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On Representations of the Elliptic Quantum Group $E_{\tau,\eta}(sl_2)$

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Abstract: We describe representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. It turns out that the representation theory is parallel to the representation theory of the Yangian $Y(sl_2)$ and the quantum loop group $U_q(\widetilde{sl}_2)$.

We introduce basic notions of representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ and construct three families of modules: evaluation modules, cyclic modules, one-dimensional modules. We show that under certain conditions any irreducible highest weight module of finite type is isomorphic to a tensor product of evaluation modules and a one-dimensional module. We describe fusion of finite dimensional evaluation modules. In particular, we show that under certain conditions the tensor product of two evaluation modules becomes reducible and contains an evaluation module, in this case the imbedding of the evaluation module into the tensor product is given in terms of elliptic binomial coefficients. We describe the determinant element of the elliptic quantum group. Representation theory becomes special if $N\eta = m + l\tau$, where N, m, l are integers. We indicate some new features in this case.

1. Introduction

The elliptic quantum group is an algebraic structure underlying the elliptic solutions of the Star–Triangle relation in statistical mechanics and connected with the Knizhnik–Zamolodchikov–Bernard equation on tori. In this paper we consider the elliptic quantum group $E_{\tau,\eta}(sl_2)$ defined in [Fe1-2] and discuss its representation theory. It turns out that representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is parallel to representation theory of the Yangian $Y(sl_2)$ and the quantum loop group $U_q(\widetilde{sl}_2)$ [T], cf. [CP].

We introduce basic notions of representation theory of the elliptic quantum group (notions of the operator algebra, a highest weight module, an irreducible module,

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a module of finite type, a singular vector). Essentially all the notions are formulated in terms of the associated operator algebra.

We construct three families of $E_{\tau,\eta}(sl_2)$-modules: evaluation Verma modules $V_\Lambda(z)$, cyclic modules, and one-dimensional modules.

An evaluation Verma module is an infinite dimensional module determined by an evaluation point $z \in \mathbb{C}$ and a weight $\Lambda \in \mathbb{C}$. If $\Lambda = n + (m + l\tau)/2\eta$, where n, m, l are integers, $n \geqq 0$, then the Verma module has a submodule and the quotient module $L_\Lambda(z)$ has finite dimension $n+1$. For generic η the evaluation Verma module $V_\Lambda(z)$ is irreducible unless $\Lambda = n + (m + l\tau)/2\eta$. If Λ has this form, then $L_\Lambda(z)$ is irreducible.

A tensor product of irreducible evaluation modules is an irreducible highest weight module of finite type, if η is generic and the evaluation points of factors are generic. We show that under certain conditions every irreducible highest weight module of finite type is isomorphic to a tensor product of irreducible evaluation modules and a one-dimensional module.

We give necessary and sufficient conditions for the tensor product of two finite dimensional modules $L_{\Lambda_1}(z_1) \otimes L_{\Lambda_2}(z_2)$ to be reducible. In particular, $L_{\Lambda_1}(z_1) \otimes L_{\Lambda_2}(z_2)$ becomes reducible, if $z_2 - z_1 = (\Lambda_1 + \Lambda_2)\eta$. In this case the tensor product contains a submodule isomorphic to $L_{\Lambda_1+\Lambda_2}(z_2 - \Lambda_1\eta)$. The imbedding of this module into the tensor product is given in terms of elliptic binomial coefficients.

We indicate the determinant element of the elliptic quantum group. It is a group-like central element, see precise statements in Sect. 10.

The elliptic quantum group depends on complex parameters τ and η. We show that the elliptic quantum groups with parameters (τ, η), $(\tau + 1, \eta)$, and $(-1/\tau, -\eta/\tau)$ are essentially isomorphic.

Representation theory of the elliptic quantum group is special, if $N\eta = m + l\tau$, where N, m, l are integers. We indicate some new features in this case.

Now we briefly sketch the definition of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ and will give a detailed account in Sects. 2 and 3.

To define the elliptic quantum group $E_{\tau,\eta}(sl_2)$ we fix two nonzero complex numbers τ and η, $\operatorname{Im}\tau > 0$, and start from a 4×4-matrix $R(\lambda, w) \in \operatorname{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$,

$$\begin{aligned} R(\lambda, w, \eta, \tau) &= E_{1,1} \otimes E_{1,1} + E_{2,2} \otimes E_{2,2} + \alpha(\lambda, w, \eta, \tau) E_{1,1} \otimes E_{2,2} \\ &\quad + \beta(\lambda, w, \eta, \tau) E_{1,2} \otimes E_{2,1} + \gamma(\lambda, w, \eta, \tau) E_{2,1} \otimes E_{1,2} \\ &\quad + \delta(\lambda, w, \eta, \tau) E_{2,2} \otimes E_{1,1} , \end{aligned}$$

where the functions $\alpha, \beta, \gamma, \delta$ are defined in Sect. 2, $E_{i,j}$ is the 2×2-matrix with the only nonzero element 1 at the intersection of the i^{th} row and j^{th} column.

Let h be the diagonal 2×2 matrix $\operatorname{Diag}(1, -1)$. The R-matrix $R(w, \lambda)$ satisfies the dynamical (or modified) Yang–Baxter equation in $\operatorname{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$,

$$\begin{aligned} & R^{(12)}(\lambda - 2\eta h^{(3)}, w_{12}) R^{(13)}(\lambda, w_{13}) R^{(23)}(\lambda - 2\eta h^{(1)}, w_{23}) \\ &= R^{(23)}(\lambda, w_{23}) R^{(13)}(\lambda - 2\eta h^{(2)}, w_{13}) R^{(12)}(\lambda, w_{12}) . \end{aligned}$$

Here $w_{ij} = w_i - w_j$; $R^{(12)}(\lambda - 2\eta h^{(3)}, w_{12})$ means that if $a \otimes b \otimes c \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $hc = \mu c$, $\mu \in \mathbb{C}$, then $R^{(12)}(\lambda - 2\eta h^{(3)}, w_{12}) a \otimes b \otimes c = R^{(12)}(\lambda - 2\eta\mu, w_{12})(a \otimes b) \otimes c$, and the other symbols have a similar meaning.

The elliptic quantum group $E_{\tau,\eta}(sl_2)$ is an algebra generated by meromorphic functions of a variable h and the matrix elements of a matrix $L(\lambda, w) \in \mathrm{End}(\mathbb{C}^2)$ with non-commutative entries, subject to the relations

$$R^{(12)}(\lambda - 2\eta h, w_{12})L^{(1)}(\lambda, w_1)L^{(2)}(\lambda - 2\eta h^{(1)}, w_2)$$
$$= L^{(2)}(\lambda, w_2)L^{(1)}(\lambda - 2\eta h^{(2)}, w_1)R^{(12)}(\lambda, w_{12})\,. \tag{1}$$

Here h is considered as a generator of a one dimensional commutative Lie algebra $\mathfrak{h}$.

An $E_{\tau,\eta}(sl_2)$-module is a diagonalizable $\mathfrak{h}$-module V together with a meromorphic function $L(\lambda, w)$ on $\mathfrak{h} \times \mathbb{C}$ with values in $\mathrm{End}(\mathbb{C}^2 \otimes V)$ such that identity (1) holds in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes V)$ and so that L is of weight zero:

$$[h^{(1)} + h^{(2)}, L(\lambda, w)] = 0\,.$$

If V and U are $E_{\tau,\eta}(sl_2)$-modules, then $V \otimes U$ is an $E_{\tau,\eta}(sl_2)$-module with an $\mathfrak{h}$-module structure $h(a \otimes b) = ha \otimes b + a \otimes hb$ and an L-operator

$$L^{(12)}(\lambda - 2\eta h^{(3)}, w)L^{(13)}(\lambda, w)\,.$$

If V, U, W are $E_{\tau,\eta}(sl_2)$-modules, then the modules $(V \otimes U) \otimes W$ and $V \otimes (U \otimes W)$ are isomorphic with the obvious isomorphism, see [Fe1-2].

Similarly one could define the elliptic quantum group associated with a simple Lie algebra of type A, B, C, D, see [Fe2].

2. The Elliptic Quantum Group $E_{\tau,\eta}(sl_2)$

Let

$$\theta(z, \tau) = -\sum_{j=-\infty}^{\infty} e^{\pi i (j+\frac{1}{2})^2 \tau + 2\pi i (j+\frac{1}{2})(z+\frac{1}{2})}$$

be the Jacobi theta function, and let functions $\alpha, \beta, \gamma, \delta$ be given by

$$\alpha(w, \lambda, \eta, \tau) = \frac{\theta(w)\theta(\lambda + 2\eta)}{\theta(w - 2\eta)\theta(\lambda)}\,, \qquad \beta(w, \lambda, \eta, \tau) = \frac{\theta(-w - \lambda)\theta(2\eta)}{\theta(w - 2\eta)\theta(\lambda)}\,,$$

$$\gamma(w, \lambda, \eta, \tau) = \frac{\theta(w - \lambda)\theta(2\eta)}{\theta(w - 2\eta)\theta(\lambda)}\,, \qquad \delta(w, \lambda, \eta, \tau) = \frac{\theta(w)\theta(\lambda - 2\eta)}{\theta(w - 2\eta)\theta(\lambda)}\,.$$

Let η and τ be nonzero complex numbers, $\mathrm{Im}\,\tau > 0$.

The elliptic quantum group $E_{\tau,\eta}(sl_2)$ is the algebra over $\mathbb{C}$ with generators of two types. The generators of the first type are labelled by meromorphic functions $f(h)$ of one complex variable with period $1/\eta$, $f(h + 1/\eta) = f(h)$. The generators of the second type are $a(\lambda, w), b(\lambda, w), c(\lambda, w), d(\lambda, w)$. They are labelled by elements $\lambda \in \mathbb{C}/\mathbb{Z}$ and complex numbers $w \in \mathbb{C}$. The generators of the algebra satisfy the following two groups of relations.

Relations involving h:

$$f(h)g(h) = g(h)f(h)\,,$$
$$f(h)a(\lambda, w) = a(\lambda, w)f(h), \qquad f(h)d(\lambda, w) = d(\lambda, w)f(h)\,,$$
$$f(h)b(\lambda, w) = b(\lambda, w)f(h - 2), \qquad f(h)c(\lambda, w) = c(\lambda, w)f(h + 2)\,, \tag{2}$$

where $f(h), g(h)$ are generators of the first type.

The remaining relations have the form:

$$
\begin{aligned}
&a_1(\lambda)a_2(\lambda-2\eta) = a_2(\lambda)a_1(\lambda-2\eta)\,,\\
&a_1(\lambda)b_2(\lambda-2\eta) = b_2(\lambda)a_1(\lambda+2\eta)\alpha(\lambda) + a_2(\lambda)b_1(\lambda-2\eta)\gamma(\lambda)\,,\\
&b_1(\lambda)a_2(\lambda+2\eta) = b_2(\lambda)a_1(\lambda+2\eta)\beta(\lambda) + a_2(\lambda)b_1(\lambda-2\eta)\delta(\lambda)\,,\\
&b_1(\lambda)b_2(\lambda+2\eta) = b_2(\lambda)b_1(\lambda+2\eta)\,,\\
&\beta(\lambda-2\eta h)c_1(\lambda)a_2(\lambda-2\eta) + \alpha(\lambda-2\eta h)a_1(\lambda)c_2(\lambda-2\eta) = c_2(\lambda)a_1(\lambda-2\eta)\,,\\
&\beta(\lambda-2\eta h)c_1(\lambda)b_2(\lambda-2\eta) + \alpha(\lambda-2\eta h)a_1(\lambda)d_2(\lambda-2\eta)\\
&\quad = d_2(\lambda)a_1(\lambda+2\eta)\alpha(\lambda) + c_2(\lambda)b_1(\lambda-2\eta)\gamma(\lambda)\,,\\
&\beta(\lambda-2\eta h)d_1(\lambda)a_2(\lambda+2\eta) + \alpha(\lambda-2\eta h)b_1(\lambda)c_2(\lambda+2\eta)\\
&\quad = d_2(\lambda)a_1(\lambda+2\eta)\beta(\lambda) + c_2(\lambda)b_1(\lambda-2\eta)\delta(\lambda)\,,\\
&\beta(\lambda-2\eta h)d_1(\lambda)b_2(\lambda+2\eta) + \alpha(\lambda-2\eta h)b_1(\lambda)d_2(\lambda+2\eta) = d_2(\lambda)b_1(\lambda+2\eta)\,,\\
&\delta(\lambda-2\eta h)c_1(\lambda)a_2(\lambda-2\eta) + \gamma(\lambda-2\eta h)a_1(\lambda)c_2(\lambda-2\eta) = a_2(\lambda)c_1(\lambda-2\eta)\,,\\
&\delta(\lambda-2\eta h)c_1(\lambda)b_2(\lambda-2\eta) + \gamma(\lambda-2\eta h)a_1(\lambda)d_2(\lambda-2\eta)\\
&\quad = b_2(\lambda)c_1(\lambda+2\eta)\alpha(\lambda) + a_2(\lambda)d_1(\lambda-2\eta)\gamma(\lambda)\,,\\
&\delta(\lambda-2\eta h)d_1(\lambda)a_2(\lambda+2\eta) + \gamma(\lambda-2\eta h)b_1(\lambda)c_2(\lambda+2\eta)\\
&\quad = b_2(\lambda)c_1(\lambda+2\eta)\beta(\lambda) + a_2(\lambda)d_1(\lambda-2\eta)\delta(\lambda)\,,\\
&\delta(\lambda-2\eta h)d_1(\lambda)b_2(\lambda+2\eta) + \gamma(\lambda-2\eta h)b_1(\lambda)d_2(\lambda+2\eta) = b_2(\lambda)d_1(\lambda+2\eta)\,,\\
&c_1(\lambda)c_2(\lambda-2\eta) = c_2(\lambda)c_1(\lambda-2\eta)\,,\\
&c_1(\lambda)d_2(\lambda-2\eta) = d_2(\lambda)c_1(\lambda+2\eta)\alpha(\lambda) + c_2(\lambda)d_1(\lambda-2\eta)\gamma(\lambda)\,,\\
&d_1(\lambda)c_2(\lambda+2\eta) = d_2(\lambda)c_1(\lambda+2\eta)\beta(\lambda) + c_2(\lambda)d_1(\lambda-2\eta)\delta(\lambda)\,,\\
&d_1(\lambda)d_2(\lambda+2\eta) = d_2(\lambda)d_1(\lambda+2\eta)\,,
\end{aligned}
\tag{3}
$$

where $\alpha, \beta, \gamma, \delta$ are functions of $w = w_1 - w_2, h, \lambda, \eta, \tau$, the symbol $a_j(\lambda)$ denotes the generator $a(\lambda, w_j)$, and similarly for b_j, c_j, d_j.

Note, for example, that $\alpha(w_1 - w_2, \lambda - 2\eta h, \eta, \tau)$ as a function of h is meromorphic and $1/\eta$-periodic, therefore, it is an element of the elliptic quantum group.

The functions $\alpha, \beta, \gamma, \delta$ have a pole at $w_1 - w_2 = 2\eta$. In this special case the relations take the form

$$
\begin{aligned}
a(\lambda, w+2\eta)a(\lambda-2\eta, w) &= a(\lambda, w)a(\lambda-2\eta, w+2\eta)\,,\\
b(\lambda, w+2\eta)b(\lambda+2\eta, w) &= b(\lambda, w)b(\lambda+2\eta, w+2\eta)\,,\\
c(\lambda, w+2\eta)c(\lambda-2\eta, w) &= c(\lambda, w)c(\lambda-2\eta, w+2\eta)\,,\\
d(\lambda, w+2\eta)d(\lambda+2\eta, w) &= d(\lambda, w)d(\lambda+2\eta, w+2\eta)\,,
\end{aligned}
$$

$$c(\lambda, w+2\eta)a(\lambda-2\eta, w) = a(\lambda, w+2\eta)c(\lambda-2\eta, w)\,,$$

$$d(\lambda, w+2\eta)b(\lambda+2\eta, w) = b(\lambda, w+2\eta)d(\lambda+2\eta, w)\,,$$

$$\theta(\lambda+2\eta)b(\lambda, w)a(\lambda+2\eta, w+2\eta) = \theta(\lambda-2\eta)a(\lambda, w)b(\lambda-2\eta, w+2\eta)\,,$$

$$\theta(\lambda+2\eta)d(\lambda, w)c(\lambda+2\eta, w+2\eta) = \theta(\lambda-2\eta)c(\lambda, w)d(\lambda-2\eta, w+2\eta)\,,$$

$$\begin{aligned}
& d(\lambda, w+2\eta)a(\lambda+2\eta, w) - b(\lambda, w+2\eta)c(\lambda+2\eta, w) \\
&= a(\lambda, w+2\eta)d(\lambda-2\eta, w) - c(\lambda, w+2\eta)b(\lambda-2\eta, w) \\
&= \frac{\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}(d(\lambda, w+2\eta)a(\lambda+2\eta, w) - b(\lambda, w+2\eta)c(\lambda+2\eta, w)) \\
&= \frac{\theta(\lambda-2\eta)}{\theta(\lambda)}a(\lambda, w)d(\lambda-2\eta, w+2\eta) - \frac{\theta(\lambda+2\eta)}{\theta(\lambda)}b(\lambda, w)c(\lambda+2\eta, w+2\eta)\,, \\
& \frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}(a(\lambda, w+2\eta)d(\lambda-2\eta, w) - c(\lambda, w+2\eta)b(\lambda-2\eta, w)) \\
&= \frac{\theta(\lambda+2\eta)}{\theta(\lambda)}d(\lambda, w)a(\lambda+2\eta, w+2\eta) - \frac{\theta(\lambda-2\eta)}{\theta(\lambda)}c(\lambda, w)b(\lambda-2\eta, w+2\eta)\,.
\end{aligned} \tag{4}$$

These relations are obvious regularizations of relations (3). Similar relations hold for other poles of the functions $\alpha, \beta, \gamma, \delta$.

An $E_{\tau,\eta}(sl_2)$-*module structure* on a complex vector space V is a direct sum decomposition

$$V = \bigoplus_{\mu \in \mathbb{C}/\frac{1}{\eta}\mathbb{Z}} V[\mu]$$

and endomorphisms $a(\lambda, w), b(\lambda, w), c(\lambda, w), d(\lambda, w) \in \mathrm{End}(V)$ which meromorphically depend on $\lambda \in \mathbb{C}/\mathbb{Z}$, $w \in \mathbb{C}$. The direct sum decomposition allows us to define endomorphisms $f(h) \in \mathrm{End}(V)$ by the rule $f(h)v = f(\mu)v$, if $v \in V[\mu]$. We assume that the endomorphisms $f(h)$, $a(\lambda, w), b(\lambda, w), c(\lambda, w), d(\lambda, w)$ satisfy the relations of the elliptic quantum group.

We allow the fact that for a given module structure the action of not all elements of the elliptic quantum group is well defined.

A module is of *finite type* if each space $V[\mu]$ has finite dimension.

If V, W are $E_{\tau,\eta}(sl_2)$-modules, then the tensor product $V \otimes W$ has an $E_{\tau,\eta}(sl_2)$-module structure, where the action of $f(h), a(\lambda, w), b(\lambda, w), c(\lambda, w), d(\lambda, w)$ is given by

$$f(h^{(1)} + h^{(2)})\,,$$

$$a(\lambda - 2\eta h^{(2)}, w) \otimes a(\lambda, w) + b(\lambda - 2\eta h^{(2)}, w) \otimes c(\lambda, w)\,,$$

$$a(\lambda - 2\eta h^{(2)}, w) \otimes b(\lambda, w) + b(\lambda - 2\eta h^{(2)}, w) \otimes d(\lambda, w)\,,$$

$$c(\lambda - 2\eta h^{(2)}, w) \otimes a(\lambda, w) + d(\lambda - 2\eta h^{(2)}, w) \otimes c(\lambda, w)\,,$$

$$c(\lambda - 2\eta h^{(2)}, w) \otimes b(\lambda, w) + d(\lambda - 2\eta h^{(2)}, w) \otimes d(\lambda, w)\,,$$

here $a(\lambda-2\eta h^{(2)},w)\otimes b(\lambda,w)$ means that if $s\otimes t\in V\otimes W$ and $b(\lambda,w)t\in W[\mu]$, then $a(\lambda-2\eta h^{(2)},w)\otimes b(\lambda,w)s\otimes t=a(\lambda-2\eta\mu,w)s\otimes b(\lambda,w)t$; if $s\otimes t\in V[\nu]\otimes W[\mu]$, then $f(h^{(1)}+h^{(2)})s\otimes t=f(\nu+\mu)s\otimes t$; and the other terms have similar meaning.

The tensor product is associative.

3. The Operator Algebra

In this section we introduce the operator algebra of the elliptic quantum group, cf. [ABB, BBB].

For a complex vector space V let $\mathrm{Fun}(V)$ be the space of meromorphic functions of λ with values in V and 1-periodic, $F(\lambda+1)=F(\lambda)$. The space $\mathrm{Fun}(V)$ has a natural structure of a vector space over the field $\mathrm{Fun}(\mathbb{C})$.

For an $E_{\tau,\eta}(sl_2)$-module V define its *operator algebra* $A(V)$ as the algebra of the following operators $f(\widetilde{h}),\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)$ acting on $\mathrm{Fun}(V)$. Introduce endomorphisms $f(\widetilde{h}),\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)\in\mathrm{End}(\mathrm{Fun}(V))$ by the rule

$$(f(\widetilde{h})F)(\lambda)=f(h)F(\lambda)\,,$$

$$(\widetilde{a}(w)F)(\lambda)=a(\lambda,w)F(\lambda-2\eta)\,,\qquad(\widetilde{c}(w)F)(\lambda)=c(\lambda,w)F(\lambda-2\eta)\,,$$

$$(\widetilde{b}(w)F)(\lambda)=b(\lambda,w)F(\lambda+2\eta)\,,\qquad(\widetilde{d}(w)F)(\lambda)=d(\lambda,w)F(\lambda+2\eta)\,,$$

where $F\in\mathrm{Fun}(V)$.

The relations for the generators in the elliptic quantum group induce some universal relations for operators $f(\widetilde{h}),\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)$ and motivate the following definition of the *operator algebra* $A_{\tau,\eta}(sl_2)$ of the elliptic quantum group. The operator algebra $A_{\tau,\eta}(sl_2)$ is an algebra over $\mathbb{C}$ generated by $f(\lambda,\widetilde{h}),\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)$, where $f(\lambda,\widetilde{h})$ runs through the space of meromorphic functions of two variables λ and $\widetilde{h}$ which are 1-periodic in λ and $1/\eta$-periodic in $\widetilde{h}$,

$$f(\lambda+1,\widetilde{h})=f(\lambda,\widetilde{h}),\qquad f(\lambda,\widetilde{h}+1/\eta)=f(\lambda,\widetilde{h})\,.$$

The generators of the algebra satisfy the following two groups of relations.

Relations involving $f(\lambda,\widetilde{h})$:

$$f(\lambda,\widetilde{h})g(\lambda,\widetilde{h})=g(\lambda,\widetilde{h})f(\lambda,\widetilde{h})\,,$$

$$f(\lambda-2\eta,\widetilde{h})\widetilde{a}(w)=\widetilde{a}(w)f(\lambda,\widetilde{h}),\qquad f(\lambda+2\eta,\widetilde{h})\widetilde{d}(w)=\widetilde{d}(w)f(\lambda,\widetilde{h})\,,$$

$$f(\lambda+2\eta,\widetilde{h}+2)\widetilde{b}(w)=\widetilde{b}(w)f(\lambda,\widetilde{h}),\qquad f(\lambda-2\eta,\widetilde{h}-2)\widetilde{c}(w)=\widetilde{c}(w)f(\lambda,\widetilde{h})\,.$$

The remaining relations have the form:

$$\widetilde{a}(w_1)\widetilde{a}(w_2)=\widetilde{a}(w_2)\widetilde{a}(w_1)\,,$$

$$\widetilde{a}(w_1)\widetilde{b}(w_2)=\alpha(\lambda)\widetilde{b}(w_2)\widetilde{a}(w_1)+\gamma(\lambda)\widetilde{a}(w_2)\widetilde{b}(w_1)\,,$$

$$\widetilde{b}(w_1)\widetilde{a}(w_2)=\beta(\lambda)\widetilde{b}(w_2)\widetilde{a}(w_1)+\delta(\lambda)\widetilde{a}(w_2)\widetilde{b}(w_1)\,,$$

$$\widetilde{b}(w_1)\widetilde{b}(w_2)=\widetilde{b}(w_2)\widetilde{b}(w_1)\,,$$

$$\beta(\lambda-2\eta\widetilde{h})\widetilde{c}(w_1)\widetilde{a}(w_2)+\alpha(\lambda-2\eta\widetilde{h})\widetilde{a}(w_1)\widetilde{c}(w_2)=\widetilde{c}(w_2)\widetilde{a}(w_1)\,,$$

$$\beta(\lambda-2\eta\widetilde{h})\widetilde{c}(w_1)\widetilde{b}(w_2)+\alpha(\lambda-2\eta\widetilde{h})\widetilde{a}(w_1)\widetilde{d}(w_2)=\alpha(\lambda)\widetilde{d}(w_2)\widetilde{a}(w_1)+\gamma(\lambda)\widetilde{c}(w_2)\widetilde{b}(w_1)\,,$$

$$\beta(\lambda-2\eta\widetilde{h})\widetilde{d}(w_1)\widetilde{a}(w_2)+\alpha(\lambda-2\eta\widetilde{h})\widetilde{b}(w_1)\widetilde{c}(w_2)=\beta(\lambda)\widetilde{d}(w_2)\widetilde{a}(w_1)+\delta(\lambda)\widetilde{c}(w_2)\widetilde{b}(w_1)\,,$$

$$\beta(\lambda-2\eta\widetilde{h})\widetilde{d}(w_1)\widetilde{b}(w_2)+\alpha(\lambda-2\eta\widetilde{h})\widetilde{b}(w_1)\widetilde{d}(w_2)=\widetilde{d}(w_2)\widetilde{b}(w_1)\,,$$

$$\delta(\lambda-2\eta\widetilde{h})\widetilde{c}(w_1)\widetilde{a}(w_2)+\gamma(\lambda-2\eta\widetilde{h})\widetilde{a}(w_1)\widetilde{c}(w_2)=\widetilde{a}(w_2)\widetilde{c}(w_1)\,,$$

$$\delta(\lambda-2\eta\widetilde{h})\widetilde{c}(w_1)\widetilde{b}(w_2)+\gamma(\lambda-2\eta\widetilde{h})\widetilde{a}(w_1)\widetilde{d}(w_2)=\alpha(\lambda)\widetilde{b}(w_2)\widetilde{c}(w_1)+\gamma(\lambda)\widetilde{a}(w_2)\widetilde{d}(w_1)\,,$$

$$\delta(\lambda-2\eta\widetilde{h})\widetilde{d}(w_1)\widetilde{a}(w_2)+\gamma(\lambda-2\eta\widetilde{h})\widetilde{b}(w_1)\widetilde{c}(w_2)=\beta(\lambda)\widetilde{b}(w_2)\widetilde{c}(w_1)+\delta(\lambda)\widetilde{a}(w_2)\widetilde{d}(w_1)\,,$$

$$\delta(\lambda-2\eta\widetilde{h})\widetilde{d}(w_1)\widetilde{b}(w_2)+\gamma(\lambda-2\eta\widetilde{h})\widetilde{b}(w_1)\widetilde{d}(w_2)=\widetilde{b}(w_2)\widetilde{d}(w_1)\,,$$

$$\widetilde{c}(w_1)\widetilde{c}(w_2)=\widetilde{c}(w_2)\widetilde{c}(w_1)\,,$$

$$\widetilde{c}(w_1)\widetilde{d}(w_2)=\alpha(\lambda)\widetilde{d}(w_2)\widetilde{c}(w_1)+\gamma(\lambda)\widetilde{c}(w_2)\widetilde{d}(w_1)\,,$$

$$\widetilde{d}(w_1)\widetilde{c}(w_2)=\beta(\lambda)\widetilde{d}(w_2)\widetilde{c}(w_1)+\delta(\lambda)\widetilde{c}(w_2)\widetilde{d}(w_1)\,,$$

$$\widetilde{d}(w_1)\widetilde{d}(w_2)=\widetilde{d}(w_2)\widetilde{d}(w_1)\,,$$

where $\alpha,\beta,\gamma,\delta$ are functions of $w=w_1-w_2,\widetilde{h},\lambda,\eta,\tau$.

An $A_{\tau,\eta}(sl_2)$*-module structure* on a vector space M over Fun($\mathbb{C}$) is a direct sum decomposition

$$M=\bigoplus_{\mu\in\mathbb{C}/\frac{1}{\eta}\mathbb{Z}} M[\mu] \tag{5}$$

and endomorphisms $\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)\in \mathrm{End}_{\mathbb{C}}(M)$ which meromorphically depend on the parameter $w\in\mathbb{C}$. The direct sum decomposition allows us to define endomorphisms $f(\lambda,\widetilde{h})\in \mathrm{End}(M)$ by the rule $f(\lambda,\widetilde{h})m=f(\lambda,\mu)m$, if $m\in M[\mu]$. We assume that the endomorphisms $f(\lambda,\widetilde{h})$, $\widetilde{a}(w),\widetilde{b}(w),\widetilde{c}(w),\widetilde{d}(w)$ satisfy the relations of the operator algebra.

An $A_{\tau,\eta}(sl_2)$-module M is of *finite type*, if each space $M[\mu]$ has finite dimension over Fun($\mathbb{C}$).

An $E_{\tau,\eta}(sl_2)$-module structure on V determines an $A_{\tau,\eta}(sl_2)$-module structure on Fun(V). The converse is also true.

For a finite type $A_{\tau,\eta}(sl_2)$-module M, we construct an $E_{\tau,\eta}(sl_2)$-module V. Let $e_j[\mu], j\in J_\mu$, be a basis of $M[\mu]$ over Fun($\mathbb{C}$). Let $V[\mu]$ be a complex vector space with a basis denoted by $e'_j[\mu], j\in J_\mu$. Set $V=\oplus_\mu V[\mu]$. Define an $E_{\tau,\eta}(sl_2)$-action on V. Set $f(h)e'_j[\mu]=f(\mu)e'_j[\mu]$. To define an action of the other operators write

$$\widetilde{a}(w)e_j[\mu]=\sum_k A_j^k(\lambda,w)e_k[\mu],\qquad \widetilde{b}(w)e_j[\mu]=\sum_k B_j^k(\lambda,w)\,e_k[\mu-2]\,,$$

$$\widetilde{c}(w)e_j[\mu]=\sum_k C_j^k(\lambda,w)\,e_k[\mu+2],\qquad \widetilde{d}(w)e_j[\mu]=\sum_k D_j^k(\lambda,w)e_k[\mu]\,,$$

for suitable functions $A_j^k(\lambda, w), B_j^k(\lambda, w), C_j^k(\lambda, w), D_j^k(\lambda, w) \in \mathrm{Fun}(\mathbb{C})$ and then set

$$a(\lambda, w)e'_j[\mu] = \sum_k A_j^k(\lambda, w)e'_k[\mu], \qquad b(\lambda, w)e'_j[\mu] = \sum_k B_j^k(\lambda, w)e'_k[\mu - 2]\,,$$

$$c(\lambda, w)e'_j[\mu] = \sum_k C_j^k(\lambda, w)e'_k[\mu + 2], \qquad d(\lambda, w)e'_j[\mu] = \sum_k D_j^k(\lambda, w)e'_k[\mu]\,.$$

These formulae define an $E_{\tau,\eta}(sl_2)$-module structure on V.

Let V and W be complex vector spaces. Any meromorphic 1-periodic function $\varphi(\lambda)$ with values in $\mathrm{Hom}_{\mathbb{C}}(V, W)$ induces a homomorphism $\mathrm{Fun}(V) \to \mathrm{Fun}(W)$, $F(\lambda) \mapsto \varphi(\lambda)F(\lambda)$.

A *morphism* of an $E_{\tau,\eta}(sl_2)$-module V to an $E_{\tau,\eta}(sl_2)$-module W is a 1-periodic meromorphic function $\varphi(\lambda)$ with values in $\mathrm{Hom}_{\mathbb{C}}(V, W)$ such that the induced homomorphism $\mathrm{Fun}(V) \to \mathrm{Fun}(W)$ commutes with the action of the operators $f(\lambda, \widetilde{h}), \widetilde{a}(w), \widetilde{b}(w), \widetilde{c}(w), \widetilde{d}(w)$. A morphism is an *isomorphism*, if the homomorphism $\varphi(\lambda)$ is nondegenerate for generic λ.

If $\varphi_1(\lambda) \in \mathrm{Hom}(V_1, W_1)$ and $\varphi_2(\lambda) \in \mathrm{Hom}(V_2, W_2)$ are morphisms, then

$$(\varphi_1 \otimes \varphi_2)(\lambda) = \varphi_1^{(1)}(\lambda - 2\eta h^{(2)})\varphi_2^{(2)}(\lambda)$$

is a morphism from $V_1 \otimes V_2$ to $W_1 \otimes W_2$.

An $E_{\tau,\eta}(sl_2)$-module W is *irreducible*, if for all non-trivial morphisms $\varphi(\lambda)$: $V \to W$ the map $\varphi(\lambda)$ is surjective for generic λ. A module is *reducible*, if it is not irreducible.

A *singular vector* in an $E_{\tau,\eta}(sl_2)$-module V is a non-zero element $v \in \mathrm{Fun}(V)$ such that $\widetilde{c}(w)v = 0$ for all w. An element $v \in \mathrm{Fun}(V)$ is of *h-weight* μ, if $f(\lambda, \widetilde{h})v = f(\lambda, \mu)v$ for all $f(\lambda, \widetilde{h})$. An element $v \in \mathrm{Fun}(V)$ is of *weight* $(\mu, A(\lambda, w), D(\lambda, w))$, if it is of h-weight μ and $\widetilde{a}(w)v = A(\lambda, w)v$, $\widetilde{d}(w)v = D(\lambda, w)v$ for all w.

Let $A(\lambda, w)$ and $D(\lambda, w)$ be two functions, $\Lambda \in \mathbb{C}/\frac{1}{\eta}\mathbb{Z}$. An $E_{\tau,\eta}(sl_2)$-module is a *highest weight module* with highest weight $(\Lambda, A(\lambda, w), D(\lambda, w))$ and highest weight vector $v \in \mathrm{Fun}(V)$, if v is a singular vector of weight $(\Lambda, A(\lambda, w), D(\lambda, w))$ and if $\mathrm{Fun}(V)$ is generated over $\mathrm{Fun}(\mathbb{C})$ by elements of the form $\widetilde{b}(w_1) \cdot \cdots \cdot \widetilde{b}(w_n)v$, where $w_1, \ldots, w_n$ is an arbitrary finite set.

In this paper we will consider only the highest weight modules with highest weights $(\Lambda, A(\lambda, w), D(\lambda, w))$ such that the functions $A(\lambda, w)$ and $D(\lambda, w)$ are not identically equal to zero.

Theorem 1. *A highest weight $E_{\tau,\eta}(sl_2)$-module is reducible, if and only if the module has a singular vector of h-weight $\Lambda - 2k$ for a positive integer k.*

Theorem 2. *Let V and W be irreducible highest weight $E_{\tau,\eta}(sl_2)$-modules with highest weight vectors $v \in \mathrm{Fun}(V)$ and $w \in \mathrm{Fun}(W)$ and the same highest weight. Then any isomorphism $V \to W$ sends the highest weight vector of the first module to the highest weight vector of the second module multiplied by a scalar meromorphic doubly periodic function $g(\lambda)$ with periods 1 and 2η. Moreover, for any such a function $g(\lambda)$ there exists a unique isomorphism $V \to W$ sending v to $g(\lambda)w$.*

4. Evaluation Modules

For complex numbers Λ and z we describe an infinite dimensional module $V_\Lambda(z)$ of the elliptic quantum group which we call the *evaluation Verma module.* If $\Lambda = n + (m + l\tau)/2\eta$, where n, m, l are integers, $n \geqq 0$, then the module has a submodule such that the quotient module $L_\Lambda(z)$ has finite dimension $n + 1$.

Let $V_\Lambda(z)$ be an infinite dimensional complex vector space with a basis e_k, $k \in \mathbb{Z}_{\geqq 0}$. Define an action of $f(h)$ by

$$f(h)e_k = f(\Lambda - 2k)e_k ,$$

and an action of the other generators by

$$a(\lambda, w)e_k = \frac{\theta(z - w + (\Lambda + 1 - 2k)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(\lambda + 2k\eta)}{\theta(\lambda)} e_k ,$$

$$b(\lambda, w)e_k = -\frac{\theta(-\lambda + z - w + (\Lambda - 1 - 2k)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(2\eta)}{\theta(\lambda)} e_{k+1} ,$$

$$c(\lambda, w)e_k = -\frac{\theta(-\lambda - z + w + (\Lambda + 1 - 2k)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(2(\Lambda + 1 - k)\eta)}{\theta(\lambda)} \frac{\theta(2k\eta)}{\theta(2\eta)} e_{k-1}$$

$$d(\lambda, w)e_k = \frac{\theta(z - w + (-\Lambda + 1 + 2k)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(\lambda - 2(\Lambda - k)\eta)}{\theta(\lambda)} e_k .$$

Theorem 3. *These formulae define an $E_{\tau,\eta}(sl_2)$-module structure on $V_\Lambda(z)$. If $\Lambda = n + (m + l\tau)/2\eta$, where n, m, l are integers, $n \geqq 0$, then the subspace spanned by e_k, $k > n$, is a submodule. The quotient space $L_\Lambda(z)$ is a module of dimension $n + 1$.*

The module is a highest weight module with highest weight vector e_0 and highest weight $(\Lambda, A(\lambda, w), D(\lambda, w))$, where $A(\lambda, w) = 1$ and the function $D(\lambda, w)$ is determined by Λ and z, namely,

$$D(\lambda, w) = \frac{\theta(z - w + (-\Lambda + 1)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(\lambda - 2\Lambda\eta)}{\theta(\lambda)} .$$

Here e_0 is considered as a constant function in $\mathrm{Fun}(V_\Lambda(z))$.

Remark. Let $z(\lambda)$ be a doubly periodic meromorphic function with periods 1 and 2η. Let Λ be a complex number. Then the above formulae define an $E_{\tau,\eta}(sl_2)$-module structure on a complex space $V_\Lambda(z(\lambda))$ with a basis e_k, if we substitute $z = z(\lambda)$. If $\Lambda = n + (m + l\tau)/2\eta$, where n, m, l are integers, $n \geqq 0$, then the subspace spanned by e_k, $k > n$, is a submodule. The quotient space $L_\Lambda(z(\lambda))$ is a module of dimension $n + 1$.

Theorem 4. *For generic η the evaluation module $V_\Lambda(z)$ is reducible if and only if $\Lambda = n + (m + l\tau)/2\eta$, where n, m, l are integers, $n \geqq 0$.*

For generic η the finite dimensional module $L_{n+(m+l\tau)/2\eta}(z)$ is irreducible.

Denote by $W_\Lambda(z)$ the finite dimensional module $L_\Lambda(z)$, if $\Lambda = n + (m + l\tau)/2\eta$ where m, l are integers, $n \geqq 0$, and the infinite dimensional module $V_\Lambda(z)$ if Λ does not have this form.

Theorem 5. *Let η be generic, $\Lambda_1, \dots, \Lambda_n$ arbitrary. Then for generic $z_1, \dots, z_n$ the module*

$$W_{\Lambda_1,\dots,\Lambda_n}(z_1, \dots, z_n) = W_{\Lambda_1}(z_1) \otimes \cdots \otimes W_{\Lambda_n}(z_n)$$

is an irreducible highest weight $E_{\tau,\eta}(sl_2)$-module with highest weight $(\Lambda_1 + \cdots + \Lambda_n, 1, D(z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n))$,

$$D(z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n) = \frac{\theta(\lambda - 2(\Lambda_1 + \cdots + \Lambda_n)\eta)}{\theta(\lambda)} \prod_{k=1}^{n} \frac{\theta(z_k - w + (-\Lambda_k + 1)\eta)}{\theta(z_k - w + (\Lambda_k + 1)\eta)},$$

and highest weight vector $e_0(1) \otimes \cdots \otimes e_0(n)$, where $e_0(j)$ is the highest weight vector of the j^{th} factor.

Corollary of Theorems 2 and 5. *Two irreducible tensor products of evaluation modules are isomorphic, if the highest weights of the products are equal (while their factors could have different z_k and Λ_k).*

5. One Dimensional Modules

Theorem 6. *If an $E_{\tau,\eta}(sl_2)$-module is one dimensional, then $b(\lambda, w)$ and $c(\lambda, w)$ act by zero for all λ, w and the module is a highest weight module with highest weight (Λ, A, D). The highest weight is of one of the following three types.*

If A and D are nonzero, then $2\eta\Lambda = l\tau$, where l is an integer,

$$A(\lambda, w) = f(\lambda, w)g(\lambda), \qquad D(\lambda, w) = e^{2\pi i l(\lambda - w)} f(\lambda, w) j(\lambda)/g(\lambda + 2\eta),$$

where $f(\lambda, w), g(\lambda), j(\lambda)$ are arbitrary meromorphic functions such that $f(\lambda, w)$ and $j(\lambda)$ are 2η-periodic in λ and $A(\lambda, w), D(\lambda, w)$ are 1-periodic in λ.

If $A \equiv 0$, then Λ is arbitrary and $D(\lambda, w) = f(\lambda, w)g(\lambda)$, where $D(\lambda, w)$, $f(\lambda, w)$, $g(\lambda)$ have the same periodicity properties as before.

If $D \equiv 0$, then Λ is arbitrary and $A(\lambda, w) = f(\lambda, w)g(\lambda)$, where $A(\lambda, w)$, $f(\lambda, w)$, $g(\lambda)$ have the same periodicity properties as before.

Denote the module corresponding to nonzero A and D by $U_{l\tau,f,g,j}$ and by $U_{l\tau,j}$, if $f = g = 1$.

An important special case is formed by the modules with $f = g = 1, j = \text{const}$. We have

$$U_{l_1\tau,f_1,g_1,j_1} \otimes U_{l_2\tau,f_2,g_2,j_2} = U_{(l_1+l_2)\tau,f,g,j},$$

where

$$f(\lambda, w) = f_1(\lambda - l_2\tau, w) f_2(\lambda, w), \qquad g(\lambda) = g_1(\lambda - l_2\tau) g_2(\lambda),$$

$$j(\lambda) = e^{-2\pi i l_1 l_2 \tau} j_1(\lambda - l_2\tau) j_2(\lambda).$$

The set of one dimensional modules forms a group with respect to the tensor product; $U_{0,1}$ is the unit element.

6. Isomorphism of Evaluation Modules

Theorem 7. *For any complex numbers Λ, z and an integer m, the evaluation modules $V_{\Lambda+m/\eta}(z)$ and $V_\Lambda(z)$ are isomorphic with an isomorphism map $\varphi(\lambda): V_{\Lambda+m/\eta}(z) \to V_\Lambda(z)$, $e_k \mapsto e_k$.*

For any Λ, z and an integer l, the evaluation modules $V_{\Lambda+l\tau/\eta}(z)$ and $U_{l\tau,j_L} \otimes V_\Lambda(z) \otimes U_{l\tau,j_R}$ are isomorphic with an isomorphism map

$$\varphi(\lambda): V_{\Lambda+l\tau/\eta}(z) \to U_{l\tau,j_L} \otimes V_\Lambda(z) \otimes U_{l\tau,j_R}, \qquad e_k \mapsto v_L \otimes e_k \otimes v_R\,,$$

where

$$j_L = e^{2\pi i(z+(\Lambda+1)\eta)l}, \qquad j_R = e^{2\pi i(z+3(\Lambda-1)\eta)l+4\pi i l^2\tau}\,.$$

and v_L, v_R are generating vectors of the one dimensional modules.

Remark. Theorems 3 and 7 show that for every non-negative integer n there are at most four potentially non-isomorphic $n+1$-dimensional evaluation modules modulo tensoring with one dimensional representations.

7. Highest Weights of Finite Type Modules

In this section we discuss highest weights of highest weight $E_{\tau,\eta}(sl_2)$-modules of finite type. Recall that we consider only the modules with highest weights $(\Lambda, A(\lambda,w), D(\lambda,w))$ such that both functions A and D are not identically equal to zero.

Lemma 8. *If $(\Lambda, A(\lambda,w), B(\lambda,w))$ is a highest weight, then*

$$A = f(\lambda,w)g(\lambda), \qquad D(\lambda,w) = \frac{\theta(\lambda-2\eta\Lambda)}{\theta(\lambda)}\frac{h(\lambda,w)}{g(\lambda+2\eta)}\,,$$

where $f(\lambda,w), g(\lambda), h(\lambda,w)$ are meromorphic functions such that $f(\lambda,w)$ and $h(\lambda,w)$ are 2η-periodic in λ and $A(\lambda,w)$ and $D(\lambda,w)$ are 1-periodic in λ.

Considering highest weights we could restrict ourselves to the case $A(\lambda,w)=1$. In fact, let V be a highest weight $E_{\tau,\eta}(sl_2)$-module of finite type with highest weight $(\Lambda, A(\lambda,w), D(\lambda,w))$. There is a one dimensional $E_{\tau,\eta}(sl_2)$-module U with highest weight $(0, 1/A(\lambda,w), C(\lambda,w))$ with suitable $C(\lambda,w)$. Then $V \otimes U$ is a highest weight module of finite type with highest weight $(\Lambda, 1, D(\lambda,w)C(\lambda,w))$.

Theorem 9. *Let η be real and irrational. Let a highest weight $E_{\tau,\eta}(sl_2)$-module be of finite type and have highest weight of the form $(\Lambda, 1, D(\lambda,w))$ with nonzero function $D(\lambda,w)$. Then $D(\lambda,w)$ has the form*

$$D(\lambda,w) = F\frac{\theta(\lambda-2\Lambda\eta)}{\theta(\lambda)}\prod_{k=1}^{n}\frac{\theta(s_k-w)}{\theta(t_k-w)}\,, \tag{6}$$

where $n \geqq 0$ and F, s_k, t_k are suitable constants.

Note, that after tensoring the highest weight module with a suitable one dimensional module, we can set $F = 1$ in (6).

Consider the submodule of a tensor product of evaluation modules generated by the tensor product of highest weight vectors of factors. Consider the quotient of the submodule by its maximal proper submodule. The resulting irreducible highest weight module will be called the *irreducible module associated with the tensor product.*

Corollary. *Let η be real and irrational, then any irreducible highest weight module of finite type with highest weight $(\Lambda, 1, D(\lambda, w))$, where $D(\lambda, w)$ is given by* (6) *with $F = 1$, is isomorphic to the irreducible highest weight module associated with a tensor product of evaluation representations.*

8. Fusion of Evaluation Modules

Theorem 10. *The tensor product of two evaluation modules $V_{\Lambda_1}(z_1) \otimes V_{\Lambda_2}(z_2)$ contains the evaluation submodule $V_{\Lambda_1+\Lambda_2}(z_2 - \Lambda_1\eta)$, if $z_2 - z_1 = (\Lambda_1 + \Lambda_2)\eta$. The imbedding $V_{\Lambda_1+\Lambda_2}(z_2 - \Lambda_1\eta) \hookrightarrow V_{\Lambda_1}(z_1) \otimes V_{\Lambda_2}(z_1 + (\Lambda_1 + \Lambda_2)\eta)$ is defined by the following formula, involving elliptic binomial coefficients,*

$$e_j \mapsto \sum_{l=0}^{j} \frac{\theta(2j\eta)\,\theta(2(j-1)\eta)\cdots\theta(2(l+1)\eta)}{\theta(2\eta)\cdots\theta(2(j-l)\eta)} e_l \otimes e_{j-l}\,. \tag{7}$$

Theorem 11. *Let η be generic. Let Λ_1, Λ_2 have the form $\Lambda_k = n_k + (m_k + l_k\tau)/2\eta$, where $k = 1, 2$ and n_k, m_k, l_k are integers, $n_k \geqq 0$. Let $V = L_{\Lambda_1}(z_1) \otimes L_{\Lambda_2}(z_2)$ be the tensor product of two finite dimensional irreducible evaluation modules. The tensor product is reducible if and only if*

$$z_1 - z_2 \text{ or } z_2 - z_1 = (\Lambda_1 + \Lambda_2 - 2j + 2)\eta + m + l\tau$$

for some integers $j, l, m, 0 < j \leqq min\{n_1, n_2\}$. In this case the tensor product has a unique proper submodule W. Moreover:

If $z_2 - z_1 = (\Lambda_1 + \Lambda_2 - 2j + 2)\eta + m + l\tau$, then

$$W \simeq L_{j-1}(z_1 + (\Lambda_1 - j + 1)\eta) \otimes L_{\Lambda_1+\Lambda_2-j+1}(z_2 - (\Lambda_1 - j + 1)\eta) \otimes U_1\,,$$

$$V/W \simeq L_{\Lambda_1-j}(z_1 - j\eta) \otimes L_{\Lambda_2-j}(z_2 + j\eta) \otimes U_2\,,$$

where U_1, U_2 are suitable one dimensional modules.

If $z_1 - z_2 = (\Lambda_1 + \Lambda_2 - 2j + 2)\eta + m + l\tau$, then

$$W \simeq L_{\Lambda_1-j}(z_1 + j\eta) \otimes L_{\Lambda_2-j}(z_2 - j\eta) \otimes U_1\,,$$

$$V/W \simeq L_{j-1}(z_1 - (\Lambda_1 - j + 1)\eta) \otimes L_{\Lambda_1+\Lambda_2-j+1}(z_2 + (\Lambda_1 - j + 1)\eta) \otimes U_2\,,$$

where U_1, U_2 are suitable one dimensional modules.

If $z_1 - z_2 = (\Lambda_1 + \Lambda_2 - 2j + 2)\eta + m + l\tau$, then the imbedding $W \hookrightarrow V$ sends the highest weight vector of W to a singular vector of V of h-weight $\Lambda_1 + \Lambda_2 - 2j$. The singular vector is unique up to multiplication by a scalar function of λ. The singular vector has the form $\sum_l (-1)^l A_l(\lambda) e_l \otimes e_{j-l}$, where

$$A_l(\lambda) = \frac{\theta(2j\eta)\theta(2(j-1)\eta)\cdots\theta(2(l+1)\eta)}{\theta(2\eta)\theta(4\eta)\cdots\theta(2(j-l)\eta)}$$

$$\times \prod_{i=0}^{l-1} \frac{\theta(-\lambda + 2\eta(\Lambda_1 + \Lambda_2 - 2j + l - i + 1))}{\theta(-\lambda + 2\eta(-j + l - i))}$$

$$\times \left[\prod_{i=0}^{l-1} \theta(2(\Lambda_1 - i)\eta) \cdot \prod_{i=0}^{j-l-1} \theta(2(\Lambda_2 - i)\eta) \right]^{-1} .$$

If $z_2 - z_1 = (\Lambda_1 + \Lambda_2)\eta$, then $W \simeq L_{\Lambda_1+\Lambda_2}(z_2 - \Lambda_1\eta)$, and the imbedding $L_{\Lambda_1+\Lambda_2}(z_2 - \Lambda_1\eta) \hookrightarrow L_{\Lambda_1}(z_1) \otimes L_{\Lambda_2}(z_1 + (\Lambda_1 + \Lambda_2)\eta)$ is given by formula (7).

9. The Universal Evaluation Module

The formulae defining evaluation representations admit the following generalization. Let $\mathcal{V}$ be the complex vector space of all functions in λ, h, z, Λ. Define an action of the operator algebra $A_{\tau,\eta}(sl_2)$ on $\mathcal{V}$. Namely, define an action of operators $f(\lambda, \widetilde{h})$ by the rule

$$(f(\lambda, \widetilde{h})v)(\lambda, h, z, \Lambda) = f(\lambda, h)v(\lambda, h, z, \Lambda) ,$$

where $v(\lambda, h, z, \Lambda) \in \mathcal{V}$, and an action of the other operators by the rule

$$(\widetilde{a}(w)v)(\lambda, h, z, \Lambda) = \frac{\theta(z - w + (h+1)\eta)}{\theta(z - w + (\Lambda+1)\eta)} \frac{\theta(\lambda - (h - \Lambda)\eta)}{\theta(\lambda)} v(\lambda - 2\eta, h, z, \Lambda) ,$$

$$(\widetilde{b}(w)v)(\lambda, h, z, \Lambda) = -\frac{\theta(-\lambda + z - w + (h-1)\eta)}{\theta(z - w + (\Lambda+1)\eta)} \frac{\theta(2\eta)}{\theta(\lambda)} v(\lambda + 2\eta, h - 2, z, \Lambda) ,$$

$$(\widetilde{c}(w)v)(\lambda, h, z, \Lambda) = -\frac{\theta(-\lambda - z + w + (h+1)\eta)}{\theta(z - w + (\Lambda+1)\eta)} \frac{\theta((h + \Lambda + 2)\eta)}{\theta(\lambda)} \frac{\theta((\Lambda - h)\eta)}{\theta(2\eta)}$$

$$\times v(\lambda - 2\eta, h + 2, z, \Lambda) ,$$

$$(\widetilde{d}(w)v)(\lambda, h, z, \Lambda) = \frac{\theta(z - w + (-h+1)\eta)}{\theta(z - w + (\Lambda+1)\eta)} \frac{\theta(\lambda - (h + \Lambda)\eta)}{\theta(\lambda)} v(\lambda + 2\eta, h, z, \Lambda) .$$

Theorem 12. *The operators* $f(\lambda, \widetilde{h}), \widetilde{a}(w), \widetilde{b}(w), \widetilde{c}(w), \widetilde{d}(w) \in \mathrm{End}(\mathcal{V})$ *satisfy the relations of the elliptic operator algebra* $A_{\tau,\eta}(sl_2)$.

$\mathcal{V}$ is called the *universal generalized evaluation* $A_{\tau,\eta}(sl_2)$*-module*.
Precisely speaking $\mathcal{V}$ is not a module in the sense defined in Sect. 3.

The universal evaluation module has many invariant subspaces constructed in the following way. Let $\mathbb{C}^4$ be the complex space with coordinates λ, h, z, Λ, and $X \subset \mathbb{C}^4$ a subset invariant with respect to the following four transformations:

$$(\lambda, h, z, \Lambda) \mapsto (\lambda \pm 2\eta, h, z, \Lambda)\,,$$
$$(\lambda, h, z, \Lambda) \mapsto (\lambda, h \pm 2, z, \Lambda)\,.$$

Then the subspace $\mathscr{V}(X) \subset \mathscr{V}$ of all functions with support in X is invariant with respect to the $A_{\tau,\eta}(sl_2)$-action.

For example, the set X of lines $z = \text{const}$, $\Lambda = \text{const}$, $h = \Xi - 2k$, where Ξ is a fixed number and $k \in \mathbb{Z}$, leads to the following *cyclic* $E_{\tau,\eta}(sl_2)$*-module* $V_{\Lambda,\Xi}(z)$.

$V_{\Lambda,\Xi}(z)$ is an infinite dimensional complex vector space with a basis e_k, $k \in \mathbb{Z}$. The action of $f(h)$ is defined by

$$f(h)e_k = f(\Xi - 2k)e_k\,,$$

and the action of the other generators is defined by

$$a(\lambda, w)e_k = \frac{\theta(z - w + (\Xi - 2k + 1)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(\lambda - (\Xi - \Lambda - 2k)\eta)}{\theta(\lambda)} e_k\,,$$

$$b(\lambda, w)e_k = -\frac{\theta(-\lambda + z - w + (\Xi - 2k - 1)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(2\eta)}{\theta(\lambda)} e_{k+1}\,,$$

$$c(\lambda, w)e_k = -\frac{\theta(-\lambda - z + w + (\Xi - 2k + 1)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \times \frac{\theta((\Xi + \Lambda + 2 - 2k)\eta)}{\theta(\lambda)} \frac{\theta((\Lambda - \Xi + 2k)\eta)}{\theta(2\eta)} e_{k-1}\,,$$

$$d(\lambda, w)e_k = \frac{\theta(z - w + (-\Xi + 2k + 1)\eta)}{\theta(z - w + (\Lambda + 1)\eta)} \frac{\theta(\lambda - (\Xi + \Lambda - 2k)\eta)}{\theta(\lambda)} e_k\,.$$

If $\Lambda = \Xi$, then the subspace of $V_{\Lambda,\Xi}(z)$ spanned by the vectors e_k, $k \geqq 0$, forms a submodule which is the evaluation module $V_\Lambda(z)$.

The module $V_\Lambda(z(\lambda))$ described in Sect. 4 also could be obtained by this construction.

Theorem 12 is an easy corollary of Theorem 3.

Remark. One can construct universal evaluation modules for the Yangian $Y(sl_2)$ and the quantum loop group $U_q(\widetilde{sl_2})$ by similar formulae. It might be that these modules are new.

10. Determinant

The element

$$\mathrm{Det}(\lambda, w) = \frac{\theta(\lambda)}{\theta(\lambda - 2\eta h)}(d(\lambda, w + 2\eta)a(\lambda + 2\eta, w) - b(\lambda, w + 2\eta)c(\lambda + 2\eta, w))$$

$$= \frac{\theta(\lambda)}{\theta(\lambda - 2\eta h)}(a(\lambda, w + 2\eta)d(\lambda - 2\eta, w) - c(\lambda, w + 2\eta)b(\lambda - 2\eta, w))$$

is called the *determinant element* of the elliptic quantum group $E_{\tau,\eta}(sl_2)$, the corresponding element

$$\widetilde{\mathrm{Det}}(w) = \frac{\theta(\lambda)}{\theta(\lambda - 2\eta\widetilde{h})}(\widetilde{d}(w+2\eta)\widetilde{a}(w) - \widetilde{b}(w+2\eta)\widetilde{c}(w))$$

$$= \frac{\theta(\lambda)}{\theta(\lambda - 2\eta\widetilde{h})}(\widetilde{a}(w+2\eta)\widetilde{d}(w) - \widetilde{c}(w+2\eta)\widetilde{b}(w))$$

is called the *determinant element* of the elliptic quantum algebra $A_{\tau,\eta}(sl_2)$.

Theorem 13. *The determinant element* $\widetilde{\mathrm{Det}}(w)$ *is a central element in the elliptic quantum algebra* $A_{\tau,\eta}(sl_2)$.

Theorem 14. *The determinant element* $\mathrm{Det}(\lambda, w)$ *is a group-like element in the elliptic quantum group* $E_{\tau,\eta}(sl_2)$. *Namely, if* V *and* W *are* $E_{\tau,\eta}(sl_2)$*-modules, then* $\mathrm{Det}(\lambda, w)$ *acts in the module* $V \otimes W$ *as*

$$\mathrm{Det}(\lambda - 2\eta h^{(2)}, w) \otimes \mathrm{Det}(\lambda, w)\,.$$

Note that other formulae for the determinant element could be deduced from relations (4).

11. Dual Modules

Define a homomorphism $E_{\tau,\eta}(sl_2) \to \mathbb{C}$, the *counit*, by the rule $f(h) \mapsto f(0)$,

$$a(\lambda, w) \mapsto 1, \qquad b(\lambda, w) \mapsto 0\,,$$

$$c(\lambda, w) \mapsto 0, \qquad d(\lambda, w) \mapsto 1\,.$$

The counit defines an $E_{\tau,\eta}(sl_2)$-module structure on $\mathbb{C}$.

Let $E_{\tau,\eta}(sl_2) \to \mathrm{End}(V)$ be an $E_{\tau,\eta}(sl_2)$-module structure on a complex vector space V. Assume that the determinant element $\mathrm{Det}(\lambda, w) \in \mathrm{End}(V)$ is nondegenerate for generic λ and w and denote by $\mathrm{Det}^{-1}(\lambda, w)$ its inverse. Let

$$V^* = \bigoplus_{\mu} V[\mu]^*$$

be the restricted dual space to V. We introduce an $E_{\tau,\eta}(sl_2)$-module structure on the restricted dual. The restricted dual space with this $E_{\tau,\eta}(sl_2)$-module structure is called the module *dual* to V.

Introduce linear maps $Sf(h), Sa(\lambda, w), Sb(\lambda, w), Sc(\lambda, w), Sd(\lambda, w) \in \mathrm{End}(V)$ by the rule

$$Sf(h) = f(-h)$$

$$Sa(\lambda, w) = \frac{\theta(\lambda + 2\eta h - 2\eta)}{\theta(\lambda - 2\eta)}\mathrm{Det}^{-1}(\lambda + 2\eta h - 2\eta, w)d(\lambda + 2\eta h - 2\eta, w + 2\eta)\,,$$

$$Sb(\lambda, w) = -\frac{\theta(\lambda + 2\eta h + 2\eta)}{\theta(\lambda + 2\eta)}\mathrm{Det}^{-1}(\lambda + 2\eta h + 2\eta, w)b(\lambda + 2\eta h - 2\eta, w + 2\eta)\,,$$

$$Sc(\lambda,w)=-\frac{\theta(\lambda+2\eta h-2\eta)}{\theta(\lambda-2\eta)}\mathrm{Det}^{-1}(\lambda+2\eta h-2\eta,w)c(\lambda+2\eta h+2\eta,w+2\eta)\,,$$

$$Sd(\lambda,w)=\frac{\theta(\lambda+2\eta h+2\eta)}{\theta(\lambda+2\eta)}\mathrm{Det}^{-1}(\lambda+2\eta h+2\eta,w)a(\lambda+2\eta h+2\eta,w+2\eta)\,. \quad (8)$$

Let $Sf(h)^*, Sa(\lambda,w)^*, Sb(\lambda,w)^*, Sc(\lambda,w)^*, Sd(\lambda,w)^* \in \mathrm{End}(V^*)$ be their dual maps, respectively.

Theorem 15. *Let V be an $E_{\tau,\eta}(sl_2)$-module. Then the linear maps $Sf(h)^*, Sa(\lambda,w)^*$, $Sb(\lambda,w)^*, Sc(\lambda,w)^*, Sd(\lambda,w)^* \in \mathrm{End}(V^*)$ define an $E_{\tau,\eta}(sl_2)$-module structure on V^*.*

Consider $V\otimes V^*$ and $V^*\otimes V$ as $E_{\tau,\eta}(sl_2)$-modules. Then the natural maps $\mathbb{C}\to V\otimes V^*$ and $V^*\otimes V\to\mathbb{C}$ are homomorphisms of $E_{\tau,\eta}(sl_2)$-modules.

Note that $\mathrm{Det}(\lambda,w)$ acts in V^* as $\mathrm{Det}^{-1}(\lambda+2\eta h,w)^*$.

Formulae (8) play the role of the *antipode* for the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

Repeating the above construction we introduce a new module structure on the initial vector space $V=(V^*)^*$. In this case the generators $f(h)$, $a(\lambda,w)$, $b(\lambda,w)$, $c(\lambda,w)$, $d(\lambda,w)$ act as $f(h)$,

$$\frac{\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}\,\frac{\theta(\lambda)}{\theta(\lambda-2\eta)}\,\frac{\mathrm{Det}(\lambda,w)}{\mathrm{Det}(\lambda,w+2\eta)}a(\lambda,w+4\eta)\,,$$

$$\frac{\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}\,\frac{\theta(\lambda)}{\theta(\lambda+2\eta)}\,\frac{\mathrm{Det}(\lambda,w)}{\mathrm{Det}(\lambda,w+2\eta)}b(\lambda,w+4\eta)\,,$$

$$\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}\,\frac{\theta(\lambda)}{\theta(\lambda-2\eta)}\,\frac{\mathrm{Det}(\lambda,w)}{\mathrm{Det}(\lambda,w+2\eta)}c(\lambda,w+4\eta)\,,$$

$$\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}\,\frac{\theta(\lambda)}{\theta(\lambda+2\eta)}\,\frac{\mathrm{Det}(\lambda,w)}{\mathrm{Det}(\lambda,w+2\eta)}d(\lambda,w+4\eta)\,,$$

respectively, and the determinant element $\mathrm{Det}(\lambda,w)$ acts as $\mathrm{Det}(\lambda,w)$.

Remark. These formulae inspire the following two constructions of automorphisms of the elliptic quantum group. Namely, for any 1-periodic nonzero meromorphic function $g(\lambda)$ one can define two automorphisms $I,J : E_{\tau,\eta}(sl_2)\to E_{\tau,\eta}(sl_2)$ by

$$I: f(h)\mapsto f(h)\,,$$

$$I: a(\lambda,w)\mapsto g(\lambda-2\eta)a(\lambda,w),\qquad I: b(\lambda,w)\mapsto g(\lambda)^{-1}b(\lambda,w)\,,$$

$$I: c(\lambda,w)\mapsto g(\lambda-2\eta)c(\lambda,w),\qquad I: d(\lambda,w)\mapsto g(\lambda)^{-1}d(\lambda,w)\,,$$

and

$$J: f(h)\mapsto f(h)\,,$$

$$J: a(\lambda,w)\mapsto g(\lambda-2\eta h-2\eta)^{-1}a(\lambda,w),\qquad J: b(\lambda,w)\mapsto g(\lambda-2\eta h-2\eta)^{-1}b(\lambda,w)\,,$$

$$J: c(\lambda,w)\mapsto g(\lambda-2\eta h)c(\lambda,w),\qquad J: d(\lambda,w)\mapsto g(\lambda-2\eta h)d(\lambda,w)\,.$$

These automorphisms preserve the determinant element, $I,J:\mathrm{Det}(\lambda,w)\mapsto\mathrm{Det}(\lambda,w)$.

12. Weyl Group

The map

$$f(h) \mapsto f(-h),$$
$$a(\lambda, w) \mapsto d(-\lambda, w), \qquad b(\lambda, w) \mapsto c(-\lambda, w),$$
$$c(\lambda, w) \mapsto b(-\lambda, w), \qquad d(\lambda, w) \mapsto a(-\lambda, w)$$

defines an automorphism of order two of the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

This fact follows from identities $\alpha(-\lambda) = \delta(\lambda)$ and $\beta(-\lambda) = \gamma(\lambda)$, where $\alpha, \beta, \gamma, \delta$ are defined in Sect. 2.

13. Solutions to the Dynamical Yang–Baxter Equation

In this section we construct solutions to the dynamical Yang–Baxter equation.

Let V_k be an irreducible highest weight module with highest weight vector $v_k \in \mathrm{Fun}(V_k)$ and highest weight $(\Lambda_k, 1, D_k(\lambda, w))$, $k = 1, 2, 3$. Assume that all tensor products $V_i \otimes V_j$ and $V_i \otimes V_j \otimes V_k$ are irreducible highest weight modules, where i, j, k are pair-wise distinct. Fix highest weight vectors $v_i(\lambda - 2\eta\Lambda_j) \otimes v_j(\lambda)$ in $\mathrm{Fun}(V_i \otimes V_j)$ and $v_i(\lambda - 2\eta(\Lambda_j + \Lambda_k)) \otimes v_j(\lambda - 2\eta\Lambda_k) \otimes v_k(\lambda)$ in $\mathrm{Fun}(V_i \otimes V_j \otimes V_k)$.

Assume that the highest weights of the highest weight vectors $v_i(\lambda - 2\eta\Lambda_j) \otimes v_j(\lambda)$ in $\mathrm{Fun}(V_i \otimes V_j)$ and $v_j(\lambda - 2\eta\Lambda_i) \otimes v_i(\lambda)$ in $\mathrm{Fun}(V_j \otimes V_i)$ are the same and, moreover, the highest weights of the highest weight vectors $v_i(\lambda - 2\eta(\Lambda_j + \Lambda_k)) \otimes v_j(\lambda - 2\eta\Lambda_k) \otimes v_k(\lambda)$ do not depend on the order of the numbers i, j, k.

Note that irreducible tensor products of evaluation modules have all these properties, cf. Theorem 5. These properties also hold, if Λ_j are integers, see Lemma 8.

Let $R^{\vee}_{V_j V_i}(\lambda) : V_j \otimes V_i \to V_i \otimes V_j$ be the unique isomorphism of the modules $V_j \otimes V_i$ and $V_i \otimes V_j$ sending the distinguished highest weight vector to the distinguished highest weight vector. Set $R_{V_i V_j}(\lambda) = R^{\vee}_{V_j V_i}(\lambda) P$, where $P : V_i \otimes V_j \to V_j \otimes V_i$ is the permutation of factors.

Theorem 16. *The linear operators $R_{V_i V_j}(\lambda) \in \mathrm{Hom}(V_i, V_j)$ satisfy the dynamical Yang–Baxter relation,*

$$R_{V_1 V_2}(\lambda - 2\eta h^{(3)}) R_{V_1 V_3}(\lambda) R_{V_2 V_3}(\lambda - 2\eta h^{(1)}) = R_{V_2 V_3}(\lambda) R_{V_1 V_3}(\lambda - 2\eta h^{(2)}) R_{V_1 V_2}(\lambda),$$

and the relation

$$R^{(12)}_{V_1 V_2}(\lambda) R^{(21)}_{V_2 V_1}(\lambda) = Id .$$

Let η be generic. Let $W_{\Lambda'}(z')$ and $W_{\Lambda''}(z'')$ be two irreducible evaluation modules such that $W_{\Lambda'}(z') \otimes W_{\Lambda''}(z'')$ and $W_{\Lambda''}(z'') \otimes W_{\Lambda'}(z')$ are irreducible. Then the unique isomorphism

$$R^{\vee}(\lambda) : W_{\Lambda'}(z') \otimes W_{\Lambda''}(z'') \to W_{\Lambda''}(z'') \otimes W_{\Lambda'}(z')$$

sending $e'_0 \otimes e''_0$ to $e''_0 \otimes e'_0$ can be constructed rather explicitly. Namely, introduce elements $\widetilde{b}^{[k]}(w)$ and $\widetilde{b}_{[k]}(w)$ of the operator algebra by formulae

$$\widetilde{b}^{[k]}(w) = \widetilde{b}(w - 2(k-1)2\eta) \cdots \widetilde{b}(w - 2\eta)\widetilde{b}(w),$$
$$\widetilde{b}_{[k]}(w) = \widetilde{b}(w + 2(k-1)2\eta) \cdots \widetilde{b}(w + 2\eta)\widetilde{b}(w).$$

For any $k \geqq 0$, $p > 0$ set $\widetilde{b}_{k,p} = \widetilde{b}^{[k]}(z' + (\Lambda' + 1)\eta)\widetilde{b}_{[p]}(z'' + (-\Lambda'' + 1)\eta)$ and set

$$\begin{aligned}\widetilde{b}_{k,0} &= \widetilde{b}^{[k-1]}(z' + (\Lambda' - 1)\eta) \cdot \Bigg(\widetilde{b}(z' + (\Lambda' - 1)\eta) \\ &\quad - \frac{\theta(-\lambda + z'' - z' + (\Lambda'' - \Lambda')\eta)}{\theta(z'' - z' + (\Lambda'' - \Lambda' + 2)\eta)} \frac{\theta(2\Lambda''\eta)}{\theta(-\lambda + 2(\Lambda'' - 1)\eta)} \\ &\quad \times \widetilde{b}(z'' + (-\Lambda'' + 1)\eta) \Bigg) .\end{aligned}$$

Then for all k, p we have $\widetilde{b}_{k,p}(e'_0 \otimes e''_0) = f_{k,p}(\lambda, z' - z'')e'_k \otimes e''_p$, where f is a scalar function. The scalar function is equal to an alternating product of theta functions which can be written explicitly using formulae of Sect. 4. Then

$$R^\vee(\lambda) : e'_k \otimes e''_p \mapsto \frac{1}{f_{k,p}(\lambda, z' - z'')} \widetilde{b}_{k,p}(e''_0 \otimes e'_0) .$$

14. Commuting Elements

Theorem 17. *For an $A_{\tau,\eta}(sl_2)$-module M the endomorphisms $\widetilde{t}(w) = \widetilde{a}(w) + \widetilde{d}(w)$ pair-wise commute for all w on the zero h-weight subspace $M[0]$.*

This fact is related to the integrability of the interaction-round-a-face models of statistical mechanics connected with the elliptic quantum group $E_{\tau,\eta}(sl_2)$, see [JMO, JKMO] and references therein. We will describe the Bethe ansatz for these models in the next paper [FTV].

15. The Case $2N\eta = 1$

Representation theory of the elliptic quantum group becomes special, if the parameter η has the form $2N\eta = m + l\tau$, where N, m, l are integers. We will discuss this subject in a separate paper. Here we make remarks about the case $2N\eta = 1$, where N is a natural number. First we construct two families of N-dimensional $E_{\tau,\eta}(sl_2)$-modules $T_\Lambda(z)$ and $T_{\Lambda,\Xi}(z)$, where $\Lambda, \Xi \in \mathbb{C}/\mathbb{Z}$ and $z \in \mathbb{C}$, and then we indicate some central elements of the operator algebra $A_{\tau,\eta}(sl_2)$.

Consider an evaluation module $V_\Lambda(z)$. If $2N\eta = 1$, then the subspace spanned by e_k, $k \geqq N$, is a submodule. The quotient space $T_\Lambda(z)$ is an N-dimensional module.

Consider a cyclic module $V_{\Lambda,\Xi}(z)$ with its distinguished basis e_k, $k \in \mathbb{Z}$. Introduce a new basis v_k, $k \in \mathbb{Z}$. Namely, if N is odd, then set

$$v_k = (-1)^{k(k+1)/2} e_k ,$$

and if N is even, then set

$$v_k = (-1)^{k(k+1)/4} e_k .$$

With respect to the new basis, the formulae for the action of all operators of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ become N-periodic in k. Identifying v_k and v_{k+N} we get an N-dimensional $E_{\tau,\eta}(sl_2)$-module $T_{\Lambda,\Xi}(z)$.

Let $2\eta = 1$. Then the operator algebra $A_{\tau,\eta=1/2}(sl_2)$ is commutative.

There is the following generalization of this fact. Let N be a natural number, z a complex number. Introduce elements $\widetilde{a}^{(N)}(z), \widetilde{b}^{(N)}(z), \widetilde{c}^{(N)}(z), \widetilde{d}^{(N)}(z)$ of the operator algebra $A_{\tau,\eta}(sl_2)$ by the formulae

$$\widetilde{a}^{(N)}(z) = \widetilde{a}(z)\widetilde{a}(z-2\eta)\widetilde{a}(z-4\eta)\cdots\widetilde{a}(z-2(N-1)\eta)\,,$$

$$\widetilde{b}^{(N)}(z) = \widetilde{b}(z)\widetilde{b}(z-2\eta)\widetilde{b}(z-4\eta)\cdots\widetilde{b}(z-2(N-1)\eta)\,,$$

$$\widetilde{c}^{(N)}(z) = \widetilde{c}(z)\widetilde{c}(z-2\eta)\widetilde{c}(z-4\eta)\cdots\widetilde{c}(z-2(N-1)\eta)\,,$$

$$\widetilde{d}^{(N)}(z) = \widetilde{d}(z)\widetilde{d}(z-2\eta)\widetilde{d}(z-4\eta)\cdots\widetilde{d}(z-2(N-1)\eta)\,.$$

For any element of the operator algebra $A_{\tau,\eta}(sl_2)$ of the form $f(\lambda,\widetilde{h})$, where f is a meromorphic function 1-periodic in λ and $1/2\eta$-periodic in $\widetilde{h}$, introduce a new element $f^{(N)}(\lambda,\widetilde{h})$ by the formula $f^{(N)}(\lambda,\widetilde{h}) = f(N\lambda, N\widetilde{h})$.

Let $V(\eta)$ be an $A_{\tau,\eta}(sl_2)$-module meromorphically depending on η and such that for $\eta = 1/(2N)$ the module is well defined.

As an example of such a module we can consider the $A_{\tau,\eta}(sl_2)$-module associated to a tensor product of evaluation or cyclic $E_{\tau,\eta}(sl_2)$-modules.

Theorem 18. *Let $2N\eta = 1$. Consider the action of the operator algebra $A_{\tau,1/2N}(sl_2)$ in the module $V(1/2N)$. Then each of the operators $f^{(N)}(\lambda,\widetilde{h}), \widetilde{a}^{(N)}(z), \widetilde{b}^{(N)}(z), \widetilde{c}^{(N)}(z), \widetilde{d}^{(N)}(z)$, $z \in \mathbb{C}$, commutes with all of the operators of the operator algebra.*

The proof of Theorem 18 is based on the following two facts.

Let $R(\lambda, w, \eta, \tau) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be the R-matrix defined in the introduction, then the image of the linear map $R(\lambda, -2\eta, \eta, \tau)$ is three dimensional and is generated by vectors $v_+ \otimes v_+$, $v_+ \otimes v_- + v_- \otimes v_+$, $v_- \otimes v_-$, where $v_+ = (1,0)$ and $v_- = (0,1)$.

If $2N\eta = 1$, then the vectors $v_+ \otimes v_+$ and $v_- \otimes v_-$ lie in the kernel of the linear map $\mathrm{Res}_{w=-2(N-1)\eta} R(\lambda, w, \eta, \tau)$.

Now the proof of Theorem 18 follows from analysis of two Yang–Baxter-type equations. If $4\eta = 1$, then these two equations are

$$\begin{aligned} &R^{(21)}(\lambda - 2\eta h, -2\eta, \eta, \tau) L^{(2)}(\lambda, z-2\eta) L^{(1)}(\lambda - 2\eta h^{(2)}, z) \\ &\quad = L^{(1)}(\lambda, z) L^{(2)}(\lambda - 2\eta h^{(1)}, z - 2\eta) R^{(21)}(\lambda, -2\eta, \eta, \tau) \end{aligned}$$

and

$$\begin{aligned} &R^{(13)}(\lambda - 2\eta h, z - w, \eta, \tau) R^{(23)}(\lambda - 2\eta(h + h^{(1)}), z - 2\eta - w, \eta, \tau) \\ &\quad \times L^{(1)}(\lambda, z) L^{(2)}(\lambda - 2\eta h^{(1)}, z - 2\eta) L^{(3)}(\lambda - 2\eta(h^{(1)} + h^{(2)}), w) \\ &\quad = L^{(3)}(\lambda, w) L^{(1)}(\lambda - 2\eta h^{(3)}, z) L^{(2)}(\lambda - 2\eta(h^{(1)} + h^{(3)}), z - 2\eta) \\ &\quad \times R^{(13)}(\lambda, z - w, \eta, \tau) R^{(23)}(\lambda - 2\eta h^{(1)}, z - 2\eta - w, \eta, \tau)\,. \end{aligned}$$

If $2N\eta = 1$, then the equations are similar.

16. The Extended Elliptic Quantum Group

The *extended* elliptic quantum group $\widetilde{E}_{\tau,\eta}$ is the algebra over $\mathbb{C}$ with generators of two types. The generators of the first type are labelled by meromorphic functions $f(h)$ of one complex variable with period $1/\eta$. The generators of the second type $a(\lambda,w), b(\lambda,w), c(\lambda,w), d(\lambda,w)$ are labelled by complex numbers $\lambda, w \in \mathbb{C}$ (without any periodicity requirements). The generators of the algebra satisfy the relations (2)–(4), cf. Sect. 2.

Theorem 19. *The extended elliptic quantum groups $\widetilde{E}_{\tau,\eta}$ and $\widetilde{E}_{\tau+1,\eta}$ are isomorphic with an isomorphism $\widetilde{E}_{\tau,\eta} \to \widetilde{E}_{\tau+1,\eta}$ given by $f(h) \mapsto f(h)$,*

$$(a(\lambda,w), b(\lambda,w), c(\lambda,w), d(\lambda,w)) \mapsto (a(\lambda,w), b(\lambda,w), c(\lambda,w), d(\lambda,w)) .$$

Let $\tau' = -1/\tau$. Then the extended elliptic quantum groups $\widetilde{E}_{\tau,\eta}$ and $\widetilde{E}_{\tau',\eta\tau'}$ are isomorphic with an isomorphism $\widetilde{E}_{\tau,\eta} \to \widetilde{E}_{\tau',\eta\tau'}$ given by $f(h) \mapsto f(h)$,

$$\begin{aligned}(a(\lambda,w), b(\lambda,w), c(\lambda,w), d(\lambda,w)) \mapsto & (a(\tau'\lambda,\tau'w)e^{2\pi i\tau' A(\lambda,h,w)}, b(\tau'\lambda,\tau'w) \\ & \times e^{2\pi i\tau' B(\lambda,h,w)},\ c(\tau'\lambda,\tau'w)\,e^{2\pi i\tau' C(\lambda,h,w)}, d(\tau'\lambda,\tau'w)e^{2\pi i\tau' D(\lambda,h,w)}) ,\end{aligned} \tag{9}$$

where A, B, C, D are polynomials in λ, h, w,

$$A = (h-\Lambda)\eta(-w+h+3\eta) ,$$

$$B = w(\lambda+2\eta-\eta h+\eta\Lambda) - (\lambda+2\eta)(h-1)\eta + \eta^2((h+1)^2-(\Lambda+1)^2)/2 ,$$

$$\begin{aligned}C =\ & w(-\lambda+(h+\Lambda+2)\eta) + (h+1)(-\eta\lambda+\eta^2(h+\Lambda+2)) \\ & +\eta^2(\Lambda-h-2)(\Lambda-h+2)/2 ,\end{aligned}$$

$$D = \eta(h+\Lambda)(w+\eta h-\lambda-\eta) .$$

Here $\Lambda \in \mathbb{C}$ is a parameter of the isomorphism.

Say that an $\widetilde{E}_{\tau,\eta}(sl_2)$-module V is *modular* with parameter $\Lambda \in \mathbb{C}$, if in this module the operators $a(\lambda,w), b(\lambda,w), c(\lambda,w), d(\lambda,w)$ are 1-periodic with respect to w,

$$a(\lambda,w+1) = a(\lambda,w), \qquad b(\lambda,w+1) = b(\lambda,w) ,$$

$$c(\lambda,w+1) = c(\lambda,w), \qquad d(\lambda,w+1) = d(\lambda,w) ,$$

and, moreover,

$$a(\lambda,w+\tau) = a(\lambda,w)e^{2\pi i\eta(h-\Lambda)} ,$$

$$b(\lambda,w+\tau) = b(\lambda,w)e^{2\pi i(-\lambda+\eta(h-\Lambda-2))} ,$$

$$c(\lambda,w+\tau) = c(\lambda,w)e^{2\pi i(\lambda-\eta(h+\Lambda+2))} ,$$

$$d(\lambda,w+\tau) = d(\lambda,w)e^{2\pi i\eta(-h-\Lambda)} ,$$

for some constant $\Lambda \in \mathbb{C}$.

Example. An evaluation module $V_\Lambda(z)$ and a cyclic module $V_{\Lambda,\Xi}(z)$ are modular with parameter Λ.

If V_k is modular with parameter Λ_k, $k = 1, 2$, then $V_1 \otimes V_2$ is modular with parameter $\Lambda_1 + \Lambda_2$.

Let $\tau' = -1/\tau$. Let an $\widetilde{E}_{\tau',\eta\tau'}$-module V be modular with parameter Λ. Consider the isomorphism $\widetilde{E}_{\tau,\eta} \to \widetilde{E}_{\tau',\eta\tau'}$ given by (9) with the same parameter Λ. This isomorphism induces an $\widetilde{E}_{\tau,\eta}(sl_2)$-module structure on V.

Theorem 20. *This $\widetilde{E}_{\tau,\eta}(sl_2)$-module structure on V is modular with parameter Λ.*

17. Concluding Remarks

Consider the elliptic quantum group $E_{\tau,\eta}(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$ of A, B, C, D type. For this elliptic quantum group one defines the notions of the operator algebra, an irreducible module, a highest weight module, a finite type module, a singular vector, the Weyl group in the same way as for the elliptic quantum group associated to sl_2. The basic interrelations between these notions (like Theorems 1 and 2) remain true after obvious changes. We plan to describe evaluation modules of the elliptic quantum group $E_{\tau,\eta}(sl_N)$ in a future paper.

In [FTV] we will describe solutions to the quantum Knizhnik–Zamolodchikov–Bernard equations with values in a tensor product of evaluation $E_{\tau,\eta}(sl_2)$-modules.

The proofs of all formulated theorems are straightforward and will be published elsewhere.

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