

Localization Expansions. I. Functions of the “Background” Configurations

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Abstract: Expansions of the type described in the inductive hypothesis (H.5) in the paper [1] are constructed for local functions of the “background” configurations, i.e., solutions of the variational problems studied in the previous paper [3]. A main part of this construction is a further analysis of a local structure of the solutions.

1. Introduction

In this paper we discuss a new kind of problem connected with expansions assumed in the inductive hypothesis (H.5) in [1], but methodologically it is a continuation of the paper [3] on the variational problems for background configurations. Our main concern here is to “localize” properly these configurations, and to do this we use extensively the results and methods of [3]. One of the most important technical problems in the renormalization group approach is to construct expansions of non-local functions of basic variables into sums of localized functions, like the expansions in (H.5) [1]. There are several types of non-localities and non-local functions, we have to consider. In this paper we construct such expansions for a simplest and most frequently occurring type of non-local functions given by compositions of localized functions with one of the background configurations, like for example terms $\mathcal{E}^{(j)}(X; \psi_k^{(j)})$ of the effective actions. Obviously values $\phi_k(x; \psi)$, $\psi^{(j)}(y; \psi)$ at points x, y of corresponding lattices are non-local functions of ψ , in fact they depend on ψ on the whole lattice, or on a generating set \mathbb{B}_k determining the functions.

Let us describe now in detail basic goals of this paper. Consider a generating set \mathbb{B}_k , which we can identify with the sequence of domains $\{\Omega_1, \dots, \Omega_k\}$, see the definitions (1.1)–(1.3) in [3], and assume that a next domain Ω_{k+1} is given, such that adding it to the sequence yields a new generating set \mathbb{B}_{k+1} . We denote $A = B(A_{k+1}) = \Omega_{k+1}^{(k)}$, i.e., $A \subset T^{(k)}$. Taking the first j domains in the sequence determines a generating set \mathbb{B}_j . Consider a function $\mathcal{E}(X; \psi_j)$, where X is a localization domain, $X \in \mathcal{D}_j$, and ψ_j is a spin variable on the lattice $T^{(j)}$. We assume that this function has the same properties as a term of the localization expansion in

(H.5) [1], and for simplicity we assume that $X \subset \Omega_{k+1}$. After k steps the variable ψ_j is replaced by the function $\psi(\mathbb{B}_j, \mathbb{B}_k; \psi_k)$, and in the $k+1$ st step ψ_k is split into the background field $\psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta)$ associated with the new spin configuration θ , and the linear function $\beta_k^{-\frac{1}{2}} C_A^{(k)\frac{1}{2}} \psi$ of the ultralocal spin fluctuation variable ψ that is about to be integrated out. In this situation we consider the function

$$\mathcal{E}(X; \psi(\mathbb{B}_j, \mathbb{B}_k; \psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta) + \beta_k^{-\frac{1}{2}} C_A^{(k)\frac{1}{2}} \psi)), \quad (1.1)$$

where θ is a spin configuration defined on \mathbb{B}_{k+1} , ψ is the so-called spin fluctuation variable defined on A , and $C_A^{(k)\frac{1}{2}}$ is a positive square root of a positive operator $C_A^{(k)}$, which is an operator of a quadratic form connected with a fluctuation integral on A . We will write an explicit formula for this operator and we will analyze it in Sect. 3. Now let us mention only that a kernel of this operator is defined on $A \times A$, and is an analytic function of the configurations $\phi(\mathbb{B}_{k+1}; \theta), \alpha(\mathbb{B}_{k+1}; \theta)$. Functions of the form (1.1) appear naturally in a fluctuation integral, which will be discussed in the next paper. Obviously (1.1) is a non-local function of the variables θ, ψ , and the main goal of this paper is to construct its localization expansion in both of the variables. To explain a form of this expansion we need some new definitions. We will use systematically domains in the continuous space T of a type described in the definition

$$\begin{aligned} \mathcal{D}(\mathbb{B}_j) &= \{Y : Y \subset T, Y \text{ is a connected domain, connected components of} \\ &Y \cap \Omega_n \cap \Omega_{n+1}^c \text{ belong to } \mathcal{D}_n, n \leq j, \\ &\text{where we put here } \Omega_0 = T, \Omega_{j+1} = \emptyset\}. \end{aligned} \quad (1.2)$$

This definition holds for any $j \leq k+1$. For such domains we generalize the concept of the linear size function d_j introduced in (2.27) [1], and we define

$$\begin{aligned} \tilde{d}_j(Y) &= \inf \left\{ \frac{1}{M} |\Gamma|_{\mathbb{B}_j} : \Gamma \text{ is a continuous tree graph contained in } Y \right. \\ &\text{and intersecting all cubes } \square^l \text{ contained in one of the domains} \\ &Y \cap \Omega_n \cap \Omega_{n+1}^c \text{ and belonging to the cover } \pi'_n, n \leq j\}, \\ &\text{where } |\Gamma|_{\mathbb{B}_j} = \sum_{n=1}^j (L^n \xi)^{-1} |\Gamma \cap \Omega_n \cap \Omega_{n+1}^c|_{\xi}. \end{aligned} \quad (1.3)$$

The length in the last equality is in the scale $\xi = L^{-j}$, which means that we take the original lattice as T_ξ , and $T^{(j)}$ as the unit lattice $T_1^{(j)}$.

To simplify the formula below let us denote the function $\psi(\mathbb{B}_j, \mathbb{B}_k)$ by $\psi_k^{(j)}$, and $\psi(\mathbb{B}_k, \mathbb{B}_{k+1})$ by $\psi^{(k)}$. The main goal of this paper is to prove the following statement:

The function (1.1) has a localization expansion of the form

$$\mathcal{E}(X; \psi_k^{(j)}(\psi^{(k)}(\theta) + \beta_k^{-\frac{1}{2}} C_A^{(k)\frac{1}{2}} \psi)) = \sum_{Y \in \mathcal{D}(\mathbb{B}_{k+1}) : Y \supset X} \mathcal{E}(X, Y; \theta, \psi), \quad (1.4)$$

where the terms of the expansion depend on the variables θ, ψ restricted to the corresponding domains Y and satisfy the bounds

$$|\mathcal{E}(X, Y; \theta, \psi)| < E_0 \exp(-\kappa_0 d_j(X) - 2\kappa \tilde{d}_{k+1}(Y)). \quad (1.5)$$

The formula (1.4) has prepared for integrating out ψ in such a way that localization in θ will be manifest afterwards, because ψ is an ultralocal variable. The formulation of the above statement is not complete yet because we have not described analyticity domains for the terms of the expansion, but we postpone formulations and a discussion of these more technical issues to Sect. 4. The statement gives a precise formulation of main aspects of these localization expansions, and it establishes a pattern for all related propositions in this and subsequent papers.

Let us explain briefly what is a main problem connected with constructing the expansion (1.4). It should be quite clear that it has to do with appropriate localization expansions of the functions $\psi_k^{(j)}, \psi^{(k)}, C_A^{(k)\frac{1}{2}}$, because they introduce a non-locality into the function (1.1). However, it is not enough to localize them in an arbitrary way, the point is that we have to use representations and bounds of Sect. 3 [1], in particular Corollary 3.11 [1], and they hold on rather small subspaces of natural analyticity domains, like subspaces $\mathcal{P}_j^c(X; C_0 L^j \eta, \varepsilon_k)$ defined in (3.13), (3.14) [1]. These subspaces incorporate in their definitions various scaling properties, some of them “anomalous,” related to the singular nature of the basic variational equations for the problem (1.6) [3]. We have to localize the function $\psi_k^{(j)}$ in such a way that the properties are preserved at least on the domain X , so its values are in the subspace above. This problem is solved in two steps, actually in three, the first step was the introduction of the auxiliary variable α and the variational equations (1.10) [3], in which the parameter λ occurs only as $\frac{1}{\lambda}$. The important step is to use the fundamental composition formula (4.39) [3] together with the representation (4.21) [3], and to write $\psi_k^{(j)}$ as a composition of such a function localized in a neighbourhood of X , and a suitable non-local function, which enters only into “boundary conditions” of the local one, and may satisfy more relaxed conditions described by corresponding spaces $\tilde{\mathcal{P}}^c$ in (1.11), (1.12) [3]. Such a representation of the function $\psi_k^{(j)}$ is presented in the formula (4.16), which is one of the most important formulas in the paper, a key to the localization procedure. It reduces the problem to a simpler one of localizing the functions of the type $\psi(\mathbb{B}_j, \mathbb{B}'_k)$ with fewer and simpler conditions to be satisfied. This is reduced again, through the representation (4.21) [3], to the problem of localizing the function $\phi(\mathbb{B}_k)$, which is still not quite standard, in particular the second non-linear variational equation in (1.14) [3] has to be treated with some care, and this is the third of the steps mentioned above, but it is resolved in a number of small technical details not worthwhile, or even impossible to describe here.

Let us describe now the content of the following three sections. We reverse in a sense the steps described above, and we start with localizations of the functions $\phi(\mathbb{B}_k), \psi(\mathbb{B}_j, \mathbb{B}_k)$, which is the subject of Sect. 2. It is achieved by introducing suitable interpolating parameters into their definitions and studying analytic extensions in the parameters, their regularity properties and bounds. We do the same for the covariance operator $C_A^{(k)}$, and for some functions of this operator, in Sect. 3. In Sect. 4 we start with a derivation of the representation (4.16), and then we construct the expansion (1.4) using the results of the previous sections and applying some simple interpolation formulas. All results of this paper are very technical and it is more convenient to start with definitions and constructions, and then to formulate the most important conclusions as propositions; for some of them it is even not possible to do it another way.

Finally let us remark that the size M of large cubes in the following two sections does not have to be equal to the size of large cubes used in the renormalization

group procedure. In these sections we consider $M = L^m$ as a parameter which must satisfy appropriate conditions, in particular M^{-1} is one of the fundamental factors controlling convergence of all expansions we will construct in this paper. These issues will be discussed in the last section.

2. Localization Expansions for the “Background” Configurations

In this section we introduce the localization parameters into the functions $\phi(\mathbb{B}_k)$, $\psi(\mathbb{B}_j, \mathbb{B}'_k)$, and other functions we use in these papers. This construction for functions $\phi(\mathbb{B}_k)$ is a basis of all other constructions, so we start with its detailed description. We use here extensively the constructions and the results of the papers [2, 3].

We fix now a generating set \mathbb{B}_k , and we denote the functions $\phi(\mathbb{B}_k)$, $\alpha(\mathbb{B}_k)$ simply by ϕ_k, α_k . Our basic goal is to construct extensions of the functions ϕ_k, α_k in the interpolating parameters, and to investigate their analyticity properties and bounds. This is based on the proof of Proposition 3.1 given in the third section of [3]. We start with the representation $\phi_k = \phi_0 + \delta\phi_k$, $\alpha_k = \alpha_0 + \delta\alpha_k$. The functions ϕ_0, α_0 are already almost local functions of $\psi + \psi'$, $h + h'$, as it follows from their definitions (3.2), (3.3) [3]. More precisely ϕ_0, α_0 restricted to $\Delta_j(y)$, $y \in \Delta_j$, depend on $\psi + \psi', h + h'$ restricted to $\tilde{\Delta}_j(y)$. Thus the basic problem mentioned above concerns the functions $\delta\phi_k, \delta\alpha_k$. They are constructed as solutions of Eqs. (3.19), and we consider again the more general system (3.20), to cover some other applications. The function $\delta\phi_k$ is represented in the form (3.23), to which we apply the identity (3.25). We write explicitly this representation for future reference

$$\begin{aligned} \delta\phi_k &= G(\alpha_0)(Q^*a\delta\psi + f_1) - G(\alpha_0)\phi_0\delta\alpha_k - G(\alpha_0)\delta\alpha_k G(\alpha_0)(Q^*a\delta\psi + f_1) \\ &\quad + G(\alpha_0)\delta\alpha_k G(\alpha_0)\phi_0\delta\alpha_k + G(\alpha_0)\delta\alpha_k G(\alpha_0)\delta\alpha_k G(\alpha_0 + \delta\alpha_k)(Q^*a\delta\psi + f_1) \\ &\quad - G(\alpha_0)\delta\alpha_k G(\alpha_0)\delta\alpha_k G(\alpha_0 + \delta\alpha_k)\phi_0\delta\alpha_k. \end{aligned} \quad (2.1)$$

The function $\delta\alpha_k$ satisfies Eqs. (3.24), or (3.26). We consider at first the function $\delta\phi_k$, or a more general function given by the above formula with $\delta\alpha_k$ replaced by some $\delta\alpha$. We introduce the interpolating parameters based on the generalized random walk expansion for $G(\alpha)$ constructed in Sect. A of [2b], see in particular the formula (2.40). Let us describe this expansion. We start with a construction of a cover of the domain Ω_1 . Each Ω_j is a localization domain from \mathcal{D}_j , so it is a union of large cubes in L^{-j} -scale, or a union of $ML^j\eta$ -cubes in the η -scale, with centers at points of the lattice $T_{ML^j\eta}^{(j+m)}$, where $M = L^m$. We take the domain Ω_j^\sim , i.e. the domain obtained by adding one layer of the large cubes touching Ω_j , and the set

$$A_j^{(m)+} = (\Omega_j^\sim \setminus \Omega_{j+1}) \cap T_{ML^j\eta}^{(j+m)},$$

which is the set of centers of the large cubes contained in Ω_j^\sim and disjoint with Ω_{j+1} . For each point z in this set we take a cube of the size $3ML^j\eta$ with a center at z and denote it by \square_z . The family of cubes

$$\left\{ \square_z : z \in \bigcup_{j=1}^k A_j^{(m)+} \right\} \text{ is a cover of the domain } \Omega_1.$$

We take a decomposition of unity corresponding to this cover and similar to the decomposition in (3.1) [3]. More precisely we take a decomposition of unity $\{h_z\}$ with functions h_z having the properties

$$\begin{aligned} h_z \in C_0^2(\square_z), \quad h_z \geq 0, \quad |\partial h_z| < 2(ML^j\eta)^{-1}, \quad |\partial\partial h_z| < 4(ML^j\eta)^{-2}, \\ \sum_z h_z^2 = 1 \quad \text{on a neighborhood of } \Omega_1. \end{aligned} \quad (2.2)$$

It is easy to construct such a decomposition starting with a function $h \in C_0^2(] - 1, 1[)$, $0 \leq h \leq 1$, $|h'| < 2$, $|h''| < 4$, and such that $h^2(t) + h^2(t - 1) = 1$ on a neighborhood of the interval $[0, 1]$. Then almost all functions h_z can be defined by the formula $h_z(x) = \prod_{\mu=1}^d h((ML^j\eta)^{-1}(x_\mu - z_\mu))$, that is for all z except those in the boundary layers $\Omega_j^c \setminus \Omega_j$, where two adjacent scales meet. It is easy to see how to adjust the definition for those points, so we do not write the rather awkward formulas here.

For each cube \square_z of the cover we take the basic operator $-\Delta_{\Omega_1}^{D,\eta} + Q^*aQ + v + \alpha$ and restrict it to \square_z taking the Neumann boundary conditions on the part of the boundary contained in Ω_1 . Denoting the corresponding Laplace operator by $(\Delta_{\Omega_1}^{D,\eta})_{\square_z}^N$ we define

$$G(\square_z, \alpha) = -(\Delta_{\Omega_1}^{D,\eta})_{\square_z}^N + (Q^*aQ + v + \alpha)|_{\square_z \cap \Omega_1}^{-1}. \quad (2.3)$$

For all \square_z which do not intersect the boundary $\partial\Omega_1$ the Laplace operator is simply $\Delta_{\square_z}^{N,\eta}$. Also the operator Q^*aQ restricted to \square_z is either equal to an operator $a_j(L^j\eta)^{-2}Q_j^*Q_j$ if \square_z is contained in $\Omega_j \setminus \Omega_{j+1}$, or to a sum of two such operators for j and $j + 1$, restricted to the subdomains $\square_z \cap \Omega_{j+1}^c$ and $\square_z \cap \Omega_{j+1}$, if \square_z intersects the boundary $\partial\Omega_{j+1}$. The inverse operators (2.3) have been investigated thoroughly in the paper [2b], see in particular the bound (2.33) there. We reformulate this bound here, but at first we have to recall the definition of the scaled distance $d(y, y')$ between two points $y, y' \in \mathbb{B}_k$. Take the set of all paths $\Gamma_{y,y'}$ connecting the points y, y' , contained in Ω_1 , and such that the intersections $\Gamma_{y,y'} \cap (\Omega_j \setminus \Omega_{j+1})$ are unions of paths in the lattice A_j , i.e. are unions of bonds of this lattice. Define

$$d(y, y') = \inf_{\Gamma_{y,y'}} |\Gamma_{y,y'}|_{\mathbb{B}_k}, \quad |\Gamma|_{\mathbb{B}_k} = \sum_{j=1}^k (L^j\eta)^{-1} |\Gamma \cap \Omega_j \cap \Omega_{j+1}^c|. \quad (2.4)$$

This function is determined by the set \mathbb{B}_k , and it is obviously a distance on this set. We will need all the properties of this function proved in [2b], in particular the bounds (2.50)–(2.53) in Lemma 2.1. Notice that if $y, y' \in \square_z \subset \Omega_j \setminus \Omega_{j+1}$, then $d(y, y') = (L^j\eta)^{-1}|y - y'|$, where $|y - y'| = |y - y'|_1 = \sum_{\mu=1}^d |y_\mu - y'_\mu|$ is the distance in the η -scale.

If \square_z intersects the boundary $\partial\Omega_{j+1}$, then we have the bounds $(L^{j+1}\eta)^{-1}|y - y'| \leq d(y, y') \leq (L^j\eta)^{-1}|y - y'|$. Now we can formulate the following property of the operators (2.3): there exist positive constants B_0, γ_0, γ_1 such that if α has complex values and satisfies $|\alpha|_2 < \gamma_1$, the norm restricted to \square_z , $y \in \square_z \cap A_j$, $y' \in \square_z \cap A_{j'}$, then

$$\begin{aligned} L^j\eta |\chi_{A_j(y)} \partial^\eta G(\square_z, \alpha) \chi_{A_{j'}(y')} f| + |\chi_{A_j(y)} G(\square_z, \alpha) \chi_{A_{j'}(y')} f| \\ \leq B_0 e^{-\gamma_0 d(y, y')} (L^{j'}\eta)^2 |\chi_{A_{j'}(y')} f| \end{aligned} \quad (2.5)$$

for any function f defined on $\square_z \cap \Omega_1$. This is the bound (2.33) in [2b], after rescaling to the η -lattice and using the distance (2.4). Next, we define an operator $K_z = K(h_z)$ by the formula

$$\begin{aligned} (K_z f)(x) &= \sum_{b \in st(x)} (\partial^\eta h_z)(b)(\partial^\eta f)(b) + (\Delta^\eta h_z)(x)f(x) \\ &\quad - a_j(L^j \eta)^{-2} \sum_{x' \in B_j(y)} L^{-jd}(h_z(x') - h_z(x))f(x') \\ &\text{for } x \in B_j(y), y \in \square_z \cap A_j, \end{aligned} \quad (2.6)$$

or in the operator form

$$K(h_z) = 2\nabla h_z \cdot \nabla + \Delta h_z - [Q^* a Q, h_z].$$

We will use this definition also in cases in which h_z is replaced by some other functions, possibly vector valued. Using the properties (2.2) of the functions h_z we obtain the following simple bound for the operators K_z :

$$\begin{aligned} |\chi_{\Delta_j(y)} K_z f| &\leq 4d(ML^j \eta)^{-1} |\chi_{\Delta_j(y)} \partial^\eta f| + 4d(ML^j \eta)^{-2} |\chi_{\Delta_j(y)} f| \\ &\quad + \frac{3}{2}(L^j \eta)^{-2} 2dM^{-1} |\chi_{\Delta_j(y)} f| \\ &\leq 4dM^{-1} ((L^j \eta)^{-1} |\chi_{\Delta_j(y)} \partial^\eta f| + (L^j \eta)^{-2} |\chi_{\Delta_j(y)} f|) \end{aligned} \quad (2.7)$$

for $y \in \square_z \cap A_j$.

At first we construct the generalized random walk expansion for the operator $G(\alpha)$ defined by (3.22) [3]. These random walks are sequences $\omega = (\omega_0, \omega_1, \dots, \omega_n)$, $n = 0, 1, \dots$ such that $\omega_i \in \bigcup_{j=1}^k A_j^{(m)^+}$, i.e. ω_i are centers of cubes belonging to the considered cover, and $\square_{\omega_i} \cap \square_{\omega_{i+1}} \neq \emptyset$. The expansion has the following form:

$$G(\alpha) = \sum_{\omega} h_{\omega_0} G(\square_{\omega_0}, \alpha) h_{\omega_0} K_{\omega_1} G(\square_{\omega_1}, \alpha) h_{\omega_1} \cdot \dots \cdot K_{\omega_n} G(\square_{\omega_n}, \alpha) h_{\omega_n}. \quad (2.8)$$

We have proved in [2b], see the proof of Proposition 2.2, that the series above is convergent in all the relevant norms, like the norms $|\cdot|_0$, $|\partial^\eta \cdot|_1$, $|\partial^\eta \cdot|_{(\alpha), 2-\alpha}$, $|\Delta^\eta \cdot|_2$, and the operator $G(\alpha)$ satisfies the bounds (2.57) in that theorem. We have used these bounds in Sect. 3 [3]. Now we need some stronger estimates for the series. Let us simplify the notation and denote the cube $\Delta_j(y)$ for $y \in A_j$ by $\Delta(y)$. We have the equality $1 = \sum_{y \in \mathbb{B}_k} \chi_{\Delta(y)}$ on Ω_1 , and we insert these decompositions between the factors h_{ω_i} and $K_{\omega_{i+1}}$ in (2.8). Denoting $|\omega| = n$, where ω is a sequence $(\omega_0, \omega_1, \dots, \omega_n)$, we have

$$\begin{aligned} \chi_{\Delta(y)} G(\alpha) f &= \sum_{n=0}^{\infty} \sum_{\omega: |\omega|=n} \sum_{(y_1, y_2, \dots, y_n, y')} \chi_{\Delta(y)} h_{\omega_0} G(\square_{\omega_0}, \alpha) h_{\omega_0} \chi_{\Delta(y_1)} \\ &\quad \cdot \chi_{\Delta(y_1)} K_{\omega_1} G(\square_{\omega_1}, \alpha) h_{\omega_1} \chi_{\Delta(y_2)} \\ &\quad \cdot \dots \cdot \chi_{\Delta(y_n)} K_{\omega_n} G(\square_{\omega_n}, \alpha) h_{\omega_n} \chi_{\Delta(y')} f. \end{aligned} \quad (2.9)$$

At first we estimate the number of walks ω with a fixed $|\omega| = n$. The number of ω_0 such that $\Delta(y) \subset \square_{\omega_0}$ is not greater than $2 \cdot 2^d$, because there are at most 2^d

cubes of a fixed scale containing $\Delta(y)$. For ω_i fixed there are less than $(2L + 1)^d$ points ω_{i+1} such that $\text{supp } h_{\omega_{i+1}} \cap \text{supp } h_{\omega_i} \neq \emptyset$, and generally, if ω_i is not in one of the boundary layers, there are 3^d such points. Thus the number of walks does not exceed $2^{d+1}(2L + 1)^{nd}$. Next, consider one factor in (2.9) enclosed between two characteristic functions. From the bounds (2.7), (2.5), we get

$$(L^{j_1} \eta)^2 |\chi_{\Delta(y_1)} K_z G(\square_z, \alpha) h_z \chi_{\Delta(y_2)} f| \leq 4dB_0 M^{-1} e^{-\gamma_0 d(y_1, y_2)} (L^{j_2} \eta)^2 |\chi_{\Delta(y_2)} f|. \quad (2.10)$$

Applying this bound and (2.5) to the sum in (2.9) we obtain

$$\begin{aligned} |\chi_{\Delta(y)} G(\alpha) f| &\leq \sum_{n=0}^{\infty} \sum_{\omega: |\omega|=n, y \in \square_{\omega_0}} \sum_{(y_1, y_2, \dots, y_n, y'): y_i \in \square_{\omega_{i-1}} \cap \square_{\omega_i}, y' \in \square_{\omega_n}} B_0 (4dB_0 M^{-1})^n \\ &\quad \cdot e^{-\gamma_0 d(y, y_1)} e^{-\gamma_0 d(y_1, y_2)} \\ &\quad \cdot \dots \cdot e^{-\gamma_0 d(y_{n-1}, y_n)} e^{-\gamma_0 d(y_n, y')} (L^{j'} \eta)^2 |\chi_{\Delta(y')} f| \\ &\leq \sum_{n=0}^{\infty} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) \left(4dB_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) M^{-1} \right)^n \\ &\quad \cdot \sum_{\omega: |\omega|=n, y \in \square_{\omega_0}} e^{-\frac{1}{2} \gamma_0 d(y; \omega)} \sup_{y' \in \square_{\omega_n}} (L^{j'} \eta)^2 |\chi_{\Delta(y')} f| \\ &\leq \sum_{n=0}^{\infty} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) \left(8dB_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) (2L + 1)^d M^{-1} \right)^n \\ &\quad \cdot \sum_{\omega: |\omega|=n, y \in \square_{\omega_0}} \frac{1}{2^n} \cdot \frac{1}{(2L + 1)^{dn}} \cdot e^{-\frac{1}{2} \gamma_0 d(y; \omega)} |f|_2 \\ &\leq 2^{d+2} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) |f|_2, \text{ where} \end{aligned}$$

$$d(y; \omega) = \inf(d(y, y_1) + d(y_1, y_2) + \dots + d(y_m, y')),$$

the infimum is taken over $(y_1, y_2, \dots, y_n, y')$:

$$y_i \in \square_{\omega_{i-1}} \cap \square_{\omega_i}, y' \in \square_{\omega_n}, \text{ assuming that } y \in \square_{\omega_0};$$

$$\begin{aligned} c_1(d, \gamma) &= 12c_0^d(\gamma), c_0(\gamma) = \sum_{x \in \mathbb{Z}} e^{-\gamma|x|} \\ &= \frac{1 + e^{-\gamma}}{1 - e^{-\gamma}} < \frac{4}{\gamma} \text{ if } 0 < \gamma \leq 1; \end{aligned} \quad (2.11)$$

and assuming $8dB_0 c_1(d, \frac{1}{4} \gamma_0) (2L + 1)^d M^{-1} \leq 1$.

In the second inequality above we have used Lemma 2.1 in [2b]. We have written these inequalities in such detailed form because it is not the final bound which is most interesting for us, but bounds and expressions in the intermediate stages, which indicate possible improvements. In particular we have not made proper use of the factors M^{-n} and $\exp(-\frac{1}{2} \gamma_0 d(y; \omega))$, for example we may replace M^{-1} by $M^{-\frac{1}{2}}$ in the last assumption, and use the factor $M^{-\frac{1}{2}n}$ for some other bounds. This remark will be important when we will study analytic extensions in interpolating parameters.

Another possibility is to represent the sum in (2.9) as a sum over domains X containing the cube $\Delta(y)$, with terms equal to the sum of terms in (2.9) determined by random walks ω such that $X_\omega = \bigcup \square_{\omega_i} = X$. These terms are localized in the domain X in the sense that they depend on α and f restricted to X , and we can use the exponential factor to get a bound of the form (H.5). This way we obtain a localization expansion for the operator $G(\alpha)$, but it is not enough for our purposes, so we do not discuss any details.

Consider the representation (2.1) of the function $\delta\phi_k(\delta\alpha)$ obtained by replacing the function $\delta\alpha_k$ by the variable $\delta\alpha$. We expand each operator $G(\alpha_0)$ and $G(\alpha_0 + \delta\alpha)$ in this formula into the generalized random walk expansions (2.8). It is easy to see that multiplying and summing these expansions yield an expansion of basically the same form as (2.8), but with some small modifications. Some operators K_{ω_i} may be replaced by a multiplication operator, the multiplication by the function $-\delta\alpha h_{\omega_i}$. We can have at most two such replacements. The localized Green's operators depend on α_0 , except the case when there are exactly two replacements, then the operators occurring after the last one depend on $\alpha_0 + \delta\alpha$. Finally, at the end of each term of the expansion there is a function f , which is equal either to $Q^*a\delta\psi + f_1$, or to $-\phi_0\delta\alpha$. We can represent this expansion in the form of a generalized random walk expansion, like in (2.8), if we introduce random walks $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ whose elements ω_i are not points, but pairs $\omega_i = (z_i, \sigma_i)$, where $z_i \in \bigcup_{j=1}^k \Lambda_j^{(m)+}$, $\sigma_i \in \{0, 1\}$. We define operators K_{ω_i} , variables ϑ_i , and functions $g_\sigma, \sigma \in \{0, 1\}$, as follows:

$$\begin{aligned} K_{\omega_i} &= K_{z_i} \quad \text{if } \sigma_i = 0, & K_{\omega_i} &= -\delta\alpha h_{z_i} \quad \text{if } \sigma_i = 1; \\ \vartheta_i &= 0 \quad \text{if } \sigma_1 + \dots + \sigma_i < 2, & \vartheta_i &= 1 \quad \text{if } \sigma_1 + \dots + \sigma_i \geq 2, \quad i \geq 1; \\ g_0 &= Q^*a\delta\psi + f_1, & g_1 &= -\phi_0\delta\alpha. \end{aligned} \quad (2.12)$$

The random walks ω are restricted by the conditions: $\square_{z_{i-1}} \cap \square_{z_i} \neq \emptyset$, $i = 1, \dots, k$, $\square_{z_{i-1}} = \square_{z_i}$, or $z_{i-1} = z_i$ if $\sigma_i = 1, \sigma_1 + \dots + \sigma_k \leq 2$. The first condition actually has the stronger form: if both cubes $\square_{z_{i-1}}, \square_{z_i}$ are in the same scale, then $\square_{z_i} \subset \square_{z_{i-1}}$, hence also $\square_{z_{i-1}} \subset \square_{z_i}$, and if they are in adjacent scales, then the cube in the smaller scale is contained in the larger cube. We will use sometimes this stronger condition in the future. The expansion of the function $\delta\phi_k(\delta\alpha)$ can be written now in the following form:

$$\delta\phi_k(\delta\alpha) = \sum_{\omega} h_{z_0} G(\square_{z_0}, \alpha_0) h_{z_0} \left(\prod_{i=1}^{|\omega|} K_{\omega_i} G(\square_{z_i}, \alpha_0 + \vartheta_i \delta\alpha) h_{z_i} \right) g_{\sigma_0}. \quad (2.13)$$

Let us remark that there is an even simpler expansion of this function, exactly of the form (2.8), if we use the representation (3.23) [3] directly, i.e. we write $\delta\phi_k(\delta\alpha) = G(\alpha_0 + \delta\alpha)(Q^*a\delta\psi + f_1 - \phi_0\delta\alpha)$, and we use the expansion (2.8) for $G(\alpha_0 + \delta\alpha)$. This is enough for a localization of the function $\delta\phi_k(\delta\alpha)$, but we have to study and localize the equation for $\delta\alpha$, and for this we need the more detailed form (2.12). This expansion can be bounded as in (2.11), with some small modifications only. We will discuss this bound later, after introducing interpolating parameters. To do this we have to describe at first a geometric setting.

Consider two domains X_0, X_1 such that

$$X_0, X_1 \subset \Omega_k, X_0, X_1 \in \mathcal{D}_k, X_0 \sim^{r_0} \subset X_1, r_0 \text{ is a positive integer to be fixed later.} \quad (2.14)$$

We could consider more general domains from $\mathcal{D}(\mathbb{B}_k)$, but in future applications we will use only the above simpler case. We would like to localize the function $\delta\phi_k(\delta\alpha)$ restricted to the domain X_0 , i.e. to represent it as a sum over domains $X \supset X_1$ of functions which depend on $\phi_0, \alpha_0, \delta\alpha, \delta\psi, f_1$ restricted to X . We do it by introducing interpolating, or localization parameters into the formula (2.13) for this function. Let us introduce at first the notations:

for a walk $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ we denote

$$\begin{aligned} \tilde{\omega} &= (\omega_1, \dots, \omega_n) \quad \text{and} \quad |\tilde{\omega}| = |\omega| = n; \\ (KG)(\tilde{\omega}) &= \prod_{i=1}^{|\omega|} K_{\omega_i} G(\square_{z_i}, \alpha_0 + \vartheta_i \delta\alpha) h_{z_i} \quad \text{if } |\tilde{\omega}| > 0, \\ (KG)(\tilde{\omega}) &= 1 \quad \text{if } |\tilde{\omega}| = 0. \end{aligned} \tag{2.15}$$

The domains $\Omega_j \setminus \Omega_{j+1}$ are unions of disjoint large cubes in the L^{-j} -scale, i.e., the cubes from the partition π_j , where π_j was defined in [1] between (2.25) and (2.26). We denote these cubes by \square , and to each cube we assign a variable s_\square . The set $s = \{s_\square\}$ is the set of interpolating, or localization parameters. We define

$$\delta\phi_k(\delta\alpha, s) = \sum_{\omega} h_{z_0} G(\square_{z_0}, \alpha_0) h_{z_0} (KG)(\tilde{\omega}) g_{\sigma_0} \prod_{\square \subset X_\omega \cap X_1^c} s_\square, \tag{2.16}$$

where $X_\omega = \bigcup_{i=0}^{|\omega|} \square_{z_i}$. This function has the following simple properties:

$$\begin{aligned} \delta\phi_k(\delta\alpha, 1) &= \delta\phi_k(\delta\alpha), \quad \text{where } s = 1 \text{ means that all } s_\square = 1; \\ \chi_{X_0} \delta\phi_k(\delta\alpha, 0) &\text{ is the sum in (2.13) over random walks } \omega \\ &\text{ satisfying } \square_{z_0} \cap X_0 \neq \emptyset, X_\omega \subset X_1, \text{ hence it is localized in } X_1. \end{aligned}$$

It is obviously an analytic function of s , actually it is a polynomial in s , and we consider it on the polydisc $\{s: |s| \leq e^{\kappa_1}\}$, where κ_1 is a sufficiently large positive number. We would like to prove bounds (3.47) in Proposition 3.1 [3] for the function (2.16), assuming that M is chosen sufficiently large, depending on κ_1 . We can obtain them in the same way as the bound (2.11), and we have for $y \in \mathbb{B}_k$,

$$\begin{aligned} |\chi_{\mathcal{A}(y)} \delta\phi_k(\delta\alpha, s)| &\leq B_0 c_1 \left(d, \frac{1}{4} \gamma_0\right) \sum_{\omega: \mathcal{A}(y) \subset \square_{z_0}} \frac{1}{2^{|\omega|}} \frac{1}{(2L+1)^{d|\omega|}} e^{-\frac{1}{2} \gamma_0 d(y; \omega)} \\ &\cdot \left[M^{-\frac{1}{2}|\omega|} + |\omega| M^{-\frac{1}{2}(|\omega|-1)} 2B_0 c_1 \left(d, \frac{1}{4} \gamma_0\right) (2L+1)^d |\delta\alpha|_2 \theta_+(|\omega|-1) \right. \\ &\quad \left. + \frac{1}{2} |\omega| (|\omega|-1) M^{-\frac{1}{2}(|\omega|-2)} \left(2B_0 c_1 \left(d, \frac{1}{4} \gamma_0\right) (2L+1)^d |\delta\alpha|_2 \right)^2 \theta_+(|\omega|-2) \right] \\ &\cdot \left(\frac{3}{2} |\delta\psi| + |f_1|_2 + \frac{3}{2} |\delta\alpha|_2 \right) \exp \kappa_1 \sum_{j=0}^k \frac{1}{(ML^j \eta)^d} |X_\omega \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c|, \end{aligned} \tag{2.17}$$

where we use the function $\theta_+(t) = 1$ for $t \geq 0$, $\theta_+(t) = 0$ for $t < 0$.

Here we have used the first two inequalities in (2.11) together with the assumption $8dB_0c_1(d, \frac{1}{4}\gamma_0)(2L+1)^d M^{-\frac{1}{2}} \leq 1$. We assume further that $4B_0c_1(d, \frac{1}{4}\gamma_0)(2L+1)^d |\delta\alpha|_2 < 1$. This assumption is almost the same as $|\delta\alpha|_2 < \frac{1}{2}\gamma_1$, because γ_1 is an inverse of the norm of the operator G , and this may be taken as the constant on the right-hand side of the last inequality in (2.11). Now the expression in the first parenthesis can be bounded by $1 + \frac{1}{2} + \frac{1}{4} < 2$, because $M^{-\frac{1}{2}|\omega|} \leq 1$ for $|\omega| \geq 0$, $M^{-\frac{1}{2}(|\omega|-1)}|\omega| \leq 1$ for $|\omega| \geq 1$, and $M^{-\frac{1}{2}(|\omega|-2)}\frac{1}{2}|\omega|(|\omega|-1) \leq 1$ for $|\omega| \geq 2$. The expression in the second parenthesis is bounded by $\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2$. Consider the exponential in the third parenthesis and assume that $X_\omega \cap X_1^c \neq \emptyset$, otherwise the exponential is equal to 1. We combine the exponential with the first exponential on the right-hand side of (2.17), and we would like to estimate the product. At first we estimate it in a general case, for an arbitrary walk ω , and then in the case considered here.

Take a walk ω and the corresponding sequence of cubes $(\square_{z_0}, \square_{z_1}, \dots, \square_{z_n})$, where $y \in \square_{z_0} \cap \mathbb{B}_k$. By the definition of $d(y, \omega)$ in (2.11) we take the infimum over a finite set, so it is equal to a value of the function at some elements of the set. Let the sequence $y_0 = y, y_1, \dots, y_n$ be one of them, thus

$$d(y, \omega) = d(y, y_1) + d(y_1, y_2) + \dots + d(y_{n-1}, y_n), y_i \in \square_{z_{i-1}} \cap \square_{z_i} \cap \mathbb{B}_k, \\ i = 1, \dots, n, y_n \in \square_{z_n} \cap \mathbb{B}_k.$$

We have $y_i, y_{i+1} \in \square_{z_i}$ and we take a path $\Gamma(y_i, y_{i+1})$ connecting the two points, contained in \square_{z_i} , and satisfying the equality $d(y_i, y_{i+1}) = |\Gamma(y_i, y_{i+1})|_{\mathbb{B}_k}$, where the scaled length is defined in (2.4). Combining these paths we get

$$d(y, \omega) = |\Gamma|_{\mathbb{B}_k}, \quad \text{where } \Gamma = \Gamma(y, y_1) \cup \Gamma(y_1, y_2) \cup \dots \cup \Gamma(y_{n-1}, y_n).$$

We assume that $\square_{z_i} \cap X_1^c \neq \emptyset$ for some i , and we denote by r a minimal i with this property, i.e. $\square_{z_0}, \dots, \square_{z_{r-1}}$ does not intersect X_1^c , and $\square_{z_r} \cap X_1^c \neq \emptyset$. Generally we can have $r = 0$. The point y_r divides the path Γ into two parts, the one from y to y_r is denoted by Γ^0 , and the other from y_r to y_n is denoted by Γ' . The index r divides also the walk ω into two walks: $\omega^0 = (\omega_0, \dots, \omega_{r-1})$, which may be empty, and $\omega' = (\omega_r, \dots, \omega_n)$. Obviously we have $X_\omega \cap X_1^c = X_{\omega'} \cap X_1^c$. Let us divide now the path Γ' into a union of subpaths $\Gamma_i, i = 1, \dots, p$, such that

$$\Gamma' = \bigcup_{i=1}^p \Gamma_i, \quad |\Gamma'|_{\mathbb{B}_k} = \sum_{i=1}^p |\Gamma_i|_{\mathbb{B}_k}, \quad |\Gamma_i|_{\mathbb{B}_k} = M \quad \text{for } i < p, \quad |\Gamma_p|_{\mathbb{B}_k} \leq M.$$

Each path Γ_i contains some number of points from the sequence (y_0, \dots, y_n) ; the number may be 0. Let us fix Γ_i and denote the points by $y_s, y_{s+1}, \dots, y_{t-1}, s \leq t$, the set is empty if $s = t$. The points determine the cubes $\square_{z_s}, \square_{z_{s+1}}, \dots, \square_{z_{t-1}}$. The scaled length of the path Γ_i is $\leq M$, so it may intersect at most two of the domains $\{\Omega_j \setminus \Omega_{j+1}\}$, and the same is true for the cubes. Therefore there exists an index j such that $\square_{z_i} \subset \Omega_{j-1} \setminus \Omega_{j+1}$ for $i = s, s+1, \dots, t-1$. The ordinary length of Γ_i in the η -scale is $\leq ML^j \eta$, hence the projections of Γ_i onto the coordinate axis are intervals of lengths $\leq ML^j \eta$, and Γ_i is contained in a continuous space cube \square_i of the size $ML^j \eta$. Take a cube \square'_i of the size $7ML^j \eta$, which has the same center as the cube \square_i . The cubes $\square_{z_s}, \dots, \square_{z_{t-1}}$ have either the size $3ML^j \eta$, or the size $3ML^{j-1} \eta$, and they have non-empty intersections with \square_i , hence they are contained in \square'_i .

Thus the sum in the second exponential in (2.17) for the domain $\square_{z_s} \cup \dots \cup \square_{z_{t-1}}$ is reduced to two terms with indices $j-1, j$, and we have

$$\begin{aligned} & \frac{1}{(ML^{j-1}\eta)^d} \left| \left(\bigcup_{i=s}^{t-1} \square_{z_i} \right) \cap \Omega_{j-1} \cap \Omega_j^c \right| + \frac{1}{(ML^j\eta)^d} \left| \left(\bigcup_{i=s}^{t-1} \square_{z_i} \right) \cap \Omega_j \cap \Omega_{j+1}^c \right| \\ & \leq \frac{1}{(ML^{j-1}\eta)^d} |\square'_i| = \frac{(7ML^j\eta)^d}{(ML^{j-1}\eta)^d} = (7L)^d = \frac{(7L)^d}{M} |\Gamma_i|_{\mathbb{B}_k}, \end{aligned}$$

the last equality holds for $i < p$, for $i = p$ we have simply the bound $(7L)^d$. Summing the above inequalities over i from 1 to p we get

$$\begin{aligned} & \sum_{j=0}^k \frac{1}{(ML^j\eta)^d} |X_{\omega'} \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c| \\ & \leq \frac{(7L)^d}{M} \sum_{i=1}^{p-1} |\Gamma_i|_{\mathbb{B}_k} + (7L)^d \leq \frac{(7L)^d}{M} |\Gamma'|_{\mathbb{B}_k} + (7L)^d. \end{aligned} \quad (2.18)$$

This is a basic estimate in the general case. We have also $d(y, \omega) = |\Gamma|_{\mathbb{B}_k} = |\Gamma^0|_{\mathbb{B}_k} + |\Gamma'|_{\mathbb{B}_k}$, hence the product of the two exponentials in (2.17) can be bounded as follows:

$$\begin{aligned} & e^{-\frac{1}{2}\gamma_0 d(y, \omega)} \exp \kappa_1 \sum_{j=1}^k \frac{1}{(ML^j\eta)^d} |X_{\omega} \cap X_1^0 \cap \Omega_j \cap \Omega_{j+1}^c| \\ & \leq \exp \left(\frac{(7L)^d \kappa_1}{M} - \frac{1}{2}\gamma_0 \right) |\Gamma'|_{\mathbb{B}_k} \cdot \exp \left(-\frac{1}{2}\gamma_0 |\Gamma^0|_{\mathbb{B}_k} + (7L)^d \kappa_1 \right) \\ & \leq \exp \left(-\frac{1}{2}\gamma_0 |\Gamma^0|_{\mathbb{B}_k} + (7L)^d \kappa_1 \right), \quad \text{if } \frac{(7L)^d \kappa_1}{M} - \frac{1}{2}\gamma_0 \leq 0, \\ & \text{or } M \geq 2(7L)^d \kappa_1 \frac{1}{\gamma_0}. \end{aligned} \quad (2.19)$$

We assume that M satisfies this second restriction, together with the first formulated just after (2.17). We would like to estimate the length $|\Gamma^0|_{\mathbb{B}_k}$ by a quantity independent of the walk ω . Let us recall that Γ^0 is a path starting at the point y and ending at the point $y_r \in \square_{z_{r-1}} \cap \square_{z_r}$, where the cubes $\square_{z_0}, \dots, \square_{z_{r-1}}$ are contained in X_1 , and \square_{z_r} intersects both domains X_1, X_1^c . From this it follows that all these cubes are from the cover π'_k , and that $\square_{z_r} \subset (X_1^c)^{\sim 2}$, hence y_r belongs to this domain and the path Γ^0 connects the point y with the point $y_r \in (X_1^c)^{\sim 2}$. Thus we have

$$|\Gamma^0|_{\mathbb{B}_k} \geq d(y, y_r) \geq \inf_{y' \in (X_1^c)^{\sim 2} \cap \mathbb{B}_k} d(y, y') = d(y, (X_1^c)^{\sim 2}), \quad (2.20)$$

and we obtain finally the inequality

$$\begin{aligned} & e^{-\frac{1}{2}\gamma_0 d(y, \omega)} \exp \kappa_1 \sum_{j=1}^k \frac{1}{(ML^j\eta)^d} |X_{\omega} \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c| \\ & \leq \exp \left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2}) + (7L)^d \kappa_1 \right), \quad \text{if } X_{\omega} \cap X_1^c \neq \emptyset. \end{aligned} \quad (2.21)$$

Generally the distance in the last exponential above is equal to 0, so the exponential can be estimated further only by $\exp(7L)^d \kappa_1$.

Consider now the special case of (2.17), when $\Delta(y) \subset \square_{z_0}$ and $\square_{z_0} \cap X_0 \neq \emptyset$. Take the cube $\square \in \pi_k$ containing $\Delta(y)$, then $\square \subset X_0^{\sim 2}$, $\square^{\sim r} \subset X_0^{\sim 2+r}$ and $\square^{\sim r} \cap (X_1^c)^{\sim 2} = \emptyset$ if $2+r \leq r_0-2$, or $r \leq r_0-4$, by the condition (2.14). From this we obtain the bound

$$d(y, (X_1^c)^{\sim 2}) > (r_0 - 4)M. \quad (2.22)$$

In this case the product of the exponentials on the left-hand side of (2.21) can be bounded by

$$\exp\left(-\frac{1}{2}\gamma_0(r_0-4)M + (7L)^d \kappa_1\right) \leq \exp\left(-\frac{1}{2}\gamma_0(r_0-5)M\right) \leq 1$$

if $r_0 \geq 5$, which we assume from now on.

Combining all the above estimates, we obtain bounds for the expression in (2.17) for all possible cases, and we obtain the inequalities

$$\begin{aligned} |\delta\phi_k(\delta\alpha, s)| &< 2^{d+3} B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) e^{(7L)^d \kappa_1} \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right), \\ |\chi_{\Delta(y)} \delta\phi_k(\delta\alpha, s)| &< 2^{d+3} B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right) \text{ for } \Delta(y) \subset X_0, \\ |\chi_{\Delta(y)}(\delta\phi_k(\delta\alpha, s) - \delta\phi_k(\delta\alpha, 1))| &< 2^{d+4} B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) e^{(7L)^d \kappa_1} \\ &\cdot \exp\left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right), \end{aligned} \quad (2.23)$$

and the same inequalities for the norms $|\partial^\eta \cdot|_1, |\partial^\eta \cdot|_{(\alpha), 2-\alpha}, |\Delta^\eta \cdot|_2$ of the functions on the left-hand sides above. The function $\delta\phi_k(\delta\alpha, s)$ has also some localization properties dependent on zeros of s , but these will be discussed below.

Consider now the function $\delta\alpha_k$ determined by the second equation in (3.20) [3]. It is a non-linear equation, and we construct an analytic extension $\delta\alpha_k(s)$ of $\delta\alpha_k$ by introducing properly the localization parameters s into this equation. Let us recall that it is

$$\phi_0 \cdot \delta\phi_k(\delta\alpha) + \frac{1}{2}\delta\phi_k(\delta\alpha) \cdot \delta\phi_k(\delta\alpha) - \frac{1}{\lambda}\delta\alpha = \frac{1}{\lambda}\alpha_1. \quad (2.24)$$

Unfortunately we cannot do it simply by putting $\delta\phi_k(\delta\alpha, s)$ in place of $\delta\phi_k(\delta\alpha)$ above, and solving the resulting equation. A problem is a bit subtler, it is a singular equation in $\delta\alpha$, the inverse of the operator determining a linear part of it is unbounded, and we have to use quite a lot of a structure of the function $\delta\phi_k(\delta\alpha)$ to deal with this problem. Fortunately enough of this structure is preserved in terms of the random walk expansion (2.13) of this function, so we substitute it in the above equation, and using the definition (2.15) we write it in the form

$$\begin{aligned} \sum_{\omega} \phi_0 \cdot h_{z_0} G(\square_{z_0}, \alpha_0) h_{z_0}(KG)(\tilde{\omega}) g_{\sigma_0} + \frac{1}{2} \sum_{\omega', \omega''} h_{z'_0} G(\square_{z'_0}, \alpha_0) h_{z'_0}(KG)(\tilde{\omega}') g_{\sigma'_0} \\ \cdot h_{z''_0} G(\square_{z''_0}, \alpha_0) h_{z''_0}(KG)(\tilde{\omega}'') g_{\sigma''_0} - \frac{1}{\lambda}\delta\alpha = \frac{1}{\lambda}\alpha_1. \end{aligned} \quad (2.25)$$

The equation has been analyzed in Sect. 3 [3], and we can apply this analysis to the above particular form of the equation without any essential changes, only with some slight changes of constants in the bounds. At first, we apply the operator $G^{-1}(\alpha_0)$ to the above equation and we obtain, after simple transformations and using the identities after (3.33)[3] and the definition (2.6), the following one:

$$\begin{aligned}
 & \delta\alpha + \frac{1}{\lambda}(-A_{\Omega_1}^{D,\eta} + Q^*aQ + \nu)\delta\alpha + (\phi_0^2 - 1)\delta\alpha + \frac{1}{\lambda}\alpha_0\delta\alpha - \phi_0 \cdot g_0 \\
 & + \sum_{\omega} K(h_{z_0}\phi_0) \cdot G(\square_{z_0}, \alpha_0)h_{z_0}(KG)(\tilde{\omega})g_{\sigma_0} - \sum_{\tilde{\omega}:|\tilde{\omega}|>0} \phi_0 \cdot (KG)(\tilde{\omega})(g_0 + g_1) \\
 & + \frac{1}{2} \sum_{\omega', \tilde{\omega}''} K(h_{z_0'}h_{z_0''}G(\square_{z_0'}, \alpha_0)(KG)(\tilde{\omega}')g_{\sigma_0'}) \cdot G(\square_{z_0'}, \alpha_0)h_{z_0''}(KG)(\tilde{\omega}'')g_{\sigma_0''} \\
 & - \frac{1}{2} \sum_{\omega', \tilde{\omega}''} h_{z_0'}G(\square_{z_0'}, \alpha_0)h_{z_0''}(KG)(\tilde{\omega}')g_{\sigma_0'} \cdot (KG)(\tilde{\omega}'')(g_0 + g_1) \\
 & = -\frac{1}{\lambda}(-A_{\Omega_1}^{D,\eta} + Q^*aQ + \nu + \alpha_0)\alpha_1. \tag{2.26}
 \end{aligned}$$

The notation used above for terms of the third sum is rather awkward and does not give justice to simplicity of these expressions. We can rearrange terms of the equation, introduce some new definitions, and write it in a simpler form. Using the condition $|\tilde{\omega}| > 0$ we can write $\tilde{\omega} = (\omega_1, \dots, \omega_n)$, $n \geq 1$, as $\omega' = (\omega'_0, \dots, \omega'_{n-1})$, $n-1 \geq 0$, with obvious identification of elements of the two walks, and we have

$$\begin{aligned}
 & \sum_{\tilde{\omega}:|\tilde{\omega}|>0} \phi_0 \cdot (KG)(\tilde{\omega})(g_0 + g_1) \\
 & = \sum_{(z_0', \tilde{\omega}')} \phi_0 K(h_{z_0'}) \cdot G(\square_{z_0'}, \alpha_0)h_{z_0'}(KG)(\tilde{\omega}')(g_0 + g_1) \\
 & - \sum_{(z_0', \tilde{\omega}')} \phi_0 \cdot \delta\alpha h_{z_0'}G(\square_{z_0'}, \alpha_0)h_{z_0'}(KG)(\tilde{\omega}')(g_0 + g_1).
 \end{aligned}$$

We combine the first sum on the right-hand side above with the first sum in (2.26) replacing the operator $K(h_{z_0}\phi_0)$ there by $K(\phi_0 h_{z_0}) - \phi_0 K(h_{z_0})$. Let us introduce a new operator $V(h_z)$ acting on a pair of two functions by the following formula:

$$\begin{aligned}
 V(h_z)(f, g) & = K(h_z f) \cdot g - f \cdot K(h_z)g = 2\nabla^n(h_z f) \cdot \nabla^n g \\
 & - 2f \cdot (\nabla^n h_z \cdot \nabla^n g) + 2(\nabla^n h_z \cdot \nabla^n f) \cdot g + (\Delta^n f) \cdot h_z g \\
 & - [Q^*aQ, f] \cdot h_z g = 2\nabla^n f \cdot \nabla^n(h_z g) + (\Delta^n f) \cdot h_z g \\
 & - [Q^*aQ, f] \cdot h_z g = K(f) \cdot h_z g. \tag{2.27}
 \end{aligned}$$

Notice that the definition is not symmetric in f, g , so the operator acts on ordered pairs. Next, we define operators V_{ω_0} for $\omega_0 = (z_0, \sigma_0)$ by the formulas

$$\begin{aligned}
 V_{\omega_0}(f, g) & = V(h_{z_0})(f, g) \quad \text{if } \sigma_0 = 0, \\
 V_{\omega_0}(f, g) & = f \cdot \delta\alpha h_{z_0}g \quad \text{if } \sigma_0 = 1,
 \end{aligned}$$

and we change slightly the definition of the variables ϑ_i in (2.12) by including σ_0 also, i.e.

$$\vartheta_i = 0 \quad \text{if } \sigma_0 + \sigma_1 + \dots + \sigma_i < 2, \quad \vartheta_i = 1 \quad \text{if } \sigma_0 + \sigma_1 + \dots + \sigma_i \geq 2.$$

Thus the first two sums in (2.26) combined can be written as follows:

$$\sum_{\omega} V_{\omega_0}(\phi_0, G(\square_{z_0}, \alpha_0) h_{z_0}(KG)(\tilde{\omega})(g_0 + g_1)). \quad (2.28)$$

In the same way we combine the last two sums, except that we have to take the last sum with the restriction $|\tilde{\omega}''| > 0$, so we get

$$\begin{aligned} & \frac{1}{2} \sum_{\omega', \omega''; \sigma'_0=0} V_{\omega'_0}(h_{z'_0} G(\square_{z'_0}) h_{z'_0}(KG)(\tilde{\omega}')(g_0 + g_1), G(\square_{z''_0}, \alpha_0) h_{z''_0}(KG)(\tilde{\omega}'')) \\ & \times (g_0 + g_1) - \frac{1}{2} \sum_{\omega; \sigma_0=0} (g_0 + g_1) \cdot h_{z_0} G(\square_{z_0}, \alpha_0) h_{z_0}(KG)(\tilde{\omega})(g_0 + g_1). \end{aligned} \quad (2.29)$$

Notice that the walks ω', ω'' must begin with points z'_0, z''_0 such that $\text{supp} h_{z'_0} \cap \text{supp} h_{z''_0} \neq \emptyset$, which means that either $\square_{z'_0} \subset \square_{z''_0}$, or $\square_{z''_0} \subset \square_{z'_0}$, or both, depending on the sizes of the cubes. We obtain the equation in which the four sums in (2.26) are replaced by the sum of the above two expressions. The next operation is to apply the operator

$$\left[1 + \frac{1}{\lambda} (-\Delta_{\Omega_1}^{D_s, \eta} + Q^* a Q + \nu) \right]^{-1} = \lambda G(\lambda)$$

to both sides of this equation. The operator $G(\lambda)$ is a special case of $G(\alpha)$; we take simply the constant $\alpha = \lambda$, and therefore it has the random walk expansion (2.8). We combine this expansion with the expansions in (2.28), (2.29), and we obtain sums which can be interpreted as sums over random tree graphs of a simple form. We have two classes of tree graphs. The first consists of graphs with a trunk, a vertex, and one branch. The trunk is a random walk τ of the simplest type appearing in (2.28), the branch is a random walk ω as in (2.28), and the vertex corresponds to ω_0 . We denote these tree graphs by t_1 , i.e. $t_1 = (\tau, \omega)$. The second class is formed by tree graphs with a trunk τ and two branches ω', ω'' , $\sigma'_0 = 0$ and the vertex at ω''_0 . We denote them by t_2 , i.e. $t_2 = (\tau, \omega', \omega'')$. We have also sums corresponding to simpler graphs, like random walks τ or ω . Now, to every graph t described above we assign a domain X_t . This domain is equal to X_τ , or X_ω , defined in (2.16), if $t = \tau$ or ω . For $t = t_1$ we put $X_{t_1} = X_\tau \cup X_\omega$, and for $t = t_2$ $X_{t_2} = X_\tau \cup X_{\omega'} \cup X_{\omega''}$. We introduce parameters s into the terms of the equation as in the first sum of (2.16), and we obtain the following equation:

$$\begin{aligned} & \delta\alpha + \sum_{\tau} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) h_{\tau_0}(KG(\lambda))(\tilde{\tau}) \left((\phi_0^2 - 1) + \frac{1}{\lambda} \alpha_0 \right) \delta\alpha \prod_{\square \subset X_\tau \cap X_1^c} s_{\square} \\ & + \sum_{t_1=(\tau, \omega)} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) h_{\tau_0}(KG(\lambda))(\tilde{\tau}) V_{\omega_0}(\phi_0, G(\square_{z_0}, \alpha_0) h_{z_0}(KG)(\tilde{\omega})) \end{aligned}$$

$$\begin{aligned}
 & \times (g_0 + g_1) \prod_{\square \subset X_{\tau_1} \cap X_1^c} s_{\square} + \frac{1}{2} \sum_{t_2=(\tau, \omega', \omega'')} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) h_{\tau_0} (KG(\lambda))(\tilde{\tau}) \\
 & \times V_{\omega''} (h_{z'_0} G(\square_{z'_0}, \alpha_0) h_{z'_0} (KG)(\tilde{\omega}')(g_0 + g_1), G(\square_{z'_0}, \alpha_0) h_{z'_0} (KG)(\tilde{\omega}'')(g_0 + g_1)) \\
 & \times \prod_{\square \subset X_{\tau_2} \cap X_1^c} s_{\square} - \frac{1}{2} \sum_{t'_1=(\tau, \omega')} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) h_{\tau_0} \cdot (KG(\lambda))(\tilde{\tau}) [(g_0 + g_1) \\
 & \cdot h_{z'_0} G(\square_{z'_0}, \alpha_0) h_{z'_0} (KG)(\tilde{\omega}')(g_0 + g_1)] \prod_{\square \subset X_{\tau_1} \cap X_1^c} s_{\square} \\
 & = \sum_{\tau} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) h_{\tau_0} (KG(\lambda))(\tilde{\tau}) \phi_0 \cdot g_0 \prod_{\square \subset X_{\tau} \cap X_1^c} s_{\square} - \sum_{\tau} h_{\tau_0} \lambda G(\square_{\tau_0}, \lambda) \\
 & \times h_{\tau_0} (KG(\lambda))(\tilde{\tau}) \frac{1}{\lambda} (-\Delta_{\Omega_1}^{D, \eta} + Q^* a Q + \nu + \alpha_0) \alpha_1 \prod_{\square \subset X_{\tau} \cap X_1^c} s_{\square}, \tag{2.30}
 \end{aligned}$$

where ω' in the third and fourth sums satisfy $\sigma'_0 = 0$. Arranging properly terms in the above equation we can write it in the form (3.43) [3], but the functions H_0, H_1, H_2 depending on s also. We consider them again on the complex polydisc $\{s: |s| \leq e^{\kappa_1}\}$. Of course they are analytic functions on this polydisc, and we would like to obtain their bounds. They follow from the basic bounds (2.11), (2.17), and the geometric combinatorial bounds (2.18), (2.19), but we need also additional ones for the operator $\lambda G(\square_z, \lambda)$, ones which are uniform in λ .

The operator $G(\square_z, \lambda)$ alone satisfies the bound (2.5) with the same constants independent of λ , and for all $\lambda \geq 0$, but $\lambda G(\square_z, \lambda)$ does not satisfy such a bound. It is easy to see that this operator approaches 1 if $\lambda \rightarrow +\infty$, so it may satisfy only bounds which hold for the identity operator. We need the following one:

$$(L^j \eta)^2 |\chi_{A_j(y)} \lambda G(\square_z, \lambda) \chi_{A_{j'}}(y') f| \leq B_0 e^{-\gamma_0 d(y, y')} (L^{j'} \eta)^2 |\chi_{A_{j'}}(y') f|, \tag{2.31}$$

where f is an arbitrary function defined on $\square_z \cap \Omega_1$, $y \in \square_z \cap A_j$, $y' \in \square_z \cap A_{j'}$. This bound may be proved for arbitrary $\lambda \geq 0$, but for simplicity, we prove it for $\lambda = \lambda_k$ only, i.e. for λ such that $\lambda \eta^2 \geq 1$. The proof follows from several simple remarks. At first notice that j, j' may differ at most by 1, $|j - j'| \leq 1$, so (2.31) follows from the same inequality without the scaling factors, with a possible increase of the constant B_0 by the factor L^2 . Assume that \square_z intersects A_j , and possibly A_{j+1} , and rescale the operators and the domain to the natural ξ -scale, $\xi = L^{-j}$. Then (2.31) follows from the rescaled inequality

$$|\chi_{A_j(y)} \lambda' G^{\xi}(\square_z, \lambda') \chi_{A_{j'}}(y') f| \leq B'_0 e^{-\gamma_0 d(y, y')} |\chi_{A_{j'}}(y') f|, \tag{2.32}$$

where $\lambda' = \lambda(L^j \eta)^2$. Notice that $d(y, y')$ is properly rescaled, in particular if \square_z intersects A_j only, then $d(y, y') = |y - y'|_{\xi}$, the distance in the ξ -scale. Consider the operator $-(\Delta_{\Omega_1}^{D, \xi})_{\square_z}^N + \nu' + \lambda'$, which is equal to the inverse of the operator $G^{\xi}(\square_z, \lambda')$ without $Q^* a Q$. Denote its inverse by $C_0^{\xi}(\square_z, \lambda')$ and assume that

$$|\chi_{A_j(y)} \lambda' C_0^{\xi}(\square_z, \lambda') \chi_{A_{j'}}(y') f| \leq B''_0 e^{-\gamma'_0 |y - y'|_{\xi}} |\chi_{A_{j'}}(y') f|. \tag{2.33}$$

It is easy to see that the above inequality implies (2.32) for λ' sufficiently large, because then the following Neumann series is convergent:

$$\begin{aligned} \lambda' G^\xi(\square_z, \lambda') &= \sum_{n=0}^{\infty} \lambda' C_0^\xi(\square_z, \lambda') \left[\left(-\frac{a_j}{\lambda'} Q_j^* Q_j \chi_{\square_z \cap \Omega_{j+1}^c} \right. \right. \\ &\quad \left. \left. - \frac{a_{j+1}}{\lambda'} L^{-2} Q_{j+1}^* Q_{j+1} \chi_{\square_z \cap \Omega_{j+1}} \right) \cdot \lambda' C_0^\xi(\square_z, \lambda') \right]^n. \end{aligned}$$

We can bound it using (2.9), (2.33) and (2.11) with the simpler exponential functions coming from (2.33). Using also the simple inequality $d(y, y') \leq |y - y'|_\xi$ for $y, y' \in \square_z$ we obtain (2.32) with $\gamma_0 = \frac{1}{2}\gamma'_0$, $B'_0 = 2B''_0 c_0^d(\frac{1}{2}\gamma'_0)$, if $\frac{3}{2\lambda'} B''_0 c_0^d(\frac{1}{2}\gamma'_0) \leq \frac{1}{2}$, or $\lambda' \geq 3B''_0 c_0^d(\frac{1}{2}\gamma'_0)$. If $\lambda' < 3B''_0 c_0^d(\frac{1}{2}\gamma'_0)$, then (2.32) follows from the corresponding bound (2.5) for $G^\xi(\square_z, \lambda')$ multiplied by the bound for λ' . Thus the inequality (2.31) is reduced to (2.33). This can be rescaled further to the unit lattice, because of our assumption $\lambda'' = \lambda' \xi^2 = \lambda \eta^2 \geq 1$. The rescaled operator is equal to

$$\begin{aligned} \lambda'' C_0^1(\square_z, \lambda'') &= \lambda'' (-\Delta_{\Omega_1}^{D,1})_{\square_z}^N + v'' + \lambda'')^{-1} = \lambda'' (\lambda'' + v'' + 2d_z - t_z)^{-1} \\ &= \frac{\lambda''}{\lambda'' + v'' + 2d_z} \left(1 - t_z \frac{1}{\lambda'' + v'' + 2d_z} \right)^{-1} \\ &= \frac{\lambda''}{\lambda'' + v'' + 2d_z} \sum_{n=0}^{\infty} \left(t_z \frac{1}{\lambda'' + v'' + 2d_z} \right)^n, \end{aligned}$$

where $2d_z$ and $-t_z$ are diagonal and off-diagonal parts of the operator $-\Delta_{\Omega_1}^{D,1}$. The above series gives the usual random walk expansion of the kernel of the operator, and from this we obtain easily the bound

$$\begin{aligned} &|\lambda'' C_0^1(\square_z, \lambda''; x, x')| \\ &\leq \frac{\lambda''}{\lambda'' + v''} \cdot \frac{\lambda'' + v'' + 2d}{\lambda'' + v'' + d} \exp \left(- \left(\log \frac{\lambda'' + v'' + 2d}{2d} \right) |x - x'|_1 \right) \\ &< 2e^{-\frac{1}{2d+1} |x - x'|_1}. \end{aligned}$$

It implies (2.33) with $\gamma'_0 = \frac{1}{4d+2}$ and $B''_0 = 2c_0^d(\frac{1}{4d+2}) \exp(\frac{Ld}{4d+2})$, which concludes the proof of (2.31).

Let us finish the proof of the inequality (3.41) used in Sect. 3 [3]. We take the random walk expansion (2.8) for $G(\lambda)$, and apply (2.9), (2.11) to it, using the inequalities (2.31) and (2.10) with λ instead of α . We obtain

$$|\lambda G(\lambda) f|_2 \leq 2^{d+2} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) |f|_2, \quad (2.34)$$

which is the inequality (3.41) [3] with the precisely defined constant.

Now we estimate the expressions in Eq. (2.30) in the norm $|\cdot|_2$. We estimate at first the whole sums in (2.30) by polynomials in $|\delta\alpha|_2$ and from such inequalities we obtain estimates for the functions $H_1(s), H_2(s; \delta\alpha), H_0(s)$. We start with the first

sum in (2.30). Estimating as in (2.11) and using (2.31), (2.19) we obtain

$$\begin{aligned}
 & |\chi_{\mathcal{A}(y)} \text{ (the first sum)}|_2 \\
 & \leq B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) \sum_{\tau; y \in \square_{\tau_0}} \frac{1}{2^{|\tau|}} \cdot \frac{1}{(2L+1)^{d|\tau|}} e^{-\frac{1}{2} \gamma_0 d(y; \tau)} \cdot M^{-\frac{1}{2} |\tau|} \\
 & \quad \times \left(\delta_0 |\delta \alpha|_2 + \left| \frac{\alpha_0}{\lambda} \delta \alpha \right|_2 \right) \exp \kappa_1 \sum_{j=0}^k \frac{1}{(ML^j \eta)^d} |X_\tau \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c| \\
 & \leq B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) \sum_{\tau; y \in \square_{\tau_0}} \frac{1}{2^{|\tau|}} \cdot \frac{1}{(2L+1)^{d|\tau|}} (e^{-\frac{1}{2} \gamma_0 d(y, (X_1^c)^{\sim 2})}) e^{(7L)^d \kappa_1} \\
 & \quad \times (\text{if } X_\tau \cap X_1^c \neq \emptyset) + 1 (\text{if } X_\tau \cap X_1^c = \emptyset) M^{-\frac{1}{2} |\tau|} \left(\delta_0 + \frac{1}{\lambda \eta^2} |\alpha_0|_2 \right) |\delta \alpha|_2 \\
 & \leq 2^{d+3} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) e^{(7L)^d \kappa_1} \delta_0 |\delta \alpha|_2. \tag{2.35}
 \end{aligned}$$

This holds for all $y \in \mathbb{B}_k$. If $\mathcal{A}(y) \subset X_0$, then we can remove the factor $e^{(7L)^d \kappa_1}$ from the bound, which is not important. Again for arbitrary y we have the bound

$$\begin{aligned}
 & |\chi_{\mathcal{A}(y)} ((\text{the first sum}) - (\text{the first sum for } s = 1))|_2 \\
 & \leq 2^{d+4} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) e^{(7L)^d \kappa_1} \exp \left(-\frac{1}{2} \gamma_0 d(y, (X_1^c)^{\sim 2}) \right) \delta_0 |\delta \alpha|_2 \tag{2.36}
 \end{aligned}$$

which is very important. The two sums on the right-hand side of (2.30) can be estimated in the same way and we obtain the above inequalities for them, but with $2\delta_0 |\delta \alpha|_2$ replaced by

$$\begin{aligned}
 & |\phi_0 \cdot g_0|_2 + \left| \frac{1}{\lambda} (-A_{\Omega_1}^{D, \eta} + Q^* a Q + v + \alpha_0) \alpha_1 \right|_2 < \frac{3}{2} \cdot \frac{5}{2} \delta_1 \\
 & + \frac{1}{\lambda \eta^2} \left(4d + \frac{1}{6} + 1 + \delta_0 \right) |\alpha_1|_2 < (4d + 5) \delta_1.
 \end{aligned}$$

To estimate the second and third sums we have to obtain a bound for the operator $V(h_z)$. From (2.27) we have for $y \in \mathbb{B}_k$,

$$\begin{aligned}
 & |\chi_{\mathcal{A}(y)} V(h_z)(f, g)|_2 \leq 2d |\chi_{\mathcal{A}(y)} \partial f|_1 |\chi_{\mathcal{A}(y)} \partial(h_z g)|_1 \\
 & \quad + |\chi_{\mathcal{A}(y)} \Delta f|_2 |\chi_{\mathcal{A}(y)} h_z g| + d |\chi_{\mathcal{A}(y)} \partial f|_1 |\chi_{\mathcal{A}(y)} h_z g|.
 \end{aligned}$$

For simplicity let us drop the characteristic functions $\chi_{\mathcal{A}(y)}$ in the inequalities below, but we always mean the localized inequalities. Taking $f = \phi_0$ we get

$$\begin{aligned}
 & |V(h_z)(\phi_0, g)|_2 \leq 2d |\partial \phi_0|_1 (|\partial h_z \cdot \eta \partial g|_1 + |(\partial h_z) g|_1 + |h_z \partial g|_1) \\
 & \quad + |\Delta \phi_0|_2 |h_z g| + d |\partial \phi_0|_1 |h_z g| \\
 & \leq 2d \delta_0 \left(\frac{2}{9} M^{-1} |\partial g|_1 + 2M^{-1} |g| + |\partial g|_1 \right) + (d+1) \delta_0 |g| \\
 & \leq (2d+1) \delta_0 |\partial g|_1 + (d+2) \delta_0 |g| \leq (2d+1) \delta_0 (|g| + |\partial g|_1), \tag{2.37}
 \end{aligned}$$

where we have used the assumptions (3.21) [3] on ϕ_0 . Finally, using the same elementary bounds we get

$$|V(h_{z''})(h_{z'}, f, g)|_2 \leq (2d+1)(|f| + |\partial f|_1 + |\Delta f|_2)(|g| + |\partial g|_1). \quad (2.38)$$

Consider now the second sum in (2.30). Using the estimates (2.11), (2.17), (2.19) we obtain

$$\begin{aligned} & |\chi_{\mathcal{A}(y)}(\text{the second sum})|_2 \\ & \leq B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) \sum_{t_1=(\tau, \omega): y \in \square_{t_0}} \frac{1}{2^{|\tau|+|\omega|+1}} \cdot \frac{1}{(2L+1)^{d(|\tau|+|\omega|+1)}} \\ & \quad \cdot \left((2d+1)\delta_0 + \frac{3}{2}|\delta\alpha|_2\right) 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d e^{-\frac{1}{2}\gamma_0 d(y; t_1)} \\ & \quad \times M^{-\frac{1}{2}|\tau|} \left[M^{-\frac{1}{2}|\omega|+|\omega|} M^{-\frac{1}{2}(|\omega|-1)} 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d |\delta\alpha|_2 \theta_+(|\omega|-1) \right. \\ & \quad \left. + \frac{1}{2}|\omega|(|\omega|-1) M^{-\frac{1}{2}(|\omega|-2)} \left(2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d |\delta\alpha|_2\right)^2 \theta_+(|\omega|-2) \right] \\ & \quad \cdot \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right) \exp \kappa_1 \sum_{j=0}^k \frac{1}{(ML^j \eta)^d} |X_{t_1} \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c| \\ & \leq B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) \cdot 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d e^{(7L)^d \kappa_1} \cdot \left((2d+1)\delta_0 + \frac{3}{2}|\delta\alpha|_2\right) \\ & \quad \cdot \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right) \left[1 + 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d |\delta\alpha|_2 \right. \\ & \quad \left. + \left(2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d |\delta\alpha|_2\right)^2 \right] \cdot \sum_{t_1=(\tau, \omega): y \in \square_{t_0}} \frac{1}{2^{|\tau|}} \\ & \quad \cdot \frac{1}{(2L+1)^{d|t_1|}} \left(\exp\left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) + (7L)^d \kappa_1\right) \text{ (if } X_{t_1} \cap X_1^c \neq \emptyset\text{)} \\ & \quad + 1 \text{ (if } X_{t_1} \cap X_1^c = \emptyset\text{)} \leq 2^{d+2} B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) B_3 \left((2d+1)\delta_0 + \frac{3}{2}|\delta\alpha|_2\right) \\ & \quad \times \left(\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2\right) (1 + B_2 |\delta\alpha|_2 + B_2^2 |\delta\alpha|_2^2), \quad (2.39) \end{aligned}$$

where we have denoted $|t_1| = |\tau| + |\omega| + 1$, and

$$B_2 = 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d,$$

$$B_3 = 2B_0 c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d e^{(7L)^d \kappa_1} = B_2 e^{(7L)^d \kappa_1}.$$

The last bound holds for all $y \in \mathbb{B}_k$. The difference of the second sum and this sum with $s = 1$ can be estimated by the right-hand side above multiplied by $2 \exp(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2}))$.

The fourth sum in (2.30) has the same form as the second sum, but the operator V_{ω_0} in the vertex is replaced by a simpler multiplication operator by the function $g_0 + g_1$. Therefore this sum can be bounded as in (2.39), but with the factor $(2d+1)\delta_0 + \frac{3}{2}|\delta\alpha|_2$ replaced by $\frac{5}{2}\delta_1 + \frac{3}{2}|\delta\alpha|_2$, i.e. we have

$$|\chi_{\mathcal{A}(y)}(\text{the fourth sum})|_2 \leq 2^{d+1} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) B_3 \left(\frac{5}{2} \delta_1 + \frac{3}{2} |\delta\alpha|_2 \right)^2 \\ (1 + B_2 |\delta\alpha|_2 + B_2^2 |\delta\alpha|_2^2). \quad (2.40)$$

The difference of the sum and the sum with $s=1$ can be estimated again by the above bound multiplied by $2 \exp(-\frac{1}{2} \gamma_0 d(y, (X_1^c)^{\sim 2}))$.

Consider finally the third sum in (2.30). It has a more complicated structure, the summation is not just over random walks, but over tree graphs with one vertex and two branches. We estimate it again as in (2.11), (2.17), (2.39), and we obtain the following preliminary bound:

$$|\chi_{\mathcal{A}(y)}(\text{the third sum})|_2 \\ \leq \frac{1}{2} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) (2d+1 + |\delta\alpha|_2) 2 \left(2B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) (2L+1)^d \right)^2 \\ \cdot (1 + B_2 |\delta\alpha|_2 + B_2^2 |\delta\alpha|_2^2)^2 \left(\frac{5}{2} \delta_1 + \frac{3}{2} |\delta\alpha|_2 \right)^2 \sum_{t_2=(\tau, \omega', \omega''): y \in \square_{t_0}} \frac{1}{2|t_2|} \\ \cdot \frac{1}{(2L+1)^{d|t_2|}} \cdot e^{-\frac{1}{2} \gamma_0 d(y, t_2)} \exp \kappa_1 \sum_{j=0}^k \frac{1}{(ML^j \eta)^d} |X_{t_2} \cap X_1^c \cap \Omega_j \cap \Omega_{j+1}^c|, \quad (2.41)$$

where $|t_2| = |\tau| + |\omega'| + |\omega''| + 2$, and $d(y; t_2)$ is defined in the same way as $d(y; \omega)$ in (2.11), but now we take points along two branches ω', ω'' of t_2 . It is easy to see that there is a tree graph Γ in the domain X_{t_2} , consisting of a path starting at y and ending at some point y_n of the last cube in τ , and of two paths starting at y_n and ending at points of the last cubes in ω', ω'' , such that $d(y; t_2) = |\Gamma|_{\mathbb{B}_i}$. We decompose the graph Γ into two disjoint parts Γ^0, Γ' as in the proof of (2.19), and applying this inequality to the two branches of Γ' we obtain an inequality of the form (2.19), but with the term $(7L)^d \kappa_1$ in the last exponential multiplied by 2. Thus in a general case we obtain the bound

$$|\chi_{\mathcal{A}(y)}(\text{the third term})|_2 \leq 2^{d+2} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) B_3^2 (2d+1 + |\delta\alpha|_2) \\ \times \left(\frac{5}{2} \delta_1 + \frac{3}{2} |\delta\alpha|_2 \right)^2 \cdot (1 + B_2 |\delta\alpha|_2 + B_2^2 |\delta\alpha|_2^2)^2. \quad (2.42)$$

If X_{t_2} intersects X_1^c , then we can choose a maximal path in Γ^0 , and the arguments leading to the inequalities (2.20), (2.21) can be applied to this maximal path.

Thus the difference of the third sum and the third sum for $s = 1$ can be estimated by the right-hand side of (2.42) multiplied again by $2 \exp(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2}))$.

Equation (2.30) can be written in the form

$$\delta\alpha + H_1(s)\delta\alpha + H_2(s; \delta\alpha) = H_0(s), \quad (2.43)$$

where the functions H_0, H_1, H_2 are defined by gathering together terms in (2.30) of the corresponding orders in $\delta\alpha$. We obtain bounds for these functions by gathering together terms of the corresponding orders in $|\delta\alpha|_2$ in bounds discussed above. For simplicity we introduce the following preliminary restrictions on $\delta_0, \delta_1, |\delta\alpha|_2$:

$$B_2\delta_0 \leq \frac{1}{10}, \quad B_2B_3\delta_1 \leq \frac{1}{10}, \quad B_2|\delta\alpha|_2 \leq \frac{1}{10}.$$

Our final restrictions will be much stronger. Using these restrictions we obtain, after some simple estimates, the inequalities

$$\begin{aligned} |H_0(s)|_2 &< B_3\delta_1, \quad |\chi_{\mathcal{A}(y)}(H_0(s) - H_0(1))|_2 < \exp\left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) B_3\delta_1, \\ |H_1(s)\delta\alpha|_2 &\leq B_1(\delta_0 + \delta_1)|\delta\alpha|_2, \quad |\chi_{\mathcal{A}(y)}(H_1(s)\delta\alpha - H_1(1)\delta\alpha)|_2 \\ &\leq \exp\left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) B_1(\delta_0 + \delta_1)|\delta\alpha|_2, \\ |H_2(s; \delta\alpha)|_2 &\leq B_1|\delta\alpha|_2^2, \quad |\chi_{\mathcal{A}(y)}(H_2(s; \delta\alpha) - H_2(1; \delta\alpha))|_2 \\ &\leq \exp\left(-\frac{1}{2}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) B_1|\delta\alpha|_2^2, \end{aligned} \quad (2.44)$$

where we may take $B_1 = 27B_2B_3^2$. The functions above are analytic functions of s and $\delta\alpha$ on the spaces $\{s: |s| \leq e^{c_1}\}$ and $\{\delta\alpha: |\delta\alpha|_2 \leq \gamma_2\}$, where $\gamma_2 = \min\{\frac{1}{10}B_2^{-1}, \frac{1}{2}\gamma_1\}$. We repeat the reasoning leading to Lemma 3.2 [3], and we conclude that Eq. (2.43) has a unique solution in $\{\delta\alpha: |\delta\alpha|_2 < c\}$ if $B_3\delta_1 \leq \frac{1}{2}c$, $B_1\delta_0 \leq \frac{1}{8}$, $B_1c \leq \frac{1}{8}$, $c \leq \gamma_2$. A maximal such c is denoted by c_6 , and is given by $c_6 = \min\{\frac{1}{8}B_1^{-1}, \gamma_2\}$. The solution $\delta\alpha_k(s)$ is an analytic function of $s, \phi_0, \alpha_0, \delta\psi, f_1, \alpha_1$ on the corresponding domains, and it satisfies the bound

$$|\delta\alpha_k(s)|_2 \leq 2|H_0(s)|_2 < 2B_3\delta_1. \quad (2.45)$$

This ends the construction of the ‘‘localization extension’’ of the function $\delta\alpha_k$.

We also need a bound for the difference $\delta\alpha(s) - \delta\alpha(1)$ of a type of the second bound in (2.44). We subtract Eqs. (2.43) for $\delta\alpha(s), \delta\alpha(1)$, and we obtain the following equation for the difference:

$$\begin{aligned} &(\delta\alpha_k(s) - \delta\alpha_k(1)) + H_1(1)(\delta\alpha_k(s) - \delta\alpha_k(1)) \\ &+ \int_0^1 dt \left\langle \left(\frac{\delta}{\delta(\delta\alpha)} H_2 \right) (1; t\delta\alpha_k(s) + (1-t)\delta\alpha_k(1)), (\delta\alpha_k(s) - \delta\alpha_k(1)) \right\rangle \\ &= (H_0(s) - H_0(1)) - (H_1(s)\delta\alpha_k(s) - H_1(1)\delta\alpha_k(s)) \\ &\quad - (H_2(s; \delta\alpha_k(s)) - H_2(1; \delta\alpha_k(s))). \end{aligned} \quad (2.46)$$

In order to be able to use the inequalities for the differences in (2.44) we have to localize the above equation, i.e. we multiply it by $\chi_{\mathcal{A}(y)}$, $y \in \mathbb{B}_k$, and we introduce the decomposition $1 = \sum_{y' \in \mathbb{B}_k} \chi_{\mathcal{A}(y')}$ before the differences $\delta\alpha(s) - \delta\alpha(1)$ in the

second and third terms on the left-hand side. We have to obtain bounds for the localized expressions $\chi_{\mathcal{A}(y)}H_1(s)\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}$ and $\chi_{\mathcal{A}(y)}\left\langle\left(\frac{\delta}{\delta(\delta\alpha)}H_2\right)(s;\delta\alpha),\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}\right\rangle$. We need them for $s = 1$ only, but we use all the bounds proved until now for the general admissible s , so we may consider these general expressions also. Let us start with the operator $H_1(s)$. The function $H_1(s)\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}$ is represented as a sum of terms over the class of random tree graphs t described before. We estimate it making only a few simple changes in the proof of the third inequality in (2.44). The basic steps yielding the exponential factors for each term are described in (2.9), (2.11). We replace each factor by a product of three factors, one with γ_0 replaced by $\frac{1}{2}\gamma_0$, and two with γ_0 replaced by $\frac{1}{4}\gamma_0$. The product of the factors with $\frac{1}{2}\gamma_0$ is bounded by $\exp(-\frac{1}{2}\gamma_0d(y,y'))$, one product of the factors with $\frac{1}{4}\gamma_0$ is bounded by $\exp(-\frac{1}{4}\gamma_0d(y,t))$, where t is the graph connected with the term, and the remaining factors with $\frac{1}{4}\gamma_0$ are used to control the sum over the sequences of points (y_i) , as in (2.11). We obtain the same bounds as before, but with the constants $c_1(d,\frac{1}{8}\gamma_0)$ instead of $c_1(d,\frac{1}{4}\gamma_0)$, and with the condition on M stronger than the condition in (2.19) by the factor 2. Also, there is no summation over y' , hence one factor $c_1(d,\frac{1}{8}\gamma_0)$ is missing. Thus we obtain the inequality

$$\begin{aligned} & |\chi_{\mathcal{A}(y)}H_1(s)\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2 \\ & \leq \exp\left(-\frac{1}{2}\gamma_0d(y,y')\right)B_1c_1^{-1}\left(d,\frac{1}{8}\gamma_0\right)(\delta_0+\delta_1)|\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2. \end{aligned} \quad (2.47)$$

For $s = 1$ the constant B_1 may be improved by replacing the constant B_3 by B_2 in all the estimates, but it is not important. Consider now the second expression, the differential of H_2 . The function H_2 is also a sum of terms of the class of random tree graphs, but now we have at least two vertices with the function $\delta\alpha$, and at most seven, including the expressions at end-points of the graphs. The differentiation replaces one of the functions $\delta\alpha$ by $\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}$, or one of the operators $G(\square_z,\alpha_0+\delta\alpha)$ by its differential equal to $-G(\square_z,\alpha_0+\delta\alpha)\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}G(\square_z,\alpha_0+\delta\alpha)$. The first operation results in an increase of the number of terms by the factor 7 at most, each term is bounded in the same way as in the proof of (2.47). The second operation results in replacing the terms $\frac{1}{2}|\omega|(|\omega|-1)\cdot M^{-\frac{1}{2}(|\omega|-2)}(B_2|\delta\alpha|_2)^2$ in the bounds (2.39), (2.40), (2.41) by

$$\frac{1}{6}(|\omega|+1)|\omega|(|\omega|-1)M^{-\frac{1}{2}(|\omega|-2)}(B_2|\delta\alpha|_2)^2B_0|\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2 \leq B_2^2|\delta\alpha|_2^2B_0|\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2,$$

and the rest of the bound is again the same as in the proof of (2.47). We obtain a bound in which $B_1|\delta\alpha|_2$ is replaced by $(70B_1B_2c_1^{-1}(d,\frac{1}{8}\gamma_0)+B_1B_0)|\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2$, and we denote the new constant by $B_1c_1^{-1}(d,\frac{1}{8}\gamma_0)$. Obviously this new B_1 is bigger than the one in (2.44). We obtain the inequality

$$\begin{aligned} & \left|\chi_{\mathcal{A}(y)}\left\langle\left(\frac{\delta}{\delta(\delta\alpha)}H_2\right)(s;\delta\alpha),\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}\right\rangle\right|_2 \\ & \leq \exp\left(-\frac{1}{2}\gamma_0d(y,y')\right)B_1c_1^{-1}\left(d,\frac{1}{8}\gamma_0\right)|\delta\alpha|_2|\chi_{\mathcal{A}(y')}\delta\tilde{\alpha}|_2. \end{aligned} \quad (2.48)$$

Notice that the third and fifth inequalities in (2.44), follow from (2.47), (2.48).

Now we are ready to estimate the difference $\delta\alpha_k(s)-\delta\alpha_k(1)$ using the localized equation (2.46). We take the norm $|\cdot|_2$ of both sides, and multiply by

$\exp \frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2})$. The norm of the left-hand side is bounded from below by the norm of the first term minus the norms of the second and third terms. We estimate the norm of the second term as follows:

$$\begin{aligned}
& \exp \frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2}) \left| \sum_{y' \in \mathbb{B}_k} \chi_{\Delta(y)} H_1(1) \chi_{\Delta(y')} (\delta \alpha_k(s) - \delta \alpha_k(1)) \right|_2 \\
& \leq \sum_{y' \in \mathbb{B}_k} \exp \frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2}) \exp \left(-\frac{1}{2} \gamma_0 d(y, y') \right) \\
& \quad \times \exp \left(-\frac{1}{4} \gamma_0 d(y', (X_1^c)^{\sim 2}) \right) B_1 c_1^{-1} \left(d, \frac{1}{8} \gamma_0 \right) (\delta_0 + \delta_1) \\
& \quad \cdot \exp \frac{1}{4} \gamma_0 d(y', (X_1^c)^{\sim 2}) |\chi_{\Delta(y')} (\delta \alpha_k(s) - \delta \alpha_k(1))|_2 \\
& \leq \sum_{y' \in \mathbb{B}_k} \exp \left(-\frac{1}{4} \gamma_0 d(y, y') \right) B_1 c_1^{-1} \left(d, \frac{1}{8} \gamma_0 \right) (\delta_0 + \delta_1) \\
& \quad \times \exp \frac{1}{4} \gamma_0 d(y', (X_1^c)^{\sim 2}) |\chi_{\Delta(y')} (\delta \alpha_k(s) - \delta \alpha_k(1))|_2 \\
& \leq \frac{3}{16} \sup_y \exp \frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2}) |\chi_{\Delta(y)} (\delta \alpha_k(s) - \delta \alpha_k(1))|_2,
\end{aligned}$$

where we have used the conditions $B_1 \delta_0 \leq \frac{1}{8}$, $B_1 B_3 \delta_1 \leq \frac{1}{2} B_1 c_6 \leq \frac{1}{16}$, which imply $B_1(\delta_0 + \delta_1) < \frac{1}{8} + \frac{1}{16} = \frac{3}{16}$. We estimate in the same way the norm of the third term using the inequality (2.48), and we bound it by

$$\begin{aligned}
& B_1 \sup_{t \in [0,1]} |t \delta \alpha_k(s) + (1-t) \delta \alpha_k(1)|_2 \\
& \quad \cdot \sup_y \exp \frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2}) |\chi_{\Delta(y)} (\delta \alpha_k(s) - \delta \alpha_k(1))|_2.
\end{aligned}$$

The first supremum is bounded by $2B_3 \delta_1$, and $2B_1 B_3 \delta_1 \leq \frac{1}{8}$, hence the above expression can be bounded by $\frac{1}{8}$ times the second supremum. Thus the supremum over $y \in \mathbb{B}_k$ of the norm of the left-hand side is bounded from below by $1 - \frac{3}{16} - \frac{1}{8} = \frac{11}{16}$ times the second supremum above. Using the second, fourth and sixth inequalities in (2.44) we bound the norm of the right-hand side by

$$\begin{aligned}
& B_3 \delta_1 + B_1 (\delta_0 + \delta_1) |\delta \alpha_k(s)|_2 + B_1 |\delta \alpha_k(s)|_2^2 \\
& < B_3 \delta_1 + B_1 (\delta_0 + \delta_1) 2B_3 \delta_1 + 4B_1 B_3^2 \delta_1^2 < \frac{13}{8} B_3 \delta_1.
\end{aligned}$$

Combining the bounds we obtain

$$|\chi_{\Delta(y)} (\delta \alpha_k(s) - \delta \alpha_k(1))|_2 < 3 \exp \left(-\frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2}) \right) B_3 \delta_1. \quad (2.49)$$

The two bounds (2.45), (2.49) are the basic bounds for the solution $\delta\alpha(s)$ of the equation (2.43). It also has the following fundamental localization property: if $s = 0$ on a family \mathcal{C} of cubes \square , then $\delta\alpha_k(s) = 0$ on the domain $\bigcup_{\square \in \mathcal{C}} \square$, and $\delta\alpha_k(s)$ does not depend on $\phi_0, \alpha_0, \delta\psi, f_1, \alpha_1$ restricted to this domain. This property follows immediately from the form (2.30) of the equation.

Let us come back to the function $\delta\phi_k(\delta\alpha, s)$. For $s = 1$, it becomes a solution of Eqs. (3.20) [3] if we substitute $\delta\alpha_k(1)$ instead of $\delta\alpha$. Generally we substitute the solution $\delta\alpha_k(s)$ instead of $\delta\alpha$, and we denote $\delta\phi_k(s) = \delta\phi_k(\delta\alpha_k(s), s)$. From the inequalities (2.23) we obtain

$$\begin{aligned} |\delta\phi_k(s)| &< 2^{d+3} B_0 c_1 \left(d, \frac{1}{4} \gamma_0 \right) e^{(7L)^d \kappa_1} \left(\frac{5}{2} + 3B_3 \right) \delta_1 < B_3^2 \delta_1, \\ |\chi_{\mathcal{A}(y)}(\delta\phi_k(s) - \delta\phi_k(\delta\alpha_k(s), 1))| &< \exp\left(-\frac{1}{2} \gamma_0 d(y, (X_1^c)^{\sim 2})\right) B_3^2 \delta_1, \end{aligned} \quad (2.50)$$

and the same inequalities for the other norms listed after (2.23). To estimate the difference $\delta\phi_k(s) - \delta\phi_k(1)$ we have to estimate the difference

$$\begin{aligned} &\delta\phi_k(\delta\alpha_k(s), 1) - \delta\phi_k(\delta\alpha_k(1), 1) \\ &= \int_0^1 dt \left\langle \left(\frac{\delta}{\delta(\delta\alpha)} \delta\phi_k \right) (t\delta\alpha_k(s) + (1-t)\delta\alpha_k(1), 1), \delta\alpha_k(s) - \delta\alpha_k(1) \right\rangle. \end{aligned} \quad (2.51)$$

We do it in the same way as for (2.46), we estimate the localized differential $\chi_{\mathcal{A}(y)} \left\langle \left(\frac{\delta}{\delta(\delta\alpha)} \delta\phi_k \right) (\delta\alpha, s), \chi_{\mathcal{A}(y')} \delta\tilde{\alpha} \right\rangle$. The function $\delta\phi_k(\delta\alpha, s)$ is given by the random walk expansion (2.16) with at most three functions $\delta\alpha$ at vertices of the walks. The differential is obtained by replacing one of the functions $\delta\alpha$ by $\chi_{\mathcal{A}(y')} \delta\tilde{\alpha}$, or one of the operators $G(\square_z, \alpha_0 + \delta\alpha)$ by $-G(\square_z, \alpha_0 + \delta\alpha) \chi_{\mathcal{A}(y')} \delta\tilde{\alpha} G(\square_z, \alpha_0 + \delta\alpha)$. We estimate it in exactly the same way as in the proof of (2.48). In this simpler case we can do it a bit more carefully, as in the inequality (2.17), and we obtain the following bound:

$$\begin{aligned} &\left| \chi_{\mathcal{A}(y)} \left\langle \left(\frac{\delta}{\delta(\delta\alpha)} \delta\phi_k \right) (\delta\alpha, s), \chi_{\mathcal{A}(y')} \delta\tilde{\alpha} \right\rangle \right| \\ &\leq \exp\left(-\frac{1}{2} \gamma_0 d(y, y')\right) B_3 c_1^{-1} \left(d, \frac{1}{8} \gamma_0 \right) |\chi_{\mathcal{A}(y')} \delta\tilde{\alpha}|_2, \end{aligned} \quad (2.52)$$

holding for all $\delta\alpha$ in the space $\{\delta\alpha: |\delta\alpha|_2 < \gamma_2\}$. Combining (2.51), (2.52) and (2.49) we obtain

$$\begin{aligned} &|\chi_{\mathcal{A}(y)}(\delta\phi_k(\delta\alpha_k(s), 1) - \delta\phi_k(\delta\alpha_k(1), 1))| \\ &\leq \sum_{y'} \exp\left(-\frac{1}{2} \gamma_0 d(y, y')\right) B_3 c_1^{-1} \left(d, \frac{1}{8} \gamma_0 \right) \cdot \exp\left(-\frac{1}{4} \gamma_0 d(y', (X_1^c)^{\sim 2})\right) 3B_3 \delta_1 \\ &\leq \exp\left(-\frac{1}{4} \gamma_0 d(y, (X_1^c)^{\sim 2})\right) 3B_3^2 \delta_1. \end{aligned}$$

This inequality and the inequalities (2.50) yield finally

$$|\delta\phi_k(s)| < B_3^2\delta_1, \quad |\chi_{\Delta(y)}(\delta\phi_k(s) - \delta\phi_k(1))| < \exp\left(-\frac{1}{4}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) 4B_3^2\delta_1. \quad (2.53)$$

The function $\chi_{X_0}\delta\phi_k(s)$ has also a similar localization property as $\delta\alpha_k(s)$: if $s = 0$ on a family \mathcal{C} of cubes, then $\chi_{X_0}\delta\phi_k(s)$ does not depend on $\phi_0, \alpha_0, \delta\psi, f_1, \alpha_1$ restricted to the domain $\bigcup_{\square \in \mathcal{C}} \square$. Actually, a stronger statement holds, the function depends on the variables restricted to this connected component of the domain $(\bigcup_{\square \in \mathcal{C}} \square)^c$ which contains X_0 .

We summarize the main results obtained until now in this section in the following proposition.

Proposition 2.1. *Consider the solutions $\delta\phi_k, \delta\alpha_k$ of the system of Eqs. (3.20) [3] defined on the space (3.21) [3]. There exist constants B_1, K_3, c_6 such that $B_1 c_6 \leq \frac{1}{8}$, $c_6 \leq \gamma_2$, and if $B_1 \delta_0 \leq \frac{1}{8}$, $B_3 \delta_1 \leq \frac{1}{2} c_6$, then the solutions are extended to analytic functions $\delta\phi_k(s), \delta\alpha_k(s)$ defined on $\{s: |s| \leq e^{K_1}\}$. The function $\delta\alpha_k(s)$ satisfies Eq. (2.30), and $\delta\phi_k(s)$ is given by the formula (2.16) with $\delta\alpha = \delta\alpha_k(s)$. For $s = 1$ they coincide with the solutions $\delta\phi_k, \delta\alpha_k$, i.e. $\delta\phi_k(1) = \delta\phi_k, \delta\alpha_k(1) = \delta\alpha_k$. Let \mathcal{C} be a family of the large cubes, and let $s_\square = 0$ for $\square \in \mathcal{C}$. Denote $Y_0 = \bigcup_{\square \in \mathcal{C}} \square$, then $\chi_{X_0}\delta\phi_k(s)$ depends on the variables in the space (3.21) [3] restricted to the connected component of Y_0^c containing the domain X_0 , $\delta\alpha_k(s) = 0$ on Y_0 and $\chi_{\Delta(y)}\delta\alpha_k(s)$ depends on the variables restricted to the component of Y_0^c containing $\Delta(y)$. The functions $\delta\phi_k(s), \delta\alpha_k(s)$ satisfy the bounds*

$$\begin{aligned} |\delta\phi_k(s)| &< K_3\delta_1, & |\partial^\eta \delta\phi_k(s)|_1 &< K_3\delta_1, \\ |\partial^\eta \delta\phi_k(s)|_{(\alpha), 2-\alpha} &< C_\alpha K_3\delta_1 \quad \text{for } 0 < \alpha \leq 1, \\ |\Delta_{\Omega_1}^{D, \eta} \delta\phi_k(s)|_2 &< K_3\delta_1, & |\delta\alpha_k(s)|_2 &< K_3\delta_1, \end{aligned} \quad (2.54)$$

$$|\chi_{\Delta(y)}(\delta\phi_k(s) - \delta\phi_k(1))| < \exp\left(-\frac{1}{4}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) K_3\delta_1, \quad (2.55)$$

and the same bounds for the remaining norms written in (2.54), only with the additional factor C_α for the third norm,

$$|\chi_{\Delta(y)}(\delta\alpha_k(s) - \delta\alpha_k(1))|_2 < \exp\left(-\frac{1}{4}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) K_3\delta_1. \quad (2.56)$$

It follows from (2.53), (2.45) and (2.49) that we can take $K_3 = 4B_3^2$ in the above proposition. We can also simplify the formulation assuming $K_3 = B_1$ without any important loss of generality, although B_1 is considerably bigger than $4B_3^2$. The estimates of the two differences are particularly important in the future. We need them on the domain X_0 , and it is enough then to replace the exponential functions of y by small exponential factors. We use the bound (2.22), and we obtain the inequalities

$$|\chi_{X_0}(\delta\phi_k(s) - \delta\phi_k)| < \exp\left(-\frac{1}{4}\gamma_0 r_1 M\right) K_3\delta_1, \quad (2.57)$$

and the corresponding bounds for the remaining norms in (2.54),

$$|\chi_{X_0}(\delta\alpha_k(s) - \delta\alpha_k)|_2 < \exp\left(-\frac{1}{4}\gamma_0 r_1 M\right) K_3 \delta_1,$$

where $r_1 = r_0 - 4$ is a positive integer, e.g. we may take $r_1 = 4$.

The above proposition is an extension of Proposition 3.1 [3], and it plays a fundamental technical role in all localization expansions we construct in these papers. Let us apply it now to the solution ϕ_k, α_k of the basic system of Eqs. (1.14) [3]. This solution has been constructed in the form $\phi_k = \phi_0 + \delta\phi_k, \alpha_k = \alpha_0 + \delta\alpha_k$, where ϕ_0, α_0 is the approximate solution defined in (3.2), (3.3) [3], and the functions $\delta\phi_k, \delta\alpha_k$ satisfy the system (3.20) with the above ϕ_0, α_0 , and with $\delta\psi = 0, f_1 = -f_0, \alpha_1 = 0$. In this case $\delta_0 = C_3\delta, \delta_1 = C_3\delta$, where δ satisfies the assumptions of Proposition 2.1 [3], i.e., $0 < \delta \leq \varepsilon, \delta \leq c_4, v\varepsilon^2 \leq \delta^2$. We assume that $B_1 C_3 \delta \leq \frac{1}{8}, B_3 C_3 \delta \leq \frac{1}{2}c_6$, which means that $\delta \leq c_7$, where $c_7 = \min\{c_4, \frac{1}{8}(B_1 C_3)^{-1}, \frac{1}{2}(B_3 C_3)^{-1}c_6\}$. Then the solutions $\delta\phi_k, \delta\alpha_k$ can be extended to the analytic functions $\delta\phi_k(s), \delta\alpha_k(s)$ by Proposition 2.1, with one small modification. In the above constructions we have used the localized Green's functions $G(\square_z, \alpha), \alpha = \alpha_0$ or $\alpha = \alpha_0 + \delta\alpha$ and we have obtained the localization property with respect to α_0 . It is not good enough now, because we are interested in localization with respect to the variables $(\psi + \psi', h + h')$, and $\chi_{\Delta(y)}\alpha_0$ depends on these variables restricted to $\tilde{\Delta}(y)$, or even to $\tilde{\Delta}(y')$, where y' is a center of a block containing the point y . For $\Delta(y)$ close to the boundary of \square_z this will involve a dependence of $G(\square_z, \alpha_0)$ on $(\psi + \psi', h + h')$ beyond the cube \square_z , on some neighborhood of \square_z . It is a small neighborhood, but to have the same simple formulation as in Proposition 2.1 we modify slightly the Green's functions. We use the fact that $\text{supp}h_z \subset \square_z^0$, where \square_z^0 is a cube with a center at z and of the size $2ML^j\eta$, if $z \in \Lambda_j^{(m)+}$. From this it follows that $\alpha_0 h_z = \chi_{\square_z^0}\alpha_0 h_z$, hence we can replace the Green's functions $G(\square_z, \alpha)$ by $G(\square_z, \chi_{\square_z^0}\alpha)$, and all the formulas, bounds and statements remain the same. Now the function $\chi_{\square_z^0}\alpha_0$ depends on $(\psi + \psi', h + h')$ restricted to the small neighborhood of \square_z^0 , which is contained in \square_z . Therefore the localization property with respect to these variables holds in the same form as in Proposition 2.1. Thus we define

$$\phi_k(s) = \phi_0 + \delta\phi_k(s), \quad \alpha_k(s) = \alpha_0 + \delta\alpha_k(s), \quad (2.58)$$

and we obtain analytic extensions of the solutions ϕ_k, α_k . They do not satisfy Eqs. (1.14) [3], in particular the relation $\alpha = \frac{1}{2}(\phi^2 - 1)$ does not hold for them. Otherwise they satisfy all the properties formulated in Proposition 1.1 [3], and also the remaining ones in Proposition 2.1, with the constant $K_4 = (2K_3 + 1)C_3$. We formulate these conclusions in the following proposition.

Proposition 2.2. *Consider the solutions of Eqs. (1.14) [3] with the assumptions of Proposition 1.1 [3]. There exist positive constants c_7, K_4 such that if $\delta \leq c_7$ then the solutions have the analytic extensions $\phi_k(s), \alpha_k(s)$ defined by (2.58). They satisfy the bounds (1.15), (1.16), of Proposition 1.1 [3] with the constant K_4 instead of K_1 , and the differences $\phi_k(s) - \phi_k, \alpha_k(s) - \alpha_k$ satisfy the bounds (1.15) [3], except the first and the sixth ones, on the domain $\Omega_j \cap X_0$ with K_1 replaced by $\exp(-\frac{1}{4}\gamma_0 d(y, (X_1^c)^{\sim 2}))K_4$, or by $\exp(-\frac{1}{4}\gamma_0 r_1 M)K_4$. If $s = 0$ on the family of cubes \mathcal{C} , as in Proposition 2.1, then $\chi_{X_0}\phi_k(s), \chi_{X_0}\alpha_k(s)$ depend on the variables $(\psi + \psi', h + h')$ restricted to the component of Y_0^c containing X_0 .*

The existence and the properties of the analytic extensions $\phi_k(s), \alpha_k(s)$ imply existence and properties of $\psi_k^{(j)}(s)$, where $\psi_k^{(j)}$ is the simplified notation for the function $\psi(\mathbb{B}_j, \mathbb{B}'_k)$ constructed in Sect. 4 [3]. It is expressed by the formula (4.21) [3] in terms of the function $\phi'_k = \phi(\mathbb{B}'_k)$, and the extension of the last defines an extension of $\psi_k^{(j)}$ by that formula. We have also

$$\begin{aligned} \psi_k^{(j)}(s) - \psi_k^{(j)} &= \mathcal{Q}(\mathbb{B}_j)(\delta\phi'_k(s) - \delta\phi'_k) \\ &\quad - \frac{1}{a(\mathbb{B}_j)} \mathcal{Q}^*(\mathbb{B}'_k, \mathbb{B}_j) a(\mathbb{B}'_k) \mathcal{Q}(\mathbb{B}'_k)(\delta\phi'_k(s) - \delta\phi'_k), \end{aligned} \quad (2.59)$$

or more explicitly, for $y \in \Delta_p(y') \cap \Lambda_n, y' \in \Lambda'_p$,

$$\begin{aligned} \psi_k^{(j)}(y; s) - \psi_k^{(j)}(y) &= \mathcal{Q}_n(\delta\phi'_k(s) - \delta\phi'_k)(y) \\ &\quad - \frac{a_p}{a_n} (L^n L^{-p})^2 \mathcal{Q}_p(\delta\phi'_k(s) - \delta\phi'_k)(y'). \end{aligned}$$

This representation implies that the difference $\psi_k^{(j)}(s) - \psi_k^{(j)}$ satisfies the bound (2.55), or (2.57) on $X_0 \cap \mathbb{B}_j$. We will need to know that $\psi_k^{(j)}(s)$ satisfies also the properties (4.29), (4.37) [3]. Let us consider again the proofs of these statements. In those proofs we have used the bounds (1.15), (1.16) [3] only, with one exception. Estimating the expression $|\psi_k^{(j)}(y)| - 1$ we have used the second equation (1.14) [3]. This equation is not satisfied by the analytically extended configurations $\phi'_k(s), \alpha'_k(s)$, so we have to use instead an almost identical argument relying on properties of ϕ'_0, α'_0 . We have

$$\begin{aligned} \|\psi_k^{(j)}(y; s) - 1\| &< \|(\mathcal{Q}_n \phi'_k(s))(y)\| - 1 + (L^n L^{-p})^2 K_4 \delta < \|(\mathcal{Q}_n \phi'_0)(y)\| - 1 \\ &\quad + K_3 C_3 \delta + K_4 \delta < \|(\mathcal{Q}_n \phi'_0)(y)\|^2 - 1 + \frac{3}{2} K_4 \delta \leq \frac{1}{2} \sum_{x_1, x_2 \in B_n(y)} L^{-2nd} \\ &\quad |\phi'_0(x_1) - \phi'_0(x_2)|^2 + (\mathcal{Q}_n \|\phi'_0\|^2 - 1)(y) + \frac{3}{2} K_4 \delta \\ &< \frac{1}{2} d^2 (L^n L^{-p})^2 C_3^2 \delta^2 + \frac{2}{\lambda} (\mathcal{Q}_n |\alpha'_0|)(y) + \frac{3}{2} K_4 \delta \\ &< \frac{1}{2} d^2 C_3^2 \delta^2 + \frac{2}{\lambda} (L^p \eta)^{-2} C_3 \delta + \frac{3}{2} K_4 \delta < 2K_4 \delta, \end{aligned}$$

which is the same bound as in the proof of (4.29) [3], but with K_4 instead of K_1 . Let us explain that in the above inequalities we take a real configuration (ψ', h) , but an arbitrary complex s from the considered domain. The remaining arguments in the proofs of (4.29), (4.37) [3] are the same for $\psi_k^{(j)}(s)$, except that we get the constant K_4 instead of K_1 . This yields the following proposition.

Proposition 2.3. *The function $\psi(\mathbb{B}_j, \mathbb{B}'_k)$ defined on a space $\tilde{\Psi}^c(\mathbb{B}'_k; \delta, \varepsilon)$ with δ, ε satisfying the assumptions of Proposition 1.1 [3] and $\delta \leq c_7$ has the analytic extension $\psi(\mathbb{B}_j, \mathbb{B}'_k; s)$ defined in terms of $\phi(\mathbb{B}'_k; s)$ by the formula (4.21) [3]. It*

satisfies the properties (4.29), (4.37) [3], with the constant K_4 instead of K_1 ,

$$\begin{aligned} |\psi(\mathbb{B}_j, \mathbb{B}'_k; y; s) - \psi(\mathbb{B}_j, \mathbb{B}'_k; y)| &< \exp\left(-\frac{1}{4}\gamma_0 d(y, (X_1^c)^{\sim 2})\right) K_4 \delta \\ &< \exp\left(-\frac{1}{4}\gamma_0 r_1 M\right) K_4 \delta \quad \text{on } X_0 \cap \mathbb{B}_j, \end{aligned} \quad (2.60)$$

and the same localization property as in Proposition 2.2.

In the following sections we have to use the extensions for background functions determined by a generating set \mathbb{B}_{k+1} . They usually involve also the set \mathbb{B}_k obtained by omitting the $k+1$ st domain Ω_{k+1} , and we consider them on lattices connected with the basic lattice in the η -scale, i.e., with T_η . We construct their extensions taking the partition into large cubes determined by \mathbb{B}_k only, that is disregarding the domain Ω_{k+1} . For cubes \square_z intersecting Ω_{k+1} we take the corresponding operators $G(\square_z, \alpha)$ as in (2.3), but with the operator $aL^{-2}Q_{k+1}^*Q_{k+1}$ on the intersection $\square_z \cap \Omega_{k+1}$. These operators satisfy the same basic bounds (2.5), (2.7), (2.10) with properly chosen constants B_0, γ_0 , hence all the remaining bounds, and Propositions 2.1–2.3 hold for such background functions also.

3. Localization Expansions for Functions of the Covariance Operator

Besides the background configurations in the function (1.1) we have the operator $C_A^{(k)\frac{1}{2}}$, which depends non-locally on θ through the configurations $\phi_{k+1} = \phi(\mathbb{B}_{k+1}), \alpha_{k+1} = \alpha(\mathbb{B}_{k+1})$. It is a function of an operator defining a basic quadratic form in $k+1$ st fluctuation integrals. We will discuss these integrals in the next paper, where we will derive the following formula for the quadratic form:

$$\begin{aligned} &\langle \psi', (aL^{-2}Q^*Q + \Delta^{(k)})\psi' \rangle \\ &= aL^{-2}\|Q\psi'\|_L^2 + a_k\|\psi'\|^2 - a_k^2\langle \psi', Q_k G(\alpha_{k+1})Q_k^* \psi' \rangle \\ &\quad + a_k^2 \left\langle \phi_{k+1} \cdot G(\alpha_{k+1})Q_k^* \psi', \left[\frac{1}{\lambda_k} + \phi_{k+1} \cdot G(\alpha_{k+1})\phi_{k+1} \right]^{-1} \phi_{k+1} \right. \\ &\quad \left. \cdot G(\alpha_{k+1})Q_k^* \psi' \right\rangle. \end{aligned} \quad (3.1)$$

Here the operator $G(\alpha)$ is determined by the generating set \mathbb{B}_k and defined by (3.22) [2], and configurations ψ' have supports in $\Lambda = \Omega_{k+1}^{(k)} \subset T_1^{(k)}$. It is rather easy to see that for real configurations ϕ_{k+1}, α_{k+1} the operator of the above quadratic form is symmetric, positive and bounded and we will prove it in the next paper. We are interested in two functions of this operator, namely in

$$C_A^{(k)\frac{1}{2}} = ((aL^{-2}Q^*Q + \Delta^{(k)}) \upharpoonright_A)^{-\frac{1}{2}} \quad \text{and} \quad \log((aL^{-2}Q^*Q + \Delta^{(k)}) \upharpoonright_A).$$

The second will appear as a main contribution to logarithm of the fluctuation integral. We consider these functions for the operator with the background configurations ϕ_{k+1}, α_{k+1} replaced by arbitrary configurations ϕ_0, α_0 in the space (3.21) of [2], with δ_0 sufficiently small. Because of their importance let us write here the conditions defining the space

$$\frac{1}{2} < |\phi_0| < \frac{3}{2}, \quad |\phi_0^2 - 1| < \delta_0, \quad |\partial^n \phi_0|_1 < \delta_0, \quad |\Delta^n \phi_0|_2 < \delta_0, \quad |\alpha_0|_2 < \delta_0. \quad (3.2)$$

At the moment we consider real configurations, but in the future we will extend all constructions onto the complex space. The operator $aL^{-2}Q^*Q + \Delta^{(k)}$ defined by configurations ϕ_0, α_0 and restricted to the set \mathcal{A} is symmetric, positive and bounded. Denote a positive lower bound by γ_0 . We can define a function f of this operator, where $f(z)$ is an analytic function on a domain containing the interval $[\gamma_0, +\infty]$, by the formula

$$f(aL^{-2}Q^*Q + \Delta^{(k)}) = \frac{1}{2\pi i} \int_C dz f(z) (zI - (aL^{-2}Q^*Q + \Delta^{(k)}))^{-1}, \quad (3.3)$$

where C is a contour in the analyticity domain of $f(z)$ and surrounding the spectrum of the operator. For simplicity we omit the set \mathcal{A} in all notations; we assume that all the unit lattice operators are restricted to \mathcal{A} without writing it explicitly. We are interested in two functions: one is $f(z) = z^{-\frac{1}{2}} = \frac{1}{\sqrt{z}}$, more precisely we take a branch of \sqrt{z} having positive values for z real positive, and a cut along the real negative half-axis; another is $f(z) = \log z$, which is a branch of logarithm having real values for z real positive, and a cut along the real negative half-axis. By the formula (3.3) the localization problems for the two operators are reduced to such a problem for the resolvent of the basic operator $aL^{-2}Q^*Q + \Delta^{(k)}$. We solve it by constructing a generalized random walk expansion for the resolvent, similar to the expansions (2.8), (2.13). There are at least two ways of constructing such an expansion. We can do it using the method and the theorem on unit lattice operators in the fifth section of [2a]. This can be applied directly to the resolvent operator, but it requires an additional construction of a random walk expansion for the operator $\Delta^{(k)}$. It turns out that it is possible to obtain an explicit representation of the resolvent, which is quite similar to the formula (3.1), and we can construct a random walk expansion based on it. This is a bit shorter way, and we follow it here. To simplify properties of a representation of the resolvent we transform the integral in the formula (3.3), taking into account the particular functions we are interested in. The resolvent itself is analytic on the whole plane, except an interval $[\gamma_0, \gamma_1]$ in the positive half-axis. For the functions described above we can take the contour C as a union of a circle $|z| = R$ with a large radius R , a circle $|z| = r$ with a small radius r , and two "sides" of the interval $[-R, -r]$. For the function $f(z) = z^{-\frac{1}{2}}$ we can take the limits $R \rightarrow +\infty, r \rightarrow 0+$ in the integral, and we obtain the representation

$$C^{(k)\frac{1}{2}} = \frac{1}{\pi} \int_0^{+\infty} \frac{dx}{\sqrt{x}} (aL^{-2}Q^*Q + \Delta^{(k)} + xI)^{-1}. \quad (3.4)$$

For the function $f(z) = \log z$ it is slightly more complicated, because $|\log z| \rightarrow +\infty$ as $|z| \rightarrow +\infty$, so we cannot take the limit $R \rightarrow +\infty$. In this case we consider

$[-R, -r]$ as a union of two subintervals $[-R, -R_0]$ and $[-R_0, -r]$, and we write the integral (3.3) as a sum of two integrals, one given by a contour which is a union of the large circle and two “sides” of the interval $[-R, -R_0]$, another given by a contour which is a union of the small circle and two “sides” of the interval $[-R_0, -r]$. For the first integral we use the resolvent equation, which yields the additional factor $\frac{1}{z}$, and then we take the limit $R \rightarrow +\infty$. For the second integral we can take the limit $r \rightarrow 0+$. We obtain the following representation:

$$\begin{aligned} \log(aL^{-2}Q^*Q + \Delta^{(k)}) &= (aL^{-2}Q^*Q + \Delta^{(k)}) \int_{R_0}^{+\infty} \frac{dx}{x} (aL^{-2}Q^*Q + \Delta^{(k)} + xI)^{-1} \\ &\quad - \int_0^{R_0} dx (aL^{-2}Q^*Q + \Delta^{(k)} + xI)^{-1} + \log R_0. \end{aligned} \quad (3.5)$$

It is easy to see directly, using the resolvent equation, that the right-hand side above does not depend on R_0 . We will use it in the future. To derive the representation of the resolvent in the formulas (3.4), (3.5) we start with the formula

$$\begin{aligned} \exp \left[\frac{1}{2} \langle f, C_x^{(k)} f \rangle \right] &= \frac{1}{Z_x^{(k)}} \int d\psi \exp \left[-\frac{1}{2} aL^{-2} \|Q\psi\|_L^2 - \frac{1}{2} \langle \psi, \Delta^{(k)} \psi \rangle \right. \\ &\quad \left. - \frac{1}{2} x \|\psi\|^2 + \langle \psi, f \rangle \right], \\ C_x^{(k)} &= (aL^{-2}Q^*Q + \Delta^{(k)} + xI)^{-1}, \end{aligned} \quad (3.6)$$

where f is an arbitrary function on Λ with values in \mathbb{R}^N , and we take $x \geq 0$. Let us recall again that the variable ψ in (3.6) is restricted to Λ . Consider the exponential factor $\exp[-\frac{1}{2} \langle \psi, \Delta^{(k)} \psi \rangle]$ under the integral. It can be represented by the integrals

$$\begin{aligned} \exp \left[-\frac{1}{2} \langle \psi, \Delta^{(k)} \psi \rangle \right] &= \frac{1}{Z_k} \int d\phi \exp \left[-\frac{1}{2} \langle \psi - Q\phi, a(\psi - Q\phi) \rangle \right. \\ &\quad \left. - \frac{1}{2} \langle \phi, (-\Delta^\eta + \nu_k + \alpha_0)\phi \rangle - \frac{1}{2} \lambda_k \|\phi_0 \cdot \phi\|^2 \right] \\ &= \frac{1}{Z_k} \int d\phi \int d\alpha \exp \left[-\frac{1}{2} \langle \psi - Q\phi, a(\psi - Q\phi) \rangle \right. \\ &\quad \left. - \frac{1}{2} \langle \phi, (-\Delta^\eta + \nu_k + \alpha_0)\phi \rangle - i \langle \alpha, \phi_0 \cdot \phi \rangle \right. \\ &\quad \left. - \frac{1}{2\lambda_k} \|\alpha\|^2 \right], \end{aligned} \quad (3.7)$$

where the variables ϕ, α are restricted to the domain Ω_1 , the first domain in the sequence connected with the generating set \mathbb{B}_k , and Q denotes the corresponding averaging operation $Q(\mathbb{B}_k)$, as in (3.22) [3]. Notice that Q in the formulas (3.1), (3.3)–(3.6) denotes Q_1 , i.e., the basic 1-block averaging operation. It is easy to see that integrating first with respect to ϕ , and then with respect to α , yields the corresponding quadratic form on the right-hand side of (3.1), which represents $\Delta^{(k)}$. Now we substitute the last integral above in place of the exponential factor in

the integral (3.6), and we obtain a Gaussian integral in the variables ψ, ϕ, α . We integrate first with respect to ψ , then with respect to ϕ , and finally with respect to α . We do not discuss here these elementary Gaussian integrations. They are reduced to a simple linear algebra; let us write a final result only. We introduce the following two operators:

$$G_{k,x}(\alpha_0) = \left(-\Delta_{\Omega_1}^{D,\eta} + a_k Q_k^* \left[\frac{x}{a_k + x} (I - Q^* Q) + \frac{aL^{-2} + x}{a_k + aL^{-2} + x} Q^* Q \right] Q_k \upharpoonright_{\Omega_{k+1}} \right. \\ \left. + Q^*(\mathbb{B}_k) a(\mathbb{B}_k) Q(\mathbb{B}_k) \upharpoonright_{\Omega_{k+1}^c} + v_k + \alpha_0 \right)^{-1}, \\ A_{k,x} = \frac{1}{a_k + x} (I - Q^* Q) + \frac{1}{a_k + aL^{-2} + x} Q^* Q, \quad (3.8)$$

where, in order to avoid confusion, we have written explicitly which operators are determined by \mathbb{B}_k . The operator $C_x^{(k)}$ has the representation

$$C_x^{(k)} = A_{k,x} + a_k^2 A_{k,x} Q_k G_{k,x}(\alpha_0) Q_k^* A_{k,x} - a_k^2 A_{k,x} Q_k G_{k,x}(\alpha_0) \phi_0 \\ \otimes \left[\frac{1}{\lambda_k} + \phi_0 \cdot G_{k,x}(\alpha_0) \phi_0 \right]^{-1} \phi_0 G_{k,x}(\alpha_0) Q_k^* A_{k,x}. \quad (3.9)$$

It is very similar to the representation (3.1) for $\Delta^{(k)}$, the most important change is that the operator $G(\alpha_0)$ in (3.1) is replaced by $G_{k,x}(\alpha_0)$ given by (3.8). Let us notice that the above representation holds quite generally for the resolvent operator, and we have the identity (3.9) with x replaced by $-z$, for z in a quite large domain in the complex plane, but then we have more complicated properties and bounds of the operator $G_{k,-z}(\alpha_0)$. They become very simple for $-z = x \geq 0$. The second operator in the parenthesis in (3.8) is equal to the linear combination of two projection operators

$$\frac{a_k x}{a_k + x} Q_k^* Q_k + \frac{a_k^2 aL^{-2}}{(a_k + x)(a_k + aL^{-2} + x)} Q_{k+1}^* Q_{k+1} \\ \geq \frac{3}{4} a_{k+1} L^{-2} Q_{k+1}^* Q_{k+1} \quad \text{for } 0 \leq x \leq \frac{1}{8} a_k, \quad \text{and} \\ \geq \frac{1}{9} a_k Q_k^* Q_k \quad \text{for } x > \frac{1}{8} a_k.$$

From the above inequalities we conclude easily that the operator in the parentheses in (3.8), i.e., the operator $G_{k,x}^{-1}(\alpha_0)$, is positive for α_0 real and sufficiently small, or its real part is positive for α_0 complex and sufficiently small. We can get a more precise bound using the following well known simple inequalities:

$$c(-\Delta_{\Delta}^{\eta,N}) + c' Q_j^* Q_j \geq (L^j \eta)^{-2} \min\{4c, c'\} \quad \text{on } L^2(\Delta), \quad \text{where } \Delta \text{ is a } j\text{-block,} \\ j \leq k \text{ and } \Delta_{\Delta}^{\eta,N} \text{ is the Laplace operator on } \Delta \text{ with Neumann boundary conditions;} \\ c(-\Delta_{\Delta'}^{\eta,N}) + c' L^{-2} Q_{k+1}^* Q_{k+1} \geq L^{-2} \min\{4c, c'\} \quad \text{on } L^2(\Delta'), \quad \text{where } \Delta' \text{ is a} \\ k+1\text{-block, i.e., an } L\text{-cube of } T_{\eta}.$$

Of course we have assumed that c, c' are positive numbers. Let us take a cube $\square \subset \Omega_k$ which is a union of $k + 1$ -blocks. By quite elementary considerations, using the above inequalities, we obtain the bound

$$\begin{aligned} & \operatorname{Re} \left[e(q)^{-1} \left(-\Delta_{\square}^{\eta, N} + \left[\frac{a_k x}{a_k + x} \mathcal{Q}_k^* \mathcal{Q}_k + \frac{a_k^2 a}{(a_k + x)(a_k + aL^{-2} + x)} L^{-2} \mathcal{Q}_{k+1}^* \mathcal{Q}_{k+1} \right] \right) \right] \upharpoonright_{\square \cap \Omega_{k+1}} \\ & \quad + a_k \mathcal{Q}_k^* \mathcal{Q}_k \upharpoonright_{\square \cap \Omega_{k+1}^c} + v_k + \alpha_0 \Big] e(q) \Big] \geq \frac{1}{3} (-\Delta_{\square}^{\eta, N}) + \frac{1}{6} a_{k+1} L^{-2} \\ & \text{on } L^2(\square), \quad \text{where } e(q; x') = e^{q \cdot x'}, \quad x' \in T_{\eta}, q \in \mathbb{R}^d \\ & \text{and } |q|_{\infty} = \max_{\mu} |q_{\mu}| \leq \frac{1}{5d} L^{-2}, \quad |\alpha_0| < \frac{1}{12} a_{k+1} L^{-2}, \quad x \geq 0. \end{aligned} \quad (3.10)$$

The numerical factors above are to some extent arbitrary. If we take a cube $\square \subset \Omega_j$, $j < k$, which may intersect Ω_{j+1} , but does not intersect Ω_{j+2} , and which is a union of $j + 1$ -blocks, then the operator in the parentheses in (3.8) restricted to this cube is equal to the corresponding operator in (2.3), and for it we have the above bound with the second term on the right-hand side multiplied by $(L^j \eta)^{-2}$. Let us define an operator $G_{k,x}(\square, \alpha_0)$ by the formula (3.8), but with the operator in the parentheses restricted to a cube \square , the Laplace operator with Neumann boundary conditions on \square . Of course we have $G_{k,x}(\square, \alpha_0) = G(\square, \alpha_0)$ if \square is disjoint with Ω_{k+1} . The operators $G_{k,x}(\square_z, \alpha_0)$ play a crucial role in constructing the generalized random walk expansion (2.8) for the operator $G_{k,x}(\alpha_0)$. Convergence of this expansion depends on the bounds (2.5). They hold for the operators $G_{k,x}(\square, \alpha_0)$ also, but strictly speaking the proofs given in [2a] do not cover this case, although they can be easily extended to yield the bounds. Instead of doing this we notice that now it is enough to prove the convergence in the L^2 -norms, because we want to use the expansions for the operators in the expression on the right-hand side of (3.9), and the averaging operations \mathcal{Q}_k can be interpreted as taking scalar products with characteristic functions of k -blocks, that is with functions having L^2 -norms equal to 1. This is a crucial remark, because the bounds (2.5) in L^2 -norms are immediate consequences of the bound (3.10). We obtain

$$\begin{aligned} & L^j \eta \|\chi_{\Delta(y)} \partial^{\eta} G_{k,x}(\square, \alpha_0) \chi_{\Delta(y')} f\| + \|\chi_{\Delta(y)} G_{k,x}(\square, \alpha_0) \chi_{\Delta(y')} f\| \\ & \leq B_0 e^{-\gamma_0 d(y,y')} (L^j \eta)^2 \|\chi_{\Delta(y')} f\|, \end{aligned} \quad (3.11)$$

where $y \in A_j \cap \square$, $y' \in A_{j'} \cap \square$, f is an arbitrary function on \square , and we can take $B_0 = 15 \frac{1}{a_{k+1}} L^2$, $\gamma_0 = \frac{1}{5d} L^{-2}$. This bound implies the convergence of the corresponding expansion (2.8) in the L^2 -norms. This expansion yields an expansion of the second term in the representation (3.9). It can be written in the form

$$\begin{aligned} & \sum_z a_k^2 A_{k,x} < \chi_{\Delta(\cdot)}, h_{z_0} G_{k,x}(\square_{z_0}, \alpha_0) h_{z_0} K_x(h_{z_1}) G_{k,x}(\square_{z_1}, \alpha_0) h_{z_1} \\ & \quad \dots \cdot K_x(h_{z_n}) G_{k,x}(\square_{z_n}, \alpha_0) h_{z_n} \chi_{\Delta(\cdot)} A_{k,x}, \end{aligned} \quad (3.12)$$

where the sum is over all walks $z = (z_0, z_1, \dots, z_n)$, and the operator $K_x(f)$ is given by the formula,

$$K_x(f) = 2\nabla f \cdot \nabla + \Delta f - \left(\frac{a_k x}{a_k + x} [Q_k^* Q_k, f] + \frac{a_k^2 a L^{-2}}{(a_k + x)(a_k + a L^{-2} + x)} \right. \\ \left. \times [Q_{k+1}^* Q_{k+1}, f] \right) \upharpoonright_{\Omega_{k+1}} - a_k [Q_k^* Q_k, f] \upharpoonright_{\Omega_{k+1}^c}, \quad (3.13)$$

if $\text{supp} f$ intersects Ω_{k+1} . If it does not, then it is given by the second formula in (2.6), but with h_z replaced by f . Instead of the bound (2.10) we use now the bound

$$(L^j \eta)^2 \|\chi_{\Delta(y)} K_x(h_z) G_{k,x}(\square_z, \alpha_0) h_z \chi_{\Delta(y')} f\| \leq 4dM^{-1} B_0 e^{-\gamma_0 d(y,y')} (L^j \eta)^2 \|\chi_{\Delta(y')} f\| \quad (3.14)$$

following from (3.10), (3.11) and (3.13). It implies the convergence of (3.12) in exactly the same way as in Sect. 2, in particular the bounds (5.11) hold without any changes. We would like to construct a similar expansion for the third term on the right-hand side of (3.9). It is slightly more complicated because of the inverse operator in the middle of the expression. We have analyzed this operator in [3], so we summarize necessary results obtained there in the equalities below

$$\left[\frac{1}{\lambda_k} + \phi_0 \cdot G_{k,x}(\alpha_0) \phi_0 \right]^{-1} \\ = \left[\frac{1}{\lambda_k} G_{k,x}^{-1}(\alpha_0) + G_{k,x}^{-1}(\alpha_0) (\phi_0 \cdot G_{k,x}(\alpha_0) \phi_0) \right]^{-1} G_{k,x}^{-1}(\alpha_0) \\ = \left[\frac{1}{\lambda_k} G_{k,x}^{-1}(0) + 1 - K_x(\phi_0) \cdot G_{k,x}(\alpha_0) \phi_0 + \left(\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k} \right) \right]^{-1} \\ \times G_{k,x}^{-1}(\alpha_0) = \left[1 - \lambda_k G_{k,x}(\lambda_k) K_x(\phi_0) \cdot G_{k,x}(\alpha_0) \phi_0 + \lambda_k G_{k,x}(\lambda_k) \right. \\ \left. \left(\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k} \right) \right]^{-1} \lambda_k G_{k,x}(\lambda_k) G_{k,x}^{-1}(\alpha_0).$$

The operator $\lambda_k G_{k,x}(\lambda_k)$ has the Hilbert space norm ≤ 1 , the operator $K_x(\phi_0) \cdot G_{k,x}(\alpha_0) \phi_0$ has a small norm of the order $O(\delta_0)$, and $|\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k}| < 2\delta_0$ by (3.2), hence the operator in the last square brackets above is a small perturbation of the identity operator, and its inverse can be represented by a convergent Neumann series, for δ_0 sufficiently small. Thus the third term in (3.9) is given by the following series:

$$\sum_{n=0}^{\infty} a_k^2 A_{k,x} < \chi_{\Delta(\cdot)}, G_{k,x}(\alpha_0) \phi_0 \otimes \left(\lambda_k G_{k,x}(\lambda_k) K_x(\phi_0) \cdot G_{k,x}(\alpha_0) \phi_0 \right. \\ \left. - \lambda_k G_{k,x}(\lambda_k) \left(\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k} \right) \right)^n (\lambda_k G_{k,x}(\lambda_k) K_x(\phi_0) G_{k,x}(\alpha_0) \\ - \lambda_k G_{k,x}(\lambda_k) \phi_0) \chi_{\Delta(\cdot)} > A_{k,x}. \quad (3.15)$$

It is a convergent series, whose n^{th} term is a product of at least n small factors, so we can obtain easily various uniform bounds for its sum. Now we construct a random walk expansion of the sum by substituting in place of all the operators $G_{k,x}(\alpha_0), G_{k,x}(\lambda_k)$ their random walk expansions. We obtain an expansion of the general form (3.12), but in which the operators $K_x(h_z)G_{k,x}(\square_z, \alpha_0)h_z$ may be replaced by other possible operators. We describe this by introducing generalized random walks, with points z replaced by pairs (z, i) , where the natural number i indicates which operator we take. We combine this expansion with (3.12), and we obtain the following expansion of the sum of the second and third terms on the right-hand side of (3.9), without the operators $a_k A_{k,x}$

$$D_{k,x}(y, y') = \sum_{\omega} \left\langle \chi_{\Delta(y)}, \prod_{m=0}^{|\omega|} G_{\omega_m}(h_{z_m}, \square_{z_m}) h_{z_m} \chi_{\Delta(y')} \right\rangle, \quad (3.16)$$

where the sum is over random walks $\omega = (\omega_0, \omega_1, \dots, \omega_n), |\omega| = n, \omega_m = (z_m, i_m)$, and an operator $G_{\omega}(h_z, \square_z)$ is one of the operators

$$\begin{aligned} & h_z G_{k,x}(\square_z, \alpha_0), \quad h_z G_{k,x}(\square_z, \alpha_0) \phi_0, \quad K_x(h_z) G_{k,x}(\square_z, \alpha_0), \\ & K_x(h_z) G_{k,x}(\square_z, \alpha_0) \phi_0, \quad \lambda_k h_z G_{k,x}(\square_z, \lambda_k), \quad K_x(h_z) G_{k,x}(\square_z, \lambda_k), \\ & K_x(\phi_0) h_z G_{k,x}(\square_z, \alpha_0), \quad K_x(\phi_0) \cdot h_z G_{k,x}(\square_z, \alpha_0) \phi_0, \\ & -\lambda_k h_z G_{k,x}(\square_z, \lambda_k) \left(\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k} \right), \quad -K_x(h_z) G_{k,x}(\square_z, \lambda_k) \left(\phi_0^2 - 1 + \frac{\alpha_0}{\lambda_k} \right), \\ & -K_x(h_z) G_{k,x}(\square_z, \lambda_k) \phi_0, \quad -\lambda_k h_z G_{k,x}(\square_z, \lambda_k) \phi_0. \end{aligned} \quad (3.17)$$

Thus the indices i_m change from 1 to 12, but obviously they do not change independently; there is a set of restrictions following from the structure of the expressions in (3.12), and in the series (3.15). They should be quite clear, so we do not need to formulate them explicitly here, but let us mention a few which will be used next in bounds. The first two operators in (3.17) appear only at the beginning of the product in (3.16), i.e. for $m = 0$, and $i_0 = 1$ or 2 . The last operator may appear only at the end of the product. After the fifth operator there is in some place in the product (3.16) either the seventh, eighth, or tenth operator, except when the fifth operator appears the last time in the product. From the inequalities (3.10), (3.11), (3.14) we obtain the following bounds:

$$\begin{aligned} & \|\chi_{\Delta(y)} G_{\omega}(h_z, \square_z) h_z \chi_{\Delta(y')} f\| \leq \left\{ 1, \frac{3}{2}, 4dM^{-1}, 6dM^{-1}, 1, 4dM^{-1}, \right. \\ & \left. (2d+1)\delta_0, \frac{3}{2}(2d+1)\delta_0, 2\delta_0, 8dM^{-1}\delta_0, 6dM^{-1}, \frac{3}{2} \right\} B_0 e^{-\gamma_0 d(y,y')} \\ & (L^j \eta)^2 \|\chi_{\Delta(y')} f\|, \end{aligned} \quad (3.18)$$

where the order of constants corresponds to the order of operators in (3.17), and we have to include the factors $(L^j \eta)^2$ in the norms for the operators involving K_x . The bound of the fifth operator is not small, but we can use the fact that it is followed by one of the operators described above, whose norms are of the order $0(\delta_0)$, and assign the factor $\delta_0^{\frac{1}{2}}$ to it. Now we can estimate the expansion (3.16)

in the same way as the expansion (2.8), and we get the inequalities (2.11) with minor changes only, more precisely the numerical factor on the right-hand side of the third inequality is replaced by

$$9B_0^2c_1^2 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d \left(48dB_0c_1 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d \max \left\{M^{-1}, \frac{2}{3}\delta_0^{\frac{1}{2}}\right\}\right)^{n-1}. \quad (3.19)$$

The remaining considerations are the same as in Sect. 2, we assume that the expression in the parentheses above is ≤ 1 , and this yields the convergence of the expansion (3.16), and various bounds for the sum and certain partial sums. In particular the whole sum, hence the operator $D_{k,x}$ satisfies the bound

$$\begin{aligned} |(D_{k,x}f)(y)| &< 9B_0^2c_1^2 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d \sum_{\omega: y \in \square_{x_0}} \frac{1}{8^n} \frac{1}{(2L+1)^{dn}} e^{-\frac{1}{2}\gamma_0 d(y;\omega)} |f| \\ &\leq B_4 |f|, \quad \text{where } B_4 = 9 \cdot 2^{d+1} B_0^2 c_1^2 \left(d, \frac{1}{4}\gamma_0\right) (2L+1)^d. \end{aligned} \quad (3.20)$$

In the above bounds we have used the fact that any ω_m in a generalized random walk ω can be followed by an ω_{m+1} with at most 4 different indices i_{m+1} , and this is the reason for the number 48 in (3.19), and $\frac{1}{8^n}$ in (3.20).

Using the bound (3.20), which is uniform in $x \geq 0$, and the equality

$$C_{k,x} = A_{k,x} + a_k^2 A_{k,x} D_{k,x} A_{k,x} \quad (3.21)$$

following from the definition of $D_{k,x}$, we can simplify the formula (3.5). At first notice that the first integral on the right-hand side of (3.5) converges to 0 as $R_0 \rightarrow +\infty$, because the under-integral expression is of the order $O(\frac{1}{x^2})$. The operator $A_{k,x}$ is a very simple operator given by the formula (3.8), and it is of the order $O(\frac{1}{x})$ for large x , so the second integral in (3.5) of the second term on the right-hand side of (3.21) is convergent as $R_0 \rightarrow +\infty$. This integral of the first term on the right-hand side can be calculated explicitly, and the result combined with the term $\log R_0$ in (3.5) is convergent to $\log a_k(I - Q^*Q) + \log(a_k + aL^{-2})Q^*Q$ as $R_0 \rightarrow +\infty$. We can do a similar calculation for (3.4), and we obtain the following final formulas for the operators in (3.4), (3.5):

$$\begin{aligned} C^{(k)\frac{1}{2}} &= \frac{1}{\sqrt{a_k}}(I - Q^*Q) + \frac{1}{\sqrt{a_k + aL^{-2}}}Q^*Q + \frac{a_k^2}{\pi} \int_0^{+\infty} \frac{dx}{\sqrt{x}} A_{k,x} D_{k,x} A_{k,x}, \\ \log(aL^{-2}Q^*Q + \Delta^{(k)}) &= \log a_k(I - Q^*Q) + \log(a_k + aL^{-2})Q^*Q \\ &\quad - a_k^2 \int_0^{+\infty} dx A_{k,x} D_{k,x} A_{k,x}. \end{aligned} \quad (3.22)$$

The expansion (3.16) yields random walk expansions of these operators. We can use them now either to introduce the localization parameters s into the operators, or to construct their localization expansions directly. The introduction of the localization parameters has been discussed thoroughly in Sect. 2. We use the expansion (3.16) and construct an analytically extended operator $D_{k,x}(s)$ by the formula (2.16). Using

the bounds (3.20) and the basic inequality (2.21) we obtain

$$|D_{k,x}(s)f| \leq B_4 \exp((7L)^d \kappa_1) |f|. \quad (3.23)$$

The extension of $D_{k,x}$ defines an extension $C^{(k)\frac{1}{2}}(s)$ through the formula (3.22). It satisfies the bound

$$|C^{(k)\frac{1}{2}}(s)f| \leq \left(\frac{2}{\sqrt{a_k}} + \frac{a_k^2}{\pi} \int_0^{+\infty} \frac{dx}{\sqrt{x}} \left(\frac{1+L^{-2}}{a_k+x} \right)^2 B_4 \exp((7L)^d \kappa_1) \right) |f| = B_5 |f| \quad (3.24)$$

following from (3.22), (3.23). Notice that the bounds (3.23), (3.24) can be improved by omitting the factor $\exp((7L)^d \kappa_1)$, as in (2.23), if we restrict the supremum norms on the left-hand sides to the domain X_0 . Let us recall that the extensions in s are defined for a given pair of localization domains X_0, X_1 satisfying (2.14). The above results form an important part of the whole localization problem, and we summarize them in the proposition below.

Proposition 3.1. *The operator $C^{(k)\frac{1}{2}}$ given by the formulas (3.4), (3.9), (3.6), (3.22), (3.16), has the localization extension constructed above, which is an analytic function of (ϕ_0, α_0) on the domain (3.2) with δ_0 small enough, and of the localization parameters s in $\{s : |s| < e^{\kappa_1}\}$. This localization extension satisfies the bound (3.24), and the bound corresponding to (2.60) with $K_4\delta$ replaced by $B_5|f|$. It satisfies also the crucial localization property, which can be formulated in the following form: if $s = 0$ on a family \mathcal{C} of cubes, and if Y_1 is the connected component of $(\cup \mathcal{C})^c$ containing X_1 , then $\chi_{Y_1} C^{(k)\frac{1}{2}}(\phi_0, \alpha_0, s)f$ depends on ϕ_0, α_0, s, f restricted to the domain Y_1 .*

Obviously similar conclusions can be formulated for the second operator in (3.22). Actually we will not need this operator, only its trace which appears as a leading term in contributions to the effective action coming from the fluctuation integral. We will discuss these conclusions later, together with the remaining localization expansion.

4. A Construction of the Localization Expansion (1.4)

In this section we construct the expansion (1.4), and prove complete analyticity statements, using the results of the two preceding sections. Let us recall the following identities for the background configurations occurring in (1.1):

$$\begin{aligned} \psi(\mathbb{B}_j, \mathbb{B}_k; \psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta)) &= \psi(\mathbb{B}_j, \mathbb{B}_{k+1}; \theta), \text{ or simply} \\ \psi_k^{(j)}(\psi^{(k)}(\theta)) &= \psi_{k+1}^{(j)}(\theta), \quad \text{where } \psi^{(k)}(\theta) = \psi_{k+1}^{(k)}(\theta) = \psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta), \\ \psi_k^{(j)}(\psi^{(k)}(\theta) + \beta_k^{-\frac{1}{2}} C^{(k)\frac{1}{2}} \psi) &= \psi_{k+1}^{(j)}(\theta) + \delta \psi_k^{(j)}(\beta_k^{-\frac{1}{2}} C^{(k)\frac{1}{2}} \psi), \end{aligned} \quad (4.1)$$

where the last function on the right-hand side above is determined by the background configurations $\phi_{k+1}(\theta), \alpha_{k+1}(\theta)$, and for simplicity of notations we have omitted the dependence on Λ . These identities are special cases of the identities (4.21), (3.49) [3]. It seems that the last identity and the results of the two previous sections

provide a natural way to construct the expansion; we could introduce properly the localization parameters s into the functions on the right-hand side, and to interpolate in s between 0 and 1 in a well known way. This is formally correct and yields a localized expansion, but unfortunately the functions $\psi_{k+1}^{(j)}(s)$, $\delta\psi_k^{(j)}(s)$ do not have all necessary properties needed in our method, for example, their values do not generally belong to the space $\Psi_j^c(X; C_0 L^j \eta, \varepsilon_k)$, which we need for the bounds obtained in [1]. This is so because the introduction of the localization parameters destroys their structure, their algebraic properties and the equations satisfied by them. To overcome these difficulties we have to use a more elaborate representation than the one above. This will be done in several steps.

Let us start with geometric definitions. We have the localization domain $X \in \mathcal{D}_j$, and we would like to assign to it a “smallest” in some sense localization domain $X_1 \in \mathcal{D}_{k+1}$ containing it. Let us recall that in the considered scale, in which the basic lattice T is in the η -scale, i.e., it is T_η , a domain from \mathcal{D}_j is a connected union of $LML^j\eta$ -cubes from the cover π'_j , which are unions of L^d cubes of the size $ML^j\eta$ from the partition π_j . We define the domain $X_1 \in \mathcal{D}_{k+1}$ in the following way:

$$X_1 = \text{a union of all } L^2M\text{-cubes from the cover } \pi'_{k+1}, \text{ which contain a} \\ LML^j\eta \text{ - cube from the cover } \pi'_j \text{ contained in } X. \quad (4.2)$$

We define also a domain X' as a union of all large cubes from the partition π_{k+1} which have non-empty intersections with X . For these domains we have the inclusions

$$X_1 \subset X'^{\sim L-1}, \quad X_1 \supset X'^{\sim L-2}, \quad (4.3)$$

for example for $L = 3$ we have $X_1 \subset X'^{\sim 2}, X_1 \supset X'^{\sim 1}$. We would like to keep the possibility of having this small value of L , and we would like also to define a domain X_0 containing X and satisfying the condition (2.14). This is possible if we introduce two scales of large cubes, one smaller for the constructions and results of this paper, and another larger for the main renormalization group procedure. We take the smaller scale in such a way that all conditions related to constructions and properties of all the minimizing functions in this paper are satisfied. There are several conditions involving the size M of the large cubes, for example the conditions $8dB_0c_1(d, \frac{1}{8}\gamma_0) \cdot (2L + 1)^d M^{-\frac{1}{2}} \leq 1, M \geq 4(7L)^d \kappa_1 \frac{1}{\gamma_0}$ used in the proofs of Sect. 2 and we assume that M_1 is a number of the form L^{m_1} satisfying all these conditions. Thus all the statements proven until now hold for the number M_1 . The size M of large cubes used in the renormalization group procedure, i.e. in the paper [1], is greater than M_1 , and we will obtain some conditions on it. From now on we have to be particularly careful with the meaning of the operation “ \sim .” In this paper it generally means that the size M_1 of large cubes is used, except in the cases where we have domains arising in the renormalization group procedure and the size M is used, like in (4.2), (4.3).

Let us consider again the localization domain X . We would like to construct a minimal in some sense sequence $\{\Omega'_1, \dots, \Omega'_k\}$ of localization domains defined in terms of M_1 , satisfying (1.1) [3] and such that $X \subset \Omega'_k$. We define Ω'_k in the same way as X_1 was defined, with M_1 replacing M , we take

$$\Omega'_k = \text{a union of } LM_1\text{-cubes from the cover } \pi'_k \text{ which contain} \\ \text{a } LM_1L^j\eta\text{-cube from the cover } \pi'_j \text{ contained in } X. \quad (4.4)$$

Notice that the conditions (1.1) [3] are satisfied if $\Omega_{n+1}^{\sim R} \subset \Omega_n$, where Ω_{n+1} is considered as a domain in \mathcal{D}_n , and the operation " \sim " is defined in terms of cubes of the partition π_n . Until now we have only very mild restrictions on R , the strongest one we may impose is connected with the conditions (2.14), and may be formulated as $\Omega_{n+1}^{\sim r_0 L} \subset \Omega_n$, where we may even take $r_0 = 5$. Thus a next domain in the sequence must satisfy this condition only, so we consider Ω'_k as a localization domain in \mathcal{D}_{k-1} and we define

$$\Omega'_{k-1} = \Omega'_k{}^{\sim 5L}, \text{ where we add layers of } M_1 L^{k-1} \eta = M_1 L^{-1} \text{-cubes.} \quad (4.5)$$

Generally, having defined Ω'_{n+1} we consider it as a localization domain in \mathcal{D}_n and we define

$$\Omega'_n = \Omega'_{n+1}{}^{\sim 5L}, \text{ where we add layers of } M_1 L^j \eta \text{-cubes.} \quad (4.6)$$

This way we construct the whole sequence $\{\Omega'_1, \Omega'_2, \dots, \Omega'_k\}$ and the corresponding generating set. They are determined by the domain X , so we denote them by $\{\Omega_1(X), \Omega_2(X), \dots, \Omega_k(X)\}$ and $\mathbb{B}_k(X)$. Notice that they depend on k also, the domains $\Omega_j(X)$ vary with k , according to the above definition. We define also a domain X'' in the same way as the domain X' before, namely as a union of M_1 -cubes from π_k which have non-empty intersections with X . Obviously we have

$$X \subset X'' \subset X''^{\sim L-2} \subset \Omega_k(X) \subset X''^{\sim L-1} \subset X'^{\sim L-1},$$

where the operation " \sim " is determined by the M_1 -cubes. We define

$$\begin{aligned} X_0 &= X'^{\sim 4L}, \text{ hence } \Omega_1(X) \subset X_0, \text{ because} \\ (L-1)M_1 + 5LM_1 L^{-1} + \dots + 5LM_1 L^{-(k-1)} &< (L-1)M_1 \\ + \frac{15}{2}M_1 &< 4LM_1. \end{aligned} \quad (4.7)$$

The domain X_0 is a localization domain from the corresponding family \mathcal{D}_k defined in terms of M_1 , and we consider X_1 also as a domain from this family. We require that the two domains X_0, X_1 satisfy the condition (2.14), i.e., $X_0^{\sim r_0} \subset X_1$, the operation " \sim " is determined by the partition into M_1 -cubes. By the definition of X_0 and (4.3) this condition is satisfied if $4LM_1 + r_0 M_1 \leq LM$. For $r_0 = 5$ we may estimate $4LM_1 + r_0 M_1 < 6LM_1$, and the condition is satisfied if $6LM_1 \leq LM$, or $6M_1 \leq M$. Assuming this, and assuming also that X_1 is contained in Ω_k , we obtain

$$X \subset \Omega_k(X) \subset \dots \subset \Omega_1(X) \subset X_0, X_0^{\sim r_0} \subset X_1 \subset \Omega_k, \quad X_0, X_1 \in \mathcal{D}_k. \quad (4.8)$$

The idea how to construct a localization expansion of the function (1.1) which we follow here is to replace at first the function $\psi_k^{(j)}$ by a local function with a generating set contained in some neighborhood of X , and then to introduce the parameters s into the function which is composed with this local one. We use the generating set $\mathbb{B}_k(X)$ and the corresponding local function $\phi(\mathbb{B}_k(X))$. We have

$$\begin{aligned} \phi(\mathbb{B}_k; \psi) &= \phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi)), \\ \phi(\mathbb{B}_{k+1}; \theta) &= \phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta)). \end{aligned} \quad (4.9)$$

We take the generating set corresponding to the sequence

$$\{\Omega_1, \dots, \Omega_j, \Omega_{j+1}(X), \dots, \Omega_k(X)\}$$

and we denote it by $\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X)$. We have

$$\psi(\mathbb{B}_j, \mathbb{B}_k; \psi) = \psi(\mathbb{B}_j, \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X), \mathbb{B}_k; \psi)). \quad (4.10)$$

Consider the function $\psi(\mathbb{B}_j, \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')$. By (4.21) [3] we have

$$\begin{aligned} \psi(\mathbb{B}_j, \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') &= Q_j \phi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') + \frac{a_p}{a_j} (L^j L^{-p})^2 \\ &\times Q_{p-j}^*(\psi_p'' - Q_p \phi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')) \text{ on } B^{p-j}(A_p(X)), p > j, \\ \psi(\mathbb{B}_j, \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') &= \psi'' \text{ on } \bigcup_{n=0}^j A_n \cup (\Omega_j \setminus \Omega_{j+1}(X))^{(j)}. \end{aligned} \quad (4.11)$$

The function $\phi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')$ can be written in the form

$$\phi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') = \phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')), \quad (4.12)$$

and we have

$$\psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') = \psi'' \text{ on } \bigcup_{p=j}^k A_p(X). \quad (4.13)$$

From the above formulas we get

$$\begin{aligned} &\psi(\mathbb{B}_j, \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') \\ &= Q_j \phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')) + \frac{a_p}{a_j} (L^j L^{-p})^2 Q_{p-j}^*(\psi(\mathbb{B}_k(X), \\ &\quad \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'') - Q_p \phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi''))) \\ &= \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi'')) \text{ on } B^{p-j}(A_p(X)), \\ &\quad p \geq j, \end{aligned} \quad (4.14)$$

again using the formula (4.21) [3]. Combining (4.10), (4.14) we obtain

$$\begin{aligned} &\psi(\mathbb{B}_j, \mathbb{B}_k; \psi) \\ &= \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X); \psi(\mathbb{B}_j \cup \mathbb{B}_k^{(j)}(X), \mathbb{B}_k; \psi))) \\ &= \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi)) \text{ on } \Omega_j(X)^{(j)}, \text{ hence on } X \cap T^{(j)}. \end{aligned} \quad (4.15)$$

This yields finally the identity

$$\begin{aligned}
 & \psi(\mathbb{B}_j, \mathbb{B}_k; \psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta) + \psi') \\
 &= \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi(\mathbb{B}_k, \mathbb{B}_{k+1}; \theta) + \psi')) \\
 &= \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi')) \\
 & \quad \text{on } \Omega_j(X)^{(j)}, \text{ hence on } X \cap T^{(j)}. \tag{4.16}
 \end{aligned}$$

In the three identities above we have taken $\mathbb{B}_j(X)$ defined by (4.4)–(4.6) with j instead of k . We obtain a sequence of different, smaller domains, in particular $\Omega'_j = \Omega_j(X) = X$ by (4.4). The external function $\psi(\mathbb{B}_j(X), \mathbb{B}_k(X))$ on the right-hand side of (4.16), restricted to $X \cap T^{(j)}$, is localized in a small neighborhood of $\Omega_1(X)$ (we add one layer of points to this domain), hence in the domain X_0 . Furthermore, regularity properties of this function on $X \cap T^{(j)}$ are basically the same as of the function $\psi(\mathbb{B}_j, \mathbb{B}_k)$. This will be analyzed in detail in the future.

Because of these properties we use the representation (4.16) as a basis of our expansion method. We need to introduce the localization parameters only into the internal functions on the right-hand side of (4.16). We discuss it now. We take the localization parameters determined by the generating set \mathbb{B}_k , or by the sequence of domains $\{\Omega_1, \Omega_2, \dots, \Omega_k\}$, which are considered as localization domains with large cubes determined by M_1 . We do not take into account the domain Ω_{k+1} . The parameters s are introduced into the function $\psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta)$ in the way discussed before. This function is represented by the formula (4.21) [3] in terms of the function $\phi(\mathbb{B}_{k+1}; \theta)$, and we extend the last one to the function $\phi(\mathbb{B}_{k+1}; \theta, s)$ determined by the geometric setting constructed for \mathbb{B}_k , although on cubes intersecting Ω_{k+1} we take operators corresponding to $k + 1^{\text{st}}$ renormalization step. Thus we obtain the extension $\psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s)$ whose properties are summarized partially in Proposition 2.3. Similarly the function $\delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi')$ is defined by the formula

$$\begin{aligned}
 \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi') &= Q(\mathbb{B}_k(X))\delta\phi(\mathbb{B}_k; \psi') + \frac{1}{\alpha(\mathbb{B}_k(X))} Q^*(\mathbb{B}_k, \mathbb{B}_k(X))a(\mathbb{B}_k) \\
 & \quad \times (\psi' - Q(\mathbb{B}_k)\delta\phi(\mathbb{B}_k; \psi')), \tag{4.17}
 \end{aligned}$$

where $\delta\phi(\mathbb{B}_k; \psi')$ is the solution of the system of Eqs. (3.20) [3] with $\delta\psi = \psi'$, $\phi_0 = \phi(\mathbb{B}_{k+1}; \theta)$, $\alpha_0 = \alpha(\mathbb{B}_{k+1}; \theta)$, $f_1 = 0$, $\alpha_1 = 0$. This solution is an analytic function of ψ' , ϕ_0, α_0 on any domain (3.21) [3] satisfying the assumptions of Proposition 3.1 [3] and it satisfies the bounds (3.47) [3] of this proposition. We denote the solution by $\delta\phi(\mathbb{B}_k; \psi', \phi_0, \alpha_0)$. We introduce the parameters s into this function as in Proposition 2.1, and we define

$$\delta\phi(\mathbb{B}_k; \psi', s) = \delta\phi(\mathbb{B}_k; \psi', \phi(\mathbb{B}_{k+1}; \theta, s), \alpha(\mathbb{B}_{k+1}; \theta, s), s). \tag{4.18}$$

Let us make a few remarks on domains and analyticity properties of the above function. By Proposition 2.1 the function $\delta\phi(\mathbb{B}_k; \psi', \phi_0, \alpha_0, s)$ is analytic on domains (3.21) [3] with δ_1, δ_0 satisfying the assumptions of this proposition, in particular $B_1\delta_0 \leq \frac{1}{8}$. The constant δ_1 occurs in the restriction $|\psi'| < \delta_1$ only, and we may assume that it is sufficiently small. We have to check when $\phi_0 = \phi(\mathbb{B}_{k+1}; \theta, s)$, $\alpha_0 = \alpha(\mathbb{B}_{k+1}; \theta, s)$ belong to a domain (3.21) [3], or (3.2), with δ_0 satisfying the above condition. We consider these functions on a space $\tilde{Y}^c(\mathbb{B}_{k+1}; \delta, \varepsilon)$ with δ, ε satisfying

the assumptions of Proposition 2.2. Then they satisfy the bounds (1.15) [3] with K_4 instead of K_1 , and these yield immediately the last three bounds in (3.2) with $\delta_0 = K_4\delta$. To bound $\phi_0^2 - 1$ we write $\phi_0^2 - 1 = (\phi_0 - \theta)^2 + 2(\phi_0 - \theta) \cdot \theta + (\theta^2 - 1)$, and from the first bound in (1.15) [3] and the definition of the space $\tilde{\Psi}^c(\mathbb{B}_{k+1}; \delta, \varepsilon)$ we get

$$|\phi_0^2 - 1| \leq |\phi_0 - \theta|^2 + 2|\phi_0 - \theta||\theta| + |\theta^2 - 1| < K_4^2\delta^2 + 2K_4\delta\frac{3}{2} + 2\delta < 4K_4\delta$$

for δ sufficiently small. Similarly we get

$$||\phi_0| - 1| \leq |\phi_0 - \theta| + ||\theta| - 1| < K_4\delta + \varepsilon + 2\delta < \frac{1}{2}$$

for δ, ε sufficiently small. Thus the two configurations belong to (3.2) with $\delta_0 = 4K_4\delta$, and we assume that δ is so small that $4B_1K_4\delta \leq \frac{1}{8}$. With these assumptions the function (4.18) is an analytic function of $(\theta, h), \psi', s$ defined on corresponding domains

$$\tilde{\Psi}^c(\mathbb{B}_{k+1}; \delta, \varepsilon) \times \{\psi': |\psi'| < \delta_1\} \times \{s: |s| < e^{\kappa_1}\}. \quad (4.19)$$

It satisfies the bounds (2.54) of Proposition 2.1. Substituting it into the expression on the right-hand side of (4.17) we obtain an analytic extension $\delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s)$ of the function (4.17) defined on the same domains (4.19). From the representation (4.17) and the first bound in (2.54) we obtain the bound

$$|\delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s)| < (2K_3 + 1)\delta_1. \quad (4.20)$$

The function $\psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s)$ has properties described in Proposition 2.3, in particular it is defined on the domains (4.19) and it has values in the space $\tilde{\Psi}^c(\mathbb{B}_k(X); 2(d+2)K_4\delta, \varepsilon)$. From all the above statements it follows that we have constructed an analytic extension

$$\psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s) \quad (4.21)$$

of the internal function on the right-hand side of (4.16), defined on the domains (4.19) and having values in the space $\tilde{\Psi}^c(\mathbb{B}_k(X); 2(d+2)K_4\delta + 4(2K_3 + 1)\delta_1, \varepsilon)$. The last condition follows from (4.20) by considering the second term in (4.21) as a part of the complex configuration in the definition (1.12) [3] of this space. Assuming again that $\delta, \varepsilon, \delta_1$ are sufficiently small, so that $2(d+2)K_4\delta + 4(2K_3 + 1)\delta_1, \varepsilon$ satisfy the assumptions of Proposition 4.1 [3], we substitute the function (4.21) into the external function on the right-hand side of (4.16), and we obtain an analytic extension of the function (4.16) in the localization parameters s . By Proposition 4.1 [3] it belongs to the space $\tilde{\Psi}^c(\mathbb{B}_j(X); \delta_2, \varepsilon)$, where

$$\delta_2 = 2(d+2)K_1 \cdot (2(d+2)K_4\delta + 4(2K_3 + 1)\delta_1).$$

Let us recall now that in the function (1.1) we have $\psi' = \beta_k^{-\frac{1}{2}} C^{(k)\frac{1}{2}} \psi$, and we have to localize this expression also. We have constructed the analytic extension $C^{(k)\frac{1}{2}}(\phi_0, \alpha_0; s)$ in Proposition 3.1, and similarly to (4.18) we define

$$C^{(k)\frac{1}{2}}(s) = C^{(k)\frac{1}{2}}(\phi_{k+1}(s), \alpha_{k+1}(s); s). \quad (4.22)$$

It is an analytic function on the part of the domain (4.19) without the second factor, and it satisfies the bound (3.24). The variable ψ , which is a "fluctuation" variable connected with the $k + 1^{\text{st}}$ renormalization transformation, and which will be discussed in the next paper, satisfies the bound $|\psi| < p_1(\beta_k) = A_1(\log \beta_k)^{p_1}$, where p_1 is an even integer smaller than p_0 , hence

$$|\psi'| < \beta_k^{-\frac{1}{2}} B_5 p_1(\beta_k) \leq B_5 C_{\alpha_1} \beta_k^{-\frac{1}{2} + \alpha_1} = \delta_1. \quad (4.23)$$

Obviously δ_1 can be arbitrarily small if β_k is large enough. In the remaining formulas and estimates we will use for simplicity the variable ψ' , remembering that

$$\psi' = \beta_k^{-\frac{1}{2}} C^{(k)\frac{1}{2}}(s)\psi. \quad (4.24)$$

We will come back to a discussion of (4.23) in another paper, where we will construct and discuss all remaining localization expansions.

Thus the extended in s function inside (1.1) is analytic on quite large domains determined by some absolute constants. Unfortunately these analyticity properties do not fit well to the inductive hypotheses, as was explained already in [1], and as should be clear from the above discussion. In particular the domains used do not behave well under the compositions of various minimizing functions. Because of this we have to use spaces of the type $\mathcal{E}_j^c(\sigma, \varepsilon)$ introduced in [1], but localized properly. Nevertheless we need also almost all results of the above analysis done for the domains (4.19).

We formulate now definitions of such generalized spaces. They are simple modifications of the definitions (2.15), (2.16) and (2.20), (2.21) in [1], but because of their importance we write here their complete formulations. Take a generating set \mathbb{B}_j , and a function $\sigma(\mathbb{B}_j)$ defined on Ω_0 , having positive values in the interval $]0, 1]$, and satisfying the condition that it is constant on corresponding blocks determined by points of \mathbb{B}_j , i.e., constant on n -blocks $\Delta_n(y)$ for $y \in \Lambda_n, n \leq j$. We take Ω_0 as some "small" neighbourhood of Ω_1 , for example $\Omega_0 = \Omega_1^\sim$, or even a smaller neighbourhood obtained by adding a layer of 1-blocks to Ω_1 . We introduce the following spaces of configurations ϕ defined on Ω_0 and having values in \mathbb{R}^N , and vectors $h \in \mathbb{R}^N$

$$\begin{aligned} \Phi(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon; \lambda, \nu) &= \{(\phi, h) : |\partial^\xi \phi(x)| < \sigma(\mathbb{B}_j; x)(L^n \xi)^{-1} \varepsilon, \\ |\Delta^\xi \phi(x)|, |\alpha(x)| &< \sigma^2(\mathbb{B}_j; x)(L^n \xi)^{-2} \varepsilon, |\phi(x) - h| < \nu^{-\frac{1}{2}} \sigma(\mathbb{B}_j; x)(L^n \xi)^{-1} \varepsilon \\ &\text{on } \Delta_n(y), y \in \Lambda_n, \text{ or simply on } B_n(\Lambda_n) = \Omega_n \setminus \Omega_{n+1}, n \leq j, \\ |h^2 - 1| &< \nu^{-1} \inf_{n, x \in \Omega_n \setminus \Omega_{n+1}} \sigma^2(\mathbb{B}_j; x)(L^n \xi)^{-2} \varepsilon^2\}, \\ \alpha &= \frac{\lambda}{2}(\phi^2 - 1), \end{aligned} \quad (4.25)$$

where $\varepsilon, \lambda, \nu$ are positive numbers. We extend these spaces to complex spaces of configurations $(\phi + \phi', h + h')$, where (ϕ, h) is as above, ϕ' is defined on Ω_0 and has values in \mathbb{C}^N , h' is defined on \mathbb{B}_j and has values in \mathbb{C}^N . We take

$$\begin{aligned} \Phi^c(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon; \lambda, \nu) &= \{(\phi + \phi', h + h') : (\phi, h) \in \Phi(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon; \lambda, \nu), \\ |\phi'| < \varepsilon, |\partial^\xi \phi'(x)| &< \sigma(\mathbb{B}_j; x)(L^n \xi)^{-1} \varepsilon, |\Delta^\xi \phi'(x)|, |\delta \alpha(x)| \end{aligned}$$

$$\begin{aligned}
& < \sigma^2(\mathbb{B}_j; x)(L^n \xi)^{-2} \varepsilon, |h'(y)| < v^{-1} \sigma^2(\mathbb{B}_j; y)(L^n \xi)^{-2} \varepsilon, \\
& |\phi(x) \cdot h'(y)| < v^{-1} \sigma^2(\mathbb{B}_j; x)(L^n \xi)^{-2} \varepsilon^2 \text{ on } \Delta_n(y), y \in \Lambda_n, n \leq j \}, \\
& \delta \alpha = \frac{\lambda}{2} ((\phi + \phi')^2 - \phi^2) = \frac{\lambda}{2} (2\phi \cdot \phi' + \phi'^2). \tag{4.26}
\end{aligned}$$

The above spaces are quite natural in many constructions, in particular Propositions 1.1 [3], 3.1 [3], 2.1, 2.2 can be formulated in terms of these spaces, with $\sigma(\mathbb{B}_j) = 1$. Of fundamental importance are spaces of configurations defined on \mathbb{B}_j . For such configurations ψ with values in \mathbb{R}^N , and vectors $h \in \mathbb{R}^N$, we introduce spaces

$$\begin{aligned}
& \Xi_j(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon) = \{(\psi, h) : (\phi_j(\mathbb{B}_j; \psi, h), h) \in \Phi(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon; \lambda_j, v_j), \\
& |\psi(y) - (Q(\mathbb{B}_j)\phi_j(\mathbb{B}_j; \psi, h))(y)| < \sigma^2(\mathbb{B}_j; y)\varepsilon \text{ for } y \in \Lambda_n, n \leq j \}, \tag{4.27}
\end{aligned}$$

where $\phi_j(\mathbb{B}_j)$ is the minimizing function determined by \mathbb{B}_j and by coefficients a_j, λ_j, v_j . Finally, for complex configurations $(\psi + \psi', h + h'), (\psi, h)$ as above, ψ', h' defined on \mathbb{B}_j and having values in \mathbb{C}^N , we define

$$\begin{aligned}
& \Xi_j^c(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon) = \{(\psi + \psi', h + h') : (\psi, h) \in \Xi_j(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon), \\
& (\phi_j(\mathbb{B}_j; \psi + \psi', h + h'), h + h') = (\phi_j(\mathbb{B}_j; \psi, h) + \delta\phi_j(\mathbb{B}_j; \psi', h'), h + h') \\
& \in \Phi^c(\mathbb{B}_j; \sigma(\mathbb{B}_j), \varepsilon; \lambda_j, v_j), |\psi'(y) - (Q(\mathbb{B}_j)\delta\phi_j(\mathbb{B}_j; \psi', h'))(y)| \\
& < \sigma^2(\mathbb{B}_j; y)\varepsilon \text{ for } y \in \Lambda_n, n \leq j \}. \tag{4.28}
\end{aligned}$$

One of the most important facts relating these spaces to functions $\psi(\mathbb{B}_j, \mathbb{B}'_k)$ is formulated in the following lemma, a generalization of Lemma 2.1 [1].

Lemma 4.1. *For ε positive and small enough, and $\mathbb{B}_j \prec \mathbb{B}'_k$ (see the definition (4.1) [3]) we have*

$$\psi(\mathbb{B}_j, \mathbb{B}'_k): \Xi_k^c(\mathbb{B}'_k; \sigma(\mathbb{B}'_k), \varepsilon) \rightarrow \Xi_j^c(\mathbb{B}_j; \sigma(\mathbb{B}_j, \mathbb{B}'_k)\sigma(\mathbb{B}'_k), \varepsilon), \tag{4.29}$$

where $\sigma(\mathbb{B}_j, \mathbb{B}'_k)$ is a function defined on Ω_0 by the formula

$$\sigma(\mathbb{B}_j, \mathbb{B}'_k; x) = L^n L^{-p} \text{ for } x \in (\Omega_n \setminus \Omega_{n+1}) \cap (\Omega'_p \setminus \Omega'_{p+1}). \tag{4.30}$$

Notice that the intersection in (4.30) is empty if $n > p$, by the assumption $\mathbb{B}_j \prec \mathbb{B}'_k$, hence $n \leq p$ and $\sigma(\mathbb{B}_j, \mathbb{B}'_k) \leq 1$. The above lemma follows immediately from the composition formula (4.21) [3] by inspecting the definitions of the spaces involved, in particular corresponding weights on the intersections in (4.30). We could introduce also generalizations of other spaces considered in [1], but in this paper we will use only spaces $\Psi_j^c(X; \sigma, \varepsilon)$, which are defined by the conditions (3.13), (3.14) [1] restricted to a localization domain X , e.g. $X \in \mathcal{D}_j$.

We modify also slightly the inductive assumption (H.5) on the analyticity domains of the localized functions in the expansion (H.5). For a localization domain $X \in \mathcal{D}_j$ we construct the minimal generating set $\mathbb{B}_j(X)$ (described in (4.4)–(4.6)), and we assume that the function $\mathcal{E}^{(j)}(y, X; \psi_j, v_j h)$ has an analytic extension onto the space $\Xi_j^c(\mathbb{B}_j(X); 1, \varepsilon_j)$ satisfying the bound and the invariance property in (H.5). This space is larger than the space $\Xi_j^c(1, \varepsilon_j)$, so the above assumption

is stronger than the inductive assumption (H.5) in [1]. It implies that a localized function of the expansion (3.127) in [1], corresponding to a domain $X \in \mathcal{D}_j$, has an analytic extension onto one of the two spaces: $\Xi_j^c(\mathbb{B}_j(X); 1, \varepsilon_j)$, or the subspace of $\Psi_j^c(X; C_0 L^j \eta, \varepsilon_k)$ given by the configurations satisfying the additional condition (3.107) [1], and it satisfies the bound (3.128) [1]. We write it more generally as

$$|\mathcal{E}(X; \psi_j)| < E \exp(-(\kappa - 1)d_j(X)), \quad (4.31)$$

where E is a positive constant, which in applications will be one of the constants in various assumptions and bounds in [1].

Let us consider now the function

$$\mathcal{E}(X; \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s))). \quad (4.32)$$

For $s = 1$ and ψ' given by (4.24) it is equal to the function (1.1). We would like to prove that it can be analytically extended onto a space

$$\Xi_{k+1}^c(\mathbb{B}_{k+1}; 1, \varepsilon') \times \{\psi' : |\psi'| < \delta_1\} \times \{s : |s| < e^{\kappa_1}\} \quad (4.33)$$

for $\varepsilon' < \varepsilon_k$, e.g. $\varepsilon' = \varepsilon_{k+1}$, and for δ_1 sufficiently small. Actually we are interested in a situation where the function depends on the variables restricted to some domain Y containing X_1 , and then we can take \mathbb{B}_{k+1} as equal to $\mathbb{B}_{k+1}(Y)$, so \mathbb{B}_{k+1} in (4.33) has a more general meaning. We will discuss it again later on. To prove the above statement we have to prove that the function inside (4.32) transforms the space (4.33) into the two spaces written above as the two possible analyticity domains. At first we write

$$\begin{aligned} & \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s) \\ &= \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta) + [A_s \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s)]. \end{aligned} \quad (4.34)$$

The idea is that for the first configuration on the right-hand side we know quite precisely spaces to which it belongs, and the configuration in the square bracket should be sufficiently small. We use (4.29) for the first configuration, and we get

$$\psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta) \in \Xi_k^c(\mathbb{B}_k(X); \sigma(\mathbb{B}_k(X), \mathbb{B}_{k+1}), \varepsilon') \subset \Xi_k^c(\mathbb{B}_k(X); 1, \varepsilon'). \quad (4.35)$$

To estimate the configuration in the square bracket we notice that the proofs of Lemmas 4.3, 4.4 [3] can be easily interpreted as yielding

$$\Xi_{k+1}^c(\mathbb{B}_{k+1}; 1, \varepsilon') \subset \tilde{\Psi}^c(\mathbb{B}_{k+1}; 2(d+2)\varepsilon', 2(d+2)\varepsilon'). \quad (4.36)$$

By Proposition 2.3 the first term in the square bracket is an analytic function on the space (4.33), if $2(d+2)\varepsilon' \leq c_7$, and it satisfies the bound (2.60) with $\delta = 2(d+2)\varepsilon'$. By the analysis between (4.17) and (4.20) the second term is analytic on this space also, if ε' is sufficiently small to satisfy the conditions listed there, and it satisfies the bound (4.20). Thus the expression in the square bracket on the right-hand side of (4.34) is an analytic function on the space (4.33) and it can be bounded by $\delta_3 = 2(d+2) K_4 \exp(-\gamma_0 M_1) \varepsilon' + (2K_3 + 1)\delta_1$, if ε', δ_1 are small enough. Let us take the value of the function $\phi(\mathbb{B}_k(X))$ on the configuration (4.34). It can be written as the sum

$$\phi(\mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta)) + \delta\phi(\mathbb{B}_k(X); [\dots]),$$

where the function $\delta\phi(\mathbb{B}_k(X))$ is defined by the configurations $\phi_0 = \phi(\mathbb{B}_{k+1}; \theta)$, $\alpha_0 = \alpha(\mathbb{B}_{k+1}; \theta)$, $f_1 = 0, \alpha_1 = 0$. It satisfies the bounds (3.47) [3] with δ_3 instead of δ_1 . From this and the definition of the spaces $\Xi_k^c(\mathbb{B}_k; \sigma, \varepsilon)$ we conclude that the configuration (4.34) belongs to the space $\Xi_k^c(\mathbb{B}_k(X); 1, \varepsilon' + (K_2 + 1)\delta_3)$, or that the function (4.34) is analytic on the space (4.33) and transforms it into the above one. From this and Lemma 5.1 we obtain the following statement: the function inside (4.32) is defined and analytic on the space (4.33) with ε', δ_1 sufficiently small, and it transforms this space into the space

$$\Xi_j^c(\mathbb{B}_j(X); \sigma(\mathbb{B}_j(X), \mathbb{B}_k(X)), \varepsilon_k), \quad \text{if } \varepsilon' + (K_2 + 1)\delta_3 \leq \varepsilon_k. \quad (4.37)$$

Let us analyze the last condition in more detail for $\varepsilon' = \varepsilon_{k+1}$, where we take ε_j given by the formula $\varepsilon_j = \alpha_{0,k} \zeta^\alpha, \alpha_{0,k} = \alpha_0(1 + \sum_{n=k}^{+\infty} \frac{1}{n^2})$, $\alpha = \frac{d-2}{2} - \gamma, 0 < \gamma < \gamma_0 = \min\{1, \frac{d-2}{2}\}$. The positive constant α_0 is sufficiently small, in particular it can be chosen in such a way that all the previous conditions on ε' are satisfied. The last condition is satisfied if $2(d+2)(K_2+1)K_4 \exp(-\gamma_0 M_1) \varepsilon_k + (K_2+1)(2K_3+1)\delta_1 \leq L^{-\alpha} \alpha \varepsilon_k$, because $L^{-\alpha} \alpha \varepsilon_k \leq \varepsilon_k - \varepsilon_{k+1}$. We assume now that M_1 satisfies the additional condition $2(d+2)(K_2+1)K_4 \exp(-\gamma_0 M_1) \leq \frac{1}{2} L^{-\alpha} \alpha$, or $8dL^\alpha K_2 K_4 \frac{1}{\alpha} \leq \exp \gamma_0 M_1$. Then the last condition is satisfied if $(K_2+1)(2K_3+1)\delta_1 \leq \frac{1}{2} L^{-\alpha} \alpha \varepsilon_k$, or if

$$8L^\alpha K_2 K_3 \frac{1}{\alpha} \leq \frac{\alpha_0 \eta^\alpha}{\delta_1}, \quad 8dL^\alpha K_2 K_4 \frac{1}{\alpha} \leq \exp \gamma_0 M_1. \quad (4.38)$$

If we take δ_1 given by (4.23), and use the asymptotic properties of β_k , then we obtain the bound $\delta_1 < 2B_5 C_{\alpha_1} \beta^{-\frac{1}{2} + \alpha_1} \eta^{\frac{d-2}{2} - \alpha_1(d-2)}$, and the first condition above is satisfied if $\beta^{\frac{1}{2} - \alpha_1} \geq 16L^\alpha B_5 C_{\alpha_1} K_2 K_3 \frac{1}{\alpha \alpha_0} \eta^{\gamma - \alpha_1(d-2)}$. Taking α_1 such that $\alpha_1(d-2) \leq \gamma$ we obtain that the last condition is satisfied if β is large enough, more precisely if $\beta^{\frac{1}{2} - \alpha_1} \geq 16L^\alpha B_5 C_{\alpha_1} K_2 K_3 \frac{1}{\alpha \alpha_0}$. We will come back to these conditions in one of the following papers, where other localization expansions will be discussed. Now we assume that they are satisfied.

For the space in (4.37) we have the inclusions

$$\begin{aligned} \Xi_j^c(\mathbb{B}_j(X); \sigma(\mathbb{B}_j(X), \mathbb{B}_k(X)), \varepsilon_k) &\subset \Xi_j^c(\mathbb{B}_j(X); 1, \varepsilon_j), \\ \Xi_j^c(\mathbb{B}_j(X); \sigma(\mathbb{B}_j(X), \mathbb{B}_k(X)), \varepsilon_k) \Big|_X &\subset \Psi_j^c(X; 3L^j \eta, \varepsilon_k), \end{aligned} \quad (4.39)$$

where the last one follows from the generalization of Lemma 3.1 in [1] discussed before in connection with Lemmas 4.3, 4.4 [3]. Actually we have to be more precise now, because the representation (3.127) [1] with the bounds (3.128) [1] hold on the subspace of the space $\Psi_j^c(C_0 L^j \eta, \varepsilon_k)$ defined by the additional condition (3.107) [1], that is the condition $|\partial^2 \psi| < K_1 C_{\alpha_1} (L^j \eta)^{2 - \alpha_1} \varepsilon_k$, where $0 < \alpha_1 \leq 1$ and C_{α_1} is an absolute constant depending on α_1 only. Thus instead of the last inclusion above we have to prove that

$$\begin{aligned} \psi(\mathbb{B}_j(X), \mathbb{B}_k(X)) \Big|_X &: \Xi^c(\mathbb{B}_k(X); 1, \varepsilon_k) \\ \rightarrow \{(\psi, h): (\psi, h) \in \Psi_j^c(X; 3L^j \eta, \varepsilon_k), |\partial^2 \psi| < K_1 C_{\alpha_1} (L^j \eta)^{2 - \alpha_1} \varepsilon_k\}. \end{aligned} \quad (4.40)$$

The function $\psi(\mathbb{B}_j(X), \mathbb{B}_k(X))$ has the representation (4.21) [3] in terms of the function $\phi(\mathbb{B}_k(X))$. We use again the inclusion $\Xi^c(\mathbb{B}_k(X); 1, \varepsilon_k) \subset \tilde{\Psi}^c(\mathbb{B}_k(X); 2(d +$

$2)\varepsilon_k, 2(d+2)\varepsilon_k$, and from Proposition 1.1 [3] we obtain that $\phi(\mathbb{B}_k(X))$ satisfies the third inequality in (1.15) [3] with δ replaced by $2(d+2)\varepsilon_k$. In particular on the domain X we have the inequality with the constant $K_1 C_\alpha 2(d+2)\varepsilon_k$ on the right-hand side. Notice that other inequalities in (1.15) [3] hold with $K_1 \delta$ replaced by ε_k , by the definition of the space $\Xi^c(\mathbb{B}_k(X); 1, \varepsilon_k)$. From these inequalities and (4.21) [3] we obtain easily that on the domain X

$$|\partial^2 \psi(\mathbb{B}_j(X), \mathbb{B}_k(X))| < K_1 C_{\alpha_1} (L^j \eta)^{2-\alpha_1} 2(d+2)\varepsilon_k + 4(L^j \eta)^2 \varepsilon_k,$$

which implies (4.40) with a properly chosen absolute constant C_{α_1} depending on α_1, d only. From the inclusions (4.39), (4.40), the statement before (4.37) and the assumptions on the function $\mathcal{E}(X, \psi_j)$ we conclude the following lemma.

Lemma 4.2. *The function (4.32) is defined and analytic on the space (4.33) with $\varepsilon' \leq \varepsilon_{k+1}$ and δ_1 satisfying the condition (4.38). On this space it satisfies the bound (4.31).*

After the above preparatory analysis we can finally construct a localization expansion for the function (1.1). We start with the following simple interpolation identity used many times for similar purposes:

$$\begin{aligned} & \mathcal{E}(X; \psi_k^{(j)}(\psi^{(k)}(\theta) + \psi')) \\ &= \mathcal{E}(X; \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, 1) + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', 1))) \\ &= \sum_{\mathcal{C}} \prod_{\square \in \mathcal{C}} \int_0^1 ds_{\square} \frac{\partial}{\partial s_{\square}} \mathcal{E}(X; \psi(\mathbb{B}_j(X), \mathbb{B}_k(X); \psi(\mathbb{B}_k(X), \mathbb{B}_{k+1}; \theta, s) \\ & \quad + \delta\psi(\mathbb{B}_k(X), \mathbb{B}_k; \psi', s))) \Big|_{s(\mathcal{C}^c)=0}. \end{aligned} \quad (4.41)$$

The summation above is over all subfamilies \mathcal{C} of the family of large cubes (in the scale M_1) determined by the generating set \mathbb{B}_k and contained in X_1^c , the complement \mathcal{C}^c is with respect to this family. From now on let us assume for simplicity that $X \subset X_1 \subset \Omega_{k+1}$. Let us consider the domain

$$Y_1 = X_1 \cup \bigcup_{\square \in \mathcal{C}} \square, \quad (4.42)$$

by which we mean, as usual, the domain obtained by adding proper lower dimensional walls (i.e. the walls such that all cubes containing them in their boundaries belong to \mathcal{C}). This domain is generally a union of several components. By the localization property discussed above the function (4.32) for s such that $s = 0$ on \mathcal{C}^c , i.e. $s(\mathcal{C}^c) = 0$, depends on $(\theta, h), \psi', s$, restricted to the component of Y_1 containing the domain X , or X_1 . If there are other non-empty components of Y_1 , then the corresponding term on the right-hand side of (4.41) vanishes, because of the derivatives $\frac{\partial}{\partial s_{\square}}$ for \square contained in those other components. Thus we can write the sum as a sum over connected domains Y_1 of the form (4.42), and the product over $\square \in \mathcal{C}$ can be replaced by a product over $\square \subset Y_1 \cap X_1^c$. We obtain an expansion of the form

$$\mathcal{E}(X; \psi_k^{(j)}(\psi^{(k)}(\theta) + \psi')) = \sum_{Y_1: X_1 \subset Y_1} \mathcal{E}'(X, Y_1; \theta, \psi'). \quad (4.43)$$

A term of this expansion is equal to the corresponding term in (4.41) with \mathcal{C} related to Y_1 by (4.42). It is almost a desired localization expansion, in particular it satisfies the most important localization property: the function $\mathcal{E}'(X, Y_1; \theta, \psi')$ depends on θ, ψ' restricted to the domain Y_1 . Unfortunately one feature is still missing, the domains Y_1 in the sum are not localization domains. We correct it by doing partial resummations in the above sum.

At first we assign to each domain Y_1 in (4.43) a localization domain $Y \in \mathcal{D}(\mathbb{B}_{k+1})$ containing Y_1 , and in some sense minimal. For a cube $\square_1 \in \mathcal{C}, \square_1 \subset \Omega_j \setminus \Omega_{j+1}$ (hence a cube of the size $M_1 L^j \eta$) we take the cube $\square \in \pi_j$ (hence the cube of the size $ML^j \eta$) containing \square_1 , and we take the domain $\bigcup \{ \square' : \square' \in \pi'_j, \square' \supset \square, \square' \subset \Omega_j \setminus \Omega_{j+1} \}$. This family of cubes is non-empty because each component of $\Omega_j \setminus \Omega_{j+1}$ belongs to \mathcal{D}_j . We obtain a domain from \mathcal{D}_j containing \square , hence \square_1 , and contained in $\square^{\sim L}$ (which is the cube of the size $(2L+1)ML^j \eta$ with \square in its center). Strictly speaking this construction is described for $j \leq k$, for $j = k+1$ we still have cubes \square_1 of the size M_1 , but we take the corresponding cubes $\square \in \pi_{k+1}$, i.e., of the size ML , the rest of the construction is the same. We take a union of these domains over all $\square_1 \in \mathcal{C}$, and we obtain a domain, which is a union of connected components denoted $Y^{(1)}, \dots, Y^{(n)}$. Each component $Y^{(i)} \in \mathcal{D}(\mathbb{B}_{k+1})$, and contains at least one component of $\bigcup \mathcal{C}$, hence a cube $\square_1 \in \mathcal{C}$ touching X_1 along some $d-1$ -dimensional wall. Take one of those cubes and denote the corresponding cube \square by $\square^{(i)}$. Each component $Y^{(i)}$ is a union of large cubes in the scale M , more precisely $Y^{(i)} \cap (\Omega_j \setminus \Omega_{j+1})$ is a union of cubes from π_j . Denote this family of large cubes by $\mathcal{C}^{(i)}$, hence $\bigcup \mathcal{C}^{(i)} = Y^{(i)}$. We define

$$Y = X_1 \cup \bigcup_{i=1}^n Y^{(i)}. \quad (4.44)$$

Obviously $Y \in \mathcal{D}(\mathbb{B}_{k+1})$, and there may be many domains Y_1 determining the same domain Y . We do a partial resummation of the terms in (4.43) over the domains Y_1 determining a fixed Y , and we define

$$\mathcal{E}(X, Y; \theta, \psi') = \sum_{Y_1: Y_1 \text{ determines } Y} \mathcal{E}'(X, Y_1; \theta, \psi'). \quad (4.45)$$

Each term in the above sum depends on θ, ψ' , or rather θ, ψ , restricted to $Y_1 \subset Y$, so the sum depends on these variables restricted to Y . In order to formulate analyticity properties of the above function we have to define the generating set $\mathbb{B}_{k+1}(Y)$. It has been defined by (4.4)–(4.6) if $Y \subset \Omega_{k+1}$, but generally we have $Y \in \mathcal{D}(\mathbb{B}_{k+1})$ and Y may intersect the domains $\Omega_j \setminus \Omega_{j+1}$ for $j \leq k$, so we have to modify this definition properly. Such a modification is rather obvious, we take $\Omega'_{k+1} = Y \cap \Omega_{k+1}$, $\Omega'_k = \Omega'_{k+1} \stackrel{\sim}{\sim} \Omega_k \cup (Y \cap \Omega_k \cap \Omega^c_{k+1})$, where we add layers of M_1 -cubes, \dots , $\Omega'_n = \Omega'_{n+1} \stackrel{\sim}{\sim} \Omega_n \cup (Y \cap \Omega_n \cap \Omega^c_{n+1})$, where we add layers of $M_1 L^n \eta$ -cubes, \dots , and so on. Obviously we have $\Omega'_j \subset \Omega_j$. The sequence $\{\Omega'_1, \Omega'_2, \dots, \Omega'_k, \Omega'_{k+1}\}$ is determined by the localization domain Y , and we denote it as before by $\{\Omega_1(Y), \Omega_2(Y), \dots, \Omega_k(Y), \Omega_{k+1}(Y)\}$. The corresponding generating set is denoted by $\mathbb{B}_{k+1}(Y)$, and we have $\mathbb{B}_{k+1}(Y) \prec \mathbb{B}_{k+1}$. Consider a term in the sum (4.45) corresponding to a domain Y_1 . It is defined by the formula in (4.41) in terms of the function (4.32), in which we take $s = 0$ on \mathcal{C}^c . This function is localized completely in Y_1 , as it follows from the constructions of the two previous sections, from the way the localization parameters s have been introduced. This means that if we take a different generating set \mathbb{B}'_{k+1} , but such

that $Y_1 \cap \mathbb{B}'_{k+1} = Y_1 \cap \mathbb{B}_{k+1}$, then the function (4.32) is equal to the corresponding function with \mathbb{B}'_{k+1} instead of \mathbb{B}_{k+1} , assuming that the parameters s' connected with \mathbb{B}'_{k+1} are equal to 0 outside Y_1 , and are equal to s inside Y_1 . Obviously we have $Y_1 \cap \mathbb{B}_{k+1}(Y) = Y_1 \cap \mathbb{B}_{k+1}$ by the definition of Y and $\mathbb{B}_{k+1}(Y)$, so we may replace \mathbb{B}_{k+1} by $\mathbb{B}_{k+1}(Y)$ in the definitions of all the terms of the sum (4.45). Applying Lemma 4.2 we conclude that the functions (4.32) determining these terms are analytic on the space (4.33) with $\mathbb{B}_{k+1}(Y)$ instead of \mathbb{B}_{k+1} . The terms are defined by the expressions in (4.41) involving integrals of derivatives with respect to s_\square for $\square \in \mathcal{C}$. We replace the derivatives by the Cauchy formulas for the circles $|s_\square| = e^{\kappa_1}$, or arbitrarily close circles, and we obtain formulas for functions $\mathcal{E}'(X, Y_1; \theta, \psi')$, which imply that they are analytic functions defined on the spaces

$$\Xi_{k+1}^c(\mathbb{B}_{k+1}(Y); 1, \varepsilon') \times \{\psi': |\psi'| < \delta_1\} \quad (4.46)$$

with ε', δ_1 as discussed previously in connection with Lemma 4.2. On the above spaces they satisfy the bounds

$$|\mathcal{E}'(X, Y_1; \theta, \psi')| < Ee^{-(\kappa_1-1)|\mathcal{C}|} \exp(-(\kappa-1)d_j(X)) \quad (4.47)$$

for $\kappa_1 \geq \log(1 + \frac{1}{\sqrt{e-1}})$, or simply for $\kappa_1 \geq 2$. From this and (4.45) we obtain that $\mathcal{E}(X, Y; \theta, \psi')$ is also defined and analytic on the spaces (4.46). We would like to estimate this function in terms of exponential factors involving linear sizes of X, Y . The relevant linear size of X is $d_j(X)$, and for Y we take $\tilde{d}_{k+1}(Y)$ defined by (1.3).

In order to find a relation between $|\mathcal{C}|$ (which is a number of cubes in the family \mathcal{C}), $d_j(X)$ and $\tilde{d}_{k+1}(Y)$, we construct a proper tree graph for the domain Y . For a domain $Y^{(i)}$ we take a tree graph $\Gamma_0^{(i)}$ starting at the center of $\square^{(i)}$ and built of segments connecting centers of nearest neighbor cubes, if they are of the same size. If they have different sizes, then we take a shortest path connecting their centers and contained in their union. We can take this tree graph in such a way that to each cube in $\mathcal{C}^{(i)}$, except the cube $\square^{(i)}$, we can assign one segment, or one path. The scaled length $|\cdot|_{\mathbb{B}_{k+1}}$ of each segment is equal to 1, and the scaled length of a path can be estimated by $1 + \frac{1}{2}(d-1) = \frac{1}{2}(d+1)$. From this we obtain the following simple bound

$$|\Gamma_0^{(i)}|_{\mathbb{B}_{k+1}} < \frac{1}{2}(d+1)(|\mathcal{C}^{(i)}| - 1) \leq \frac{1}{2}(d+1)(2L+1)^d |\mathcal{C} \cap Y^{(i)}|,$$

where in the second inequality we have used the fact that a cube in \mathcal{C} may generate $(2L+1)^d$ cubes in $\mathcal{C}^{(i)}$. Now take a tree graph $\Gamma_0 \subset X$ satisfying the conditions of the definition of $d_j(X)$, and such that $\frac{1}{ML\eta}|\Gamma_0| < d_j(X) + \varepsilon$, where ε is an arbitrary positive number, and the length $|\Gamma_0|$ is taken in the η -scale. This tree graph intersects every cube from the cover π'_j contained in X , so it intersects every cube from the cover π'_{k+1} contained in X_1 . The cube $\square^{(i)}$ touches a cube from π'_{k+1} contained in X_1 along a $d-1$ -dimensional wall, and the last one intersects Γ_0 , so we connect Γ_0 with the center of $\square^{(i)}$ by a shortest path, and we combine this path with the tree graph $\Gamma_0^{(i)}$ obtaining this way a tree graph $\Gamma^{(i)}$. The scaled length of the path can be estimated by $dL + \frac{1}{2}$, so we get

$$|\Gamma^{(i)}|_{\mathbb{B}_{k+1}} < \frac{1}{2}(d+1)(2L+1)^d |\mathcal{C} \cap Y^{(i)}| + dL + \frac{1}{2} < d(2L+1)^d |\mathcal{C} \cap Y^{(i)}|.$$

We combine now all the graphs and we take the graph $\Gamma = \Gamma_0 \cup \Gamma^{(1)} \cup \dots \cup \Gamma^{(n)}$. It is a tree graph, if the paths connecting Γ_0 with the centers of the cubes $\square^{(i)}$ are disjoint, which is always possible to achieve, and it is contained in Y . It is also clear that it intersects every cube from the cover π'_{k+1} contained in $Y \cap \Omega_{k+1}$, and every cube from one of the covers π'_j contained in $Y \cap (\Omega_j \setminus \Omega_{j+1})$. Thus we have $|\Gamma|_{\mathbb{B}_{k+1}} \geq \tilde{d}_{k+1}(Y)$. On the other hand $\Gamma_0 \subset X \subset \Omega_{k+1}$, hence $|\Gamma_0|_{\mathbb{B}_{k+1}} = \frac{1}{M} L^{-1} |\Gamma_0|$, and

$$\begin{aligned} |\Gamma|_{\mathbb{B}_{k+1}} &= L^{-1} \frac{1}{M} |\Gamma_0| + \sum_{i=1}^n |\Gamma^{(i)}|_{\mathbb{B}_{k+1}} < L^{-1} L^j \eta (d_j(X) + \varepsilon) \\ &\quad + \sum_{i=1}^n d(2L+1)^d |\mathcal{C} \cap Y^{(i)}| = L^{-1} L^j \eta (d_j(X) + \varepsilon) + d(2L+1)^d |\mathcal{C}|. \end{aligned}$$

Combining the two bounds and taking the limit $\varepsilon \rightarrow 0$ we obtain the inequality

$$\tilde{d}_{k+1}(Y) \leq L^{-1} L^j \eta d_j(X) + d(2L+1)^d |\mathcal{C}|, \quad (4.48)$$

which yields the desired relation between the three quantities involved.

Now we would like to estimate the function given by the formula (4.45). A term in the sum is bounded in (4.47). For the exponent of the exponential functions on the right-hand side of this bound we have

$$\begin{aligned} &(\kappa_1 - 1) |\mathcal{C}| + (\kappa - 1) d_j(X) \\ &= \left(\frac{1}{2} \kappa_1 - 1 \right) |\mathcal{C}| + \left(\frac{1}{3} \kappa - 1 \right) d_j(X) + \frac{\kappa_1}{2d(2L+1)^d} d(2L+1)^d |\mathcal{C}| \\ &\quad + \frac{2L}{3} (L^j \eta)^{-1} \kappa L^{-1} L^j \eta d_j(X) \geq \kappa_2 |\mathcal{C}| + \kappa_0 d_j(X) + 2\kappa \tilde{d}_{k+1}(Y), \end{aligned} \quad (4.49)$$

where we have assumed that $\frac{1}{3} \kappa - 1 \geq \kappa_0$, $\frac{\kappa_1}{2d(2L+1)^d} \geq 2\kappa$, or $\kappa_1 \geq 4d(2L+1)^d \kappa$, and we have denoted $\kappa_2 = \frac{1}{2} \kappa_1 - 1$. To estimate the sum in (4.45) we use the exponential factor $e^{-\kappa_2 |\mathcal{C}|}$. We start with the following simple inequality:

$$\sum'_{Y_1} e^{-\kappa_2 |\mathcal{C}|} \leq \prod_{j=0}^{k+1} \sum'_{Y_1 \cap (\Omega_j \setminus \Omega_{j+1})} e^{-\kappa_2 |\mathcal{C} \cap (\Omega_j \setminus \Omega_{j+1})|},$$

where the ‘‘primes’’ mean that the sums are restricted to the admissible domains determining a fixed domain Y . Using again the fact that a cube in $\mathcal{C} \cap (\Omega_j \setminus \Omega_{j+1})$ may generate $(2L+1)^d$ cubes from π_j in $Y \cap X_1^c \cap (\Omega_j \setminus \Omega_{j+1})$ we obtain the bound

$$(2L+1)^d |\mathcal{C} \cap (\Omega_j \setminus \Omega_{j+1})| \geq \frac{1}{M^d} |Y \cap X_1^c \cap (\Omega_j \setminus \Omega_{j+1})|_{\xi},$$

where the volume is taken in the ξ -scale, $\xi = L^{-j}$. We take half of the exponent in the exponential function on the right-hand side of the previous inequality and we apply the above bound. The remaining half yields the product $\prod_{\square_1 \subset Y_1 \cap X_1^c \cap (\Omega_j \setminus \Omega_{j+1})} e^{-\frac{1}{2} \kappa_2}$. We can estimate the primed sum on the right-hand side by a sum over all subdomains of $Y \cap X_1^c \cap (\Omega_j \setminus \Omega_{j+1})$ which are unions of large cubes in the scale M_1 , and

the last sum of the above products is equal to $(1 + e^{-\frac{1}{2}\kappa_2})^{\frac{1}{M_1^d}|Y \cap X_1^c \cap (\Omega_j \setminus \Omega_{j+1})|_\xi}$. Thus we can bound the whole primed sum on the right-hand side by

$$\exp \left[- \left(\frac{1}{2}\kappa_2 - e^{-\frac{1}{2}\kappa_2}(2L+1)^d \frac{M^d}{M_1^d} \right) \frac{1}{(2L+1)^d M^d} \left| Y \cap X_1 \cap (\Omega_j \setminus \Omega_{j+1}) \right|_\xi \right].$$

The expression in the parenthesis is equal to a value of the function $f(x) = x - e^{-x}A$ at $x = \frac{1}{2}\kappa_2$, where $A = (2L+1)^d \frac{M^d}{M_1^d}$. This is an increasing function, because $f'(x) = 1 + e^{-x} > 0$, and $f(\log A) = \log A - 1 > 0$, hence if $\frac{1}{2}\kappa_2 \geq \log A$, then $f(\frac{1}{2}\kappa_2) > 0$. With this assumption on κ_2 the above exponential can be bounded by 1, and we obtain that

$$\sum_{Y_1: Y_1 \text{ determines } Y} e^{-\kappa_2|\mathcal{G}|} \leq 1. \quad (4.50)$$

The assumption on κ_2 translates into an assumption on κ_1 of the form $\kappa_1 \geq 4d \log((2L+1)\frac{M}{M_1}) + 2$. Let us analyze this assumption in more detail. We have only two conditions on the quotient $\frac{M}{M_1}$: that it is a power L^{m_0} , where m_0 is a natural number, and that it is $\geq 14d$. We can replace the above assumption on κ_1 by the slightly stronger one $\kappa_1 \geq 8dm_0 \log(2L+1)$, and taking the smallest number m_0 we still obtain a weaker condition than the one introduced previously, i.e., $\kappa_1 \geq 4d(2L+1)^d \kappa$. We fix κ_1 satisfying both conditions.

The expansion (4.43), the definition (4.45) and the bounds (4.47), (4.48), (4.49), (4.50) yield the expansion

$$\mathcal{E}(X; \psi_k^{(j)}(\psi^{(k)}(\theta) + \psi')) = \sum_{Y \in \mathcal{D}(\mathbb{B}_{k+1}): X_1 \subset Y} \mathcal{E}(X, Y; \theta, \psi'), \quad (4.51)$$

whose terms are analytic functions on the domains (4.46), satisfying the bounds

$$|\mathcal{E}(X, Y; \theta, \psi')| < E \exp(-\kappa_0 d_j(X) - 2\kappa \tilde{d}_{k+1}(Y)). \quad (4.52)$$

Let us formulate the most important results on the localization expansion constructed in this section in the following proposition.

Proposition 4.3. *Let us consider the function (1.1), where $\mathcal{E}(X, \psi_j)$ is analytic on a complex space containing one of the two spaces $\Xi_j^c(\mathbb{B}_j(X); 1, \varepsilon_j)$, $\Psi_j^c(X; C_0 L^j \eta, \varepsilon_k)$ with the condition (3.107) [1] and it satisfies the bound (4.31). The function (1.1) has a localization expansion of the form (4.51), whose terms are analytic functions on the spaces (4.46) with ε', δ_1 satisfying the conditions $\varepsilon' \leq \varepsilon_{k+1}$, (4.38). They depend on θ, ψ' , or θ, ψ , restricted to Y , and they satisfy the bounds (4.52).*

The construction of the localization expansion above is quite universal and we will apply it in several other situations. Notice that until now we have obtained only one condition on κ , namely $\frac{1}{3}\kappa - 1 \geq \kappa_0$, or $\kappa \geq 3\kappa_0 + 3$. Notice also that we have formulated and proved it under the simplifying assumption $X \subset X_1 \subset \Omega_{k+1}$. Later we will have to discuss also some simple variations, like $X_1 \subset \Omega_k$. We have also considered the two versions of ψ' , one as an independent complex variable, and the other given by the formula (4.24). This will be also used in the future for some simple generalizations.

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