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Solutions of Klein–Gordon and Dirac Equations on Quantum Minkowski Spaces

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Abstract: Covariant differential calculi and exterior algebras on quantum homogeneous spaces endowed with the action of inhomogeneous quantum groups are classified. In the case of quantum Minkowski spaces they have the same dimensions as in the classical case. Formal solutions of the corresponding Klein–Gordon and Dirac equations are found. The Fock space construction is sketched.

0. Introduction

It is well known that lattice-like theories serve as regularization schemes in quantum field theory. But after introducing the lattice, we no longer have the full symmetry of the original theory. On the other hand, there was a lot of interest in quantum spacetimes endowed with the actions of quantum groups which are deformations of the objects used in the standard field theory (cf. [20, 3, 8, 7, 24, 13, 9, 5, 29, 19]). There were two motivations of such a development: providing naive models of changed geometry at the Planck scale and attempts to regularize the theory while preserving the "size" of the symmetry group in such a way that the regularized theory could still be imagined as the theory of our universe. Although the present paper doesn't provide support for any of these claims, we find a lattice-like behavior of certain quantum Minkowski spaces. It has two aspects:

1. It was found [12] that in the differential calculus on **R** corresponding to the one-dimensional lattice one has

$$xdx = (dx)x + ldx ,$$

where x is the identity function and l is the lattice constant. In Sect. 1 we describe differential calculi on quantum Minkowski spaces by a very similar relation (1.7).

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2. For the above differential calculus on $\mathbf{R}$ one has $df = dx\partial(f) = \tilde{\partial}(f)dx$, where f is a function on $\mathbf{R}$, $\partial(f) = (f(x+l) - f(x))/l$, $\tilde{\partial}(f) = (f(x) - f(x-l))/l$ (cf. [12]). Setting $\Delta = \partial\tilde{\partial} = \tilde{\partial}\partial$, one gets

$$\Delta e^{-ipx} = \frac{\sin^2 k}{k^2}(-p^2)e^{-ipx},$$

where $k = pl/2$. Thus we obtain an additional factor $\frac{\sin^2 k}{k^2}$ (comparing with the action of the usual Laplacian $\partial^2/\partial x^2$). In Sect. 4 similar factors appear in the description of eigenvalues of the Laplacian on quantum Minkowski spaces.

In Sect. 1 we recall the definition of homogeneous quantum spaces M (e.g. quantum Minkowski spaces) endowed with the action of inhomogeneous quantum groups G (e.g. quantum Poincaré groups). We classify the differential calculi on M which have the same properties as in the classical case. They exist if and only if a certain matrix $\tilde{F}$ (related to the existence of quasitriangular structure on G [17]) vanishes, in which case they are unique. In Sect. 2 we prove that each such calculus has a unique natural extension to an exterior algebra of differential forms. In the case of quantum Minkowski spaces the modules of k-forms have the classical dimensions $\binom{4}{k}$. In Sect. 3 the properties of partial derivatives, Laplacian and the Dirac operator are investigated. We heuristically assert hermiticity of the momenta and Laplacian. In Sect. 4 we find formal solutions of Klein–Gordon and Dirac equations for two special classes of M. They are obtained from the plane waves $e^{-ip_a x^a}$, but the eigenvalues of the momenta are related to p_a in a complicated way in general. The sketch of the Fock space construction is provided in Sect. 5.

We sum over repeated indices (Einstein's convention). If V, W are vector spaces then $\tau : V \otimes W \longrightarrow W \otimes V$ is given by $\tau(x \otimes y) = y \otimes x$, $x \in V$, $y \in W$. We denote the unit matrix by $\mathbb{1}$, $\mathbb{1}^{\otimes k} = \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ (k times). If $\mathscr{A}$ is an algebra, $v \in M_N(\mathscr{A})$, $w \in M_K(\mathscr{A})$, then the tensor product $v \otimes w \in M_{NK}(\mathscr{A})$ is defined by

$$(v \otimes w)^{ij}{}_{kl} = v^i{}_k w^j{}_l, \quad i,k = 1,\dots,N\ ,j,l = 1,\dots,K\ .$$

We set $\dim v = N$. If $\mathscr{A}$ is a $*$-algebra then the conjugate of v is defined as $\bar{v} \in M_N(\mathscr{A})$, where $\bar{v}^i{}_j = (v^i{}_j)^*$. We also set $v^* = \bar{v}^T$ (v^T denotes the transpose of v, i.e. $(v^T)_i{}^j = v^j{}_i$).

Throughout the paper quantum groups H are abstract objects described by the corresponding Hopf ($*$-) algebras $\mathrm{Poly}(H) = (\mathscr{A}, \Delta)$. We denote by Δ, ε, S the comultiplication, counit and the coinverse of $\mathrm{Poly}(H)$. We say that v is a representation of H (i.e. $v \in$ Rep H) if $v \in M_N(\mathscr{A})$, $N \in \mathbf{N}$, and

$$\Delta v^i{}_j = v^i{}_k \otimes v^k{}_j\ , \qquad \varepsilon(v^i{}_j) = \delta^i{}_j\ , \quad i,j = 1,\dots,N\ ,$$

in which case $S(v^i{}_j) = (v^{-1})^i{}_j$. Matrix elements of all $v \in$ Rep H linearly span $\mathscr{A}$. The conjugate of a representation and tensor products of representations are also representations. The set of nonequivalent irreducible representations of H is denoted by Irr H. If $v, w \in$ Rep H, then we say that $A \in M_{\dim v \times \dim w}(\mathbf{C})$ intertwines v with w (i.e. $A \in \mathrm{Mor}(v,w)$) if $Av = wA$. For $\rho, \rho' \in \mathscr{A}'$ (the dual vector space of $\mathscr{A}$) one defines their convolution $\rho * \rho' = (\rho \otimes \rho')\Delta$. For $\rho \in \mathscr{A}'$, $a \in \mathscr{A}$, we set $\rho * a = (\mathrm{id} \otimes \rho)\Delta a$, $a * \rho = (\rho \otimes \mathrm{id})\Delta a$.

1. The Covariant Differential Calculi on Quantum Homogeneous Spaces

In this section we recall the definition of the quantum homogeneous space M endowed with the action of the inhomogeneous quantum group G [18]. The corresponding unital algebra $\mathscr{C} = \mathrm{Poly}(M)$ is generated by quantum coordinates x^i, $i = 1,\dots,N$. We prove that there exists a covariant differential calculus on M which has dx^i, $i = 1,\dots,N$, as the basis of the module of 1-forms if and only if a certain matrix $\tilde{F} = 0$. Moreover, such a calculus is unique. We specify the quantum Minkowski spaces endowed with the action of quantum Poincaré groups [19] for which $\tilde{F} = 0$.

Throughout the section $\mathrm{Poly}(H) = (\mathscr{A}, \Delta)$ is any Hopf algebra with invertible S (we need invertibility of S in the proof of Theorem 1.1, it was not needed in [18]) such that

(a) each representation of H is completely reducible,
(b) Λ is an irreducible representation of H,
(c) $\mathrm{Mor}(v \otimes w,\ \Lambda \otimes v \otimes w) = \{0\}$ for any two irreducible representations of H.

Moreover, we assume that $f^i{}_j$, $\eta^i \in \mathscr{A}'$, $T^{ij} \in \mathbf{C}$, $i,j = 1,\dots,N = \dim \Lambda$, are given and satisfy

1. $\mathscr{A} \ni a \to \rho(a) = \begin{pmatrix} f(a) & \eta(a) \\ 0 & \varepsilon(a) \end{pmatrix} \in M_{N+1}(\mathbf{C})$ is a unital homomorphism,
2. $\Lambda^s{}_t(f^t{}_r * a) = (a * f^s{}_t)\Lambda^t{}_r$ for $a \in \mathscr{A}$,
3. $R^2 = \mathbb{1}$ where $R^{ij}{}_{sm} = f^i{}_m(\Lambda^j{}_s)$,
4. $(\Lambda \otimes \Lambda)^{kl}{}_{ij}(\tau^{ij} * a) = a * \tau^{kl}$ for $a \in \mathscr{A}$, where

$$\tau^{ij} = (R - \mathbb{1})^{ij}{}_{mn}(\eta^n * \eta^m - \eta^m(\Lambda^n{}_s)\eta^s + T^{mn}\varepsilon - f^n{}_b * f^m{}_a T^{ab}) ,$$

5. $A_3\tilde{F} = 0,$ where $A_3 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} - R \otimes \mathbb{1} - \mathbb{1} \otimes R + (R \otimes \mathbb{1})(\mathbb{1} \otimes R) + (\mathbb{1} \otimes R)(R \otimes \mathbb{1}) - (R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1})$, $\tilde{F}^{ijk}{}_m = \tau^{ij}(\Lambda^k{}_m)$,
6. $A_3(Z \otimes \mathbb{1} - \mathbb{1} \otimes Z)T = 0$, $RT = -T$, where $Z^{ij}{}_k = \eta^i(\Lambda^j{}_k)$.

In particular, 4–5 are satisfied if $\tau^{ij} = 0$. The inhomogeneous quantum group G corresponds to the Hopf algebra $\mathrm{Poly}(G) = (\mathscr{B}, \Delta)$ defined (cf. Corollary 3.8.*a* of [18]) as follows: $\mathscr{B}$ is the universal unital algebra generated by $\mathscr{A}$ and y^i, $i = 1,\dots,d$, satisfying the relations $I_{\mathscr{B}} = I_{\mathscr{A}}$,

$$y^s a = (a * f^s{}_t)y^t + a * \eta^s - \Lambda^s{}_t(\eta^t * a), \quad a \in \mathscr{A} , \tag{1.1}$$

$$(R - \mathbb{1})^{kl}{}_{ij}(y^i y^j - \eta^i(\Lambda^j{}_s)y^s + T^{ij} - \Lambda^i{}_m\Lambda^j{}_n T^{mn}) = 0 . \tag{1.2}$$

Moreover, $(\mathscr{A}, \Delta)$ is a Hopf subalgebra of $(\mathscr{B}, \Delta)$ and $\Delta y^i = \Lambda^i{}_j \otimes y^j + y^i \otimes I$ (note that the y^i were denoted by p_i in [18]). We define $\mathscr{C} = \mathrm{Poly}(\mathscr{M})$ as the universal unital algebra generated by x^i, $i = 1,\dots,N$, satisfying

$$(R - \mathbb{1})^{ij}{}_{kl}(x^k x^l - Z^{kl}{}_s x^s + T^{kl}) = 0 . \tag{1.3}$$

The action of G on M is described by the unital homomorphism $\Psi : \mathscr{C} \to \mathscr{B} \otimes \mathscr{C}$ such that

$$\Psi(x^i) = \Lambda^i{}_j \otimes x^j + y^i \otimes I , \tag{1.4}$$

$(\varepsilon \otimes \mathrm{id})\Psi = \mathrm{id}$, $(\mathrm{id} \otimes \Psi)\Psi = (\Delta \otimes \mathrm{id})\Psi$. The pair $(\mathscr{C}, \Psi)$ was investigated in Sect. 5 of [18]. We assume

$$\mathrm{Mor}(I, \Lambda \otimes \Lambda \otimes \Lambda) = \{0\}\,, \tag{1.5}$$

$$\mathrm{Mor}(I, \Lambda \otimes \Lambda) \cap \ker(R + \mathbb{1}) = \{0\} \tag{1.6}$$

((1.5)–(1.6) are satisfied for G being quantum Poincaré groups [19]). Then M is called a quantum homogeneous space and has the properties analogous to the Minkowski space (cf. Sect. 5 of [18], Sect. 1 of [19]). The 'sizes' of $\mathscr{B}$ and $\mathscr{C}$ were described in Corollary 3.6 and Proposition 5.3 of [18].

Motivated by [27, 21, 15] we have

Definition 1.1. *We say that $\boldsymbol{\Gamma}^{\wedge 1} = (\Gamma^{\wedge 1}, \Psi^{\wedge 1}, d)$ is a covariant differential calculus on M if*

1. *$\Gamma^{\wedge 1}$ is a $\mathscr{C}$-bimodule, $\omega I_{\mathscr{C}} = I_{\mathscr{C}}\omega = \omega$ for $\omega \in \Gamma^{\wedge 1}$,*
2. *$\Psi^{\wedge 1} : \Gamma^{\wedge 1} \to \mathscr{B} \otimes \Gamma^{\wedge 1}$ satisfies*
   (a) $(\varepsilon \otimes \mathrm{id})\Psi^{\wedge 1} = \mathrm{id}$, $(\mathrm{id} \otimes \Psi^{\wedge 1})\Psi^{\wedge 1} = (\Delta \otimes \mathrm{id})\Psi^{\wedge 1}$,
   (b) $\Psi^{\wedge 1}(\omega a) = \Psi^{\wedge 1}(\omega)\Psi(a)$, $\Psi^{\wedge 1}(a\omega) = \Psi(a)\Psi^{\wedge 1}(\omega)$ *for* $\omega \in \Gamma^{\wedge 1}$, $a \in \mathscr{C}$,
3. *$d : \mathscr{C} \to \Gamma^{\wedge 1}$ is a linear map such that*
   (a) $d(ab) = a(db) + (da)b$, $a, b \in \mathscr{C}$,
   (b) $(\mathrm{id} \otimes d)\Psi = \Psi^{\wedge 1} d$,
   (c) $\Gamma^{\wedge 1} =$ *span*$\{(da)b : a, b \in \mathscr{C}\}$.

We say that $\Gamma^{\wedge 1}$ is N-dimensional if dx^i, $i = 1, \ldots, N$, form a basis of $\Gamma^{\wedge 1}$ (as the right $\mathscr{C}$-module).

Theorem 1.1. *There exists N-dimensional covariant differential calculus on M iff $\tilde{F} = 0$. In that case it is uniquely determined by*

$$x^i dx^j = R^{ij}{}_{kl} dx^k x^l + Z^{ij}{}_k dx^k, \quad i, j = 1, \ldots, N\,, \tag{1.7}$$

$$\Psi^{\wedge 1} dx^i = \Lambda^i{}_j \otimes dx^j, \quad i = 1, \ldots, N\,. \tag{1.8}$$

Proof. Let $\boldsymbol{\Gamma}^{\wedge 1}$ be N-dimensional covariant differential calculus on M. Using (1.4) and condition 3b) of Definition 1.1, one gets (1.8). The linear mappings $\partial_i : \mathscr{C} \to \mathscr{C}$, $\rho_i{}^j : \mathscr{C} \to \mathscr{C}$, $i, j = 1, \ldots, N$, are uniquely defined by

$$da = dx^i \partial_i(a), \qquad a dx^i = dx^j \rho_j{}^i(a)\,. \tag{1.9}$$

Using condition 3a) of Definition 1.1 and $(ab)dx^i = a(bdx^i)$, one gets that

$$\mathscr{L} : \mathscr{C} \ni a \to \begin{bmatrix} (\rho_i{}^j(a))_{i,j=1}^N & (\partial_i(a))_{i=1}^N \\ 0 & a \end{bmatrix} \in M_{N+1}(\mathscr{C}) \tag{1.10}$$

is a unital homomorphism. Conditions 2b) and 3b) of Definition 1.1 imply

$$(\mathrm{id} \otimes \partial_j)\Psi(a) = (\Lambda^i{}_j \otimes I)\Psi(\partial_i(a)), \quad j = 1, \ldots, N\,, \tag{1.11}$$

$$(\mathrm{id} \otimes \rho_k{}^j)\Psi(a)(\Lambda^i{}_j \otimes I) = (\Lambda^j{}_k \otimes I)\Psi(\rho_j{}^i(a)), \quad i, k = 1, \ldots, N\,, \tag{1.12}$$

$a \in \mathscr{C}$. Moreover,

$$\partial_j(x^i) = \delta^i{}_j, \quad i, j = 1, \ldots, N\,. \tag{1.13}$$

Conversely, any unital homomorphism (1.10) satisfying (1.11)–(1.13) for $a \in \mathscr{C}$ defines, through (1.9), a covariant N-dimensional differential calculus on M. So we need to find all such homomorphisms. Let us notice that it is sufficient to check (1.11)–(1.12) for $a = x^l$, $l = 1, \dots, N$ (they are trivial for $a = I$ and if a, b satisfy them then–using the homomorphism property–$a \cdot b$ does also). But for $a = x^l$ (1.11) is trivial. Moreover, $\mathscr{L}$ is determined by

$$\mathscr{L}(x^l) = h^l \equiv \begin{bmatrix} (K_i{}^{lj})_{i,j=1}^N & (\delta^l{}_i)_{i=1}^N \\ 0 & x^l \end{bmatrix},$$

$l = 1, \dots, N$, where $K_i{}^{lj} = \rho_i{}^j(x^l)$. Equation (1.13) and the existence of $\mathscr{L}$ (for given K) are equivalent to (1.3) with x^i replaced by h^i. This can be translated to

$$d[(R - \mathbb{1})^{ij}{}_{kl}(x^k x^l - Z^{kl}{}_s x^s + T^{kl})] = 0\,, \tag{1.14}$$

$$[(R - \mathbb{1})^{ij}{}_{kl}(x^k x^l - Z^{kl}{}_s x^s + T^{kl})]dx^m = 0\,, \tag{1.15}$$

where the left-hand sides should be expanded using condition 3a) of Definition 1.1 and (1.9) so that dx^s appear on the very left of the equations. Then the condition means that the total coefficient multiplying dx^s from the right is zero. The condition (1.12) for $a = x^l$ means

$$\Lambda^l{}_t \Lambda^i{}_j \otimes K_k{}^{tj} + y^l \Lambda^i{}_k \otimes I = (\Lambda^j{}_k \otimes I)\Psi(K_j{}^{li})\,,$$

i.e.

$$\Psi(K_m{}^{li}) = (G \otimes \Lambda \otimes \Lambda)_m{}^{li,k}{}_{tj} \otimes K_k{}^{tj} + G_m{}^k y^l \Lambda^i{}_k \otimes I\,, \tag{1.16}$$

where $(G^{-1})_i{}^j = \Lambda^j{}_i$ ($G = [S^{-1}(\Lambda)]^T = (\Lambda^T)^{-1}$). Thus we need to find K satisfying (1.14)–(1.16).

Now (1.7) is equivalent to $K_k{}^{ij} = \rho_k{}^j(x^i) = R^{ij}{}_{kl} x^l + Z^{ij}{}_k$. It is easy to check that such a K satisfies (1.14) and (1.16). Suppose there exists another $\tilde{K}$ satisfying (1.16). Then $M = K - \tilde{K}$ satisfies $\Psi(M_m{}^{li}) = (G \otimes \Lambda \otimes \Lambda)_m{}^{li,k}{}_{tj} \otimes M_k{}^{tj}$. Using Condition 2 of Sect. 5 of [18], one gets $M_m{}^{li} \in \mathbf{C}$, $G_m{}^k \Lambda^l{}_t \Lambda^i{}_j M_k{}^{tj} = M_m{}^{li}$. Multiplying from the left by $\Lambda^m{}_s = (G^{-1})_s{}^m$ and setting $U^{tj}{}_s = M_s{}^{tj}$, one gets $\Lambda^l{}_t \Lambda^i{}_j U^{tj}{}_s = U^{li}{}_m \Lambda^m{}_s$, i.e.

$$U \in \mathrm{Mor}(\Lambda, \Lambda \otimes \Lambda) = \{0\}, \quad U = 0,\ M = 0,\ \tilde{K} = K\,.$$

Therefore uniqueness follows. Expanding (1.15) and using (1.7), one gets that (after a long computation) the total coefficients multiplying dx^s from the right are zero if and only if $\tilde{F} = 0$ (we use the results of [18]: Proposition 5.3 and Remark 5.4 for $N = 1$, (3.61) and (3.30)). Thus the existence statement and (1.7) are proved. □

All the assumptions (including (1.5)–(1.6)) are fulfilled if H is a quantum Lorentz group [28], G is a quantum Poincaré group and M is the corresponding unique quantum Minkowski space [19]. According to Theorem 1.1, there exists 4-dimensional covariant differential calculus on the quantum Minkowski space iff $\tilde{F} = 0$, iff $\lambda = 0$ (see the proof of Theorem 1.6 of [19]), which holds for all cases except of the following:

1), $t = 1$, $s = 1$, $t_0 \in \mathbf{R}\backslash\{0\}$,

5), $t = 1$, $s = \pm 1$, $t_0 \in \mathbf{R}\backslash\{0\}$,
4), $s = 1$, $b \neq 0$,

(in the terminology of Theorem 1.6 and Remark 1.8 of [19]). Then $N = 4$. Such a calculus is unique.

Let $(\mathscr{A}, \Delta)$ be a Hopf $*$-algebra. Then S is always invertible. We also assume

$$\bar{\Lambda} = \Lambda,\ f^i{}_j(S(a^*)) = \overline{f^i{}_j(a)},\ \eta^i(S(a^*)) = \overline{\eta^i(a)},\ \overline{T^{ij}} = T^{ji},\ i,j = 1,\dots,N,\ a \in \mathscr{A}\ .$$

In that case [18] $(\mathscr{B}, \Delta)$ has a unique Hopf $*$-algebra structure such that $(\mathscr{A}, \Delta)$ is its Hopf $*$-subalgebra and $y^{i*} = y^i$. Moreover, $\mathscr{C}$ is a $*$-algebra with $*$ defined by $x^{i*} = x^i$ and Ψ is a $*$-homomorphism.

Proposition 1.1. *Under assumptions as above the N-dimensional covariant differential calculus on M described in Theorem* 1.1 *possesses a unique* $* : \Gamma^{\wedge 1} \to \Gamma^{\wedge 1}$ *such that*:

1. $(\omega a)^* = a^*\omega^*,\ (a\omega)^* = \omega^* a^*,\ \omega \in \Gamma^{\wedge 1},\ a \in \mathscr{C}$,
2. $(da)^* = d(a^*),\ a \in \mathscr{C}$,
3. $\Psi^{\wedge 1}(\omega^*) = (\Psi^{\wedge 1}(\omega))^{*\otimes *},\ \omega \in \Gamma^{\wedge 1}$.

Proof. We must define $* : \Gamma^{\wedge 1} \to \Gamma^{\wedge 1}$ by

$$(dx^i a_i)^* = a_i^* dx^i, \quad a_i \in \mathscr{C}\ . \tag{1.17}$$

By virtue of (1.7) of the present paper, (4.14) and the next formula of [18]

$$(x^i dx^j)^* = (R^{ij}{}_{kl} dx^k x^l + Z^{ij}{}_k dx^k)^* = R^{ji}{}_{lk}(x^l dx^k - Z^{lk}{}_s dx^s) = dx^j x^i\ .$$

But

$$(x^i dx^j)^* = (dx^k \rho_k{}^j(x^i))^* = \rho_k{}^j(x^i)^* dx^k = dx^s[\rho_s{}^k(\rho_k{}^j(x^i)^*)].$$

Therefore $\rho_s{}^k(\rho_k{}^j(a)^*) = a^*\delta^j{}_s$ for $a = x^i$ and hence (ρ is a unital homomorphism) for all $a \in \mathscr{A}$. This means $(a_j dx^j)^* = dx^j a_j^*$, $a_j \in \mathscr{C}$. This and (1.17) prove condition 1 for any $\omega = dx^i a_i \in \Gamma^{\wedge 1}$. Writing $a \in \mathscr{C}$ as a polynomial in x^j, using condition 1 and $(dx^i)^* = dx^i$ (see (1.17)), one gets condition 2. By virtue of (1.17) and (1.8) we obtain condition 3. □

In particular, all the above $*$-structures exist for quantum Poincaré groups [19], quantum Minkowski spaces [19] and 4-dimensional covariant differential calculi on them.

Remark. In the case of $Z = T = 0$ formulae (1.7), (2.4), (3.1), (3.8), (3.9), (3.13) and the second formula of (3.2) or their analogues were studied in several contexts in [21, 26, 4, 13, 24, 5, 1].

2. Exterior Algebras

In this section we construct the exterior algebras for the N-dimensional covariant differential calculi described in the previous section. In the case of quantum Minkowski spaces the right $\mathscr{C}$-module of k-forms has dimension $\binom{4}{k}$ as in the classical case.

Throughout this section $\boldsymbol{\Gamma}^{\wedge 1} = (\Gamma^{\wedge 1}, \Psi^{\wedge 1}, d)$ is an N-dimensional covariant differential calculus on quantum homogeneous space M endowed with the action of the inhomogeneous quantum group G as described in Theorem 1.1. In particular, we assume all the conditions introduced before Theorem 1.1 and that $\tilde{F} = 0$.

Definition 2.1 (cf. [27,21,15]). *We say that $\boldsymbol{\Gamma}^{\wedge} = (\Gamma^{\wedge}, \Psi^{\wedge}, d)$ is an exterior algebra on M iff*

1. $\Gamma^{\wedge} = \bigoplus_{n=0}^{\infty} \Gamma^{\wedge n}$ *is a graded algebra such that* $\Gamma^{\wedge 0} = \mathscr{C}$ *and the unit of* $\mathscr{C}$ *is the unit of* $\Gamma^{\wedge}$,
2. $\Psi^{\wedge} : \Gamma^{\wedge} \to \mathscr{B} \otimes \Gamma^{\wedge}$ *is a graded homomorphism such that*

$$(\varepsilon \otimes \mathrm{id})\Psi^{\wedge} = \mathrm{id}, \quad (\mathrm{id} \otimes \Psi^{\wedge})\Psi^{\wedge} = (\Delta \otimes \mathrm{id})\Psi^{\wedge}, \quad \Psi^{\wedge 0} = \Psi\,,$$

3. $d : \Gamma^{\wedge} \to \Gamma^{\wedge}$ *is a linear mapping such that*
   (a) $d(\Gamma^{\wedge n}) \subset \Gamma^{\wedge (n+1)}$, $n = 0, 1, 2, \dots$,
   (b) $d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta'$, $\theta \in \Gamma^{\wedge k}$, $\theta' \in \Gamma^{\wedge}$ *($\wedge$ denotes multiplication in $\Gamma^{\wedge}$)*,
   (c) $(\mathrm{id} \otimes d)\Psi^{\wedge} = \Psi^{\wedge} d$,
   (d) $dd = 0$,
4. $\Gamma^{\wedge n} =$ *span*$\{(da_1 \wedge \dots \wedge da_n)a_0 : a_0, a_1, \dots, \ a_n \in \mathscr{C}\}$ *(we omit $\wedge$ if one of multipliers belongs to $\mathscr{C}$)*,
5. $\Gamma^{\wedge 1}$, $\Psi^{\wedge 1}$, $d : \mathscr{C} \to \Gamma^{\wedge 1}$ *are as in Definition* 1.1 *and Theorem* 1.1,
6. *if* $(\tilde{\Gamma}^{\wedge}, \tilde{\Psi}^{\wedge}, \tilde{d})$ *also satisfies* 1–5 *then there exists a graded homomorphism* $\rho : \Gamma^{\wedge} \to \tilde{\Gamma}^{\wedge}$ *which is an identity on* $\mathscr{C}$ *and satisfies* $\tilde{\Psi}^{\wedge}\rho = (\mathrm{id} \otimes \rho)\Psi^{\wedge}$, $\tilde{d}\rho = \rho d$ *(universality condition).*

We set $R_{nk} = \mathbb{1}^{\otimes(k-1)} \otimes R \otimes \mathbb{1}^{\otimes(n-k-1)}$, $R_{n\pi} = R_{nk_1} \cdot \dots \cdot R_{nk_s}$ for any permutation $\pi = t_{k_1} \cdot \dots \cdot t_{k_s} \in \Pi_n$ where t_k is the transposition $k \leftrightarrow k+1$, $A_n = \frac{1}{n!}\sum_{\pi \in \Pi_n}(-1)^{\operatorname{sgn}\pi} R_{n\pi}$, $A_n^2 = A_n$, $R_{nk}A_n = A_n R_{nk} = -A_n$, $k = 1, 2, \dots, n-1$.

Let $\alpha' = \{\alpha'_i : i = 1, \dots, \dim A_n\}$ be a basis of $\operatorname{im} A_n$, $\beta' = \{\beta'_j : j = 1, \dots, \dim(\mathbb{1} - A_n)\}$ be a basis of $\operatorname{im}(\mathbb{1} - A_n)$. Then $\alpha' \sqcup \beta'$ is a basis of $(\mathbf{C}^N)^{\otimes n}$. We denote by $\alpha \sqcup \beta$ the dual basis. Therefore

$$\alpha^i A_n = \alpha^i, \qquad \beta^j A_n = 0\,, \tag{2.1}$$

$$A_n = \alpha'_i \alpha^i\,. \tag{2.2}$$

Theorem 2.1. *There exists a unique exterior algebra $\boldsymbol{\Gamma}^{\wedge}$ on M. The n-forms*

$$\omega^{\gamma} = \alpha^{\gamma n}_{k_1,\dots,k_n} dx^{k_1} \wedge \dots \wedge dx^{k_n}, \quad \gamma = 1, \dots, \dim A_n\,, \tag{2.3}$$

form a basis of the right $\mathscr{C}$-module $\Gamma^{\wedge n}$. Moreover,

$$dx^i \wedge dx^j = -R^{ij}{}_{kl} dx^k \wedge dx^l, \quad i, j = 1, \dots, N\,. \tag{2.4}$$

Remark. The first statement of the theorem follows also from the general considerations (cf. e.g. [2]).

Proof. Assume that $\mathbf{S}^\wedge = (S^\wedge, \Psi_S^\wedge, d_S)$ satisfies conditions 1–5 of Definition 2.1. Acting d_S on (1.7), one gets (2.4). Set $dx_S^J = dx^{i_1} \wedge \cdots \wedge dx^{i_n}$, where $J = (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n \equiv N^{(n)}$. They generate the right $\mathscr{C}$-module $\Gamma^{\wedge n}$. We obtain $dx_S^J = -(R_{nk})^J{}_K dx_S^K$, $dx_S^J = (-1)^{\operatorname{sgn}\pi}(R_{n\pi})^J{}_K dx_S^K$, $dx_S^J = (A_n)^J{}_K dx_S^K$, (2.3) generate the right $\mathscr{C}$-module $S^{\wedge n}$. Moreover, $S^{\wedge n} = \{dx_S^J a_J : J \in N^{(n)}\}$, formulae for $d_S(dx_S^J a_J)$, $\Psi_S^\wedge(dx_S^J a_J)$ are determined by conditions 2 and 3(b)(c)(d) of Definition 2.1. Thus it suffices to construct $\boldsymbol{\Gamma}^\wedge = (\Gamma^\wedge, \Psi^\wedge, d)$ which satisfies the conditions 1–5 of Definition 2.1 and such that ω^γ, $\gamma = 1, \ldots, \dim A_n$, are independent in the right $\mathscr{C}$-module $\Gamma^{\wedge n}$.

We set $\Gamma = \Gamma^{\wedge 1}$, $\Gamma^{\otimes 0} = \mathscr{C}$, $\Gamma^{\otimes n} = \Gamma \otimes_{\mathscr{C}} \ldots \otimes_{\mathscr{C}} \Gamma$, $\Gamma^{\otimes} = \bigoplus_{n=0}^\infty \Gamma^{\otimes n}$, $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_n}$ for $I = (i_1, \ldots, i_n) \in N^{(n)}$,

$$L_0^n = \operatorname{span}\{dx^J - (A_n)^J{}_K dx^K : J \in N^{(n)}\}\,,$$

$$L^n = L_0^n \mathscr{C}, \quad L = \bigoplus_{n=0}^\infty L^n, \quad \Gamma^{\wedge n} = \Gamma^{\otimes n}/L^n, \quad \Gamma^\wedge = \bigoplus_{n=0}^\infty \Gamma^{\wedge n} = \Gamma^{\otimes}/L\,.$$

Then $\Gamma^{\otimes n}$ are $\mathscr{C}$-bimodules, $\Gamma^{\otimes}$ is a graded algebra, ω^γ, $\gamma = 1, \ldots, \dim A_n$, form a basis of the right $\mathscr{C}$-module $\Gamma^{\wedge n}$ (cf. (2.1)). We see that $L_0^n = \operatorname{span}\{dx^J + (R_{nk})^J{}_K dx^K : J \in N^{(n)},\ k = 1, \ldots, N-1\}$. Thus L is the right ideal generated by $dx^K \otimes \omega^{ik}$, $K \in N^{(s)}$, $s = 0, 1, \ldots, i, k = 1, \ldots, N$, where $\omega^{ik} = dx^i \otimes_{\mathscr{C}} dx^j + R^{ij}{}_{kl} dx^k \otimes_{\mathscr{C}} dx^l$. Using (1.7) and (3.11), (3.61) of [18], one shows that

$$x^m \omega^{ik} = R^{sk}{}_{nb} R^{mi}{}_{js} \omega^{jn} x^b + \omega^{ab} l^{mik}{}_{ab},$$

where $l = Z \otimes \mathbb{1} + (R \otimes \mathbb{1})(\mathbb{1} \otimes Z)$. That and (1.7) prove that L is the ideal generated by ω^{ik} and condition 1. of Definition 2.1 follows. Moreover, (2.4) is satisfied.

We define a linear mapping $\Psi^{\otimes} : \Gamma^{\otimes} \to \Gamma^{\otimes}$ by

$$\Psi^{\otimes}(\omega_1 \otimes_{\mathscr{C}} \cdots \otimes_{\mathscr{C}} \omega_n) = \omega_1^{(1)} \omega_2^{(1)} \cdot \cdots \cdot \omega_n^{(1)} \otimes \omega_1^{(2)} \otimes_{\mathscr{C}} \cdots \otimes_{\mathscr{C}} \omega_n^{(2)}\,, \tag{2.5}$$

where $\omega_s \in \Gamma^{\wedge 1}$, $\Psi^{\wedge 1}(\omega_s) = \omega_s^{(1)} \otimes \omega_s^{(2)}$ (Sweedler's notation), $s = 1, \ldots, n$. Then $\Psi^{\otimes}$ is well defined, $\Psi^{\otimes} \omega^{ik} = \Lambda^i{}_j \Lambda^k{}_m \otimes \omega^{jm}$ (see (1.8)), $\Psi^{\otimes}(L) \subset \mathscr{B} \otimes L$, $\Psi^{\otimes}$ defines $\Psi^\wedge : \Gamma^\wedge = \Gamma^{\otimes}/L \to \mathscr{B} \otimes \Gamma^\wedge$ which satisfies condition 2 of Definition 2.1.

We set $d(\omega^\gamma a_\gamma) = (-1)^n \omega^\gamma \wedge da_\gamma$, i.e. (see (2.3), (2.2))

$$d(dx^J a_J) = (-1)^{|J|} dx^J \wedge da_J, \quad a_J \in \mathscr{C},\ J \in N^{(n)},\ |J| = n\,. \tag{2.6}$$

Conditions 5, 4 and 3(a)(c) follow. Moreover,

$$ddx^J = 0\,. \tag{2.7}$$

We shall prove

$$d(a_J dx^J) = da_J \wedge dx^J\,. \tag{2.8}$$

Due to (1.7), (2.6) and (2.4)

$$d(x^i dx^j) = d[R^{ij}{}_{kl}(dx^k) x^l + Z^{ij}{}_l dx^l] = -R^{ij}{}_{kl} dx^k \wedge dx^l = dx^i \wedge dx^j.$$

Thus $x^i \in E = \{a \in \mathscr{C} : d(adx^i) = da \wedge dx^i,\ i = 1,\dots,N\}$. Moreover, E is a unital algebra (we use (1.9) and (2.6) in order to find LHS in the definition of E and perform a direct computation). Hence (2.8) for $|J| = 1$ follows. According to (2.6), $d(dx^i \wedge \omega) = -dx^i \wedge d\omega$ for $\omega \in \Gamma^\wedge$. Using this, (1.9) and the mathematical induction w.r.t. $|J|$, a simple calculation proves (2.8) in the general case. Then it is easy to check condition 3(b).

We set $F = \{a \in \mathscr{C} : dda = 0\}$. Then $x^i \in F$ (see (2.7)) and F is a unital algebra (we use (2.6), (2.8)). Thus $F = \mathscr{C}$. This and (2.7) show $dd\omega = 0$ for any $\omega = dx^J a_J$ and also condition 3(d) follows. □

We also have

Proposition 2.1. *Under the assumptions of Proposition* 1.1 *there exists a unique graded antilinear involution* $* : \Gamma^\wedge \to \Gamma^\wedge$ *such that*

(a) $(\theta \wedge \theta')^* = (-1)^{kl}\theta'^* \wedge \theta^*$, $\theta \in \Gamma^{\wedge k}$, $\theta' \in \Gamma^{\wedge l}$,
(b) $\Psi^\wedge(\theta^*) = (\Psi^\wedge(\theta))^{*\otimes *}$, $\theta \in \Gamma^\wedge$,
(c) $d(\theta^*) = d(\theta)^*$, $\theta \in \Gamma^\wedge$,
(d) $*$ *on* $\Gamma^{\wedge 0}$, $\Gamma^{\wedge 1}$ *coincides with the original one.*

Proof. We use the notation of the proof of Theorem 2.1. We define $* : \Gamma^{\otimes n} \to \Gamma^{\otimes n}$ by

$$(\omega_1 \otimes_{\mathscr{C}} \cdots \otimes_{\mathscr{C}} \omega_n)^* = (-1)^{\frac{n(n-1)}{2}} \omega_n^* \otimes_{\mathscr{C}} \cdots \otimes_{\mathscr{C}} \omega_1^*$$

for $\omega_1,\dots,\omega_n \in \Gamma^{\wedge 1}$. Using (4.14) of [18], $(\omega^{ij})^* = dx^j \otimes_{\mathscr{C}} dx^i + R^{ji}{}_{lk} dx^l \otimes_{\mathscr{C}} dx^k = \omega^{ji}$. Hence $L^* \subset L$ and we get $* : \Gamma^{\wedge n} \to \Gamma^{\wedge n}$ satisfying conditions (a)(d). Condition (b) follows from (2.5). We see that conditions (a)(d) imply

$$(dx^I a_I)^* = a_I^* (dx^I)^* = (-1)^{|I|(|I|-1)/2} a_I^* dx^{I'}\ , \tag{2.9}$$

where $I' = (i_n,\dots,i_1)$ for $I = (i_1,\dots,i_n)$. Therefore $*$ is unique. Set $\theta = dx^I a_I$. According to (2.9), (2.8), condition (a) and (2.6),

$$d(\theta^*) = (-1)^{|I|(|I|-1)/2} d(a_I)^* \wedge dx^{I'} = ((-1)^{|I|} dx^I \wedge da_I)^* = (d\theta)^*\ ,$$

and condition (c) follows. □

In particular, quantum Minkowski spaces with $\tilde{F} = 0$ (described after the proof of Theorem 1.1) admit a unique exterior algebra as described in Theorem 2.1 and Proposition 2.1. Moreover, using the arguments of the proof of Theorem 1.9 of [19], we get

Proposition 2.2. *Let* M *be a quantum Minkowski space. Then* $\dim A_k = \binom{4}{k}$, $k = 0,1,2,3,4$, $\dim A_k = 0$ *for* $k > 4$.

3. Differential operators

Here we introduce and investigate the properties of the momenta $P_j = i\partial_j$, $P^j = i\partial^j$, the Laplacian $\Box$ and the Dirac operator $\not{P}$. In particular, the momenta commute with $\Box$. We heuristically show that P^j, $\Box$ are hermitian.

Throughout the section we deal with the exterior algebra on M as described in Theorems 1.1 and 2.1. Further assumptions will be made later on. Let us recall that the partial derivatives ∂_i were defined by (1.9) and satisfy (1.13), (1.11). Their values can be computed using the unital homomorphism (1.10). They satisfy the following.

Proposition 3.1. 1. *One has*

$$\partial_l \partial_k = R^{ij}{}_{kl} \partial_j \partial_i, \quad i,j = 1,\dots,N . \tag{3.1}$$

2. *There exist* $X_i{}^k$, $Y_i \in \mathscr{B}'$ *such that*

$$\rho_i{}^k = (X_i{}^k \otimes \mathrm{id})\Psi, \qquad \partial_i = (Y_i \otimes \mathrm{id})\Psi . \tag{3.2}$$

3. *One has*

$$\partial_c \rho_a{}^t = \rho_d{}^t \partial_b R^{bd}{}_{ac}, \qquad R^{st}{}_{bd} \rho_c{}^d \rho_a{}^b = \rho_d{}^t \rho_b{}^s R^{bd}{}_{ac} . \tag{3.3}$$

Proof. *Ad 1.* Let $a \in \mathscr{C}$. Using (1.9) and (2.4), we get

$$\begin{aligned} 0 &= dda = d(dx^i \partial_i(a)) = -dx^i \wedge d(\partial_i(a)) = -dx^i \wedge dx^j \partial_j \partial_i(a) \\ &= -\tfrac{1}{2}(dx^i \wedge dx^j - R^{ij}{}_{kl} dx^k \wedge dx^l) \partial_j \partial_i(a) . \end{aligned}$$

But

$$\frac{1}{2}(dx^i \wedge dx^j - R^{ij}{}_{kl} dx^k \wedge dx^l) = (A_2)^{ij}{}_{kl} dx^k \wedge dx^l = (\alpha_s')^{ij} \omega^s$$

(we have used (2.2), (2.3)). By virtue of the independence of ω^s, $(\alpha_s')^{ij}\partial_j\partial_i = 0$. Multiplying by $(\alpha^s)_{kl}$ and using (2.2), we get (3.1).

Ad 2. We set

$$X(a) = \begin{pmatrix} (f^l{}_j(S(a)))_{j,l=1}^N & 0 \\ 0 & \varepsilon(a) \end{pmatrix}, \quad a \in \mathscr{A} , \tag{3.4}$$

$$X(y^i) = \begin{pmatrix} (Z^{il}{}_j)_{j,l=1}^N & (\delta^i{}_j)_{j=1}^N \\ 0 & 0 \end{pmatrix}, \quad i = 1,\dots,N . \tag{3.5}$$

There exists a unital homomorphism $X : \mathscr{B} \to M_{N+1}(\mathbf{C})$ which satisfies (3.4)–(3.5) ($X(I_{\mathscr{A}}) = \mathbb{1}$; using $\tilde{F} = 0$ and (2.18) of [18] for $b = v^k{}_l$, $v \in$ Rep H, we show that $X(a)$, $X(y^i)$ satisfy (1.1)–(1.2) for $a = v^i{}_j$, $v \in$ Rep H). We have

$$X = \begin{pmatrix} (X_j{}^l)_{j,l=1}^N & (Y_j)_{j=1}^N \\ 0 & \varepsilon \end{pmatrix},$$

where $X_j{}^l, Y_j \in \mathscr{B}'$. By a direct computation (cf. the proof of Theorem 1.1)

$$\mathscr{L}(x) = (X \otimes \mathrm{id})\Psi(x) \tag{3.6}$$

for $x = x^i \in \mathscr{C}$. But $\mathscr{L}$, X are both unital homomorphisms. Hence (3.6) follows for all $x \in \mathscr{C}$. This proves (3.2).

Ad 3. We set $X_i{}^+ = Y_i$, $X_+{}^i = 0$, $X_+{}^+ = \varepsilon$. We claim that

$$(X_a{}^b * X_c{}^d)K_{bd}{}^{st} = K_{ac}{}^{bd}(X_b{}^s * X_d{}^t), \quad a,c,s,t = 1,\dots,N,+\,, \tag{3.7}$$

where

$$K = \begin{pmatrix} R^T & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Indeed, since X is a homomorphism, it is enough to check (3.7) on $a \in \mathscr{A}$ (which follows from (3.4) and the last formula before Proposition 3.14 of [18]), and on y^i, $i = 1,\dots,N$, (which can be obtained by a direct computation – see (3.61) of [18]). Using (3.7) and (3.2), one easily gets (3.3) and also (3.1). $\square$

Let us notice that

$$\begin{aligned} dx^i\partial_i(x^k a) &= d(x^k a) = (dx^k)a + x^k dx^l \partial_l(a) \\ &= dx^i[\delta^k{}_i a + R^{kl}{}_{in}x^n\partial_l(a) + Z^{kl}{}_i\partial_l(a)], \end{aligned}$$

i.e.

$$\partial_i x^k = \delta^k{}_i + (R^{kl}{}_{in}x^n + Z^{kl}{}_i)\partial_l\,. \tag{3.8}$$

In the following we assume that $g = (g^{ab})_{a,b=1}^N \in \mathrm{Mor}(I, \Lambda \otimes \Lambda)$ is a fixed invertible matrix. It is called the metric tensor. Then (see (1.12) of [18] and (1.6)) $Rg = g$. Moreover $g^{-1} = (g_{ab})_{a,b=1}^N \in \mathrm{Mor}(\Lambda \otimes \Lambda, I)$ ($\Lambda g \Lambda^T = g$ implies $\Lambda^{-1} = g\Lambda^T g^{-1}$, $\Lambda^T g^{-1}\Lambda = g^{-1}$). In the case with $*$ (as in Propositions 1.1 and 2.1) one has $\tilde{g} \in \mathrm{Mor}(I, \Lambda \otimes \Lambda)$, where $\tilde{g}^{ij} = \overline{g^{ji}}$. Then we also assume $\tilde{g} = g$. Such g exists e.g. for quantum Poincaré groups and in this case is given (up to a nonzero real multiplicative factor) by $q^{1/2}m$, where m is given after Theorem 1.12 of [19] (we use (2.2) of [19], $m^{-1} = (E' \otimes E'\tau)(\mathbb{1} \otimes X^{-1} \otimes \mathbb{1})(V \otimes V)$).

We set $\partial^a = g^{ab}\partial_b$, $P_a = i\partial_a$, $P^a = i\partial^a$, $\square = g^{ij}\partial_j\partial_i = g_{ij}\partial^i\partial^j$. Moreover, $\partial^i\partial^j = R^{ij}{}_{kl}\partial^k\partial^l$ (we use (3.1) and twice $g^{jb}R^{dc}{}_{ba} = R^{jd}{}_{ak}g^{kc}$, which follows from $R^{jd}{}_{ak} = (R^{-1})^{jd}{}_{ak} = f^d{}_a((\Lambda^{-1})^j{}_k)$ and $\Lambda^{-1} = g\Lambda^T g^{-1}$). Using (3.64) of [18] for $m = g$ and (3.1), one easily gets

$$\square\partial_k = \partial_k\square\,. \tag{3.9}$$

Therefore $\square$ commutes with all ∂_k, ∂^k, P_k, P^k. Moreover, (1.11) implies

$$(\mathrm{id} \otimes \square)\Psi(a) = \Psi(\square(a))\,, \tag{3.10}$$

$$(\mathrm{id} \otimes \partial^j)\Psi(a) = ((\Lambda^{-1})^j{}_i \otimes I)\Psi(\partial^i(a))\,. \tag{3.11}$$

The momenta P_j, P^j transform under the action of the inhomogeneous quantum group G in the same way as the partial derivatives ∂_j, ∂^j in (1.11), (3.11).

The Dirac gamma matrices are defined as matrices $\gamma^a \in M_d(\mathbf{C})$, $a = 1,\dots,N$, satisfying

$$\gamma^a\gamma^b + R^{ba}{}_{dc}\gamma^c\gamma^d = 2g^{ba}\mathbb{1}, \quad a,b = 1,\dots,N\,, \tag{3.12}$$

(cf. [22,23]). It seems that one can analyze irreducible representations of the relations (3.12) only case by case for particular matrices R. In the following we assume that such an irreducible representation has been chosen. Then the Dirac operator $\partial\!\!\!/ = \partial_a \gamma^a$ satisfies

$$\partial\!\!\!/^2 = \partial_c \partial_d \gamma^c \gamma^d = \tfrac{1}{2}(\partial_c \partial_d + R^{ba}{}_{dc} \partial_a \partial_b)\gamma^c \gamma^d$$

$$= \tfrac{1}{2}\partial_a \partial_b (\gamma^a \gamma^b + R^{ba}{}_{dc} \gamma^c \gamma^d) = g^{ba} \partial_a \partial_b \cdot \mathbb{1} = \Box \cdot \mathbb{1}$$

(cf. [23]). Let $P\!\!\!/ = i\partial\!\!\!/$. Then $P\!\!\!/^{\,2} = -\Box \cdot \mathbb{1}$.

Later on we shall need the following.

Proposition 3.2. *One has*

$$\partial^j x^k = g^{jk} + R^{jk}{}_{ab} x^a \partial^b - (RZ)^{jk}{}_b \partial^b \, . \tag{3.13}$$

Proof. According to (3.8), $\partial^j x^k = g^{jk} + G^{jk}{}_{ab} x^a \partial^b + V^{jk}{}_b \partial^b$, where

$$G^{jk}{}_{ab} = g^{ji} R^{kl}{}_{ia} g_{lb} = f^k{}_a (g^{ji} \Lambda^l{}_i g_{lb}) = f^k{}_a ((\Lambda^{-1})^j{}_b) = (R^{-1})^{jk}{}_{ab} = R^{jk}{}_{ab},$$

$$V^{jk}{}_b = g^{ji} Z^{kl}{}_i g_{lb} = \eta^k (g^{ji} \Lambda^l{}_i g_{lb}) = \eta^k ((\Lambda^{-1})^j{}_b) = -(RZ)^{jk}{}_b$$

(cf. the proof of Proposition 4.5.2 of [18]). □

In the remaining part of the section we shall heuristically prove that the operators P^a, $\Box$ are hermitian. A strict proof of that fact would need the existence of topological structures on the quantum spaces G, M. Therefore the considerations below serve as a motivation for such a topological approach. We shall assume the existence of a G-invariant measure μ defined on some "functions" x on M (elements of $\mathscr{C}$ are also "functions" on M but we don't expect μ to act on them; nevertheless we shall disregard such subtleties here). Thus $(\mathrm{id} \otimes \mu)\Psi x = \mu(x) I_{\mathscr{B}}$. One has the scalar product $(x \mid y) = \mu(x^* y)$. Using (3.2), we obtain

$$(I \mid \partial_i x) = \mu(\partial_i x) = \mu(Y_i \otimes \mathrm{id})\Psi(x)$$

$$= Y_i[(\mathrm{id} \otimes \mu)\Psi(x)] = Y_i(I_{\mathscr{B}})\mu(x) = 0$$

$(0 = \partial_i I = Y_i(I) I)$, $(I \mid P^a x) = 0$, $(P^a)^* I = 0$. Also $P^a I = 0$. Moreover, the hermitian conjugate of (3.13) yields

$$x^k (\partial^j)^* = g^{kj} + R^{kj}{}_{ba} (\partial^b)^* x^a + Z^{kj}{}_b (\partial^b)^* \, ,$$

i.e.

$$(\partial^b)^* x^a = -g^{ba} + R^{ba}{}_{kj} x^k (\partial^j)^* - (RZ)^{ba}{}_s (\partial^s)^* \, .$$

This means that P^b and $(P^b)^*$ commute with x^a in the same way. Thus they act in the same way on all elements of $\mathscr{C}$ and P^b is hermitian. This and $g_{ab} = \overline{g_{ba}}$ show that $\Box$ is also hermitian.

4. Solutions of Klein–Gordon and Dirac Equations

We shall consider formal solutions φ of the Klein–Gordon equation $(\Box + m^2)\varphi = 0$ and Dirac equation $\not{P}\varphi = m\varphi$ obtained using the plane waves $e^{-\imath p_a x^a}$. But now p_a are not (in a general case) eigenvalues of P_a.

Setting $F_i{}^l(x^k) = R^{kl}{}_{in}x^n + Z^{kl}{}_i$ and using $\partial_i x^k = \delta^k{}_i + F_i{}^l(x^k)\partial_l$ (see (3.8)), one gets

$$\partial_j(x^{k_1}\cdots x^{k_n}) = \sum_{m=0}^{n-1} F_j{}^{i_1}(x^{k_1})F_{i_1}{}^{i_2}(x^{k_2})\cdots F_{i_{m-1}}{}^{k_{m+1}}(x^{k_m})x^{k_{m+2}}\cdots x^{k_n}\,, \tag{4.1}$$

where the $m = 0$ term reads $\delta^{k_1}{}_j x^{k_2}\cdots x^{k_n}$. We shall consider two cases: $Z = 0$ and $R = \tau$.

1. $Z = 0$. Let $\mathscr{F}$ be a new $*$-algebra generated by p^k, $k = 1,\dots,N$, satisfying $(p^k)^* = p^k$ and $p^k p^l = R^{lk}{}_{ji}p^i p^j$ (same relations as for P^k but w.r.t. the opposite multiplication). Set $p_a = g_{ab}p^b$, $\not{p} = p_j \otimes \gamma^j$. Then

$$p_a p_b = p_k p_l R^{kl}{}_{ab}\,. \tag{4.2}$$

We put $x \otimes p = x^a \otimes p_a \in \mathscr{C} \otimes \mathscr{F}$. So in a sense we assume that p_a commute with all x^b and ∂_m. This doesn't seem to be the case in [10, 6, 11, 1] (cf. also [4, 14, 5]) although calculations in the present case 1 (up to the moment when we use π) are quite similar as in these papers. Using (4.1) and (4.2), one obtains

$$(\partial_j \otimes \mathrm{id})(x \otimes p)^n = \sum_{m=0}^{n-1}(I \otimes p_j)(x \otimes p)^{n-1} = n(I \otimes p_j)(x \otimes p)^{n-1}\,. \tag{4.3}$$

Thus in the sense of the formal power series w.r.t. t

$$(\partial_j \otimes \mathrm{id})e^{-it(x\otimes p)} = -it(I \otimes p_j)e^{-it(x\otimes p)}\,, \tag{4.4}$$

$$(\Box \otimes \mathrm{id})e^{-it(x\otimes p)} = -t^2(I \otimes s)e^{-it(x\otimes p)}\,, \tag{4.5}$$

where $s = g^{ij}p_\imath p_j = g_{im}p^m p^i = p_i p^i$ is a central, hermitian element of $\mathscr{F}$. Let π be an irreducible $*$-representation of $\mathscr{F}$ in a Hilbert space H with an orthonormal basis e_k, $k \in K$. We denote $\pi_{kl}(x) = (e_k \mid \pi(x)e_l)$, $x \in \mathscr{F}$. In the remaining part of the $Z = 0$ case we assume that all performed operations are well defined and have "good" properties. Acting $id \otimes \pi_{kl}$ on (4.5) and setting $\pi(s) = m^2 \in \mathbf{R}$, $(\mathrm{id} \otimes \pi_{kl})(e^{-it(x\otimes p)}) = \varphi^{(t)}_{kl}$, one obtains $\Box\varphi^{(t)}_{kl} = -t^2m^2\varphi^{(t)}_{kl}$. In a topological approach one should be able to put $t = 1$. Then $\varphi = \varphi^{(t=1)}_{kl}$ is a solution of the Klein–Gordon equation $\Box\varphi = -m^2\varphi$ in a usual sense (if $m \geqq 0$). Also (see (4.4))

$$P^j\varphi^{(t=1)}_{kl} = \pi_{ks}(p^j)\varphi^{(t=1)}_{sl}\,,$$

so one should get a real spectrum of P^j (p^j are selfadjoint).

Let us pass to the Dirac equation. We denote the canonical basis in $\mathbf{C}^d$ by ε_m, $m = 1,\dots,d$, $\varphi^{(t)}_{klm} = \varphi^{(t)}_{kl} \otimes \varepsilon_m$. Tensoring (4.3) by $\gamma^j\varepsilon_m$, one obtains

$$\not{\partial}_{(13)}[(x \otimes p)^n \otimes \varepsilon_m] = n\not{p}_{(23)}[(x \otimes p)^{n-1} \otimes \varepsilon_m]\,,$$

where e.g. (13) means that $\not{\partial}$ acts in the first and the third factors of the tensor product. Thus

$$\not{P}_{(13)}[e^{-it(x\otimes p)}\otimes \varepsilon_m] = t\not{P}_{(23)}[e^{-it(x\otimes p)}\otimes \varepsilon_m] .$$

Acting by $\mathrm{id}\otimes \pi_{kl}\otimes \mathrm{id}$, we get

$$\not{P}\varphi^{(t)}_{klm} = t\pi_{ks}(p_a)(\gamma^a)^i{}_m\varphi^{(t)}_{sli} . \tag{4.6}$$

Set $U^{si}{}_{kn} = \pi_{ks}(p_a)(\gamma^a)^i{}_n$. One has $U^2 = m^2\mathbb{1}$. The possible eigenvalues of U are $\pm m$. Suppose $Uv = mv$ for $v \in H\otimes \mathbf{C}^d$. Put $\varphi^{(t)}_{vl} = v^{si}\varphi^{(t)}_{sli}$. Then $\not{P}\varphi^{(t)}_{vl} = tm\varphi^{(t)}_{vl}$. In a topological approach one should be able to put $t = 1$ and find a solution $\varphi = \varphi^{(t=1)}_{vl}$ of Dirac equation $\not{P}\varphi = m\varphi$ (if $m \geqq 0$).

2. $R = \tau$, i.e. $R^{ij}{}_{kl} = \delta^i{}_l\delta^j{}_k$. Then $Rg = g$ implies $g^{ab} = g^{ba} = \overline{g^{ab}}$. According to the proof of Proposition 4.5.2 of [18], $RZ = -\tau\bar{Z}$, $Z = -\bar{Z}$, $Z^{ab}{}_c \in i\mathbf{R}$. Let $p^k \in \mathbf{R}$, $k = 1,\ldots,N$, $p_j = g_{jk}p^k \in \mathbf{R}$. Setting $U_i{}^l = Z^{kl}{}_i p_k \in i\mathbf{R}$, $x\cdot p = x^j p_j \in \mathscr{C}$, we get $F_i{}^l(x\cdot p) = \delta^l{}_i(x\cdot p) + U_i{}^l$. This and (4.1) yield $((U^k)_i{}^j = U_i{}^{i_1}\cdot U_{i_1}{}^{i_2}\cdot\cdots\cdot U_{i_{k-1}}{}^j)$

$$\begin{aligned}\partial_j(x\cdot p)^n &= \sum_{\substack{a_0+\cdots+a_r=m-r\\ r\geqq 0,\ 0\leqq m\leqq n-1}} [U^{a_0}(x\cdot p)U^{a_1}(x\cdot p)\cdot\cdots\cdot(x\cdot p)U^{a_r}]_j{}^b\\ &\quad\times p_b(x\cdot p)^{n-m-1} .\end{aligned}$$

Let $G = \sum_{s=0}^{\infty}(tU)^s$ (in the sense of formal power series w.r.t. t). Since $x\cdot p$ commutes with U,

$$\partial_j\sum_{n=0}^{\infty}(tx\cdot p)^n = \sum_{l=0}^{\infty}\sum_{r=0}^{l}(G^{r+1})_j{}^b t p_b(tx\cdot p)^l .$$

But using mathematical induction,

$$\sum_{r=0}^{l}G^{r+1} = \sum_{k=1}^{\infty}\binom{l+k}{k}(tU)^{k-1} .$$

Therefore

$$\partial_j(x\cdot p)^n = \sum_{k=1}^{n}\binom{n}{k}(U^{k-1})_j{}^b p_b(x\cdot p)^{n-k} ,$$

$$\partial_j e^{-itx\cdot p} = [\rho(-itU)]_j{}^b(-itp_b)e^{-itx\cdot p} , \tag{4.7}$$

where $\rho(x) = \sum_{k=1}^{\infty}\frac{x^{k-1}}{k!} = \frac{e^x-1}{x}$. Using (3.62) of [18] for $m = g$, one obtains

$$Z^{kl}{}_r g^{rj} = -Z^{kj}{}_s g^{ls} , \tag{4.8}$$

$U_r{}^l g^{rj} = -g^{ls}U_s{}^j$. By virtue of the mathematical induction $(U^m)_r{}^l g^{rj} = g^{ls}[(-U)^m]_s{}^j$, $\rho(-itU)_r{}^l g^{rj} = g^{ls}\rho(itU)_s{}^j$. Thus

$$\begin{aligned}\Box e^{-itx\cdot p} &= g^{mj}\rho(-itU)_m{}^a\rho(-itU)_j{}^b(-t^2p_ap_b)e^{-itx\cdot p}\\ &= g^{as}h(-itU)_s{}^b(-t^2p_ap_b)e^{-itx\cdot p} ,\end{aligned}$$

where $h(x) = \rho(-x)\rho(x) = \left(\frac{\sinh(x/2)}{x/2}\right)^2$. Set

$$m^2 = g^{as}h(-iU)_s{}^b p_a p_b \in \mathbf{R} \,. \tag{4.9}$$

In a topological approach one should be able to put $t = 1$ and find a solution $\varphi = e^{-ix \cdot p}$ of the Klein–Gordon equation $(\Box + m^2)\varphi = 0$ (if $m \geqq 0$).

Using (4.7), one would also have

$$P^j \varphi = \mathscr{P}^{j(t=1)}\varphi \,,$$

where $\mathscr{P}^{j(t=1)} = g^{js}\rho(-iU)_s{}^b p_b \in \mathbf{R}$.

Let us pass to the Dirac equation. Setting $\varphi_r = e^{-itx \cdot p} \otimes \varepsilon_r$, one gets $\not{P}\varphi_r = (\not{\mathscr{P}})^k{}_r\varphi_k$, where $\not{\mathscr{P}} = \mathscr{P}_j\gamma^j$, $\mathscr{P}_j = \rho(-itU)_j{}^b t p_b \in \mathbf{R}[[t]]$. In a topological approach one should be able to put $t = 1$. Then $\not{\mathscr{P}}^2 = g^{js}\mathscr{P}_j\mathscr{P}_s\mathbb{1}$, the possible eigenvalues of $\not{\mathscr{P}}$ are $\pm m$, $m = \sqrt{g^{js}\mathscr{P}_j\mathscr{P}_s}$. Suppose $\not{\mathscr{P}}v = mv$ for $v \in \mathbf{C}^d$. Put $\varphi = v^r\varphi_r = e^{-ix \cdot p} \otimes v$. Then φ is a solution of the Dirac equation $\not{P}\varphi = m\varphi$ (if $m \geqq 0$).

Let us notice that spectra of P^j, $\Box$ obtained (formally) above are real in agreement with hermiticity of those operators, heuristically asserted at the end of Sect. 3.

The case $R = \tau$ covers e.g. the quantum Poincaré groups of case 1), $s = 1$, $t = 1$, $t_0 = 0$. Many of them are described by Remark 1.8 of [19] and [30]. Then eigenvalues λ of $-iU$ are real or imaginary which yields factors $\left(\frac{\sinh(|\lambda|/2)}{|\lambda|/2}\right)^2$, $\left(\frac{\sin(|\lambda|/2)}{|\lambda|/2}\right)^2$ in (4.9) (when U is diagonalizable). Despite the sine factors, m^2 of (4.9) is never bounded in the cases given by [30]. Hyperbolic sine factors can improve the properties of the propagator $\frac{i}{g^{as}h(-iU)_s{}^b p_a p_b - M^2}$ $(M \geqq 0)$ (although it is never integrable in the cases given by [30]) e.g. $b = \varepsilon e_1 \wedge H$, $a = 0$, $c = 0$ in [30] ($\varepsilon \in \mathbf{R}$) corresponds to $U_0^3 = U_3^0 = i\varepsilon p_1/2$, other U_a^b are 0 and we get the propagator

$$\frac{i}{(p_0^2 - p_3^2)\left[\frac{\sinh(\varepsilon p_1/4)}{\varepsilon p_1/4}\right]^2 - p_1^2 - p_2^2 - M^2} \,.$$

A detailed analysis of Feynman diagrams is not yet possible since the measure in the p-space and interaction factors are not known. Let us also notice that for $H = SO(4)$, U is antisymmetric and we can have only factors of the second type.

Remark. In [25] differential calculi corresponding to a quantum Minkowski space of [8] are considered. Despite other choices of axioms the result is the same as here: there are no 4-dimensional covariant differential calculi in that case (due to [30] $t_0 \neq 0$ for [8]). But nevertheless a different approach suggests a similar form for the propagator as above [8].

5. The Fock Space

Here we define the Fock space for noninteracting particles on M. We assume that G has CT Hopf $*$-algebra structure $\mathscr{R}$ as in Theorem 3.1.3–4 of [17] which is the case e.g. for quantum Poincaré groups of cases 1) 2) 3) 4) (except 1), $s = 1$, $t = 1$, $t_0 \neq 0$ and 4), $s = 1$, $b \neq 0$) as proved in Theorem 3.2.3 of [17].

We follow the scheme of [16] but now we have the left (instead of right) action. The particles interchange operator $K : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$ is defined by

$$K(x \otimes y) = \mathscr{R}(y^{(1)} \otimes x^{(1)})(y^{(2)} \otimes x^{(2)}),$$

where $\Psi(x) = x^{(1)} \otimes x^{(2)}$, $\Psi(y) = y^{(1)} \otimes y^{(2)}$. We set $K^{(m)} = \mathbb{1}^{\otimes(m-1)} \otimes K \otimes \mathbb{1}^{\otimes(n-m-1)} : \mathscr{C}^{\otimes n} \to \mathscr{C}^{\otimes n}$, $m = 1, 2, \ldots, n-1$. Then one obtains the representation of the permutation group

$$\pi : \Pi_n \ni t_m = (m, m+1) \to \pi_{(m,m+1)} = K^{(m)}$$

acting in $\mathscr{C}^{\otimes n}$ which defines the boson subspace $\mathscr{C}^{\otimes_s n}$. The group G acts on $\mathscr{C}^{\otimes n}$ by the linear mapping $\Psi^{\otimes n} : \mathscr{C}^{\otimes n} \to \mathscr{B} \otimes \mathscr{C}^{\otimes n}$ defined by

$$\Psi^{\otimes n}(x_1 \otimes \cdots \otimes x_n) = x_1^{(1)} \cdot \cdots \cdot x_n^{(1)} \otimes x_1^{(2)} \otimes \cdots \otimes x_n^{(2)},$$

$x_1, \ldots, x_n \in \mathscr{C}$. One has $\Psi^{\otimes n} K^{(m)} = (\mathrm{id} \otimes K^{(m)})\Psi^{\otimes n}$, i.e. the actions of G and Π_n agree. In particular, one can restrict $\Psi^{\otimes n}$ to $\mathscr{C}^{\otimes_s n}$ getting $\Psi^{\otimes_s n} : \mathscr{C}^{\otimes_s n} \to \mathscr{B} \otimes \mathscr{C}^{\otimes_s n}$. If $W : \mathscr{C} \to \mathscr{C}$ is an operator related to a single particle then the corresponding n-particle operator is given by

$$W^{(n)} = \sum_{m=1}^{n} \pi_{(1,m)}(W \otimes \mathbb{1}^{\otimes(n-1)})\pi_{(1,m)} = \frac{1}{(n-1)!} \sum_{\sigma \in \Pi_n} \pi_\sigma(W \otimes \mathbb{1}^{\otimes(n-1)}) \times \pi_\sigma^{-1} : \mathscr{C}^{\otimes_s n} \to \mathscr{C}^{\otimes_s n}$$

(the m^{th} term in the first sum is the operator in $\mathscr{C}^{\otimes n}$ corresponding to the m^{th} particle). We can also define the Fock space $F = \bigoplus_{n=0}^{\infty} \mathscr{C}^{\otimes_s n}$ and the operator $\bigoplus_{n=0}^{\infty} W^{(n)}$ acting in F.

For particles of mass m we should consider $\ker(\Box + m^2)$ instead of $\mathscr{C}$ and a scalar product there but it would lead us beyond the scope of the present article (heuristically e.g. $W = P^k$, $k = 1, \ldots, N$, would be hermitian operators in such a space – see (3.9)).

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