}} Q_k, \quad j = 1, 2, \dots, M_0\,. \end{aligned} \tag{4.7}$$

Then E_j are disjoint and $\mathbb{R}^n = \bigcup_{j=0}^{M_0} E_j$.

Let $V_j = V\chi_{E_j}$. The same argument as in [F, pp. 157–159] shows that

$$l(Q)^2 \left(\frac{1}{|Q|} \int_Q |V_j|^p dx \right)^{1/p} \leqq C c_1 \tag{4.8}$$

for $0 \leqq j \leqq M_0$ and any cube Q in $\mathscr{A}$. Furthermore, if $1 \leqq j \leqq M_0$, (4.8) holds for any dyadic cube in $\mathbb{R}^n$. Indeed, if $Q \subset Q_j$, then $Q \in \mathscr{A}$ since $Q_j \in \mathscr{A}$. If $Q_j \subset Q$, (4.8) follows from the fact that supp $V_j \subset Q_j$. Here we have assumed that $p \leqq n/2$, since Theorem 0.11 becomes stronger as p decreases.

To prove (4.6), we first estimate the integral over E_0. By (4.8) and Theorem 3.1, we have

$$\int_{E_0} |V||g|^2\, dx \leqq C c_1 \left\{ \sum_{j=1}^n \int_{\mathbb{R}^n} \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right) g \right|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 dx \right\} \tag{4.9}$$

for $g \in C_0^1(\mathbb{R}^n)$.

To deal with the case $1 \leqq j \leqq M_0$, we use Lemma 4.4 to obtain

$$E_j = \bigcup_{k=1}^{m_j} I_k^j\,,$$

where I_k^j are cubes (not necessarily dyadic or disjoint) of $\mathbb{R}^n$. Also the number of cubes in $\{I_k^j : 1 \leqq j \leqq M_0,\ 1 \leqq k \leqq m_j\}$ is bounded by $C_n \tilde{N}_0$.

Thus, as in [K-Sa], if $x \in I_k^j$ and

$$\int_{I_k^j} f(y)\, dy = 0\,,$$

we have

$$|f(x)| \leqq C(-\Delta)^{-1/2}(|\nabla f|\chi_{I_k^j})(x) \leqq C(-\Delta)^{-1/2}(|\nabla f|\chi_{E_j})(x)\,.$$

Hence, since (4.8) holds for any dyadic cubes, we may use the trace inequality for $(-\Delta)^{-1/2}$ [K-Sa, Theorem 2.3] to obtain

$$\int_{E_j} |V|\,|f|^2\,dx \leqq C \int_{\mathbb{R}^n} |V_j|\,|(-\Delta)^{-1/2}(|\nabla f|\chi_{E_j})|^2\,dx \leqq C c_1 \int_{E_j} |\nabla f|^2\,dx\,.$$

Finally, for each Q_j, we construct $\mathbf{h}^j$ and Φ^j as in the proof of Theorem 0.26, such that

$$\mathbf{a}(x) = \mathbf{h}^j(x) + \nabla\Phi^j(x) \quad \text{for } x \in Q_j\,.$$

See (1.3–1.7).

We define

$$\mathscr{H} = \left\{ g \in L^2(\mathbb{R}^n) : \int_{I_k^j} e^{-i\Phi^j} g(x)\,dx = 0 \text{ for } j = 1,2,\ldots,M_0,\ 1 \leqq k \leqq m_j \right\}.$$

Then, $\mathscr{H}$ is a subspace of $L^2(\mathbb{R}^n)$ of codimension less than or equal to $C_n \tilde{N}_0$. If $g \in \mathscr{H} \cap \mathrm{Domain}(H)$, we have

$$\begin{aligned}
\int_{E_j} |V|\,|g|^2\,dx &= \int_{E_j} |V|\,|ge^{-i\Phi^j}|^2\,dx \leqq Cc_1 \int_{E_j} |\nabla(ge^{-i\Phi^j})|^2\,dx \\
&\leqq Cc_1 \sum_{l=1}^n \int_{E_j} \left| \left(\frac{1}{i}\frac{\partial}{\partial x_l} - \frac{\partial \Phi^j}{\partial x_l} \right) g \right|^2 dx \\
&\leqq Cc_1 \left\{ \sum_{l=1}^n \int_{E_j} \left| \left(\frac{1}{i}\frac{\partial}{\partial x_l} - a_l \right) g \right|^2 dx + \int_{E_j} |\mathbf{h}^j|^2 |g|^2 dx \right\} \\
&\leqq Cc_1 \left\{ \sum_{l=1}^n \int_{E_j} \left| \left(\frac{1}{i}\frac{\partial}{\partial x_l} - a_l \right) g \right|^2 dx + \int_{E_j} |m(x,|\mathbf{B}|)g|^2\,dx \right\},
\end{aligned}$$

where we have used $|\mathbf{h}^j(x)| \leqq C\,m(x,|\mathbf{B}|)$ on Q_j in the last inequality. This, together with (4.9), yields that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |V|\,|g|^2 dx \\
&\leqq Cc_1 \left\{ \sum_{l=1}^n \int_{\mathbb{R}^n} \left| \left(\frac{1}{i}\frac{\partial}{\partial x_l} - a_l \right) g \right|^2 dx + \int_{\mathbb{R}^n} |m(x,|\mathbf{B}|)g|^2 dx + |\lambda| \int_{\mathbb{R}^n} |g|^2 dx \right\}.
\end{aligned}$$

The desired estimate (4.6) then follows from Theorem 4.1 by choosing c_1 sufficiently small.

The proof of Theorem 0.11 is complete.

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