

Drinfeld–Sokolov Gravity

Roberto Zucchini

Dipartimento di Fisica, Università degli Studi di Bologna, V. Irnerio 46, I-40126 Bologna, Italy

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Abstract: A lagrangian euclidean model of Drinfeld–Sokolov (DS) reduction leading to general W -algebras on a Riemann surface of any genus is presented. The background geometry is given by the DS principal bundle K associated to a complex Lie group G and an $SL(2, \mathbb{C})$ subgroup S . The basic fields are a hermitian fiber metric H of K and a $(0, 1)$ Koszul gauge field A^* of K valued in a certain negative graded subalgebra \mathfrak{r} of \mathfrak{g} related to \mathfrak{s} . The action governing the H and A^* dynamics is the effective action of a DS field theory in the geometric background specified by H and A^* . Quantization of H and A^* implements on one hand the DS reduction and on the other defines a novel model of $2d$ gravity, DS gravity. The gauge fixing of the DS gauge symmetry yields an integration on a moduli space of DS gauge equivalence classes of A^* configurations, the DS moduli space. The model has a residual gauge symmetry associated to the DS gauge transformations leaving a given field A^* invariant. This is the DS counterpart of conformal symmetry. Conformal invariance and certain non-perturbative features of the model are discussed in detail.

1. Introduction

In recent years, a considerable amount of work has been devoted to the study of W -algebras [1]. The interest in W -algebras stems mainly from the fact that they are non-linear extensions of the Virasoro algebra appearing as symmetry algebras in certain critical two dimensional statistical systems as well as in W strings and W -gravity models. The latter in turn are of considerable interest in themselves as generalizations of ordinary string and gravity models with non-standard values of the critical dimension [2–5].

The construction of W -algebras can be carried out both in a hamiltonian and in a lagrangian framework. In the former approach [6–12], based on the methods of hamiltonian reduction, the currents of a Wess–Zumino–Novikov–Witten phase space with the standard Kac–Moody Poisson structure and Virasoro action are subject to a set of conformally invariant first class constraints corresponding to a certain nilpotent subalgebra of the relevant symmetry Lie algebra. Upon gauge fixing, the reduced phase space exhibits a non-linear Poisson structure and a Virasoro action, realizing

the W -algebra. Quantization is carried out in a Becchi–Rouet–Stora framework. In the latter approach [10, 13], based on lagrangian local field theory, a certain nilpotent subgroup of the relevant symmetry group of a Wess–Zumino–Novikov–Witten field theory is gauged yielding a conformally invariant gauge theory. Quantizing and gauge fixing à la Fadeev–Popov, one gets a quantum field theory whose gauge invariant operators generate the W -algebra. Underlying both approaches is the existence of an $\mathfrak{sl}(2)$ subalgebra of the symmetry Lie algebra defining a halfinteger gradation of the latter [10–12].

It seems appropriate to test the basic assumptions of such formulations in new ways and explore the consequences of the results so obtained. A possible approach in this direction consists in seeing whether W -algebras can be constructed on a topological non-trivial world sheet. In the hamiltonian framework, this has been done in refs. [14] for Drinfeld–Sokolov lowest weight reductions [15], where the conformal properties are manifest. It has not been attempted yet in the lagrangian framework. This is precisely the aim of this paper.

There are at least two reasons why this is an interesting problem. First, this is an integral part of the programme of constructing the Polyakov measure [16–19] for W -strings and W -gravity. Second, the gauge fixing of the Drinfeld–Sokolov gauge symmetry leaves in principle a residual integration on the space of Drinfeld–Sokolov gauge orbits. The existence of such Drinfeld–Sokolov moduli space is a non-trivial feature of Drinfeld–Sokolov lowest weight reduction which is manifest only in the lagrangian approach.

It is important to appreciate the salient features of the construction of the present paper by comparing it with earlier lagrangian formulations. The basic elements of the construction of ref. [10] are a split simple real Lie group G and an $SL(2, \mathbb{R})$ subgroup S of G . To these data, one can associate canonically a halfinteger grading of \mathfrak{g} and a certain negative graded subalgebra \mathfrak{r} of \mathfrak{g} . One considers then a modified minkowskian G Wess–Zumino–Novikov–Witten model and gauges the subgroup $X = \exp \mathfrak{r}$ of G . The classical action is

$$I^M(H, A_-, A_+) = KS_{WZNW}^M(H) + \frac{K}{\pi} \int d^2x \operatorname{tr}[(\partial_+ H H^{-1} - t_{+1}) A_- + (H^{-1} \partial_- H - t_{-1}) A_+ - A_- \operatorname{Ad} H A_+], \tag{1.1}$$

where tr is the Cartan–Killing form of \mathfrak{g} , the t_d are the standard generators of \mathfrak{s} and K is the level. H is the minkowskian Wess–Zumino field and $dx^+ A_+ + dx^- A_-$ is the minkowskian \mathfrak{r} gauge field. $S_{WZNW}^M(H)$ is the customary minkowskian Wess–Zumino–Novikov–Witten action integrating the variational identity

$$\delta S_{WZNW}^M(H) = \frac{1}{\pi} \int d^2x \operatorname{tr}[\delta H H^{-1} \partial_- (\partial_+ H H^{-1})]. \tag{1.2}$$

As recalled above, this field theory yields upon quantization the W -algebra associated to the data (G, S) .

On a Riemann surface, one needs a euclidean reformulation of the above. The basic algebraic data are now a simple complex Lie group G and an $SL(2, \mathbb{C})$ subgroup S of G . To these data, there is associated again a grading of \mathfrak{g} and a negative graded subalgebra \mathfrak{r} of \mathfrak{g} . The euclidean version of the action (1.1) should read:

$$I^E(H, A^*, A) = KS_{WZNW}^E(H) + \frac{K}{\pi} \int d^2z \operatorname{tr}[(\partial H H^{-1} - t_{+1}) A^* + (H^{-1} \bar{\partial} H - t_{-1}) A - A^* \operatorname{Ad} H A]. \tag{1.3}$$

H is the euclidean Wess–Zumino field and $dzA + d\bar{z}A^*$ is the euclidean \mathfrak{g} gauge field. $S_{\text{WZNW}}^E(H)$ is the “euclidean Wess–Zumino–Novikov–Witten action” integrating the variational identity

$$\delta S_{\text{WZNW}}^E(H) = \frac{1}{\pi} \int d^2z \text{tr} [\delta H H^{-1} \bar{\partial}(\partial H H^{-1})]. \tag{1.4}$$

Resorting to complex groups is unavoidable when switching from minkowskian light-cone to euclidean holomorphic geometry. However, in so doing, I have doubled the number of real field theoretic degrees of freedom and generated a complex action. To eliminate the spurious degrees of freedom and have a real positive definite action, one has to impose on the fields certain reality conditions with respect to a suitable conjugation. Such conditions are

$$H = H^\dagger, \tag{1.5}$$

$$A^* = A^\dagger, \tag{1.6}$$

and $t_d^\dagger = t_{-d}$, where \dagger is the compact conjugation of \mathfrak{g} . This leads to a reinterpretation of the model with surprising features.

The reality conditions (1.5)–(1.6) suggest that H is the fiber metric for some principal G bundle and that the $(0,1)$ gauge field A^* is the Koszul field corresponding to its holomorphic structure in the spirit of deformation theory [20]. The euclidean Wess–Zumino–Novikov–Witten action $S_{\text{WZNW}}^E(H)$ is then nothing but the Donaldson action first employed by Donaldson in his studies of Hermitian–Einstein bundles [21]. The principal bundle in question is the Drinfeld–Sokolov bundle DS discovered in ref. [14]. DS prescribes the transformation rule of a \mathfrak{g} -valued field $\Psi(z, \bar{z})$ under a coordinate change $z \rightarrow z'$, which reads

$$\Psi'(z', \bar{z}') = \exp\left(-\ln \frac{\partial z}{\partial z'} \text{ad } t_0\right) \exp\left(\frac{\partial}{\partial z'} \frac{\partial z'}{\partial z} \text{ad } t_{-1}\right) \Psi(z, \bar{z}). \tag{1.7}$$

This important relation encapsulates at once the algebraic data (G, S) defining the W -algebra and the holomorphic geometry of the underlying Riemann surface. It also provides a mathematically precise formulation of Polyakov’s ideas of soldering [22].

This is reminiscent of ordinary gravity à la Polyakov [16–19], where the basic fields are the surface metric h and the Beltrami field μ and the effective action $I(h, \mu, \bar{\mu})$ exhibits a structure analogous to the one shown above, the counterpart of the Wess–Zumino–Novikov–Witten action being the Liouville action. The resemblance is even more striking when it is realized that there are field theories whose effective action is a functional of H and A^* of the form (1.3) with (1.4)–(1.6) satisfied. Therefore, I shall call this euclidean model Drinfeld–Sokolov gravity. After gauge fixing, the model has a residual gauge symmetry associated to the gauge transformations leaving the given Koszul field invariant. This is the Drinfeld–Sokolov counterpart of conformal symmetry. It also involves an integration on a non-trivial space of Drinfeld–Sokolov gauge orbits. It must be stressed that the Drinfeld–Sokolov moduli space considered here is distinct from the W -moduli space of ref. [23] and from the moduli space studied by Hitchin in ref. [24] and later related to quantum W -gravity in ref. [25].

The plan of the paper is as follows. In Sect. 2, the basic notions concerning the holomorphic and hermitian structures and the symmetries of the Drinfeld–Sokolov

bundle necessary for the understanding of the following constructions are collected. In Sect. 3, the main properties of Drinfeld–Sokolov field theory are expounded. In Sect. 4, Drinfeld–Sokolov gravity is defined, the gauge fixing of the Drinfeld–Sokolov symmetry is illustrated and the formal construction of the measure is carried out. In Sect. 5, the Drinfeld–Sokolov ghost system is studied in detail. In Sect. 6, conformal invariance and certain non-perturbative features of the resulting theory are analyzed and the remaining unsolved problems are pointed out. Finally, the appendices explain in great detail the definition of the functional measures and the implementation of the gauge fixing for the interested reader.

2. The Drinfeld–Sokolov Bundle

In the first part this section, I review certain general results concerning the holomorphic and hermitian geometry of principal bundles on a surface [26–28]. In the second part, I define the Drinfeld–Sokolov bundle and analyze its main properties [14, 29].

1. Holomorphic Structures. Let Σ be a compact Riemann surface of genus ℓ with local holomorphic coordinates z_a , where a is a coordinate label. Σ is characterized by the holomorphic 1-cocycle k defined by $k_{ab} = \partial_a z_b$, where $\partial_a = \partial/\partial z_a$. In applications, it is necessary to choose a 1-cocycle square root of k , that is a holomorphic 1-cocycle $k^{\otimes \frac{1}{2}}$ such that $(k^{\otimes \frac{1}{2}})_{ab}^2 = k_{ab}$. For any $j \in \mathbb{Z}/2$, one can then define the holomorphic 1-cocycle $k^{\otimes j}$ by setting $k^{\otimes j}_{ab} = (k^{\otimes \frac{1}{2}})_{ab}^{2j}$. As is well known, these 1-cocycles define holomorphic line bundles on Σ , k and $k^{\otimes j}$ corresponding to the canonical line bundle and its tensor powers.

Let $w, \bar{w} \in \mathbb{Z}/2$. A conformal field ψ of weights w, \bar{w} is given as a collection of smooth complex valued maps ψ_a of domain $\text{dom } z_a$ such that, whenever defined, $\psi_a = k^{\otimes w} \otimes \bar{k}^{\otimes \bar{w}} \psi_b$. The conformal fields ψ of weights w, \bar{w} span a infinite dimensional complex linear space $\text{CF}^{w, \bar{w}}$.

The spaces $\text{CF}^{w, \bar{w}}$ and $\text{CF}^{1-w, 1-\bar{w}}$ are dual to each other. The dual pairing is given by $\langle \phi, \psi \rangle = \frac{1}{\pi} \int_{\Sigma} d^2z \phi \psi$ for $\psi \in \text{CF}^{w, \bar{w}}$ and $\phi \in \text{CF}^{1-w, 1-\bar{w}}$.

The Cauchy–Riemann operator $\bar{\partial} : \text{CF}^{w, 0} \rightarrow \text{CF}^{w, 1}$ is locally defined by $(\bar{\partial}\psi)_a = \bar{\partial}_a \psi_a$ for $\psi \in \text{CF}^{w, 0}$. The kernel of $\bar{\partial}$ is the subspace HCF^w of holomorphic elements of $\text{CF}^{w, 0}$. By the Riemann–Roch theorem, $\dim \text{HCF}^w - \dim \text{HCF}^{1-w} = (2w - 1)(\ell - 1)$.

A $(1, 0)$ affine connection γ is a collection of smooth complex valued maps γ_a of domain $\text{dom } z_a$ such that $\gamma_a = k_{ab}[\gamma_b + \partial_b \ln k_{ab}]$ whenever defined. γ is characterized by its curvature f_{γ} , given locally by $f_{\gamma a} = \bar{\partial}_a \gamma_a$. $f_{\gamma} \in \text{CF}^{1, 1}$. Let Aff be the family of all $(1, 0)$ affine connections γ .

To any $\gamma \in \text{Aff}$, one can associate the covariant derivative $\partial_{\gamma} : \text{CF}^{w, \bar{w}} \rightarrow \text{CF}^{w+1, \bar{w}}$ locally given by $(\partial_{\gamma}\psi)_a = (\partial_a - w\gamma_a)\psi_a$ for $\psi \in \text{CF}^{w, \bar{w}}$.

Let K be a holomorphic G -valued 1-cocycle on Σ , where G is a simple complex Lie group. To K , one can associate a smooth principal G -bundle P over Σ by means of a well known construction.

A holomorphic structure \mathbf{s} is specified by a collection of smooth G -valued maps $V_{\mathbf{s}a}$ of domain $\text{dom } z_a$ such that there exists a holomorphic G -valued 1-cocycle $K_{\mathbf{s}}$ such that, whenever defined, $V_{\mathbf{s}a} = K_{ab} V_{\mathbf{s}b} K_{\mathbf{s}ab}^{-1}$. Note that the 1-cocycle $K_{\mathbf{s}}$ characterizes but does not determine the holomorphic structure \mathbf{s} , since the map $\mathbf{s} \rightarrow K_{\mathbf{s}}$ is many-to-one. Two holomorphic structures \mathbf{s}_1 and \mathbf{s}_2 are said to be equivalent

if $V_{\mathfrak{s}_1 a} = V_{\mathfrak{s}_2 a} v_a$ for some holomorphic G -valued function v_a for every a . This is indeed an equivalence relation. Below, I shall not distinguish between equivalent holomorphic structures. The family of all holomorphic structures of will be denoted by Hol .

Let $\mathfrak{s} \in \text{Hol}$ and $w, \bar{w} \in \mathbb{Z}/2$. An extended \mathfrak{s} -conformal field $\Psi_{\mathfrak{s}}$ of weights w, \bar{w} is given as a collection of smooth \mathfrak{g} -valued maps $\Psi_{\mathfrak{s}a}$ of domain $\text{dom } z_a$ such that, whenever defined, $\Psi_{\mathfrak{s}a} = k^{\otimes w}{}_{ab} \bar{k}^{\otimes \bar{w}}{}_{ab} \text{Ad } K_{\mathfrak{s}ab} \Psi_{\mathfrak{s}b}$. The extended \mathfrak{s} -conformal fields Ψ of weights w, \bar{w} span an infinite dimensional complex linear space $\text{ECF}_{\mathfrak{s}}^{w, \bar{w}}$.

The spaces $\text{ECF}_{\mathfrak{s}}^{w, \bar{w}}$ and $\text{ECF}_{\mathfrak{s}}^{1-w, 1-\bar{w}}$ are dual to each other. The dual pairing is given by $\langle \Phi, \Psi \rangle_{\mathfrak{s}} = \frac{1}{\pi} \int_{\Sigma} d^2 z \text{tr}_{\text{ad}}(\Phi \Psi)_{\mathfrak{s}}$ for $\Psi_{\mathfrak{s}} \in \text{ECF}_{\mathfrak{s}}^{w, \bar{w}}$ and $\Phi_{\mathfrak{s}} \in \text{ECF}_{\mathfrak{s}}^{1-w, 1-\bar{w}}$, where tr_{ad} denotes the Cartan–Killing form of \mathfrak{g} .

The Cauchy–Riemann operator $\bar{\partial}_{\mathfrak{s}} : \text{ECF}_{\mathfrak{s}}^{w, 0} \rightarrow \text{ECF}_{\mathfrak{s}}^{w, 1}$ is locally defined by $(\bar{\partial} \Psi)_{\mathfrak{s}a} = \bar{\partial}_a \Psi_{\mathfrak{s}a}$. The kernel of $\bar{\partial}_{\mathfrak{s}}$ is the subspace $\text{HECF}_{\mathfrak{s}}^w$ of holomorphic elements of $\text{ECF}_{\mathfrak{s}}^{w, 0}$. By the Riemann–Roch theorem, $\dim \text{HECF}_{\mathfrak{s}}^w - \dim \text{HECF}_{\mathfrak{s}}^{1-w} = (2w - 1)(\ell - 1) \dim \mathfrak{g}$.

A $(1, 0)$ \mathfrak{s} -connection $\Gamma_{\mathfrak{s}}$ is a collection of smooth \mathfrak{g} -valued maps $\Gamma_{\mathfrak{s}a}$ such that $\Gamma_{\mathfrak{s}a} = k_{ab}[\text{Ad } K_{\mathfrak{s}ab} \Gamma_{\mathfrak{s}b} + \partial_b K_{\mathfrak{s}ab} K_{\mathfrak{s}ab}^{-1}]$. The connection $\Gamma_{\mathfrak{s}}$ is characterized by its curvature $F_{\Gamma_{\mathfrak{s}}}$ locally given by $F_{\Gamma_{\mathfrak{s}a}} = \bar{\partial}_a \Gamma_{\mathfrak{s}a}$. $F_{\Gamma_{\mathfrak{s}}} \in \text{ECF}_{\mathfrak{s}}^{1, 1}$. Let $\text{Conn}_{\mathfrak{s}}$ be the family of all $(1, 0)$ \mathfrak{s} -connections $\Gamma_{\mathfrak{s}}$.

To any $\gamma \in \text{Aff}$ and $\Gamma_{\mathfrak{s}} \in \text{Conn}_{\mathfrak{s}}$, one can associate the covariant derivative $\partial_{\gamma, \Gamma_{\mathfrak{s}}} : \text{ECF}_{\mathfrak{s}}^{w, \bar{w}} \rightarrow \text{ECF}_{\mathfrak{s}}^{w+1, \bar{w}}$ locally given by $(\partial_{\gamma, \Gamma} \Psi)_{\mathfrak{s}a} = (\partial_a - w\gamma_a - \text{ad } \Gamma_{\mathfrak{s}a}) \Psi_{\mathfrak{s}a}$.

In applications, the holomorphic structure \mathfrak{s} is considered as variable. The dependence on \mathfrak{s} is then to be studied.

Hol contains a natural reference holomorphic structure defined by $V_a = 1$ for all a . By convention, all geometric objects related to such structure, such as the holomorphic 1-cocycle K , the extended conformal fields Ψ , the spaces of (holomorphic) extended conformal fields $\text{ECF}^{w, \bar{w}}$ and HECF^w , the $(1, 0)$ connections Γ and their family Conn , etc. will carry no subscript \mathfrak{s} . In particular, the adjective “conformal” is always understood as “reference–holomorphic–structure–conformal.”

Let $w, \bar{w} \in \mathbb{Z}/2$. A minimal extended conformal field functional Ψ of weights w, \bar{w} is a map that associates to any $\mathfrak{s} \in \text{Hol}$ an element $\Psi_{\mathfrak{s}} \in \text{ECF}_{\mathfrak{s}}^{w, \bar{w}}$ in such a way that the condition $\Psi_a = \text{Ad } V_{\mathfrak{s}a} \Psi_{\mathfrak{s}a}$ is satisfied for any a . In this way, the dependence of $\Psi_{\mathfrak{s}}$ on \mathfrak{s} is determined entirely by $V_{\mathfrak{s}}$. The space of all minimal extended conformal field functionals Ψ of weights w, \bar{w} may thus be identified with $\text{ECF}^{w, \bar{w}}$ itself.

For any $\Psi \in \text{ECF}^{w, \bar{w}}$ and $\Phi \in \text{ECF}^{1-w, 1-\bar{w}}$, $\langle \Phi, \Psi \rangle_{\mathfrak{s}} = \langle \Phi, \Psi \rangle$ for $\mathfrak{s} \in \text{Hol}$. In this way, the dual pairing $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ of $\text{ECF}_{\mathfrak{s}}^{w, \bar{w}}$ and $\text{ECF}_{\mathfrak{s}}^{1-w, 1-\bar{w}}$ induces a dual pairing $\langle \cdot, \cdot \rangle$ of the spaces of minimal extended conformal field functionals $\text{ECF}^{w, \bar{w}}$ and $\text{ECF}^{1-w, 1-\bar{w}}$.

A minimal $(1, 0)$ connection functional Γ is a map that associates to any $\mathfrak{s} \in \text{Hol}$ an element $\Gamma_{\mathfrak{s}} \in \text{Conn}_{\mathfrak{s}}$ in such a way that the condition $\Gamma_a = \text{Ad } V_{\mathfrak{s}a} \Gamma_{\mathfrak{s}a} + \partial_a V_{\mathfrak{s}a} V_{\mathfrak{s}a}^{-1}$ is satisfied for any a . As for minimal extended conformal field functionals, this condition means that the dependence of $\Gamma_{\mathfrak{s}}$ on \mathfrak{s} is determined by $V_{\mathfrak{s}}$. The family of minimal $(1, 0)$ connection functionals Γ may be identified with Conn itself.

There exists a parametrization of Hol , the Koszul parametrization defined next, which is particularly useful in field theoretic applications.

A Koszul field A^* is simply an element of $\text{ECF}^{0, 1}$. There is a one-to-one correspondence between the family of holomorphic structures \mathfrak{s} and the family of Koszul

fields A^* [20]. The correspondence, expressed notationally as $\mathfrak{s} \equiv A^*$, is given by the relation $A^*_a = \bar{\partial}_a V_{\mathfrak{S}a} V_{\mathfrak{S}a}^{-1}$. Thus, one may view equivalently Hol as the manifold formed by all Koszul fields and cast dependence on \mathfrak{s} as dependence on A^* . Note that $A^* = 0$ for the reference holomorphic structure.

In general, field theoretic expressions are compact when written in terms of the relevant holomorphic structure \mathfrak{s} . The dependence on \mathfrak{s} is however explicit only in the Koszul parametrization provided one restricts to minimal extended conformal field functionals and minimal $(1, 0)$ connection functionals. The rules for translating from the first to the second description are the following:

$$\Psi_{\mathfrak{S}} \leftrightarrow \Psi, \tag{2.1}$$

$$\bar{\partial}_{\mathfrak{S}} \leftrightarrow \bar{\partial} - \text{ad } A^*, \tag{2.2}$$

$$F_{\Gamma_{\mathfrak{S}}} \leftrightarrow F_{\Gamma} - \hat{\partial}_{\Gamma} A^*, \tag{2.3}$$

$$\hat{\partial}_{\gamma, \Gamma_{\mathfrak{S}}} \leftrightarrow \hat{\partial}_{\gamma, \Gamma}, \tag{2.4}$$

for $\Psi \in \text{ECF}^{w, \bar{w}}$, $\gamma \in \text{Aff}$ and $\Gamma \in \text{Conn}$, where $\mathfrak{s} \equiv A^* \in \text{Hol}$. If $\Psi \in \text{ECF}^{w, 0}$ is such that $\Psi_{\mathfrak{S}} \in \text{HECF}_{\mathfrak{S}}^w$, then $(\bar{\partial} - \text{ad } A^*)\Psi = 0$.

2. Hermitian Structures. A hermitian surface metric h on Σ is a collection of smooth maps h_a of domain $\text{dom } z_a$ such that $h_a > 0$ and $h_a = k_{ab} \bar{k}_{ab} h_b$. The hermitian surface metrics h form an infinite dimensional real functional manifold Met .

Given any metric $h \in \text{Met}$, one can define a Hilbert structure on $\text{CF}^{w, \bar{w}}$ by setting $\langle \psi_1, \psi_2 \rangle_h = \frac{1}{\pi} \int_{\Sigma} d^2 z h^{\otimes 1-w-\bar{w}} \bar{\psi}_1 \psi_2$ for $\psi_1, \psi_2 \in \text{CF}^{w, \bar{w}}$.

Each metric h is characterized by a $(1, 0)$ affine connection γ_h locally given by $\gamma_{ha} = \partial_a \ln h_a$. The curvature f_h of γ_h is then given by $f_{ha} = \bar{\partial}_a \partial_a \ln h_a$. The covariant derivative of γ_h will be denoted by $\hat{\partial}_h$.

Let $\mathfrak{s} \in \text{Hol}$ be a holomorphic structure. A \mathfrak{s} -hermitian fiber metric $H_{\mathfrak{S}}$ is defined as a collection of smooth G -valued maps $H_{\mathfrak{S}a}$ of domain $\text{dom } z_a$ such that $H_{\mathfrak{S}a}^{\dagger} = H_{\mathfrak{S}a}$ and $H_{\mathfrak{S}a} = K_{\mathfrak{S}ab} H_{\mathfrak{S}b} K_{\mathfrak{S}ab}^{\dagger}$, where \dagger denotes the compact conjugation of G . The \mathfrak{s} -hermitian fiber metrics $H_{\mathfrak{S}}$ form an infinite dimensional real manifold $\text{Herm}_{\mathfrak{S}}$.

Given metrics $h \in \text{Met}$ and $H_{\mathfrak{S}} \in \text{Herm}_{\mathfrak{S}}$, one can define a Hilbert structure on $\text{ECF}_{\mathfrak{S}}^{w, \bar{w}}$ by setting $\langle \Psi_1, \Psi_2 \rangle_{h, H_{\mathfrak{S}}} = \frac{1}{\pi} \int_{\Sigma} d^2 z h^{\otimes 1-w-\bar{w}} \text{tr}_{\text{ad}}(\text{Ad } H \Psi_1^{\dagger} \Psi_2)_{\mathfrak{S}}$ for $\Psi_1, \Psi_2 \in \text{ECF}_{\mathfrak{S}}^{w, \bar{w}}$.

Each fiber metric $H_{\mathfrak{S}}$ is characterized by a $(1, 0)$ \mathfrak{s} -connection $\Gamma_{H_{\mathfrak{S}}}$ of $K_{\mathfrak{S}}$ locally given by $\Gamma_{H_{\mathfrak{S}a}} = \partial_a H_{\mathfrak{S}a} H_{\mathfrak{S}a}^{-1}$. The curvature $F_{H_{\mathfrak{S}}}$ of $\Gamma_{H_{\mathfrak{S}}}$ is given by $F_{H_{\mathfrak{S}a}} = \bar{\partial}_a (\partial_a H_{\mathfrak{S}a} H_{\mathfrak{S}a}^{-1})$. The covariant derivative associated to a surface metric $h \in \text{Met}$ and to $H_{\mathfrak{S}}$ is $\hat{\partial}_{h, H_{\mathfrak{S}}}$.

A minimal hermitian fiber metric functional is a map that associates to each holomorphic structure $\mathfrak{s} \in \text{Hol}$ a hermitian fiber metric $H_{\mathfrak{S}} \in \text{Herm}_{\mathfrak{S}}$ in such a way that $H_a = V_{\mathfrak{S}a} H_{\mathfrak{S}a} V_{\mathfrak{S}a}^{\dagger}$ holds for any a . As for minimal extended conformal field functionals, this condition means that the dependence of $H_{\mathfrak{S}}$ on \mathfrak{s} is determined by $V_{\mathfrak{S}}$. Hence, the space of minimal hermitian fiber metric functionals H may be identified with Herm .

For any $H \in \text{Herm}$ and any two $\Psi_1, \Psi_2 \in \text{ECF}^{w, \bar{w}}$, $\langle \Psi_1, \Psi_2 \rangle_{h, H_{\mathfrak{S}}} = \langle \Psi_1, \Psi_2 \rangle_{h, H}$ for $\mathfrak{s} \in \text{Hol}$. Thus, for a given minimal hermitian fiber metric functional H , the Hilbert structure $\langle \cdot, \cdot \rangle_{h, H_{\mathfrak{S}}}$ on $\text{ECF}_{\mathfrak{S}}^{w, \bar{w}}$ induces a Hilbert structure $\langle \cdot, \cdot \rangle_{h, H}$ on the space of minimal extended conformal field functionals $\text{ECF}^{w, \bar{w}}$.

For the curvature F_H and the covariant derivative $\partial_{h,H}$ associated to metrics $h \in \text{Met}$ and $H \in \text{Herm}$, (2.3)–(2.4) do not apply. To express everything in the Koszul parametrization, one has instead to perform the substitutions

$$F_{H\mathbf{s}} \leftrightarrow F_H - \partial_H A^* - \bar{\partial} \text{Ad} H A^{*\dagger} + [A^*, \text{Ad} H A^{*\dagger}], \tag{2.5}$$

$$\partial_{h,H\mathbf{s}} \leftrightarrow \partial_{h,H} + \text{ad} \text{Ad} H A^{*\dagger}, \tag{2.6}$$

with $\mathbf{s} \equiv A^* \in \text{Hol}$.

3. The Gauge Group. A gauge transformation α is a collection of smooth G -valued maps α_a of domain $\text{dom } z_a$ such that, whenever defined, $\alpha_a = K_{ab} \alpha_b K_{ab}^{-1}$. The gauge transformations form a group Gau under pointwise multiplication. $\text{Lie Gau} \cong \text{ECF}^{0,0}$ with the obvious Lie brackets. To Gau , there are associated a few relevant actions.

Gau does not act on Σ and on the spaces $\text{CF}^{w,\bar{w}}$ of conformal fields.

Gau acts on the family Hol of holomorphic structures as follows. If $\alpha \in \text{Gau}$ and $\mathbf{s} \in \text{Hol}$, then $\alpha^* \mathbf{s} \in \text{Hol}$ is the holomorphic structure specified by $V_{\alpha^* \mathbf{s}a} = \alpha_a V_{\mathbf{s}a}$. Note that $K_{\alpha^* \mathbf{s}} = K_{\mathbf{s}}$. The action of Gau on Hol is not free. The stability subgroup $\mathcal{G}(\mathbf{s})$ of a holomorphic structure $\mathbf{s} \in \text{Hol}$ in Gau is formed by all gauge transformations η such that $\eta_{\mathbf{s}a} = V_{\mathbf{s}a}^{-1} \eta_a V_{\mathbf{s}a}$ is holomorphic. In fact, for $\eta \in \mathcal{G}(\mathbf{s})$, $\eta^* \mathbf{s}$ is equivalent to \mathbf{s} , and hence is not distinguished from the latter. Note that $\text{Lie } \mathcal{G}(\mathbf{s}) \cong \text{HECF}_{\mathbf{s}}^0$.

Associated to this action is also an action on extended conformal fields defined as follows. For $\alpha \in \text{Gau}$ and $\Psi_{\mathbf{s}} \in \text{ECF}_{\mathbf{s}}^{w,\bar{w}}$, $\alpha^* \Psi_{\alpha^* \mathbf{s}}$ is the extended conformal field in $\text{ECF}_{\alpha^* \mathbf{s}}^{w,\bar{w}}$ locally defined by $\alpha^* \Psi_{\alpha^* \mathbf{s}a} = \Psi_{\mathbf{s}a}$.

The dual pairing $\langle \cdot, \cdot \rangle_{\mathbf{s}}$ of $\text{ECF}_{\mathbf{s}}^{w,\bar{w}}$ and $\text{ECF}_{\mathbf{s}}^{1-w,1-\bar{w}}$ is covariant under Gau . In fact, $\langle \alpha^* \Phi, \alpha^* \Psi \rangle_{\alpha^* \mathbf{s}} = \langle \Phi, \Psi \rangle_{\mathbf{s}}$ for $\Psi_{\mathbf{s}} \in \text{ECF}_{\mathbf{s}}^{w,\bar{w}}$ and $\Phi_{\mathbf{s}} \in \text{ECF}_{\mathbf{s}}^{1-w,1-\bar{w}}$.

There is a corresponding action of Gau on the space of minimal extended conformal field functionals $\text{ECF}^{w,\bar{w}}$. For $\alpha \in \text{Gau}$ and $\Psi \in \text{ECF}^{w,\bar{w}}$, $\alpha^* \Psi$ is the element of $\text{ECF}^{w,\bar{w}}$ locally given by $\alpha^* \Psi_a = \text{Ad} \alpha_a \Psi_a$. The value $\alpha^* \Psi_{\alpha^* \mathbf{s}}$ of $\alpha^* \Psi$ at the holomorphic structure $\alpha^* \mathbf{s}$ is the result of the action of α on $\Psi_{\mathbf{s}}$ defined above, as suggested by the notation.

The dual pairing $\langle \cdot, \cdot \rangle$ of $\text{ECF}^{w,\bar{w}}$ and $\text{ECF}^{1-w,1-\bar{w}}$ is invariant under Gau , i.e. one has $\langle \alpha^* \Phi, \alpha^* \Psi \rangle = \langle \Phi, \Psi \rangle$ for $\Psi \in \text{ECF}^{w,\bar{w}}$ and $\Phi \in \text{ECF}^{1-w,1-\bar{w}}$.

In the Koszul parametrization, the action of Gau on Hol translates into an action on the Koszul field A^* . For $\alpha \in \text{Gau}$ and $A^* \in \text{Hol}$, the action is locally given by $\alpha^* A^*_a = \bar{\partial}_a \alpha_a \alpha_a^{-1} + \text{Ad} \alpha_a A^*_a$. If $\eta \in \mathcal{G}(\mathbf{s})$ with $\mathbf{s} \equiv A^*$, then the equation $(\bar{\partial} - \text{ad} A^*) \eta \eta^{-1} = 0$ is satisfied.

Gau is inert on the space of surface metrics Met .

Gau acts on the hermitian fiber metrics as follows. For any $\alpha \in \text{Gau}$ and $H_{\mathbf{s}} \in \text{Herm}_{\mathbf{s}}$, $\alpha^* H_{\alpha^* \mathbf{s}}$ is the element of $\text{Herm}_{\alpha^* \mathbf{s}}$ locally given by $\alpha^* H_{\alpha^* \mathbf{s}a} = H_{\mathbf{s}a}$.

It is easy to verify that, for any $h \in \text{Met}$ and any $H_{\mathbf{s}} \in \text{Herm}_{\mathbf{s}}$, the Hilbert structure $\langle \cdot, \cdot \rangle_{h,H_{\mathbf{s}}}$ on $\text{ECF}_{\mathbf{s}}^{w,\bar{w}}$ defined earlier is Gau covariant, i.e. $\langle \alpha^* \Psi_1, \alpha^* \Psi_2 \rangle_{h,\alpha^* H_{\alpha^* \mathbf{s}}} = \langle \Psi_1, \Psi_2 \rangle_{h,H_{\mathbf{s}}}$, for $\Psi_{1\mathbf{s}}, \Psi_{2\mathbf{s}} \in \text{ECF}_{\mathbf{s}}^{w,\bar{w}}$.

There is a corresponding action of Gau on the space of minimal fiber metrics functionals Herm . For $\alpha \in \text{Gau}$ and $H \in \text{Herm}$, $\alpha^* H$ is the element of Herm locally given by $\alpha^* H_a = \alpha_a H_a \alpha_a^\dagger$. The value $\alpha^* H_{\alpha^* \mathbf{s}}$ of $\alpha^* H$ at the holomorphic structure $\alpha^* \mathbf{s}$ is the result of the action of α on $H_{\mathbf{s}}$ defined above.

It is easy to verify that, for any $h \in \text{Met}$ and any $H \in \text{Herm}$, the Hilbert structure $\langle \cdot, \cdot \rangle_{h,H}$ on $\text{ECF}^{w,\bar{w}}$ defined earlier is Gau invariant, i.e. $\langle \alpha^* \Psi_1, \alpha^* \Psi_2 \rangle_{h,\alpha^* H} = \langle \Psi_1, \Psi_2 \rangle_{h,H}$, for $\Psi_1, \Psi_2 \in \text{ECF}^{w,\bar{w}}$.

In the analysis of symmetries, it is much simpler to proceed at the infinitesimal level. Let Ξ be the gauge ghost. Ξ is an element of $\text{ECF}^{0,0} \otimes \wedge^1(\text{Lie Gau})^\vee$ defining a basis of $\wedge^1(\text{Lie Gau})^\vee$. The infinitesimal action of the gauge group Gau on field functionals is given by the nilpotent Slavnov operator s , $s^2 = 0$. From the Maurer–Cartan equations of Gau, one has

$$s\Xi = \frac{1}{2}[\Xi, \Xi]. \tag{2.7}$$

Further,

$$s\psi = 0, \tag{2.8}$$

$$sA^* = (\bar{\partial} - \text{ad } A^*)\Xi, \tag{2.9}$$

$$s\Psi = \text{ad } \Xi\Psi, \tag{2.10}$$

where $\psi \in \text{CF}^{w,\bar{w}}$, $A^* \in \text{Hol}$ and $\Psi \in \text{ECF}^{w,\bar{w}}$.

At the infinitesimal level, the action of Gau on Met and Herm is given by

$$s \ln h = 0, \tag{2.11}$$

$$sHH^{-1} = \Xi + \text{Ad}H\Xi^\dagger, \tag{2.12}$$

with $h \in \text{Met}$ and $H \in \text{Herm}$.

4. The Drinfeld–Sokolov Bundle. The basic data entering in the definition of the Drinfeld–Sokolov bundle are the following: i) a simple complex Lie group G ; ii) an $SL(2, \mathbb{C})$ subgroup S of G invariant under the compact conjugation \dagger of G ; iii) a Riemann surface Σ of genus ℓ with a spinor structure $k^{\otimes \frac{1}{2}}$. Let t_{-1}, t_0, t_{+1} be a set of standard generators of \mathfrak{s} , so that

$$[t_{+1}, t_{-1}] = 2t_0, \quad [t_0, t_{\pm 1}] = \pm t_{\pm 1}, \tag{2.13}$$

$$t_d^\dagger = t_{-d}, \quad d = -1, 0, +1. \tag{2.14}$$

Then,

$$K_{ab} = \exp(-\ln k_{ab}t_0) \exp(\partial_a k_{ab}^{-1}t_{-1}) \tag{2.15}$$

defines a holomorphic G -valued 1-cocycle K [14]. This in turn defines a smooth principal G -bundle, the Drinfeld–Sokolov bundle DS , whose relevance has been explained in the introduction.

The Drinfeld–Sokolov bundle has extra structures derived from a special nilpotent subalgebra \mathfrak{z} of \mathfrak{g} associated to \mathfrak{s} . Such structures will be called Drinfeld–Sokolov and will play an important role in the following. The reason for this, related to the form of anomalies, will be explained in detail in the next section.

To the Cartan element t_0 of \mathfrak{s} , there is associated a halfinteger grading of \mathfrak{g} : the subspace \mathfrak{g}_m of \mathfrak{g} of degree $m \in \mathbb{Z}/2$ is the eigenspace of $\text{ad } t_0$ with eigenvalue m . One can further define a bilinear form χ on \mathfrak{g} by $\chi(x, y) = \text{tr}_{\text{ad}}(t_{+1}[x, y])$, $x, y \in \mathfrak{g}$ [10]. The restriction of χ to $\mathfrak{g}_{-\frac{1}{2}}$ is non-singular. By the Darboux theorem, there is a direct sum decomposition $\mathfrak{g}_{-\frac{1}{2}} = \mathfrak{p}_{-\frac{1}{2}} \oplus \mathfrak{q}_{-\frac{1}{2}}$ of $\mathfrak{g}_{-\frac{1}{2}}$ into subspaces $\mathfrak{p}_{-\frac{1}{2}}$

and $\mathfrak{q}_{-\frac{1}{2}}$ of the same dimension, which are maximally isotropic and dual to each other with respect to χ . Set

$$\mathfrak{r} = \mathfrak{p}_{-\frac{1}{2}} \oplus \bigoplus_{m \leq -1} \mathfrak{g}_m. \tag{2.16}$$

\mathfrak{r} is a negative graded nilpotent subalgebra of \mathfrak{g} .

Let Hol_{DS} be the family of all holomorphic structures \mathfrak{s} such that $V_{\mathfrak{s}a}$ is $\exp \mathfrak{r}$ -valued for every a . Such structures will be called Drinfeld–Sokolov. For $\mathfrak{s} \in \text{Hol}_{\text{DS}}$, $K_{\mathfrak{s}ab} = K_{ab}L_{\mathfrak{s}ab}$, where $L_{\mathfrak{s}ab}$ is a holomorphic $\exp \mathfrak{r}$ -valued function.

Let $\mathfrak{s} \in \text{Hol}_{\text{DS}}$ and $w, \bar{w} \in \mathbb{Z}/2$. A Drinfeld–Sokolov extended \mathfrak{s} -conformal field $\Psi_{\mathfrak{s}}$ of weights w, \bar{w} is an element of $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, \bar{w}}$ such that $\Psi_{\mathfrak{s}a}$ is valued in \mathfrak{r} for any a . This definition is consistent because of the form of the 1-cocycle $K_{\mathfrak{s}}$ and the fact that $[t_0, \mathfrak{r}] \subseteq \mathfrak{r}$ and $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r}$. The Drinfeld–Sokolov fields $\Psi_{\mathfrak{s}}$ of weights w, \bar{w} span an infinite dimensional complex linear space $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, \bar{w}}$. Similarly, a dual Drinfeld–Sokolov extended \mathfrak{s} -conformal field $\Psi_{\mathfrak{s}}$ of weights w, \bar{w} is an element of $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, \bar{w}}$ such that $\Psi_{\mathfrak{s}a}$ is defined modulo a \mathfrak{r}^{\perp} -valued local function for any a , where \mathfrak{r}^{\perp} is the orthogonal complement of \mathfrak{r} with respect to the Cartan–Killing form tr_{ad} . This definition is also consistent because of the form of the 1-cocycle $K_{\mathfrak{s}}$ and the fact that $[t_0, \mathfrak{r}^{\perp}] \subseteq \mathfrak{r}^{\perp}$ and $[\mathfrak{r}, \mathfrak{r}^{\perp}] \subseteq \mathfrak{r}^{\perp}$. The dual Drinfeld–Sokolov fields $\Psi_{\mathfrak{s}}$ of weights w, \bar{w} span an infinite dimensional complex space $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, \bar{w}}$.

For $\mathfrak{s} \in \text{Hol}_{\text{DS}}$, the Cauchy–Riemann operator $\bar{\partial}_{\mathfrak{s}}$ maps $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, 0}$ into $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, 1}$. Therefore, $\bar{\partial}_{\mathfrak{s}}$ defines by restriction a Cauchy–Riemann operator $\bar{\partial}_{\text{DSS}} : \text{ECF}_{\text{DS}\mathfrak{s}}^{w, 0} \rightarrow \text{ECF}_{\text{DS}\mathfrak{s}}^{w, 1}$, the Drinfeld–Sokolov Cauchy–Riemann operator. In this way, one can consistently define a notion of holomorphy for Drinfeld–Sokolov extended \mathfrak{s} -conformal fields. The subspace of holomorphic elements $\Psi_{\mathfrak{s}}$ of $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, 0}$ will be denoted by $\text{HECF}_{\text{DS}\mathfrak{s}}^w$. Similarly, $\bar{\partial}_{\mathfrak{s}}$ maps $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, 0}$ into $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, 1}$. So, $\bar{\partial}_{\mathfrak{s}}$ induces a Cauchy–Riemann operator $\bar{\partial}_{\text{DSS}}^{\vee} : \text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, 0} \rightarrow \text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, 1}$, the dual Drinfeld–Sokolov Cauchy–Riemann operator. So, one can consistently define a notion of holomorphy also for dual Drinfeld–Sokolov extended \mathfrak{s} -conformal fields. The subspace of holomorphic elements $\Phi_{\mathfrak{s}}$ of $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee w, 0}$ will be denoted by $\text{HECF}_{\text{DS}\mathfrak{s}}^{\vee w}$. There exists an interesting Drinfeld–Sokolov version of the Riemann–Roch theorem:

$$\dim \text{HECF}_{\text{DS}\mathfrak{s}}^w - \dim \text{HECF}_{\text{DS}\mathfrak{s}}^{\vee 1-w} = \text{tr} [((2w - 1)1 - 2 \text{ad } t_0) p_{\mathfrak{r}}] (\ell - 1), \tag{2.17}$$

where $p_{\mathfrak{r}}$ is any projector of \mathfrak{g} onto \mathfrak{r} [29].

The spaces $\text{ECF}_{\text{DS}\mathfrak{s}}^{w, \bar{w}}$ and $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee 1-w, 1-\bar{w}}$ are dual to each other. The dual pairing is given by $\langle \Phi, \Psi \rangle_{\text{DS}\mathfrak{s}} = \frac{1}{\pi} \int_{\Sigma} d^2z \text{tr}_{\text{ad}} (\Phi \Psi)_{\mathfrak{s}}$ for $\Psi_{\mathfrak{s}} \in \text{ECF}_{\text{DS}\mathfrak{s}}^{w, \bar{w}}$ and $\Phi_{\mathfrak{s}} \in \text{ECF}_{\text{DS}\mathfrak{s}}^{\vee 1-w, 1-\bar{w}}$. Note that the result of the integration does not depend on the representative of $\Phi_{\mathfrak{s}}$ used.

A Drinfeld–Sokolov $(1, 0)$ \mathfrak{s} -connection $\Gamma_{\mathfrak{s}}$ is an element of $\text{Conn}_{\mathfrak{s}}$ such that $\Gamma_{\mathfrak{s}a} - \frac{1}{2}t_{+1}$ is \mathfrak{r}^{\perp} -valued for every a . This definition is consistent because of the form of the 1-cocycle $K_{\mathfrak{s}}$ and the fact that $[t_d, \mathfrak{r}] \subseteq \mathfrak{r}^{\perp}$ for $d = 0, -1$, $\mathfrak{r} \subseteq \mathfrak{r}^{\perp}$ and $[\mathfrak{r}, \mathfrak{r}^{\perp}] \subseteq \mathfrak{r}^{\perp}$. If $\Gamma_{\mathfrak{s}}$ is Drinfeld–Sokolov, then $F_{\Gamma_{\mathfrak{s}}} = 0$ in $\text{ECF}_{\text{DS}\mathfrak{s}}^{\vee 1, 1}$. Let $\text{Conn}_{\text{DS}\mathfrak{s}}$ be the family of all Drinfeld–Sokolov $(1, 0)$ \mathfrak{s} -connections $\Gamma_{\mathfrak{s}}$.

The reference holomorphic structure is obviously Drinfeld–Sokolov, since $V_a = 1$ is $\exp \mathfrak{r}$ -valued.

Let $w, \bar{w} \in \mathbb{Z}/2$. A Drinfeld–Sokolov minimal extended conformal field functional Ψ of weights w, \bar{w} is a minimal extended conformal field functional defined on Hol_{DS} and such that, for any $\mathbf{s} \in \text{Hol}_{\text{DS}}$, $\Psi_{\mathbf{s}} \in \text{ECF}_{\text{DS } \mathbf{s}}^{w, \bar{w}}$. This definition is certainly consistent, as the reference holomorphic structure is Drinfeld–Sokolov, $V_{\mathbf{s}}$ is $\exp \mathfrak{r}$ -valued and $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r}$. The space of Drinfeld–Sokolov minimal extended conformal field functionals of weights w, \bar{w} may clearly be identified with $\text{ECF}_{\text{DS}}^{w, \bar{w}}$. Similarly, a dual Drinfeld–Sokolov minimal extended conformal field functional Ψ of weights w, \bar{w} is a minimal extended conformal field functional defined on Hol_{DS} and such that, for any $\mathbf{s} \in \text{Hol}_{\text{DS}}$, $\Psi_{\mathbf{s}} \in \text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$. This definition also is consistent, for the reference holomorphic structure is Drinfeld–Sokolov, $V_{\mathbf{s}}$ is $\exp \mathfrak{r}$ -valued and $[\mathfrak{r}, \mathfrak{r}^{\perp}] \subseteq \mathfrak{r}^{\perp}$. The space of dual Drinfeld–Sokolov minimal extended conformal field functionals of weights w, \bar{w} may clearly be identified with $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}}$.

For any $\Psi \in \text{ECF}_{\text{DS}}^{w, \bar{w}}$ and $\Phi \in \text{ECF}_{\text{DS}}^{\vee 1-w, 1-\bar{w}}$, $\langle \Phi, \Psi \rangle_{\text{DS } \mathbf{s}} = \langle \Phi, \Psi \rangle_{\text{DS}}$ for any $\mathbf{s} \in \text{Hol}_{\text{DS}}$. Therefore, the dual pairing $\langle \cdot, \cdot \rangle_{\text{DS } \mathbf{s}}$ of $\text{ECF}_{\text{DS } \mathbf{s}}^{w, \bar{w}}$ and $\text{ECF}_{\text{DS } \mathbf{s}}^{\vee 1-w, 1-\bar{w}}$ induces a dual pairing $\langle \cdot, \cdot \rangle_{\text{DS}}$ of the spaces of (dual) Drinfeld–Sokolov minimal extended conformal field functionals $\text{ECF}_{\text{DS}}^{w, \bar{w}}$ and $\text{ECF}_{\text{DS}}^{\vee 1-w, 1-\bar{w}}$.

A minimal Drinfeld–Sokolov $(1, 0)$ connection functional Γ is a minimal $(1, 0)$ connection functional defined on Hol_{DS} such that, for any $\mathbf{s} \in \text{Hol}_{\text{DS}}$, $\Gamma_{\mathbf{s}} \in \text{Conn}_{\text{DS } \mathbf{s}}$. This definition is consistent again because the reference holomorphic structure is Drinfeld–Sokolov, $V_{\mathbf{s}}$ is $\exp \mathfrak{r}$ -valued and the fact that $[\mathfrak{t}_{+1}, \mathfrak{r}] \subseteq \mathfrak{r}^{\perp}$, $\mathfrak{r} \subseteq \mathfrak{r}^{\perp}$ and $[\mathfrak{r}, \mathfrak{r}^{\perp}] \subseteq \mathfrak{r}^{\perp}$. The space of dual Drinfeld–Sokolov minimal connection functionals Γ may clearly be identified with Conn_{DS} .

In the Koszul parametrization, the Drinfeld–Sokolov holomorphic structures are represented by \mathfrak{r} -valued Koszul fields A^* . Such Koszul fields are also called Drinfeld–Sokolov.

5. Hermitian Structures of the Drinfeld–Sokolov Bundle. Let $h \in \text{Met}$ and $H_{\mathbf{s}} \in \text{Herm}_{\mathbf{s}}$ be metrics. The Hilbert structure $\langle \cdot, \cdot \rangle_{h, H_{\mathbf{s}}}$ on $\text{ECF}_{\mathbf{s}}^{w, \bar{w}}$ defines by restriction a Hilbert structure $\langle \cdot, \cdot \rangle_{\text{DS } h, H_{\mathbf{s}}}$ on $\text{ECF}_{\text{DS } \mathbf{s}}^{w, \bar{w}}$. The Hilbert structure allows one to identify $\text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$ with $\text{ECF}_{\text{DS } \mathbf{s}}^{1-w, 1-\bar{w}}$. By definition, the element $\Phi_{h, H_{\mathbf{s}}} \in \text{ECF}_{\text{DS } \mathbf{s}}^{1-w, 1-\bar{w}}$ corresponding to $\Phi_{\mathbf{s}} \in \text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$ is the unique element of $\text{ECF}_{\text{DS } \mathbf{s}}^{1-w, 1-\bar{w}}$ such that $\langle \Phi | \Psi \rangle_{\text{DS } \mathbf{s}} = \langle \Phi_{h, H}, \Psi \rangle_{\text{DS } h, H_{\mathbf{s}}}$ for all $\Psi_{\mathbf{s}} \in \text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$. One may now define a Hilbert structure on $\text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$ by setting $\langle \Phi_1, \Phi_2 \rangle_{\text{DS } h, H_{\mathbf{s}}}^{\vee} = \langle \Phi_{2h, H}, \Phi_{1h, H} \rangle_{\text{DS } h, H_{\mathbf{s}}}$.

For any $H \in \text{Herm}$ and any $\Psi_1, \Psi_2 \in \text{ECF}_{\text{DS}}^{w, \bar{w}}$, $\langle \Psi_1, \Psi_2 \rangle_{\text{DS } h, H_{\mathbf{s}}} = \langle \Psi_1, \Psi_2 \rangle_{\text{DS } h, H}$ for $\mathbf{s} \in \text{Hol}_{\text{DS}}$. Similarly, for $H \in \text{Herm}$ and $\Phi_1, \Phi_2 \in \text{ECF}_{\text{DS}}^{\vee w, \bar{w}}$, $\langle \Phi_1, \Phi_2 \rangle_{\text{DS } h, H_{\mathbf{s}}}^{\vee} = \langle \Phi_1, \Phi_2 \rangle_{\text{DS } h, H}^{\vee}$. Thus, for a given minimal hermitian fiber metric functional H , the Hilbert structures $\langle \cdot, \cdot \rangle_{\text{DS } h, H_{\mathbf{s}}}$ on $\text{ECF}_{\text{DS } \mathbf{s}}^{w, \bar{w}}$ and $\langle \cdot, \cdot \rangle_{\text{DS } h, H_{\mathbf{s}}}^{\vee}$ on $\text{ECF}_{\text{DS } \mathbf{s}}^{\vee w, \bar{w}}$ induce Hilbert structures $\langle \cdot, \cdot \rangle_{\text{DS } h, H}$ and $\langle \cdot, \cdot \rangle_{\text{DS } h, H}^{\vee}$ on the spaces of (dual) Drinfeld–Sokolov minimal extended conformal field functionals $\text{ECF}_{\text{DS}}^{w, \bar{w}}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}}$, respectively.

6. The Drinfeld–Sokolov Gauge Group. The gauge group Gau does not respect Hol_{DS} . There is however a subgroup of Gau_{DS} of Gau , the Drinfeld–Sokolov gauge group, which does. Gau_{DS} is formed by those elements $\alpha \in \text{Gau}$ such that α_a is $\exp \mathfrak{r}$ -valued for every a . Clearly, $\text{Lie } \text{Gau}_{\text{DS}} \cong \text{ECF}_{\text{DS}}^{0, 0}$.

For any $\mathfrak{s} \in \text{Hol}_{\text{DS}}$, the stability subgroup $\mathcal{G}_{\text{DS}}(\mathfrak{s})$ of \mathfrak{s} in Gau_{DS} is simply the intersection $\mathcal{G}(\mathfrak{s}) \cap \text{Gau}_{\text{DS}}$. Clearly, $\text{Lie } \mathcal{G}_{\text{DS}}(\mathfrak{s}) \cong \text{HECF}_{\text{DS}}^0 \mathfrak{s}$. $\text{Lie } \mathcal{G}_{\text{DS}}(\mathfrak{s})$ is nilpotent, since $\exp \mathfrak{r}$ is.

From the definition, it is immediate to see that the action $\alpha^* : \text{ECF}_{\mathfrak{s}}^{w, \bar{w}} \rightarrow \text{ECF}_{\alpha^* \mathfrak{s}}^{w, \bar{w}}$ associated to $\alpha \in \text{Gau}_{\text{DS}}$ maps $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \mathfrak{s}$ respectively into $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \alpha^* \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \alpha^* \mathfrak{s}$.

It can also be seen that the dual pairing $\langle \cdot | \cdot \rangle_{\text{DS}} \mathfrak{s}$ of $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee 1-w, 1-\bar{w}} \mathfrak{s}$ is Gau_{DS} covariant.

Using the fact that \mathfrak{r} is a subalgebra of \mathfrak{g} such that $[\mathfrak{r}, \mathfrak{r}^\perp] \subseteq \mathfrak{r}^\perp$, it is easy to check that the action of Gau_{DS} on $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ preserves both $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \mathfrak{s}$.

The dual pairing $\langle \cdot | \cdot \rangle_{\text{DS}} \mathfrak{s}$ of $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee 1-w, 1-\bar{w}} \mathfrak{s}$ is Gau_{DS} invariant.

The Hilbert structures on $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \mathfrak{s}$ defined above are both Gau_{DS} covariant.

One can similarly show that the Hilbert structures on $\text{ECF}_{\text{DS}}^{w, \bar{w}} \mathfrak{s}$ and $\text{ECF}_{\text{DS}}^{\vee w, \bar{w}} \mathfrak{s}$ are both Gau_{DS} invariant.

To Gau_{DS} , one can consistently associate a Slavnov operator s_{DS} and a \mathfrak{r} -valued ghost field $\Xi_{\text{DS}} \in \text{ECF}_{\text{DS}}^{0,0} \otimes \bigwedge^1 (\text{Lie } \text{Gau}_{\text{DS}})^\vee$ obeying (2.7). Equations (2.8)–(2.10) also holds with A^* a Drinfeld–Sokolov Koszul field and \mathcal{P} a (dual) Drinfeld–Sokolov extended conformal field with s and Ξ replaced by s_{DS} and Ξ_{DS} . Of course, (2.11)–(2.12) continue to hold with s and Ξ replaced by s_{DS} and Ξ_{DS} .

Before completing this section, I shall state the following conventions. In what follows, *when in the same equation there appear a holomorphic structure \mathfrak{s} and a Koszul field A^* , it is implicitly assumed, unless otherwise stated, that $\mathfrak{s} \equiv A^*$. Further, all field functionals on Hol or Hol_{DS} are implicitly assumed, unless otherwise stated, to be minimal field functionals.*

3. Drinfeld–Sokolov Field Theory

A Drinfeld–Sokolov field theory is a local field theory whose basic fields are (extended) conformal fields of the Drinfeld–Sokolov bundle.

The standard classical example to have in mind is the Drinfeld–Sokolov B – C system. The basic fields B and C belong to $\mathbf{F} \otimes \text{ECF}^{1-w,0}$ and $\mathbf{F} \otimes \text{ECF}^{w,0}$, respectively, where \mathbf{F} is the fermionic Grassmann algebra. The action, for a given holomorphic structure A^* , is¹

$$S(B, C, A^*) = \frac{1}{\pi} \int_{\Sigma} d^2z \text{Re } \text{tr}_{\text{ad}}(B \bar{C})_{\mathfrak{s}}. \tag{3.1}$$

In general, the quantization of a Drinfeld–Sokolov field theory requires the introduction of a hermitian structure $(h, H) \in \text{Met} \times \text{Herm}$ for the proper definition of the adjoint of the relevant differential operators. The regularization of the ultraviolet divergencies of the corresponding functional determinants involves further the use of an ultraviolet cut-off ε . The regularization method which will be applied below is the so-called proper time method [18]. I shall restrict to Drinfeld–Sokolov field

¹In the notation of this paper, a functional $f(X)$ of a complex field X is not necessarily holomorphic. Holomorphy, when it occurs, will be explicitly stated.

theories for which the bare Gau invariant effective action $\hat{I}(h, H, A^*; \varepsilon)$ is of the form

$$\hat{I}(h, H, A^*; \varepsilon) = -\frac{r}{\pi\varepsilon} \int_{\Sigma} d^2z h + \left[\frac{n}{6\pi} \int_{\Sigma} d^2z f_h - d_{\mathbf{S}} \right] \ln \varepsilon + I_0(h, H, A^*) + O(\varepsilon). \quad (3.2)$$

Here r , n and $d_{\mathbf{S}}$ are real coefficients. $\int_{\Sigma} d^2z f_h$ is the Gauss–Bonnet topological invariant whose well known value is $2\pi(\ell - 1)$. $I_0(h, H, A^*)$ is a non-local functional of h , H and A^* such that

$$\delta I_0(h, H, A^*) = -\frac{\kappa_0}{12\pi} \int_{\Sigma} d^2z \delta \ln h f_h + \frac{K}{\pi} \int_{\Sigma} d^2z \text{tr}_{\text{ad}}(\delta H H^{-1} F_H)_{\mathbf{S}}, \quad (3.3)$$

where δ denotes variation with respect to h and H at fixed A^* [30]. κ_0 and $K > 0$ are generalized central charges. The Drinfeld–Sokolov B – C system introduced earlier is precisely of this type with $r = \dim \mathfrak{g}$, $n = (3w - 1) \dim \mathfrak{g}$, $d_{\mathbf{S}} = \dim \text{HECF}_{\mathfrak{g}}^w$, $\kappa_0 = -2(6w^2 - 6w + 1) \dim \mathfrak{g}$ and $K = 1$.

To renormalize the bare effective action, one has to add to it a counterterm of the form

$$\Delta \hat{I}(h, H, A^*; \varepsilon) = \frac{r}{\pi\varepsilon} \int_{\Sigma} d^2z h - \left[\frac{n}{6\pi} \int_{\Sigma} d^2z f_h - d_{\mathbf{S}} \right] \ln \varepsilon + \Delta I(h, H, A^*) + O(\varepsilon). \quad (3.4)$$

Here, $\Delta I(h, H, A^*)$ is a local but otherwise arbitrary functional of h , H and A^* , whose choice defines a renormalization prescription. The renormalized effective action is thus

$$I(h, H, A^*) = I_0(h, H, A^*) + \Delta I(h, H, A^*). \quad (3.5)$$

$I_0(h, H, A^*)$ is the renormalized effective action in the minimal subtraction renormalization scheme.

In what follows, $\Delta I(h, H, A^*)$ is assumed to be independent from A^* :

$$\Delta I(h, H, A^*) = \Delta I(h, H). \quad (3.6)$$

Under this hypothesis, it can be shown that $I(h, H, A^*)$ has the following structure:

$$I(h, H, A^*) = I(h, H) + L(H, A^*; A) + I_{\text{hol}}(A^*; A). \quad (3.7)$$

Here, $A \in \text{Conn}$ is a background $(1, 0)$ connection. $I(h, H)$ is the functional $I(h, H, A^*)$ evaluated at the reference holomorphic structure $A^* = 0$,

$$L(H, A^*; A) = \frac{K}{\pi} \int_{\Sigma} d^2z [2 \text{Re} \text{tr}_{\text{ad}}((\Gamma_H - A)A^*) - \text{tr}_{\text{ad}}(A^* \text{Ad} H A^{*\dagger})]. \quad (3.8)$$

$I_{\text{hol}}(A^*; A)$ is a non-local functional of A^* only depending on A . Next, I shall analyze the properties of the three terms in the right-hand side of (3.7).

In order for the counterterm $\Delta \hat{I}(h, H, A^*; \varepsilon)$ to be Gau invariant, $\Delta I(h, H)$ must satisfy

$$s \Delta I(h, H) = 0. \quad (3.9)$$

In this way, the renormalized effective action $I(h, H, A^*)$ is Gau invariant as well. When (3.9) is fulfilled, one has

$$sI(h, H) = \mathcal{W}(H), \tag{3.10}$$

$$sL(H, A^*; A) = -\mathcal{W}(H) - \mathcal{A}(A^*; A), \tag{3.11}$$

$$sI_{\text{hol}}(A^*; A) = \mathcal{A}(A^*; A), \tag{3.12}$$

where

$$\mathcal{W}(H) = \frac{K}{\pi} \int_{\Sigma} d^2z \operatorname{Re} \operatorname{tr}_{\text{ad}}(\Xi F_H), \tag{3.13}$$

$$\mathcal{A}(A^*; A) = -\frac{K}{\pi} \int_{\Sigma} d^2z \operatorname{Re} \operatorname{tr}_{\text{ad}}(\Xi(F_A - \partial_A A^*)) \tag{3.14}$$

are the gauge anomalies.

$I(h, H)$ is a non-local functional of h and H . Its dependence on h and H can be analyzed as follows. The Drinfeld–Sokolov bundle possesses a remarkable property, the possibility of lifting any surface metric $h \in \text{Met}$ to a fiber metric $H(h) \in \text{Herm}$. Explicitly, $H(h)$ is given by

$$H(h) = \exp(-\partial \ln h t_{-1}) \exp(-\ln h t_0) \exp(-\bar{\partial} \ln h t_{+1}). \tag{3.15}$$

This allows one to write $I(h, H)$ as follows:

$$I(h, H) = I_{\text{conf}}(h) + S(h, H) + \Delta I(h, H) - \Delta I(h, H(h)), \tag{3.16}$$

where

$$I_{\text{conf}}(h) = I(h, H(h)), \tag{3.17}$$

$$S(h, H) = \Omega(H, H(h)). \tag{3.18}$$

Here, for any two $H, H_0 \in \text{Herm}$, $\Omega(H, H_0)$ is the Donaldson action defined by the functional path integral

$$\Omega(H, H_0) = \frac{K}{\pi} \int_{H_0}^H \int_{\Sigma} d^2z \operatorname{tr}_{\text{ad}}(\delta H' H'^{-1} F_{H'}). \tag{3.19}$$

The right-hand side is independent from the choice of the functional integration path joining H_0 to H , since the functional 1-form on Herm integrated is closed and Herm is clearly contractible. $\Omega(H, H_0)$ can be computed explicitly. The metric $H \in \text{Herm}$ can be written as $H = \exp \Phi H_0$, where the Donaldson field Φ is an element of $\text{ECF}^{0,0}$ such that $\text{Ad}_{H_0} \Phi^\dagger = \Phi$. By direct calculation, one then finds

$$\Omega(H, H_0) = -\frac{K}{\pi} \int_{\Sigma} d^2z \operatorname{tr}_{\text{ad}} \left[\bar{\partial} \Phi \frac{\exp \operatorname{ad} \Phi - 1 - \operatorname{ad} \Phi}{(\operatorname{ad} \Phi)^2} \partial_{H_0} \Phi - \Phi F_{H_0} \right] \tag{3.20}$$

[31].

Now, $I_{\text{conf}}(h)$ is a non-local functional of h . Using (3.3), (3.5), (3.15) and (3.17), one can show that

$$\delta I_{\text{conf}}(h) = -\frac{\kappa_0 + \kappa}{12\pi} \int_{\Sigma} d^2z \delta \ln h f_h + \delta \left[\frac{\lambda}{\pi} \int_{\Sigma} d^2z h^{-1} f_h^2 + \Delta I(h, H(h)) \right], \tag{3.21}$$

where

$$\kappa = -12K \operatorname{tr}_{\text{ad}}(t_0^2), \tag{3.22}$$

$$\lambda = -2K \operatorname{tr}_{\text{ad}}(t_0^2). \tag{3.23}$$

If

$$\Delta I(h, H) = \frac{\lambda_0}{\pi} \int_{\Sigma} d^2z h^{-1} f_h^2, \tag{3.24}$$

where λ_0 is some constant, then (3.21) simplifies into

$$\delta I_{\text{conf}}(h) = -\frac{\kappa_0 + \kappa}{12\pi} \int_{\Sigma} d^2z \delta \ln h f_h + \frac{\lambda_0 + \lambda}{\pi} \delta \int_{\Sigma} d^2z h^{-1} f_h^2. \tag{3.25}$$

A counterterm $\Delta I(h, H, A^*)$ for which (3.24) holds is given by the right-hand side of (3.24) itself and clearly satisfies both (3.6) and (3.9). Setting $\lambda_0 = -\lambda$, $I_{\text{conf}}(h)$ becomes the renormalized effective action of a conformal field theory of conformal central charge $\kappa_{\text{conf}} = \kappa_0 + \kappa$. Note that *the shift κ given by (3.22) is precisely the classical central charge of the classical W -algebras associated to the pair (G, S) , if K is interpreted as the Wess–Zumino–Novikov–Witten level*. For a generic value of λ_0 , one obtains a more general renormalized effective action with a $\int \sqrt{h} R_h^2$ term yielding a model of induced $2d$ gravity of the same type as that considered in refs. [32–33].

The functional $S(h, H)$ is local. In fact, the Donaldson field $\Phi(h, H)$ relevant here, given by

$$\exp \Phi(h, H) = HH(h)^{-1}, \tag{3.26}$$

is clearly a local functional of h and H and $\Omega(H, H_0)$, given by (3.20), is a local functional of Φ and H_0 .

From the above discussion, it follows that *the suitably renormalized effective action $I(h, H)$ differs from the conformal effective action $I_{\text{conf}}(h)$ by a local functional of h and H . In particular, the H dependence is local*.

From (3.8), it is apparent that $L(H, A^*; A)$, the interaction term of H and A^* , is local.

$I_{\text{hol}}(A^*; A)$ is the real part of a holomorphic functional of A^* and A [30]. Holomorphic factorization is an important feature of the model which however will not be discussed in this paper. Its independence from H is crucial.

One has thus reached the following important conclusion. *The full suitably renormalized Gau invariant effective action $I(h, H, A^*)$ is a local functional of H .*

An important observation, related to the analysis of ref. [10], is the following. If one restricts to Drinfeld–Sokolov holomorphic structures $A^* \in \text{Hol}_{\text{DS}}$ and to Drinfeld–Sokolov background connections $A \in \text{Conn}_{\text{DS}}$, then the functionals $L(H, A^*; A)$ and $I_{\text{hol}}(A^*; A)$ are independent from A . Further, under the action of the Drinfeld–Sokolov gauge group Gau_{DS} , one has relations analogous to (3.10)–(3.12), with s , $\mathcal{W}(H)$ and $\mathcal{A}(A^*; A)$ replaced by s_{DS} , $\mathcal{W}_{\text{DS}}(H)$ and $\mathcal{A}_{\text{DS}}(A^*; A)$, respectively, where $\mathcal{W}_{\text{DS}}(H)$ and $\mathcal{A}_{\text{DS}}(A^*; A)$ are given by (3.13)–(3.14) with Ξ substituted by Ξ_{DS} . In this case, however, one has

$$\mathcal{A}_{\text{DS}}(A^*; A) = 0, \quad A \in \text{Conn}_{\text{DS}}, \quad A^* \in \text{Hol}_{\text{DS}} \tag{3.27}$$

identically by (2.16). Henceforth, *it is assumed that $A \in \text{Conn}_{\text{DS}}$* .

4. Drinfeld–Sokolov Gravity

In Polyakov’s approach to two dimensional gravity, the functional integration over all smooth metrics on the string world sheet is reduced into an integration over the conformal factor of the metric h and on the Beltrami field μ . The action governing the quantum dynamics of such fields is the diffeomorphism invariant effective action of a conformal field theory.

In many respects, the quantization of Drinfeld–Sokolov gravity parallels that of ordinary two dimensional gravity. One integrates over all fiber metrics H of Herm and on all Drinfeld–Sokolov Koszul fields A^* of Hol_{DS} . The action of such fields is the Gau_{DS} invariant bare effective action $\hat{I}(h, H, A^*)$ of a Drinfeld–Sokolov field theory of the type described in Sect. 3. The partition function is thus of the form

$$\mathcal{Z}_\Theta(h) = \int_{\text{Herm} \times \text{Hol}_{\text{DS}}} \frac{(DH) \otimes (DA^*)}{\text{vol}(\text{Gau}_{\text{DS}})} \hat{\Theta}(h, H, A^*) \exp \hat{I}(h, H, A^*), \quad (4.1)$$

where $\hat{\Theta}(h, H, A^*)$ is some bare Gau_{DS} -invariant insertion. This is of course a rather formal expression whose precise meaning is to be defined. The relation of this quantization prescription with earlier approaches, in particular with that of ref. [10], has been discussed in the introduction.

The basic configuration space is the cartesian product $\text{Herm} \times \text{Hol}_{\text{DS}}$ carrying the action of Gau_{DS} described in Sect. 2. To gauge fix, one has to transform the functional integral on $\text{Herm} \times \text{Hol}_{\text{DS}}$ into one on a configuration space containing, roughly speaking, a factor Gau_{DS} by computing the jacobian of the corresponding functional change of variables.

To properly carry out the gauge fixing, it is necessary to define a good moduli space of Drinfeld–Sokolov holomorphic structures modulo the action of the Drinfeld–Sokolov gauge group and characterize the stability group of Drinfeld–Sokolov holomorphic structures. This requires a notion of stability. A thorough geometric investigation of this issue is beyond the scope of this paper. Nevertheless, it is still possible to make an educated guess about these geometric structures by the following argument.

As is well known, every stable holomorphic structure is simple and the family SHol of stable holomorphic structures is dense in Hol and invariant under the action of the gauge group Gau [27–28]. Here, the relevant holomorphic structures are those of Hol_{DS} and the relevant symmetry group is the Drinfeld–Sokolov gauge group Gau_{DS} . No holomorphic structure $\mathfrak{s} \in \text{Hol}_{\text{DS}}$ is stable in the customary sense. It is however reasonable to assume by analogy that, for any reasonable definition of Drinfeld–Sokolov stability, a Drinfeld–Sokolov stable holomorphic structure should be Drinfeld–Sokolov simple and that the family SHol_{DS} of Drinfeld–Sokolov stable holomorphic structures should be dense in Hol_{DS} and invariant under the action of the Drinfeld–Sokolov gauge group Gau_{DS} . Recall that a holomorphic structure $\mathfrak{s} \in \text{Hol}$ is simple if the subgroup $\mathcal{G}(\mathfrak{s})$ of \mathfrak{s} -holomorphic gauge transformations of Gau is trivial [27–28], a condition equivalent to the vanishing of the space $\text{HECF}_{\mathfrak{s}}^0$, since $\text{Lie } \mathcal{G}(\mathfrak{s}) \cong \text{HECF}_{\mathfrak{s}}^0$. Similarly, a holomorphic structure $\mathfrak{s} \in \text{Hol}_{\text{DS}}$ is said to be Drinfeld–Sokolov simple if $\mathcal{G}_{\text{DS}}(\mathfrak{s})$ has minimal dimension, or, equivalently, if the space $\text{HECF}_{\text{DS } \mathfrak{s}}^0$ has minimal dimension, since $\text{Lie } \mathcal{G}_{\text{DS}}(\mathfrak{s}) \cong \text{HECF}_{\text{DS } \mathfrak{s}}^0$.

In analogy to the ordinary moduli space, the Drinfeld–Sokolov moduli space \mathcal{M}_{DS} will be defined as the quotient $\text{SHol}_{\text{DS}}/\text{Gau}_{\text{DS}}$. \mathcal{M}_{DS} is a finite dimensional complex manifold.

For \mathfrak{s} varying in SHol_{DS} , the groups $\mathcal{G}_{\text{DS}}(\mathfrak{s})$ are all isomorphic to the same complex Lie group \mathcal{G}_{DS} . In fact they all are of the form $\exp \text{HECF}_{\text{DS}}^0$, where the spaces $\text{HECF}_{\text{DS}}^0$ are all valued in the same nilpotent subalgebra of \mathfrak{r} of \mathfrak{g} and can be continuously deformed into one another by continuously varying \mathfrak{s} in SHol_{DS} . \mathcal{G}_{DS} is nilpotent, since $\exp \mathfrak{r}$ is.

In this paper, *it will be assumed that* $\dim \text{HCF}^{\frac{1}{2},0} = 0$. This holds for an even spinor structure and for a generic holomorphic structure of Σ . It is merely a technically simplifying hypothesis with a very nice consequence. If the assumption is fulfilled, all holomorphic structures are Drinfeld–Sokolov simple. This is no longer true in the generic situation, where even the reference holomorphic structure characterized by the 1-cocycle (2.15) may fail to be Drinfeld–Sokolov simple [29].

A method for computing the dimensions of \mathcal{G}_{DS} and \mathcal{M}_{DS} exploiting the Drinfeld–Sokolov simplicity has been presented in [29]. They are given by

$$\dim \mathcal{G}_{\text{DS}} = \begin{cases} 0, & \text{if } \ell = 0, \\ \dim \mathfrak{r}_{\text{int}}, & \text{if } \ell = 1, \\ \dim \mathfrak{g}_{-1} - \text{tr}[(2\text{ad } t_0 + 1)p_{\mathfrak{r}}](\ell - 1), & \text{if } \ell \geq 2, \end{cases} \quad (4.2)$$

$$\dim \mathcal{M}_{\text{DS}} = \begin{cases} -\text{tr}[(2\text{ad } t_0 + 1)p_{\mathfrak{x}}], & \text{if } \ell = 0, \\ \dim \mathfrak{r}_{\text{int}}, & \text{if } \ell = 1, \\ \dim \mathfrak{g}_{-1}, & \text{if } \ell \geq 2, \end{cases} \quad (4.3)$$

where $p_{\mathfrak{r}}$ is any projector of \mathfrak{g} onto \mathfrak{r} and $\mathfrak{r}_{\text{int}} = \bigoplus_{m \in \mathbb{Z}, m \leq -1} \mathfrak{g}_m$.

The relevant configuration space is properly $\text{Herm} \times \text{SHol}_{\text{DS}}$. A natural parametrization of $\text{Herm} \times \text{SHol}_{\text{DS}}$ is provided by

$$H(\tilde{H}, \alpha) = \alpha^* \tilde{H} = \alpha \tilde{H} \alpha^\dagger, \quad (4.4)$$

$$A^*(t, \alpha) = \alpha^* A^*(t) = \bar{\partial} \alpha \alpha^{-1} + \text{Ad } \alpha A^*(t), \quad (4.5)$$

where $t \in \mathcal{M}_{\text{DS}}$, $\tilde{H} \in \text{Herm}$, $\alpha \in \text{Gau}_{\text{DS}}$, and $t \in \mathcal{M}_{\text{DS}} \rightarrow A^*(t) \in \text{SHol}_{\text{DS}}$ is a fiducial gauge slice. The parametrization possesses a \mathcal{G}_{DS} -symmetry as follows from the following argument. Any two elements (t, \tilde{H}, α) and $(t', \tilde{H}', \alpha')$ of $\mathcal{M}_{\text{DS}} \times \text{Herm} \times \text{Gau}_{\text{DS}}$ have the same image under (4.4)–(4.5) if and only if $t' = t$ and $\tilde{H}' = \eta \tilde{H} \eta^\dagger$ and $\alpha' = \alpha \eta^{-1}$ for some $\eta \in \mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$ with $\mathfrak{s}_t \equiv A^*(t)$, since $\mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$ is the subgroup of Gau_{DS} leaving $A^*(t)$ invariant. Now, for fixed $t \in \mathcal{M}_{\text{DS}}$, the maps

$$\eta^* \tilde{H} = \eta \tilde{H} \eta^\dagger, \quad (4.6)$$

$$\eta \alpha = \alpha \eta^{-1}, \quad (4.7)$$

with $\eta \in \mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$, define an action of $\mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$ on $\text{Herm} \times \text{SHol}_{\text{DS}}$. The action (4.6)–(4.7) is free and is a symmetry of (4.4)–(4.5). Since $\mathcal{G}_{\text{DS}} \cong \mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$ for any t , it is a \mathcal{G}_{DS} symmetry. One can then construct the space $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}}) / \mathcal{G}_{\text{DS}}(\mathfrak{s}_t) = \prod_{t \in \mathcal{M}_{\text{DS}}} \{t\} \times ((\text{Herm} \times \text{Gau}_{\text{DS}}) / \mathcal{G}_{\text{DS}}(\mathfrak{s}_t))$. This provides the realization of the configuration space relevant for the implementation of the gauge fixing.

The second realization is rather unwieldy, because the meaning of the functional integration on a functional manifold of the form $(\text{Herm} \times \text{Gau}_{\text{DS}}) / \mathcal{G}_{\text{DS}}(\mathfrak{s}_t)$ for fixed $t \in \mathcal{M}_{\text{DS}}$ is not quite clear. One way of solving this problem consists in transforming the integration on such a functional manifold into an integration on

Herm \times Gau_{DS} with a residual unfixed gauge symmetry corresponding to $\mathcal{G}_{DS}(\mathfrak{s}_t)$. To do this, one employs the obvious isomorphism $\text{Herm} \times \text{Gau}_{DS} \cong ((\text{Herm} \times \text{Gau}_{DS})/\mathcal{G}_{DS}(\mathfrak{s}_t)) \times \mathcal{G}_{DS}$, where the action of $\mathcal{G}_{DS}(\mathfrak{s}_t)$ on $\text{Herm} \times \text{Gau}_{DS}$ is given by (4.6)–(4.7). Upon choosing a group isomorphism of $\zeta(\cdot; t) : \mathcal{G}_{DS} \rightarrow \mathcal{G}_{DS}(\mathfrak{s}_t)$ of \mathcal{G}_{DS} onto $\mathcal{G}_{DS}(\mathfrak{s}_t)$, the isomorphism is explicitly given by

$$H(\tilde{H}, g) = \zeta(g; t)^* \tilde{H} = \zeta(g; t) \tilde{H} \zeta(g; t)^\dagger, \tag{4.8}$$

$$\omega(\alpha, g) = \zeta(g; t) \alpha = \alpha \zeta(g; t)^{-1}, \tag{4.9}$$

where (\tilde{H}, α) varies in a slice of $\text{Herm} \times \text{Gau}_{DS}$ representing the quotient $(\text{Herm} \times \text{Gau}_{DS})/\mathcal{G}_{DS}(\mathfrak{s}_t)$ and $g \in \mathcal{G}_{DS}$.

The definition of the functional measures on the relevant field spaces and the computation of the jacobians relating such measures is carried out by means of certain formal prescriptions outlined below. It is important to realize that such prescriptions only serve the purpose of producing and justifying heuristically a definition of the measure of the gauge fixed partition $\mathcal{Z}_\Theta(h)$ and should not in any way be interpreted as a means of proving theorems about an otherwise well defined field theoretic model.

To any complex Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$, there is associated a real Hilbert space \mathcal{H}^r with inner product $\langle \cdot, \cdot \rangle^r$. \mathcal{H}^r is just \mathcal{H} seen as a real vector space by restricting the numerical field from \mathbb{C} to \mathbb{R} . $\langle x_1, x_2 \rangle^r = 2 \text{Re} \langle x_1, x_2 \rangle$ for $x_1, x_2 \in \mathcal{H}^r = \mathcal{H}$. In particular, $\|x\|^2 = 2\|x\|^2$.

To any real Hilbert space \mathcal{H} , there is associated a translation invariant functional measure (Dx) normalized so that $\int_{\mathcal{H}} (Dx) \exp(-\frac{1}{2}\|x\|^2) = 1$.

If \mathcal{F} is a real Hilbert manifold, then, for any $f \in \mathcal{F}$, the tangent space $T_f \mathcal{F}$ of \mathcal{F} at f is a Hilbert space with norm $\|\delta f\|_{|f}$ and measure $(D\delta f)_{|f}$. This defines a measure $(Df)_{|f}$ on \mathcal{F} by identifying $(Df)_{|f}$ with $(D\delta f)_{|f}$ at f . In general, $(Df)_{|f}$ is not translation invariant, depending explicitly on f .

If \mathcal{F} and \mathcal{E} are Hilbert manifolds and $\varphi : \mathcal{F} \rightarrow \mathcal{E}$ is an invertible map, then \mathcal{F} is a parameter space for \mathcal{E} and it is possible to transform functional integration on \mathcal{E} with measure $(De)_{|e}$ into functional integration on \mathcal{F} with measure $(Df)_{|f}$.

To this end, one needs the jacobian relation $(D\varphi(f))_{|\varphi(f)} = [\det(\delta\varphi(f))]^{\frac{1}{2}} (Df)_{|f}$, where, for any $f \in \mathcal{F}$, $\delta\varphi(f) : T_f \mathcal{F} \rightarrow T_{\varphi(f)} \mathcal{E}$ is the tangent map of φ at f .

Applying the above formal recipes, one can define real Hilbert structures on Herm , SHol_{DS} , Gau_{DS} , \mathcal{M}_{DS} and \mathcal{G}_{DS} and obtain in this way the corresponding functional measures $(DH)_{|h|H}$, $(DA^*)_{|H|A^*}$, $(D\alpha)_{|h,H|\alpha}$, $(Dt)_{|t}$ and $(Dg)_{|g}$. The measures depend on a background surface metric $h \in \text{Met}$ and on a fiber metric $H \in \text{Herm}$ through the underlying Hilbert structures. h is fixed whereas H is chosen to be the varying metric integrated over in the functional integral. Using these basic Hilbert structures and functional measures, one can define real Hilbert structures on the derived field spaces defined above, obtain the corresponding functional measures, implement the gauge fixing in the partition function and computing the resulting functional jacobians. The details of this analysis are rather technical and have been lumped in App. A for the interested reader. Here, I shall limit myself to illustrating the result.

By varying (4.5) with respect to α and taking (2.2) into account, it appears that the Drinfeld–Sokolov ghost kinetic operator is $\tilde{\delta}_{DS\mathfrak{s}}$, acting on $\text{ECF}_{DS\mathfrak{s}_t}^{0,0}$. Hence, the Drinfeld–Sokolov Fadeev–Popov determinant is something like $\det(\tilde{\delta}_{DS\mathfrak{s}}^* \tilde{\delta}_{DS\mathfrak{s}})$ for a Drinfeld–Sokolov holomorphic structure $\mathfrak{s} \in \text{Hol}_{DS}$. This notation is a little bit

too formal. First, the adjoint $\bar{\partial}_{\text{DS}\mathbf{s}}^*$ of $\bar{\partial}_{\text{DS}\mathbf{s}}$ is defined with respect to the Hilbert structures $\langle \cdot, \cdot \rangle_{\text{DS}h, H\mathbf{s}}$ of $\text{ECF}_{\text{DS}\mathbf{s}}^{0,0}$ and $\text{ECF}_{\text{DS}\mathbf{s}}^{0,1}$ corresponding to the fixed background surface metric h and the varying fiber metric H . Secondly, the ghost kinetic operator $\bar{\partial}_{\text{DS}\mathbf{s}}$ has zero eigenvalues which have to be removed from the determinant. Hence, the Drinfeld–Sokolov Fadeev–Popov determinant should properly be $\det'_{h,H}(\bar{\partial}_{\text{DS}\mathbf{s}}^* \bar{\partial}_{\text{DS}\mathbf{s}})$, where the dependence on the metrics h and H and the removal of the zero eigenvalues are explicitly stated. The resulting functional of h, H and \mathbf{s} is essentially the bare ghost effective action once the zero modes and comodes of $\bar{\partial}_{\text{DS}\mathbf{s}}$ are properly taken care of. Let $\{e_i(\mathbf{s})_{\mathbf{s}} | i = 1, \dots, \dim \mathcal{G}_{\text{DS}}\}$ be a basis of $\ker \bar{\partial}_{\text{DS}\mathbf{s}}$. Since $\bar{\partial}_{\text{DS}\mathbf{s}}$ is defined independently from any choice of hermitian structure, the $e_i(\mathbf{s})_{\mathbf{s}}$ can be chosen independent from h and H . Let $\{f^j(\mathbf{s})_{\mathbf{s}} | j = 1, \dots, \dim \mathcal{M}_{\text{DS}}\}$ be a basis of $\text{coker } \bar{\partial}_{\text{DS}\mathbf{s}}$. This is defined here as the annihilator of $\text{ran } \bar{\partial}_{\text{DS}\mathbf{s}}$ in $\text{ECF}_{\text{DS}\mathbf{s}}^{\vee 1,0}$ under the dual pairing $\langle \cdot | \cdot \rangle_{\text{DS}\mathbf{s}}$. $\bar{\partial}_{\text{DS}\mathbf{s}}$ being defined independently of any choice of hermitian structure, the $f^j(\mathbf{s})_{\mathbf{s}}$ can also be chosen independent from h and H . The bare effective action $\hat{I}^{\text{gh}}(h, H, A^*)$ is

$$\hat{I}^{\text{gh}}(h, H, A^*) = \ln \left[\frac{\det'_{h,H}(\bar{\partial}_{\text{DS}\mathbf{s}}^* \bar{\partial}_{\text{DS}\mathbf{s}})}{\det M_{h,H\mathbf{s}}(e(\mathbf{s})) \det M_{h,H\mathbf{s}}^{\vee}(f(\mathbf{s}))} \right], \tag{4.10}$$

where

$$M_{h,H\mathbf{s}}(e(\mathbf{s}))_{ij} = \langle e_i(\mathbf{s}), e_j(\mathbf{s}) \rangle_{\text{DS}h, H\mathbf{s}}, \quad i, j = 1, \dots, \dim \mathcal{G}_{\text{DS}}, \tag{4.11}$$

$$M_{h,H\mathbf{s}}^{\vee}(f(\mathbf{s}))^{kl} = \langle f^k(\mathbf{s}), f^l(\mathbf{s}) \rangle_{\text{DS}h, H\mathbf{s}}^{\vee}, \quad k, l = 1, \dots, \dim \mathcal{M}_{\text{DS}} \tag{4.12}$$

are the Gramian matrices of the bases $\{e_i(\mathbf{s})_{\mathbf{s}}\}$ and $\{f^j(\mathbf{s})_{\mathbf{s}}\}$.

Below, I shall make some reasonable assumptions on the gauge slice function $A^*(t)$ and the group isomorphism $\zeta(g; t)$. Though they are not strictly necessary for the formal manipulations of functional integrals required by the gauge fixing, as such they guarantee the holomorphic factorization on \mathcal{M}_{DS} of all finite dimensional factors entering in the measure of the gauge fixed partition function $\mathcal{Z}_{\Theta}(h)$, a property known to hold in ordinary string theory which one would like to keep also in the present context.

As a first assumption, the gauge slice function $t \rightarrow A^*(t)$ is assumed to be analytic:

$$\bar{\partial}_t A^*(t) = 0. \tag{4.13}$$

It is not known to me whether it is possible to find a gauge slice function $A^*(t)$ globally holomorphic on \mathcal{M}_{DS} . In general, $A^*(t)$ may develop singularities on a submanifold of \mathcal{M}_{DS} of non-zero codimension, where $A^*(t)$ fails to be transverse to the action of the gauge group Gau_{DS} on SHol_{DS} . The singularities may eventually entail divergencies in the modular integration.

Equation (4.13) implies that the family of elliptic operators $t \rightarrow \bar{\partial}_{\text{DS}\mathbf{s}_t}$ is complex analytic. So, setting $e_i(t) = e_i(\mathbf{s}_t)$ and $f^j(t) = f^j(\mathbf{s}_t)$, one also has $\bar{\partial}_t e_i(t) = 0$ and $\bar{\partial}_t f^j(t) = 0$.

For fixed $t \in \mathcal{M}_{\text{DS}}$, define

$$\sigma_j^*(t) = \partial_{\nu} A^*(t), \quad j = 1, \dots, \dim \mathcal{M}_{\text{DS}}. \tag{4.14}$$

Since $\text{SHol}_{\text{DS}} \subseteq \text{ECF}_{\text{DS}}^{0,1}$, $\sigma_j^*(t) \in \text{ECF}_{\text{DS}}^{0,1}$. The $\sigma_j^*(t)$ are analytic, since $A^*(t)$ is. They are also linearly independent, since $A^*(t)$ defines a gauge slice, except perhaps on the submanifold of \mathcal{M}_{DS} , where $A^*(t)$ is singular. Using the $\sigma_j^*(t)$, one can build the matrix

$$F(t, f)_j^i = \langle f^i(t) | \sigma_j^*(t) \rangle_{\text{DS}}, \quad i, j = 1, \dots, \dim \mathcal{M}_{\text{DS}}. \quad (4.15)$$

$F(t, f)$ is analytic on \mathcal{M}_{DS} .

As a second assumption, the map $\zeta(g; t)$ is assumed to be analytic in both arguments:

$$\zeta(g; t)^{-1} \bar{\partial}_g \zeta(g; t) = 0, \quad (4.16)$$

$$\zeta(g; t)^{-1} \bar{\partial}_t \zeta(g; t) = 0. \quad (4.17)$$

As a function of t , $\zeta(g; t)$ may develop singularities on some submanifold of \mathcal{M}_{DS} of non-zero codimension, where $\zeta(g; t)$ fails to be a group isomorphism.

For fixed $t \in \mathcal{M}_{\text{DS}}$, define

$$\tau_i(t) = \zeta(1; t)^{-1} \bar{\partial}_g \zeta(1; t), \quad i = 1, \dots, \dim \mathcal{G}_{\text{DS}}. \quad (4.18)$$

$\tau_i(t) \in \text{ECF}_{\text{DS}}^{0,0}$, since $\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t) \subseteq \text{ECF}_{\text{DS}}^{0,0}$. The $\tau_i(t)$ are analytic, since $\zeta(g; t)$ is. They are also linearly independent, since $\zeta(g; t)$ is a group isomorphism, except perhaps on the submanifold of \mathcal{M}_{DS} where $\zeta(g; t)$ is singular. Away from that submanifold, they span $\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t) \cong \ker \bar{\partial}_{\text{DS } \mathbf{s}_t}$. One then picks vectors $\{\tau^{\vee i}(t) | i = 1, \dots, \dim \mathcal{G}_{\text{DS}}\}$ in $\text{ECF}_{\text{DS}}^{\vee 1,1}$ defining a basis dual to $\{\tau_i(t) | i = 1, \dots, \dim \mathcal{G}_{\text{DS}}\}$ with respect to the dual pairing $\langle \cdot | \cdot \rangle_{\text{DS}}$ and depending analytically on t . Using the $\tau^{\vee i}(t)$, one can build the matrix

$$E(t, e)_j^i = \langle \tau^{\vee i}(t) | e_j(t) \rangle_{\text{DS}}, \quad i, j = 1, \dots, \dim \mathcal{G}_{\text{DS}}. \quad (4.19)$$

$E(t, e)$ does not depend on the choice of the $\tau^{\vee i}(t)$. $E(t, e)$ is clearly analytic on \mathcal{M}_{DS} .

Let $\nu(g)$ be a left invariant positive $(\dim \mathcal{G}_{\text{DS}}, \dim \mathcal{G}_{\text{DS}})$ form on \mathcal{G}_{DS} . Hence, $L_f^* \nu(g) = \nu(g)$, for any $f \in \mathcal{G}_{\text{DS}}$. Using $\nu(g)$, one can define the volume $v_\nu = \int_{\mathcal{G}_{\text{DS}}} (Dg)_{|g} \nu(g)$ of \mathcal{G}_{DS} . This is actually divergent, as \mathcal{G}_{DS} is non-compact. The gauge fixed partition function $\mathcal{Z}_\Theta(h)$ reads

$$\begin{aligned} \mathcal{Z}_\Theta(h) &= \int_{\mathcal{M}_{\text{DS}}} (Dt)_{|t} |\det F(t, f) \det E(t, e)|^2 \frac{\nu(1)}{v_\nu} \int_{\text{Herm}} (DH)_{h|H} \\ &\quad \times \hat{\Theta}(h, H, A^*(t)) \exp(\hat{I}(h, H, A^*(t)) + \hat{I}^{\text{gh}}(h, H, A^*(t))). \end{aligned} \quad (4.20)$$

The denominator $v_\nu(t)$ reflects the residual unfixed \mathcal{G}_{DS} gauge symmetry, as mentioned earlier. In fact, $\Theta(h, H, A^*)$, $\hat{I}(h, H, A^*)$ and $\hat{I}^{\text{gh}}(h, H, A^*)$ are $\mathcal{G}_{\text{DS}}(\mathbf{s})$ invariant as functionals of H , the former two by Gau_{DS} invariance, the latter as a consequence of (4.10)–(4.12) and the nilpotence of $\mathcal{G}_{\text{DS}}(\mathbf{s})$. By (4.14)–(4.15), the measure is a $(\dim \mathcal{M}_{\text{DS}}, \dim \mathcal{M}_{\text{DS}})$ form on \mathcal{M}_{DS} so that the t integration is well defined. From (4.10)–(4.12), (4.15) and (4.19), it is immediate to see that the measure is independent from the choice of the bases $\{e_i(t)\}$ and $\{f^j(t)\}$. Gauge invariance ensures the measure is independent from the choice of the gauge slice $A^*(t)$. It may also

be shown that it is independent from the choice of the group isomorphism $\zeta(g; t)$. The measure is also independent from the choice of ν , since left invariance entails that ν is determined up to a positive constant. Finally, the measure is independent from the choice of the coordinates of \mathcal{G}_{DS} at 1, provided of course one uses the same coordinates for the $\tau(t)_i$ and $\nu(1)$.

The contribution of the Drinfeld–Sokolov ghosts has a functional integral representation. Let \mathbf{G} be the ghost Grassmann algebra. The ghost fields are $\beta \in \mathbf{G} \otimes (\text{Lie Gau}_{DS})^\vee$ and $\gamma \in \mathbf{G} \otimes \text{Lie Gau}_{DS}$. The isomorphisms $(\text{Lie Gau}_{DS})^\vee \cong \text{ECF}_{DS}^{\vee 1,0}$ and $\text{Lie Gau}_{DS} \cong \text{ECF}_{DS}^{0,0}$ allow one to construct the appropriate ghost functional measures $(D\beta)_{H|\beta}$ and $(D\gamma)_{h,H|\gamma}$. The Drinfeld–Sokolov ghost action is

$$S(\beta, \gamma, A^*) = 2 \text{Re} \langle \beta | \bar{\partial}_{DS} \gamma \rangle_{DS \mathbf{s}} = \frac{1}{\pi} \int_{\Sigma} d^2z 2 \text{Re} \text{tr}_{\text{ad}}(\beta \bar{\partial}_{DS} \gamma)_{\mathbf{s}}. \tag{4.21}$$

Then, by standard functional techniques, one can show that

$$\begin{aligned} & |\det F(t, f) \det E(t, e)|^2 \exp \hat{I}^{\text{gh}}(h, H, A^*(t)) \\ &= \int_{\mathbf{G} \otimes (\text{Lie Gau}_{DS})^\vee \times \mathbf{G} \otimes \text{Lie Gau}_{DS}} (D\beta)_{H|\beta} \otimes (D\gamma)_{h,H|\gamma} \exp(-S(\beta, \gamma, A^*(t))) \\ & \times \left| \prod_i \langle \beta | \sigma_i^*(t) \rangle_{DS} \prod_j \langle \tau^{\vee j}(t) | \gamma \rangle_{DS} \right|^2. \end{aligned} \tag{4.22}$$

The formal similarities with the construction of the Polyakov measure for ordinary strings are evident [16–19]. A detailed study of the Drinfeld–Sokolov ghost system is now in order.

5. The Drinfeld–Sokolov Ghost System

The study of the Drinfeld–Sokolov ghost effective action is problematic. For any Drinfeld–Sokolov holomorphic structure $\mathbf{s} \in \text{SHol}_{DS}$, the Cauchy–Riemann operator $\bar{\partial}_{DS \mathbf{s}}$ acts on the Drinfeld–Sokolov space $\text{ECF}_{DS \mathbf{s}}^{0,0}$. However, the hermitian structure is defined in terms of a metric $H_{\mathbf{s}} \in \text{Herm}_{\mathbf{s}}$, which does not respect the \mathfrak{r} -valuedness of the Drinfeld–Sokolov fields, since, for $\Psi_{\mathbf{s}} \in \text{ECF}_{DS \mathbf{s}}^{0,0}$, $(\text{Ad} H \Psi^\dagger)_{\mathbf{s}}$ is not \mathfrak{r} -valued in general. This renders the application of standard field theoretic techniques to the study of the Drinfeld–Sokolov ghost system impossible. This problem has been solved in a general context in ref. [29] by using the method of local projectors which now I shall briefly recall.

Given a metric $H_{\mathbf{s}} \in \text{Herm}_{\mathbf{s}}$, one can introduce the orthogonal projector $\varpi(H)_{\mathbf{s}}$ of $\text{ECF}_{\mathbf{s}}^{w,\bar{w}}$ onto $\text{ECF}_{DS \mathbf{s}}^{w,\bar{w}}$ with Hilbert structures corresponding to $H_{\mathbf{s}}$ defined in Sect. 2. $\varpi(H)_{\mathbf{s}}$ is given as a collection of local maps $\varpi(H)_{\mathbf{s}a}$ valued in the endomorphisms of \mathfrak{g} with range \mathfrak{r} such that $\varpi(H)_{\mathbf{s}a} = \text{Ad} \text{Ad} K_{\mathbf{s}ab} \varpi(H)_{\mathbf{s}b}$ whenever defined and that $\varpi(H)_{\mathbf{s}}^2 = \varpi(H)_{\mathbf{s}}$ and $(\text{Ad} H \varpi(H)^\dagger \text{Ad} H^{-1})_{\mathbf{s}} = \varpi(H)_{\mathbf{s}}$, where $\varpi(H)^\dagger$ is the pointwise adjoint of $\varpi(H)$ with respect to the hermitian inner product on \mathfrak{g} defined by $(x, y) = \text{tr}_{\text{ad}}(x^\dagger y)$ for $x, y \in \mathfrak{g}$.

Recall that the Cauchy Riemann operator $\bar{\partial}_S$ maps ECF_{DSS}^w , into $ECF_{DSS}^{w,1}$. It can be shown that this implies that $\varpi(H)_S$ obeys the relation

$$(\bar{\partial}\varpi(H)\varpi(H))_S = 0. \tag{5.1}$$

Projectors $\varpi(H)_S$ satisfying (5.1) were introduced earlier in the mathematical literature in the analysis of Hermitian–Einstein and Higgs bundles [34, 35].

The dependence of $\varpi(H)_S$ on S is minimal in the sense explained in Sect. 2, i.e. $\varpi(H)_a = \text{Ad } V_{Sab} \varpi(H)_{bS} \text{Ad } V_{Sab}^{-1}$.

The independence of the range of $\varpi(H)$ from H implies that

$$\delta\varpi(H)\varpi(H) = 0. \tag{5.2}$$

By combining the $\text{Ad } H$ hermiticity of $\varpi(H)$ and (5.2), one obtains

$$\delta\varpi(H) = -\varpi(H)\text{ad}(\delta HH^{-1})(1 - \varpi(H)). \tag{5.3}$$

This identity is a functional differential equation constraining the dependence of $\varpi(H)$ on H and shows that $\varpi(H)$ is a local functional of H .

Let H_0 be a reference fiber metric in Herm . As explained in Sect. 3, any other fiber metric $H \in \text{Herm}$ can be written as $H = \exp \Phi H_0$, where the Donaldson field Φ is an element of $ECF^{0,0}$ such that $\text{Ad } H_0 \Phi^\dagger = \bar{\Phi}$. Using (5.3), it is straightforward to show that $\varpi(H)$ has a local Taylor expansion in Φ of the form

$$\varpi(H) = \sum_{r=0}^{\infty} \frac{1}{r!} \varpi^{(r)}(\Phi, H_0), \tag{5.4}$$

where, for each $r \geq 0$, $\varpi^{(r)}(\Phi, H_0)$ transforms as $\varpi(H)$ under coordinate changes and is a homogeneous degree r polynomial in Φ :

$$\begin{aligned} \varpi^{(0)}(\Phi, H_0) &= \varpi(H_0), \\ \varpi^{(1)}(\Phi, H_0) &= -\varpi(H_0)\text{ad } \Phi(1 - \varpi(H_0)), \\ \varpi^{(2)}(\Phi, H_0) &= \varpi(H_0)\text{ad } \Phi(1 - 2\varpi(H_0))\text{ad } \Phi(1 - \varpi(H_0)), \\ \varpi^{(3)}(\Phi, H_0) &= \varpi(H_0)[\text{ad } \Phi(3\varpi(H_0) - 1)\text{ad } \Phi(1 - \varpi(H_0))\text{ad } \Phi \\ &\quad + \text{ad } \Phi(2 - 3\varpi(H_0))\text{ad } \Phi\varpi(H_0)\text{ad } \Phi](1 - \varpi(H_0)), \\ &\vdots \end{aligned} \tag{5.5}$$

It is not difficult to show that the projector $\varpi(H(h))$ corresponding to the metric $H(h)$ given in (3.15) is given by

$$\varpi(H(h)) = \exp(-\partial \ln h \text{ad } t_{-1}) p_{\mathfrak{r}} \exp(\partial \ln h \text{ad } t_{-1}), \tag{5.6}$$

where $p_{\mathfrak{r}}$ is the orthogonal projector of \mathfrak{g} onto \mathfrak{r} with respect to the hermitian inner product (\cdot, \cdot) of \mathfrak{g} defined above.

Next, consider the Gau_{DS} invariant unrenormalized Drinfeld–Sokolov ghost effective action $\hat{I}^{\text{gh}}(h, H, A^*)$ with $A^* \in \text{SHol}_{DS}$ a Drinfeld–Sokolov holomorphic structure. Because of the unboundedness of the ghost kinetic operator $\bar{\partial}_{DSS}$, $\hat{I}^{\text{gh}}(h, H, A^*)$ suffers ultraviolet divergencies which have to be regularized by means of an ultraviolet cut-off ε . As in Sect. 3, I shall adopt here proper time regularization

[18]. Next, I shall analyze the main properties of this effective action. It turns out that the Drinfeld–Sokolov ghost system is not a Drinfeld–Sokolov field theory of the type discussed in Sect. 3. In spite of this, it shares many of the qualitative features of a Drinfeld–Sokolov field theory, as is shown below.

Using the methods of [29], it can be seen that $\hat{I}^{\text{gh}}(h, H, A^*; \varepsilon)$ has the following expansion as $\varepsilon \rightarrow 0$:

$$\begin{aligned} \hat{I}^{\text{gh}}(h, H, A^*; \varepsilon) = & -\frac{r^{\text{gh}}}{\pi\varepsilon} \int_{\Sigma} d^2z h + \left[\frac{-r^{\text{gh}}}{6\pi} \int_{\Sigma} d^2z f_h \right. \\ & + \frac{1}{2\pi} \int_{\Sigma} d^2z \text{tr}((\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H))_{\mathbf{S}} - q \left. \right] \ln \varepsilon \\ & + I_0^{\text{gh}}(h, H, A^*) + O(\varepsilon). \end{aligned} \tag{5.7}$$

Here, $r^{\text{gh}} = \dim \mathfrak{r}$ and $q = \dim \mathcal{G}_{\text{DS}}$. $\partial_H\varpi(H) = \partial\varpi(H) - [\text{ad}\Gamma_H, \varpi(H)]$. The first two terms of the coefficient of $\ln \varepsilon$ are topological invariants. In fact, $\int_{\Sigma} d^2z f_h = 2\pi(\ell - 1)$ is the Guass–Bonnet invariant, already encountered in Sect. 3, and $\int_{\Sigma} d^2z \text{tr}((\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H))_{\mathbf{S}} = -2\pi \text{tr}[\text{ad}t_0 p_{\mathfrak{r}}](\ell - 1)$ is the Chern–Weil invariant of DS , where $p_{\mathfrak{r}}$ is defined below (5.6). $I_0^{\text{gh}}(h, H, A^*)$ is a non local functional of h, H and A^* such that

$$\begin{aligned} \delta I_0^{\text{gh}}(h, H, A^*) = & \frac{r^{\text{gh}}}{6\pi} \int_{\Sigma} d^2z \delta \ln h f_h \\ & - \frac{1}{2\pi} \int_{\Sigma} d^2z [\delta \ln h \text{tr}((\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H))_{\mathbf{S}} \\ & + \text{tr}(\text{ad}(\delta HH^{-1})\varpi(H))_{\mathbf{S}} f_h] \\ & + \frac{1}{\pi} \int_{\Sigma} d^2z \text{tr}(\text{ad}(\delta HH^{-1})(\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H))_{\mathbf{S}}, \end{aligned} \tag{5.8}$$

where δ denotes variation with respect to h and H at fixed A^* .

To renormalize the bare effective action $\hat{I}^{\text{gh}}(h, H, A^*; \varepsilon)$, one has to add to it a counterterm of the form

$$\begin{aligned} \Delta I^{\text{gh}}(h, H, A^*; \varepsilon) = & \frac{r^{\text{gh}}}{\pi\varepsilon} \int_{\Sigma} d^2z h - \left[\frac{-r^{\text{gh}}}{6\pi} \int_{\Sigma} d^2z f_h \right. \\ & + \frac{1}{2\pi} \int_{\Sigma} d^2z \text{tr}((\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H))_{\mathbf{S}} - q \left. \right] \ln \varepsilon \\ & + \Delta I^{\text{gh}}(h, H, A^*) + O(\varepsilon). \end{aligned} \tag{5.9}$$

Here, $\Delta I^{\text{gh}}(h, H, A^*)$ is a local but otherwise arbitrary functional of h, H and A^* , whose choice defines a renormalization prescription, as in Drinfeld–Sokolov field theory. The renormalized effective action is thus

$$I^{\text{gh}}(h, H, A^*) = I_0^{\text{gh}}(h, H, A^*) + \Delta I^{\text{gh}}(h, H, A^*). \tag{5.10}$$

Below, $\Delta I^{\text{gh}}(h, H, A^*)$ is assumed to be independent from A^* :

$$\Delta I^{\text{gh}}(h, H, A^*) = \Delta I^{\text{gh}}(h, H). \tag{5.11}$$

It can be shown that, if this condition is fulfilled, $I^{\text{gh}}(h, H, A^*)$ has the following structure

$$I^{\text{gh}}(h, H, A^*) = I^{\text{gh}}(h, H) + L^{\text{gh}}(H, A^*; A, \rho) + I_{\text{hol}}^{\text{gh}}(A^*; A, \rho). \tag{5.12}$$

Here, $A \in \text{Conn}_{\text{DS}}$ is a background Drinfeld–Sokolov $(1, 0)$ connection. ρ is a background local projector on \mathfrak{r} . In analogy to $\varpi(H)$, ρ is given as a collection of maps ρ_a valued in the endomorphisms of \mathfrak{g} with range \mathfrak{r} such that $\rho_a = \text{Ad Ad}K_{ab}\rho_b$ whenever defined and that $\rho^2 = \rho$. $I^{\text{gh}}(h, H)$ is the functional $I^{\text{gh}}(h, H, A^*)$ evaluated at the reference holomorphic structure $A^* = 0$,

$$L^{\text{gh}}(H, A^*; A, \rho) = \frac{1}{\pi} \int_{\Sigma} d^2z [2 \text{Re tr}((\varpi(H) \text{ad} \Gamma_H - \rho \text{ad} A) \text{ad} A^*) - \text{tr}(\text{ad} A^* \varpi(H) \text{ad Ad} H A^{*\dagger})]. \tag{5.13}$$

Using (2.15) and the fact that \mathfrak{r} is a nilpotent subalgebra of \mathfrak{g} such that $[t_d, \mathfrak{r}] \subseteq \mathfrak{r}$ for $d = 0, -1$, it is straightforward to verify that the integrand belongs to $\text{CF}^{1,1}$ so that the integration can be carried out. $I_{\text{hol}}^{\text{gh}}(A^*; A, \rho)$ is a non-local functional of A^* depending on A and ρ . Next, I shall study the properties of the three contributions in the right-hand side of (5.12).

In order for the counterterm $\Delta I^{\text{gh}}(h, H, A^*; \varepsilon)$ to be Gau_{DS} invariant, $\Delta I^{\text{gh}}(h, H)$ must satisfy

$$s_{\text{DS}} \Delta I^{\text{gh}}(h, H) = 0. \tag{5.14}$$

This ensures that the renormalized effective action $I^{\text{gh}}(h, H, A^*)$ is also Gau_{DS} invariant. Under this assumption, one has

$$s_{\text{DS}} I^{\text{gh}}(h, H) = \mathcal{W}_{\text{DS}}^{\text{gh}}(H), \tag{5.15}$$

$$s_{\text{DS}} L^{\text{gh}}(H, A^*; A, \rho) = -\mathcal{W}_{\text{DS}}^{\text{gh}}(H) - \mathcal{A}_{\text{DS}}^{\text{gh}}(A^*; A, \rho), \tag{5.16}$$

$$s_{\text{DS}} I_{\text{hol}}^{\text{gh}}(A^*; A, \rho) = \mathcal{A}_{\text{DS}}^{\text{gh}}(A^*; A, \rho), \tag{5.17}$$

where

$$\mathcal{W}_{\text{DS}}^{\text{gh}}(H) = \frac{1}{\pi} \int_{\Sigma} d^2z 2 \text{Re tr}(\text{ad } \Xi_{\text{DS}} \bar{\partial}(\varpi(H) \text{ad } \Gamma_H)), \tag{5.18}$$

$$\begin{aligned} \mathcal{A}_{\text{DS}}^{\text{gh}}(A^*; A, \rho) = & -\frac{1}{\pi} \int_{\Sigma} d^2z 2 \text{Re tr}(\text{ad } \Xi_{\text{DS}}(\bar{\partial}(\rho \text{ad} A) \\ & - \partial(\text{ad} A^* \rho) + [\rho \text{ad} A, \text{ad} A^* \rho])) \end{aligned} \tag{5.19}$$

are the ghost gauge anomalies. Using (2.15) and the properties of \mathfrak{r} recalled below (5.13), it is straightforward to verify that the integrand belongs to $\text{CF}^{1,1}$ so that the integration can be carried out. As a check, I have verified that the restriction of $\mathcal{A}_{\text{DS}}^{\text{gh}}(A^*; A, \rho)$ to $\text{Lie } \mathcal{G}_{\text{DS}}(\mathfrak{s})$ vanishes as it should.

$I^{\text{gh}}(h, H)$ is a non local functional of h and H . Its dependence on h and H can be analyzed as follows. Using the fiber metric $H(h)$ defined in (3.15), one has

$$\begin{aligned} I^{\text{gh}}(h, H) = & I_{\text{conf}}^{\text{gh}}(h) + S^{\text{gh}}(h, H) + F(h, H, H(h)) \\ & + \Delta I^{\text{gh}}(h, H) - \Delta I^{\text{gh}}(h, H(h)), \end{aligned} \tag{5.20}$$

where

$$I_{\text{conf}}^{\text{gh}}(h) = I^{\text{gh}}(h, H(h)), \tag{5.21}$$

$$S^{\text{gh}}(h, H) = \Omega^{\text{gh}}(h, H(h)). \tag{5.22}$$

Here, for any two metrics $H, H_0 \in \text{Herm}$, $\Omega^{\text{gh}}(H, H_0)$ is the Drinfeld–Sokolov generalization of the Donaldson action defined by functional path integral

$$\Omega^{\text{gh}}(H, H_0) = \frac{1}{\pi} \int_{H_0}^H \int_{\Sigma} d^2z \text{tr}(\text{ad}(\delta H' H'^{-1})(\text{ad} F_{H'} + \bar{\partial} \partial_{H'} \varpi(H')) \varpi(H')). \tag{5.23}$$

$F(h, H, H_0)$ is the functional

$$F(h, H, H_0) = \frac{-1}{2\pi} \int_{H_0}^H \int_{\Sigma} d^2z \text{tr}(\text{ad}(\delta H' H'^{-1}) \varpi(H')) f_h. \tag{5.24}$$

The right-hand sides of (5.23) and (5.24) are both independent from the choice of the functional integration path joining H_0 to H , since the functional 1-forms on Herm integrated are closed and Herm is contractible. This can easily be verified using (5.1) and (5.3). $\Omega^{\text{gh}}(H, H_0)$ can be computed in terms of the Donaldson field Φ of H relative to H_0 by using the local Taylor expansion (5.4)–(5.5) of $\varpi(H)$. The result is

$$\Omega^{\text{gh}}(H, H_0) = -\frac{1}{\pi} \int_{\Sigma} d^2z \text{tr} \left[K^*(\Phi, \bar{\partial} \Phi, H_0) \text{ad} \partial_{H_0} \Phi - D(\Phi, H_0) \text{ad} F_{H_0} + T(\Phi, H_0) \right], \tag{5.25}$$

where

$$D(\Phi, H_0) = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \sum_{n=0}^m \binom{m}{n} \varpi^{(m-n)}(\Phi, H_0) \text{ad} \Phi \varpi^{(n)}(\Phi, H_0), \tag{5.26}$$

$$T(\Phi, H_0) = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \sum_{n=0}^m \binom{m}{n} \partial_{H_0} \varpi^{(m-n)}(\Phi, H_0) \text{ad} \Phi \bar{\partial} \varpi^{(n)}(\Phi, H_0), \tag{5.27}$$

$$\begin{aligned} K^*(\Phi, \bar{\partial} \Phi, H_0) &= \sum_{m=0}^{\infty} \frac{1}{(m+2)!} \sum_{n=0}^m \binom{m+1}{n} (-\text{ad ad} \Phi)^{m-n} \\ &\quad \times \sum_{k=0}^n \binom{n}{k} \bar{\partial}(\varpi^{(n-k)}(\Phi, H_0) \text{ad} \Phi) \varpi^{(k)}(\Phi, H_0). \end{aligned} \tag{5.28}$$

By a similar and simpler calculation, one finds

$$F(h, H, H_0) = \frac{-1}{2\pi} \int_{\Sigma} d^2z \text{tr} J(\Phi, H_0) f_h, \tag{5.29}$$

where

$$J(\Phi, H_0) = \sum_{r=0}^{\infty} \frac{1}{(r+1)!} \text{ad} \Phi \varpi^{(r)}(\Phi, H_0). \tag{5.30}$$

Now, $I_{\text{conf}}^{\text{gh}}(h)$ is a non-local functional of h . By using (3.15), (5.6) and (5.8), one can obtain the variational relation obeyed by $I_{\text{conf}}^{\text{gh}}(h)$. This can be written in a

rather explicit form, because of the simple dependence of $H(h)$ and $\varpi(H(h))$ on h . By a somewhat lengthy but straightforward calculation, one finds

$$\delta I_{\text{conf}}^{\text{gh}}(h) = -\frac{\kappa^{\text{gh}}}{12\pi} \int d^2z \delta \ln h f_h + \delta \left[\frac{\lambda_0^{\text{gh}}}{\pi} \int d^2z h^{-1} f_h^2 + \Delta I^{\text{gh}}(h, H(h)) \right], \quad (5.31)$$

where

$$\kappa^{\text{gh}} = -2\text{tr}[(6(\text{ad } t_0)^2 + 6\text{ad } t_0 + 1)p_{\mathfrak{r}}], \quad (5.32)$$

$$\lambda_0^{\text{gh}} = -\text{tr}(\text{ad } t_{+1} \text{ad } t_{-1} p_{\mathfrak{r}}). \quad (5.33)$$

If

$$\Delta I^{\text{gh}}(h, H) = \Delta' I^{\text{gh}}(h, H) + \frac{\lambda_0^{\text{gh}}}{\pi} \int d^2z h^{-1} f_h^2, \quad (5.34)$$

where λ_0^{gh} is some constant and $\Delta' I^{\text{gh}}(h, H)$ is a local functional of h and H such that

$$\Delta' I^{\text{gh}}(h, H(h)) = 0, \quad (5.35)$$

(5.31) becomes simply

$$\delta I_{\text{conf}}^{\text{gh}}(h) = -\frac{\kappa^{\text{gh}}}{12\pi} \int d^2z \delta \ln h f_h + \frac{\lambda_0^{\text{gh}} + \lambda^{\text{gh}}}{\pi} \delta \int d^2z h^{-1} f_h^2. \quad (5.36)$$

A counterterm $\Delta I^{\text{gh}}(h, H, A^*)$ for which (5.34) holds is given by the right-hand side of (5.34) with $\Delta' I^{\text{gh}}(h, H)$ satisfying (5.14) and (5.35) and clearly satisfies both (5.11) and (5.14). Choosing $\lambda_0^{\text{gh}} = -\lambda^{\text{gh}}$ yields a renormalized effective action $I_{\text{conf}}^{\text{gh}}(h)$ describing a conformal field theory of central charge $\kappa_{\text{conf}}^{\text{gh}} = \kappa^{\text{gh}}$. This is precisely the central charge of the Drinfeld–Sokolov ghost system of the W -algebra associated to the pair (G, S) as computed with the methods of hamiltonian reduction and conformal field theory [10]². For a generic value of λ_0 , one obtains a renormalized effective action with a $\int \sqrt{h} R_h^2$ term yielding a model of induced $2d$ gravity of the same type as that considered in refs. [32, 33], as in Sect. 3.

The functional $S^{\text{gh}}(h, H)$ and $F(h, H, H(h))$ are local. In fact, the Donaldson field relevant here is $\Phi(h, H)$, defined in (3.26). From (5.6), the locality of $\Phi(h, H)$ as a functional of h and H and Eqs. (5.25)–(5.30) showing that $\Omega^{\text{gh}}(H, H_0)$ and $F(h, H, H_0)$ are local functionals of Φ and H_0 , the statement is evident.

From the above discussion, it follows that the suitably renormalized Drinfeld–Sokolov ghost effective action $I^{\text{gh}}(h, H)$ differs from the conformal effective action $I_{\text{conf}}^{\text{gh}}(h)$ by a local functional of h and H . In particular, the H dependence is local.

From (5.13), it appears that $L^{\text{gh}}(H, A^*; A, \rho)$, the interaction term of H and A^* , is local.

It is also likely, though no proof is available at present, that $I_{\text{hol}}^{\text{gh}}(A^*; A, \rho)$ is the real part of a holomorphic functional of A^* and A and ρ , entailing holomorphic factorization. Its crucial property, however, is its independence from H .

One has thus reached the following important conclusion. The full suitably renormalized Gau_{DS} invariant Drinfeld–Sokolov ghost effective action $I^{\text{gh}}(h, H, A^*)$ is a local functional of H .

²The odd looking sign of the mid term in the right-hand side of (5.32) is due to the fact that \mathfrak{r} is negative graded.

One could choose $\Delta'I^{\text{gh}}(h, H) = 0$ above. There is however a different more interesting choice, namely

$$\Delta'I^{\text{gh}}(h, H) = -F(h, H, H(h)). \tag{5.37}$$

Using (5.3) and (3.15), one can show that

$$\begin{aligned} \delta\Delta'I^{\text{gh}}(h, H) &= \frac{1}{2\pi} \int_{\Sigma} d^2z [\delta \ln h \text{tr}((\text{ad} F_H + \bar{\partial}\partial_H \varpi(H))\varpi(H)) \\ &\quad + \text{tr}(\text{ad}(\delta HH^{-1})\varpi(H))f_h] \\ &\quad - \frac{1}{2\pi} \int_{\Sigma} d^2z [\delta \ln h \text{tr}((\text{ad} F_{H_0} + \bar{\partial}\partial_{H_0} \varpi(H_0))\varpi(H_0)) \\ &\quad + \text{tr}(\text{ad}(\delta H_0 H_0^{-1})\varpi(H_0))f_h]|_{H_0=H(h)}. \end{aligned} \tag{5.38}$$

Hence, the counterterm $\Delta'I^{\text{gh}}(h, H)$ has the nice property of cancelling the mid term of (5.8) separating the $\delta \ln h$ and δHH^{-1} terms in $\delta I^{\text{gh}}(h, H, A^*)$.

6. Conformal Invariance

Let us go back to Eq. (4.20) providing the expression of the gauge fixed partition function $\mathcal{Z}_{\Theta}(h)$. Here, I shall assume that the insertion $\hat{\Theta}(h, H, A^*)$ contains only the counterterms necessary to absorb the ultraviolet divergencies of the bare effective actions $\hat{I}(h, H, A^*)$ and $\hat{I}^{\text{gh}}(h, H, A^*)$. Thus, $\hat{\Theta}(h, H, A^*)$ has the structure

$$\hat{\Theta}(h, H, A^*) = \exp(\Delta\hat{I}(h, H, A^*) + \Delta\hat{I}^{\text{gh}}(h, H, A^*))\hat{\theta}(h, A^*), \tag{6.1}$$

where $\Delta\hat{I}(h, H, A^*)$ and $\Delta\hat{I}^{\text{gh}}(h, H, A^*)$ are given by (3.4) and (5.9) in the proper time regularization scheme and $\hat{\theta}(h, A^*)$ is a Gau_{DS} invariant functional of h and A^* . Then, after cancellation of matter and ghost ultraviolet divergencies, (4.20) may be written as

$$\mathcal{Z}(h) = \int_{\mathcal{M}_{\text{DS}}} (Dt)_{|t} |\det F(t, f) \det E(t, e)|^2 \frac{v(1)}{v_v} \hat{\theta}(h, A^*(t)) \mathcal{Z}^{\text{herm}}(h, A^*(t)), \tag{6.2}$$

where

$$\mathcal{Z}^{\text{herm}}(h, A^*) = \int_{\text{Herm}} (DH)_{h|H} \exp I^{\text{tot}}(h, H, A^*), \tag{6.3}$$

$$I^{\text{tot}}(h, H, A^*) = I(h, H, A^*) + I^{\text{gh}}(h, H, A^*). \tag{6.4}$$

The problem to tackle next is the study of the partition function $\mathcal{Z}^{\text{herm}}(h, A^*)$. By the discussion of Sects. 3 and 5, the underlying H field theory is local.

Before proceeding, an important remark is in order. Using the results of Sect. 3 of ref. [29], it is easy to show that, for fixed $\mathfrak{s} \equiv A^* \in \text{SHol}_{\text{DS}}$, the action $I^{\text{tot}}(h, H, A^*)$ is invariant under the subgroup $\mathcal{G}'_{\text{DS}}(\mathfrak{s})$ of $\exp \mathfrak{n}_{\mathfrak{r}}$ -valued elements of $\mathcal{G}(\mathfrak{s})$, where $\mathfrak{n}_{\mathfrak{r}}$ is the normalizer \mathfrak{r} . $\mathcal{G}'_{\text{DS}}(\mathfrak{s})$ is larger than $\mathcal{G}_{\text{DS}}(\mathfrak{s})$. For varying $\mathfrak{s} \in \text{SHol}_{\text{DS}}$, the groups $\mathcal{G}'_{\text{DS}}(\mathfrak{s})$ are all isomorphic to the same complex Lie group \mathcal{G}'_{DS} containing \mathcal{G}_{DS} . Therefore, even after formally dividing by the volume v_v of \mathcal{G}_{DS} , the partition

function $\mathcal{L}^{\text{herm}}(h, A^*)$ is still divergent. This problem can be solved either by insertions that break the extra gauge symmetry or by further gauge fixing. The following analysis of conformal invariance is not affected by this.

In the method used here, the H functional integration is viewed as the integration on a suitable manifold of classical H configurations times the functional integration on the quantum H fluctuations around each of the corresponding H vacua.

The classical action for the H field is $I^{\text{tot}}(h, H, A^*)$. The classical H equation obtained from $I^{\text{tot}}(h, H, A^*)$ is

$$[F_H + K^{-1}\Pi(H)(\text{ad}F_H + \bar{\delta}\partial_H\varpi(H))\varpi(H)]_{\mathbf{s}} = 0. \tag{6.5}$$

Here, $\Pi(H)_{\mathbf{s}}$ is defined as follows. Consider the real vector space of local fields $\tilde{X}_{\mathbf{s}}$ valued in the endomorphisms of \mathfrak{g} such that $\tilde{X}_{\mathbf{s}a} = \text{Ad Ad}K_{\mathbf{s}ab}\tilde{X}_{\mathbf{s}b}$ and that $(\text{Ad Ad}H\tilde{X}^{\dagger})_{\mathbf{s}} = \tilde{X}_{\mathbf{s}}$, equipped with the pointwise Hilbert norm $\tilde{X}_{\mathbf{s}} \rightarrow \text{tr}(\tilde{X}^2)_{\mathbf{s}}$. Let $\tilde{\Pi}(H)_{\mathbf{s}}$ be the orthogonal projector of such a space onto its subspace of elements $\tilde{X}_{\mathbf{s}}$ of the form $\tilde{X}_{\mathbf{s}} = \text{ad}X_{\mathbf{s}}$ for some local field $X_{\mathbf{s}}$ such that $X_{\mathbf{s}a} = \text{Ad}K_{\mathbf{s}ab}X_{\mathbf{s}b}$ and that $(\text{Ad}HX^{\dagger})_{\mathbf{s}} = X_{\mathbf{s}}$. $\tilde{\Pi}(H)_{\mathbf{s}}$ is a field valued in the endomorphisms of the space of endomorphisms of \mathfrak{g} such that $\tilde{\Pi}(H)_{\mathbf{s}a} = \text{Ad Ad}K_{\mathbf{s}ab}\tilde{\Pi}(H)_{\mathbf{s}b}$ and depending locally on $H_{\mathbf{s}}$ since the $H_{\mathbf{s}}$ hermiticity condition is local. Since \mathfrak{g} is simple, the adjoint representation ad is faithful so that ad^{-1} is defined. By definition, $\Pi(H)_{\mathbf{s}} = \text{ad}^{-1}\tilde{\Pi}(H)_{\mathbf{s}}$. Equation (6.5) is easily obtained by using the variational identities (3.3), (5.8) and (5.38). I do not have any proof that Eq. (6.5) admits solutions. I shall assume anyway that solutions exist.

Equation (6.5) does not contain the surface metric h . It is therefore conformally invariant. This is a consequence of the renormalization prescription of the Drinfeld–Sokolov ghost sector used corresponding to the choice (5.37) of the finite part of the ghost counterterm.

The general solution of Eq. (6.5) is a function $H_{\text{cl}}(n; \mathbf{s})$ depending on \mathbf{s} of a set of parameters n varying in some finite dimensional real manifold \mathcal{N} . The n label the different solutions. For fixed \mathbf{s} , the metrics $H_{\text{cl}}(n; \mathbf{s})$ span a finite dimensional submanifold $\text{Herm}_{\text{cl}}(\mathbf{s})$ of Herm .

Since $I^{\text{tot}}(h, H, A^*)$ is $\mathcal{G}'_{\text{DS}}(\mathbf{s})$ invariant, if $\eta \in \mathcal{G}'_{\text{DS}}(\mathbf{s})$ and $H \in \text{Herm}_{\text{cl}}(\mathbf{s})$, then also $\eta^*H \in \text{Herm}_{\text{cl}}(\mathbf{s})$. So, the space of solutions of Eq. (6.5) for fixed \mathbf{s} is $\mathcal{G}'_{\text{DS}}(\mathbf{s})$ invariant. There exists therefore a free action $n \rightarrow {}^g n$, $g \in \mathcal{G}'_{\text{DS}}$, of \mathcal{G}'_{DS} on \mathcal{N} such that $H_{\text{cl}}({}^g n; \mathbf{s}) = \zeta'(g; \mathbf{s})^* H_{\text{cl}}(n; \mathbf{s})$ for some isomorphism $\zeta'(\cdot; \mathbf{s}) : \mathcal{G}'_{\text{DS}} \rightarrow \mathcal{G}'_{\text{DS}}(\mathbf{s})$.

To carry out the functional integration of the H quantum fluctuations around the classical vacua, one needs a fibration $\varphi(\cdot; \mathbf{s}) : \text{Herm} \rightarrow \mathcal{N}$ depending parametrically on a holomorphic structure \mathbf{s} . The fibration yields a parametrization of Herm of the form

$$H(\Phi, n; \mathbf{s}) = \exp \Phi H_{\text{cl}}(n; \mathbf{s}), \tag{6.6}$$

where $n \in \mathcal{N}$ and $\Phi \in \text{ECF}^{0,0}$ with $\text{Ad}H_{\text{cl}}(n; \mathbf{s})\Phi^{\dagger} = \Phi$ subject to the constraint that $\exp \Phi H_{\text{cl}}(n; \mathbf{s}) \in \varphi^{-1}(n; \mathbf{s})$. Such Donaldson fields Φ form a real manifold obviously isomorphic to $\varphi^{-1}(n; \mathbf{s})H_{\text{cl}}(n; \mathbf{s})^{-1}$.

The fibration $\varphi(n; \mathbf{s})$ must have the following properties. For any $n \in \mathcal{N}$ and any $H \in \varphi^{-1}(n; \mathbf{s})$, $T_H \text{Herm} = T_H \varphi^{-1}(n; \mathbf{s}) \oplus \mathcal{H}_H(n; \mathbf{s})$, where $\mathcal{H}_H(n; \mathbf{s})$ is some subspace of $T_H \text{Herm}$ of dimension equal to that of \mathcal{N} and the direct sum is orthogonal with respect to the Hilbert structure in Herm (cf. App. A). Further, $T_{H(\Phi, n; \mathbf{s})}\varphi^{-1}(n; \mathbf{s}) = \exp(\text{ad} \Phi/2)T_{H_{\text{cl}}(n; \mathbf{s})}\varphi^{-1}(n; \mathbf{s})$ and $\mathcal{H}_{H(\Phi, n; \mathbf{s})}(n; \mathbf{s}) = \exp(\text{ad} \Phi/2)\mathcal{H}_{H_{\text{cl}}(n; \mathbf{s})}(n; \mathbf{s})$. Finally, one has $\mathcal{H}_{H_{\text{cl}}(n; \mathbf{s})}(n; \mathbf{s}) = T_{H_{\text{cl}}(n; \mathbf{s})} \text{Herm}_{\text{cl}}(\mathbf{s})$.

The fibration $\varphi(n; \mathbf{s})$ must also be \mathcal{G}'_{DS} covariant, i.e. $\varphi^{-1}(g n; \mathbf{s}) = \zeta'(g; \mathbf{s})^* \varphi^{-1}(n; \mathbf{s})$ for any $g \in \mathcal{G}'_{\text{DS}}$. This implies the \mathcal{G}'_{DS} covariance of the parametrization (6.6), being $H(\zeta'(g; \mathbf{s})^* \Phi, g \mathbf{n} \mathbf{s}) = \zeta'(g; \mathbf{s})^* H(\Phi, \mathbf{n} \mathbf{s})$. One must also have that $T_{\zeta'(g; \mathbf{s})^* H} \varphi^{-1}(g n; \mathbf{s}) = \zeta'(g; \mathbf{s})^* T_H \varphi^{-1}(n; \mathbf{s})$ and $\mathcal{H}_{\zeta'(g; \mathbf{s})^* H}(g n; \mathbf{s}) = \zeta'(g; \mathbf{s})^* \mathcal{H}_H(n; \mathbf{s})$.

One clearly has the isomorphism $\text{Herm} \cong \mathcal{N} \times \varphi^{-1}(\cdot; \mathbf{s})$, where $\mathcal{N} \times \varphi^{-1}(\cdot; \mathbf{s}) = \prod_{n \in \mathcal{N}} \{n\} \times \varphi^{-1}(n; \mathbf{s})$. One can use the isomorphism to transform the functional integration on Herm into one on $\mathcal{N} \times \varphi^{-1}(\cdot; \mathbf{s})$. To this end, one has to provide \mathcal{N} and each $\varphi^{-1}(n; \mathbf{s})$ with the appropriate real Hilbert structure and construct the corresponding functional measures $(Dn)_{|n}$ and $(D\Phi)_{h, H_{\text{cl}}(n; \mathbf{s}) | \Phi}$. Details may be found in App. B.

Using the fibration $\varphi(\cdot; \mathbf{s})$, the partition function $\mathcal{Z}^{\text{herm}}(h, A^*)$ can be written as

$$\mathcal{Z}^{\text{herm}}(h, A^*) = \int_{\mathcal{N}} (Dn)_{|n} \exp I^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*) \mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n), \tag{6.7}$$

where

$$\begin{aligned} \mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n) &= [\det J(h, A^*; n)]^{\frac{1}{2}} \int_{\varphi^{-1}(n; \mathbf{s}) H_{\text{cl}}(n; \mathbf{s})^{-1}} (D\Phi)_{h, H_{\text{cl}}(n; \mathbf{s}) | \Phi} \\ &\times \exp I_{\text{qu}}^{\text{tot}}(\exp \Phi H_{\text{cl}}(n; \mathbf{s}), A^*; n), \end{aligned} \tag{6.8}$$

$$\begin{aligned} J(h, A^*; n)_{rs} &= \langle \partial_{n^r} H_{\text{cl}}(n; \mathbf{s}) H_{\text{cl}}(n; \mathbf{s})^{-1}, \partial_{n^s} H_{\text{cl}}(n; \mathbf{s}) H_{\text{cl}}(n; \mathbf{s})^{-1} \rangle_{h, H_{\text{cl}}(n; \mathbf{s})}, \\ r, s &= 1, \dots, \dim \mathcal{N}, \end{aligned} \tag{6.9}$$

$$I_{\text{qu}}^{\text{tot}}(H, A^*; n) = I^{\text{tot}}(h, H, A^*) - I^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*). \tag{6.10}$$

$I_{\text{qu}}^{\text{tot}}(H, A^*; n)$ is the quantum fluctuation action. The independence of $I_{\text{qu}}^{\text{tot}}(H, A^*; n)$ from h follows straightforwardly from (6.4), (3.5), (5.10), (3.3), (3.24), (5.8), (5.34) and (5.38). Details about the derivation of this formula are provided in App. B. Equation (6.7) may be cast in more suggestive form as follows.

Define

$$\begin{aligned} V^{\text{tot}}(h, H, A^*; A, \rho) &= I^{\text{tot}}(h, H, A^*) - I_{\text{hol}}(A^*; A) - I_{\text{hol}}^{\text{gh}}(A^*; A, \rho) \\ &= I(h, H) + I^{\text{gh}}(h, H) + L(H, A^*; A) + L^{\text{gh}}(H, A^*; A, \rho) \end{aligned} \tag{6.11}$$

(cf. Eqs. (6.4), (3.7) and (5.12)). Now, for a fixed A^* , one can impose the constraint $\delta V^{\text{tot}}(h, H, A^*; A, \rho) / \delta A^* = 0$ on the solutions of Eq. (6.5). This can be written in the form

$$[\Gamma_H - A + K^{-1} \Pi^c(H)(\varpi(H) \text{ad} \Gamma_H - \rho \text{ad} A)]_{\mathbf{s}} = 0 \quad \text{in ECF}_{\text{DSS}}^{\vee, 1, 0}. \tag{6.12}$$

Here, $\Pi^c(H)$ is defined similarly to $\Pi(H)$ below (6.5), by considering instead the complex vector space of local fields $\tilde{Z}_{\mathbf{s}}$ valued in the endomorphisms of \mathfrak{g} such that $\tilde{Z}_{\mathbf{s}a} = \text{Ad Ad} K_{\mathbf{s}ab} \tilde{Z}_{\mathbf{s}b}$ equipped with the pointwise Hilbert norm $\tilde{Z}_{\mathbf{s}} \rightarrow \text{tr}(\text{Ad Ad} H \tilde{Z}^\dagger \tilde{Z})_{\mathbf{s}}$. The above equation depends on the background fields A and ρ at order $O(K^{-1})$, except when the grading of \mathfrak{g} induced by \mathfrak{s} is integer. Below, I assume that, for any $\mathbf{s} \in \text{SHol}_{\text{DS}}$, there are common solutions of the dynamical

equation (6.5) and the constraint (6.12) at least for some choice A_0 and ρ_0 of the backgrounds. I further assume that such solutions are of the form $H_{\text{cl}}(n; \mathbf{s})$ for n varying in some submanifold \mathcal{N}_{DS} of \mathcal{N} .

$V^{\text{tot}}(h, H, A^*; A, \rho)$ is $\mathcal{G}'_{\text{DS}}(\mathbf{s})$ invariant, as $I^{\text{tot}}(h, H, A^*)$, $I_{\text{hol}}(A^*; A)$ and $I_{\text{hol}}^{\text{gh}}(A^*; A, \rho)$ are. Hence, if $\eta \in \mathcal{G}'_{\text{DS}}(\mathbf{s})$ and H satisfies (6.12), then also η^*H does. So, \mathcal{N}_{DS} is invariant under the action of \mathcal{G}'_{DS} on \mathcal{N} defined earlier.

Consider the classical action $I^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*)$. If $n \in \mathcal{N}_{\text{DS}}$, $H_{\text{cl}}(n; \mathbf{s})$ satisfies both (6.5) and (6.12). Then, by (6.4), (3.7), (5.12), (6.5) and (6.12), the functional $V^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*)$ is independent from A^* . Thus, one can evaluate it by setting $A^* = 0$. From here, using (6.11), (3.8), (5.13), (3.16), (3.24), (5.20), (5.34) and (5.37), one finds

$$I^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*) = I_{\text{conf}}^{\text{tot}}(h) + \Delta I_{\text{conf}}^{\text{tot}}(h; n) + I_{\text{hol}}^{\text{tot}}(A^*; A_0, \rho_0), \tag{6.13}$$

where

$$I_{\text{conf}}^{\text{tot}}(h) = I_{\text{conf}}(h) + I_{\text{conf}}^{\text{gh}}(h), \tag{6.14}$$

$$\Delta I_{\text{conf}}^{\text{tot}}(h; n) = S(h, H_{\text{cl}}(n)) + S^{\text{gh}}(h, H_{\text{cl}}(n)), \tag{6.15}$$

$$I_{\text{hol}}^{\text{tot}}(A^*; A, \rho) = I_{\text{hol}}(A^*; A, \rho) + I_{\text{hol}}^{\text{gh}}(A^*; A, \rho), \tag{6.16}$$

$S(h, H_{\text{cl}}(n))$ and $S^{\text{gh}}(h, H_{\text{cl}}(n))$ being given (3.18) and (5.22) and $H_{\text{cl}}(n)$ being $H_{\text{cl}}(n; \mathbf{s})$ evaluated at the reference holomorphic structure. By (3.25) and (5.36), $I_{\text{conf}}^{\text{tot}}(h)$ is the effective action of a conformal field theory of central charge $\kappa_{\text{conf}0}^{\text{tot}} = \kappa_0 + \kappa + \kappa^{\text{gh}}$, where κ and κ^{gh} are given respectively by (3.22) and (5.32). $\Delta I_{\text{conf}}^{\text{tot}}(h; n)$ is a local functional of h , since the two terms in the right-hand side of (6.15) are, as is explained in Sects. 3 and 5.

By the classical H equation (6.5), $I^{\text{tot}}(h, H_{\text{cl}}(n; \mathbf{s}), A^*)$ is constant as a function of n on each connected component \mathcal{N}_i of \mathcal{N} . Thus, it may be evaluated at any point $n_i \in \mathcal{N}_i \cap \mathcal{N}_{\text{DS}}$, which I assume to be non-empty. Then, on account of (6.13), (6.7) may be written as

$$\begin{aligned} \mathcal{Z}^{\text{herm}}(h, A^*) &= \sum_i \exp(I_{\text{conf}}^{\text{tot}}(h) + \Delta I_{\text{conf}}^{\text{tot}}(h; n_i) + I_{\text{hol}}^{\text{tot}}(A^*; A_0, \rho_0)) \\ &\times \int_{\mathcal{N}_i} (Dn)|_n \mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n). \end{aligned} \tag{6.17}$$

Next, one has to study the partition function $\mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n)$, but before doing that a few important remarks are in order.

Equations (6.5) and (6.12) are rather complicated because of the Drinfeld–Sokolov ghost contributions proportional to K^{-1} . In the limit $K \rightarrow \infty$, however, the ghosts decouple and they simplify considerably. Calling H_∞ the corresponding H configuration, the equations become

$$(F_{H_\infty})_{\mathbf{s}} = 0, \tag{6.18}$$

$$(\Gamma_{H_\infty} - A)_{\mathbf{s}} = 0 \quad \text{in ECF}_{\text{DSS}}^{\vee, 1,0}. \tag{6.19}$$

So, H_∞ is a flat fiber metric such that Γ_{H_∞} is Drinfeld–Sokolov, since A is. Equations of this form were found in [10] on a minkowskian cylindrical world sheet and shown to be equivalent to the non-abelian Toda equations associated to the

pair (G, S) . On a euclidean topologically non-trivial world sheet, however, one has to take into account further constraints coming from global definedness and non-singularity. One then finds that the above equations admit solutions H_∞ of Toda type at genus $\ell = 0$. For instance, $H(h_{cc})$, where h_{cc} is the constant curvature surface metric with $-2h_{cc}^{-1} f_{h_{cc}} = 1$ and $H(h)$ is given by (3.15), satisfies (6.18)–(6.19) for the reference holomorphic structure. In higher genus there still are solutions of Toda type but ones which are hermitian with respect to a non-compact conjugation of the Lie algebra \mathfrak{g} [36, 37]. The use of the compact conjugation \dagger however cannot be avoided since positivity of the various Hilbert structures in the construction of the measures is indispensable. If the Toda solutions are the only solutions available, then it will be necessary to introduce some type of insertion in the H functional integral providing extra terms in the classical equations compensating for the problem. Unfortunately, very little is known at present about these equations on a Riemann surface.

The partition function $\mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n)$ can be computed to leading order in a semiclassical expansion with expansion parameter $\hbar \equiv K^{-1}$. To this end, one rescales the Donaldson field Φ into $K^{-\frac{1}{2}} \Phi^0$ and expands in powers of $K^{-\frac{1}{2}}$. In so doing, one must take into account that the classical solution $H_{\text{cl}}(n; \mathbf{s})$, the fibration $\varphi(n, \mathbf{s})$ and the functional measure $(D\Phi)_{h, H_{\text{cl}}(n; \mathbf{s})|\Phi}$, also, depend on K .

Below, it is assumed that the metric $H_{\text{cl}}(n; \mathbf{s})$ has a well defined limit $H_{\text{cl}\infty}(n; \mathbf{s})$ in Herm as $K \rightarrow +\infty$ for every $n \in \mathcal{N}$ satisfying (6.18) and that such $K \rightarrow +\infty$ solutions span a submanifold $\text{Herm}_{\text{cl}\infty}(\mathbf{s})$ of Herm.

It can be seen that, in the limit $K \rightarrow +\infty$, one has $(D\Phi)_{h, H_{\text{cl}}(n; \mathbf{s})|\Phi} = z_K(h)[1 + O(K^{-1})](D\Phi^0)_{h, H_{\text{cl}\infty}(n; \mathbf{s})}$. Here, Φ^0_∞ varies in $\text{Don}(H_{\text{cl}\infty}(n; \mathbf{s}))$, where $\text{Don}(H_{\text{cl}\infty})$ is the space of Donaldson fields $\Phi^0_\infty \in \text{ECF}^{0,0}$ satisfying $\text{Ad } H_{\text{cl}\infty} \Phi^{0\dagger}_\infty = \Phi^0_\infty$ and orthogonal in $\text{ECF}^{0,0r}$ with Hilbert structure $\langle \cdot, \cdot \rangle_{h, H_{\text{cl}\infty}}^r$ to the kernel of the operator $\Delta_{H_{\text{cl}\infty}} \mathbf{s} = -(\bar{\partial} \partial_{H_{\text{cl}\infty}}) \mathbf{s}$. This follows from the properties of the fibration and the fact that the tangent vectors $\delta H_\infty H_\infty^{-1}$ to $\text{Herm}_{\text{cl}\infty}(\mathbf{s})$ at $H_{\text{cl}\infty}$ satisfy $(\Delta_{H_{\text{cl}\infty}}(\delta H_\infty H_\infty^{-1})) \mathbf{s} = 0$. $(D\Phi^0)_{h, H_{\text{cl}\infty}}$ is the translation invariant measure on $\text{Don}(H_{\text{cl}\infty})$ obtained from the obvious real Hilbert structure. $z_K(h) = [\det_{h, H_{\text{cl}\infty}(n; \mathbf{s})} (K^{-1} 1)]^{\frac{1}{2}}$ is a constant arising because of the different normalization of the fields Φ and Φ^0 related by $\Phi = K^{-\frac{1}{2}} \Phi^0$. It depends on h because of the h dependence of the measure.

Proceeding in this way, one finds

$$\begin{aligned} \mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n) &= z_K(h) [\det J_\infty(h, A^*; n)]^{\frac{1}{2}} \int_{\text{Don}(H_{\text{cl}\infty}(n; \mathbf{s}))} (D\Phi^0)_{h, H_{\text{cl}\infty}(n; \mathbf{s})} \\ &\times \exp(-S_{\text{qu}\infty}^{\text{tot}}(\Phi^0_\infty, A^*; H_{\text{cl}\infty}(n; \mathbf{s}))) [1 + O(K^{-1})], \end{aligned} \tag{6.20}$$

where $J_\infty(h, A^*; n)$ is given by (6.9) with $H_{\text{cl}}(n; \mathbf{s})$ replaced by $H_{\text{cl}\infty}(n; \mathbf{s})$ and

$$S_{\text{qu}\infty}^{\text{tot}}(\Phi^0_\infty, A^*; H_{\text{cl}\infty}) = \frac{1}{2\pi} \int_{\Sigma} d^2z \text{tr}_{\text{ad}}(\Phi^0_\infty \Delta_{H_{\text{cl}\infty}} \Phi^0_\infty) \mathbf{s} \tag{6.21}$$

is the Gaussian fluctuation action. The effective action $\hat{I}_{\text{qu}}^{\text{herm}}(h, A^*; n) = \ln \mathcal{Z}_{\text{qu}}^{\text{herm}}(h, A^*; n)$ is therefore

$$\hat{I}_{\text{qu}}^{\text{herm}}(h, A^*; n) = -\frac{1}{2} \ln \left[\frac{\det'_{h, H_{\text{cl}\infty}(n; \mathbf{s})} \Delta_{h, H_{\text{cl}\infty}(n; \mathbf{s})}}{\det J_\infty(h, A^*; n)} \right] + \ln z_K(h) + O(K^{-1}), \tag{6.22}$$

where $\Delta_{h, H_{cl\infty} \mathbf{s}} = h^{-1} \Delta_{H_{cl\infty} \mathbf{s}}$. Here, I shall use again the proper time regularization scheme. Then, the effective action becomes dependent on the proper time cut off ε . Taking into account that the vectors $\partial_n H_{cl\infty}(n; \mathbf{s}) H_{cl\infty}(n; \mathbf{s})^{-1}$ span $\ker \Delta_{h, H_{cl\infty}(n; \mathbf{s})}$, one finds, using standard heat kernel techniques,

$$\begin{aligned} I_{qu}^{\text{herm}}(h, A^*; n; \varepsilon) &= \frac{(1 - \ln K) \dim \mathfrak{g}}{\pi \varepsilon} \int_{\Sigma} d^2 z h + \frac{1}{2} \left[\frac{\dim \mathfrak{g}}{6\pi} \int_{\Sigma} d^2 z f_h + \dim \mathcal{N} \right] \ln \varepsilon \\ &\quad + W_{\text{conf}}(h) + \Lambda(A^*; n) + \ln K^{c'} + O(\varepsilon) + O(K^{-1}), \\ c' &= \frac{1}{6} \dim \mathfrak{g}(\ell - 1) + \frac{1}{2} \dim \mathcal{N}. \end{aligned} \tag{6.23}$$

Here, $W_{\text{conf}}(h)$ is a non-local functional of h such that

$$\delta W_{\text{conf}}(h) = - \frac{\dim \mathfrak{g}}{12\pi} \int_{\Sigma} d^2 z \delta h f_h. \tag{6.24}$$

$\Lambda(A^*; n)$ is a non-local functional of A^* depending on n . The ultraviolet divergencies can be cancelled by adding to the bare effective action the counterterm

$$\begin{aligned} \Delta I_{qu}^{\text{herm}}(h, A^*; \varepsilon) &= - \frac{(1 - \ln K) \dim \mathfrak{g}}{2\pi \varepsilon} \int_{\Sigma} d^2 z h \\ &\quad - \frac{1}{2} \left[\frac{\dim \mathfrak{g}}{6\pi} \int_{\Sigma} d^2 z f_h + \dim \mathcal{N} \right] \ln \varepsilon + O(\varepsilon) + O(K^{-1}). \end{aligned} \tag{6.25}$$

This must be independent from n , since the divergent terms of $\ln \mathcal{Z}^{\text{herm}}(h, A^*)$ depend only on h and A^* . The renormalized effective action is thus

$$\begin{aligned} I_{qu}^{\text{herm}}(h, A^*; n) &= I_{qu}^{\text{herm}}(h, A^*; n; \varepsilon) + \Delta I_{qu}^{\text{herm}}(h, A^*; \varepsilon) \\ &= W_{\text{conf}}(h) + \Lambda(A^*; n) + \ln K^{c'} + O(K^{-1}). \end{aligned} \tag{6.26}$$

From (6.24), the variation of $I_{qu}^{\text{herm}}(h, A^*; n)$ with respect to h at fixed A^* and n is

$$\delta I_{qu}^{\text{herm}}(h, A^*; n) = - \frac{\kappa_{\text{conf}}^{\text{herm}}}{12\pi} \int_{\Sigma} d^2 z \delta h f_h + O(K^{-1}), \tag{6.27}$$

where

$$\kappa_{\text{conf}}^{\text{herm}} = \dim \mathfrak{g} + O(K^{-1}). \tag{6.28}$$

It appears from here that, to order $O(K^0)$, the renormalized effective action $I_{qu}^{\text{herm}}(h, A^*; n)$ is that of a conformal field theory of central charge $\kappa_{\text{conf}}^{\text{herm}}$ given by (6.28). This is in agreement with the exact result obtained by conformal field theory techniques for the Wess–Zumino–Novikov–Witten model

$$\kappa_{\text{conf}}^{\text{herm}} = \frac{K \dim \mathfrak{g}}{K + c^{\vee}}, \tag{6.29}$$

where c^{\vee} is the dual Coxeter number. It remains to be seen if the agreement continues to hold at higher orders in K^{-1} , though physical intuition would seem to suggest so since the short distance structure of Drinfeld–Sokolov gravity is essentially the same as that of the Wess–Zumino–Novikov–Witten model.

From (6.2), (6.17) and (6.26), choosing

$$\hat{\theta}(h, A^*) = \exp \Delta I_{\text{qu}}^{\text{herm}}(h, A^*) [1 + O(K^{-1})], \tag{6.30}$$

where $\Delta I_{\text{qu}}^{\text{herm}}(h, A^*)$ is given by (6.25) in the proper time regularization scheme, one has

$$\begin{aligned} \mathcal{Z}(h) &= K^{c_\ell} \sum_i \exp(I_{\text{conf}}^{\text{tot}}(h) + \Delta I_{\text{conf}}^{\text{tot}}(h; n_i) + W_{\text{conf}}(h)) \\ &\times \int_{\mathcal{M}_{\text{DS}}} (Dt)_{|t} |\det F(t, f) \det E(t, e)|^2 \exp I_{\text{hol}}^{\text{tot}}(A^*(t); A_0, \rho_0) \frac{v(1)}{v_V} \\ &\times \int_{\mathcal{N}_i} (Dn)_{|n} \exp \Lambda(A^*(t); n) [1 + O(K^{-1})]. \end{aligned} \tag{6.31}$$

This is the final form of the partition function. To order $O(K^0)$, conformal invariance is manifest.

Several issues remain to be investigated. The analysis expounded is to some extent formal due to the lack of detailed geometric information about the Drinfeld–Sokolov moduli space \mathcal{M}_{DS} , the Drinfeld–Sokolov stability group \mathcal{G}_{DS} and the parameter space \mathcal{N} . A thorough investigation of these spaces is desirable. Also, the holomorphic structure on the Riemann surface Σ has been kept fixed throughout. One may try to deform the complex structure and study the resulting effects in the framework of deformation theory using the Beltrami parametrization. Such deformations should be a special subset of more general deformations parametrized by generalized Beltrami differentials [38, 39]. The study of this matter requires a better understanding of W geometry, which at present is lacking. This issue is also related to that of the analysis of the Gau_{DS} invariant content of the model. In fact, the generalized Beltrami differentials should be the sources of a suitable basis of Gau_{DS} invariant operators including the energy momentum tensor. At this level, W symmetries are expected to emerge.

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Appendix A

In this appendix, I shall provide the basic details about the derivation of the measure (4.20). The notation used here is the same as that defined in Sect. 4. I also set $q = \dim \mathcal{G}_{\text{DS}}$ and $m = \dim \mathcal{M}_{\text{DS}}$.

Let us construct the basic Hilbert manifolds. All such Hilbert manifolds are real, though as ordinary manifolds, they may be complex. Below, $h \in \text{Met}$ is a generic surface metric on Σ , which will be kept fixed throughout.

Consider first Herm . For any $H \in \text{Herm}$, the tangent space $T_H \text{Herm}$ is the subspace of $\text{ECF}^{0,0r}$ spanned by the elements $\delta H H^{-1}$ such that $\text{Ad} H (\delta H H^{-1})^\dagger = \delta H H^{-1}$ and equipped with the Hilbert structure $\frac{1}{2} \langle \cdot, \cdot \rangle_{h, H}^r$. The factor $\frac{1}{2}$ is conventional. Hence, one has $\|\delta H H^{-1}\|_{h, H}^2 = \|\delta H H^{-1}\|_{h, H^2}^2$, where the norm in the right-hand side is that of $\text{ECF}^{0,1}$. In this way, Herm becomes a real Hilbert manifold.

Next, consider SHol_{DS} . For any $A^* \in \text{SHol}_{\text{DS}}$, the tangent space $T_{A^*} \text{SHol}_{\text{DS}}$ is just $\text{ECF}_{\text{DS}}^{0,1}$ with the Hilbert structure $\langle \cdot, \cdot \rangle_{\text{DS}h,H}^r$ depending on a fiber metric $H \in \text{Herm}$. This is actually independent from h . Denoting by δA^* a generic element of $T_{A^*} \text{SHol}_{\text{DS}}$, one has $\|\delta A^*\|_{H|A^*}^2 = 2\|\delta A^*\|_{\text{DS}h,H}^2$, the norm in the right-hand side being that of $\text{ECF}_{\text{DS}}^{0,1}$. In this way, SHol_{DS} becomes a real Hilbert manifold.

Next, consider Gau_{DS} . For any $\alpha \in \text{Gau}_{\text{DS}}$, the tangent space $T_\alpha \text{Gau}_{\text{DS}}$ is just $\text{ECF}_{\text{DS}}^{0,0r}$ with the Hilbert structure $\langle \cdot, \cdot \rangle_{\text{DS}h,H}^r$ depending on a fiber metric $H \in \text{Herm}$. A generic element of $T_\alpha \text{Gau}_{\text{DS}}$ is of the form $\alpha^{-1} \delta \alpha$. One thus has $\|\alpha^{-1} \delta \alpha\|_{h,H|\alpha}^2 = 2\|\alpha^{-1} \delta \alpha\|_{\text{DS}h,H}^2$, the norm in the right-hand side being that of $\text{ECF}_{\text{DS}}^{0,1}$. In this way, Gau_{DS} too becomes a real Hilbert manifold. Because of the form of the tangent vectors, the Hilbert manifold structure defined is left invariant.

Consider now \mathcal{M}_{DS} . For any $t \in \mathcal{M}_{\text{DS}}$, $T_t \mathcal{M}_{\text{DS}}$ is just $(\mathbb{C}^m)^r$ with the standard euclidean inner product $\langle \cdot, \cdot \rangle^r$. So, $\|\delta t\|_t^2 = 2|\delta t|^2$ for $\delta t \in T_t \mathcal{M}_{\text{DS}}$. \mathcal{M}_{DS} becomes thus a real Hilbert manifold of dimension $2m$.

Finally, consider \mathcal{G}_{DS} . For any $g \in \mathcal{G}_{\text{DS}}$, $T_g \mathcal{G}_{\text{DS}}$ is just $(\mathbb{C}^q)^r$ with the standard euclidean inner product $\langle \cdot, \cdot \rangle^r$. So, $\|\delta g\|_g^2 = 2|\delta g|^2$ for $\delta g \in T_g \mathcal{G}_{\text{DS}}$. In this way, \mathcal{G}_{DS} becomes a real $2q$ dimensional Hilbert manifold.

The first problem to tackle is the definition of the Hilbert manifold structures of the two realizations $\text{Herm} \times \text{SHol}_{\text{DS}}$ and $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot)$ of the configuration space.

$\text{Herm} \times \text{SHol}_{\text{DS}}$ can be given naturally the structure of real Hilbert manifold as follows. For $(H, A^*) \in \text{Herm} \times \text{SHol}_{\text{DS}}$, $T_{(H,A^*)} \text{Herm} \times \text{SHol}_{\text{DS}} \cong T_H \text{Herm} \oplus T_{A^*} \text{SHol}_{\text{DS}}$ with the Hilbert norm

$$\|\delta H H^{-1} \oplus \delta A^*\|_{h,H|(H,A^*)}^2 = \|\delta H H^{-1}\|_{h|H}^2 + \|\delta A^*\|_{H|A^*}^2. \tag{A.1}$$

Providing $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot)$ with a Hilbert manifold structure is slightly trickier because of the quotient by the action of $\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot)$. For any $(t, \tilde{H}, \alpha) \in \mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot)$, one has $T_{(t,\tilde{H},\alpha)} \mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot) \cong T_t \mathcal{M}_{\text{DS}} \oplus ((T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}})/T \text{bit}_{(\tilde{H},\alpha)}(1;t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t))$. $\text{bit}_{(\tilde{H},\alpha)}(\cdot;t) : \mathcal{G}_{\text{DS}}(\mathbf{s}_t) \rightarrow \text{Herm} \times \text{Gau}_{\text{DS}}$ is the orbit map associated to the $\mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ action (4.6)–(4.7). Its tangent map $T \text{bit}_{(\tilde{H},\alpha)}(1;t)$ maps $\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ into the subspace of $T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}$ spanned by the vectors of the form $(\delta \eta + \text{Ad } \tilde{H} \delta \eta^\dagger) \oplus (-\delta \eta)$ with $\delta \eta \in \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$. This follows from the linearization of (4.6)–(4.7). The tangent space can be given a Hilbert structure as follows. One equips $T_t \mathcal{M}_{\text{DS}} \oplus T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}$ with the Hilbert norm

$$\|\delta t \oplus \delta \tilde{H} \tilde{H}^{-1} \oplus \alpha^{-1} \delta \alpha\|_{h,\tilde{H}|(t,\tilde{H},\alpha)}^2 = \|\delta t\|_t^2 + \|\delta \tilde{H} \tilde{H}^{-1}\|_{h|\tilde{H}}^2 + \|\alpha^{-1} \delta \alpha\|_{h,\tilde{H}|\alpha}^2. \tag{A.2}$$

Then, one has the identification $T_{(t,\tilde{H},\alpha)} \mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot) \cong T_t \mathcal{M}_{\text{DS}} \oplus ((T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}) \ominus T \text{bit}_{(\tilde{H},\alpha)}(1;t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t))$. The right-hand side carries the Hilbert structure induced by that of $T_t \mathcal{M}_{\text{DS}} \oplus T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}$. In this way, $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_\cdot)$ becomes a real Hilbert manifold. The above construction is independent from the choice of the representative (\tilde{H}, α) of the corresponding equivalence class modulo the $\mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ action (4.6)–(4.7). Indeed, different choices lead to unitarily equivalent realizations of the Hilbert tangent space, as is straightforward to check.

One has to compute now the jacobian $J(t, h, \tilde{H})$ of the map (4.4)–(4.5) relating the functional measures of $\text{Herm} \times \text{SHol}_{\text{DS}}$ and $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s} \cdot)$:

$$\begin{aligned} & (D\delta H(\tilde{H}, \alpha)H(\tilde{H}, \alpha)^{-1})_{h|H(\tilde{H}, \alpha)} \otimes (D\delta A^*(t, \alpha))_{H(\tilde{H}, \alpha)|A^*(t, \alpha)} \\ &= J(t, h, \tilde{H}) \left[(D\delta t)|_t \otimes (D\delta \tilde{H} \tilde{H}^{-1})_{h|\tilde{H}} \right. \\ & \quad \left. \otimes (D\alpha^{-1} \delta \alpha)_{h, \tilde{H}|\alpha} \right] \Big|_{(T \text{bit}_{(\tilde{H}, \alpha)}(1; t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t))^\perp}. \end{aligned} \tag{A.3}$$

By explicit calculation, one finds

$$J(t, h, \tilde{H}) = 2^q \Delta(t, h, \tilde{H}) \det P(t, \tilde{H}), \tag{A.4}$$

where

$$\Delta(t, h, \tilde{H}) = \det'_{h, \tilde{H}}((\bar{\partial} - \text{ad} A^*(t))^* (\bar{\partial} - \text{ad} A^*(t))) \tag{A.5}$$

is the functional determinant of the Laplacian associated to the operator $(\bar{\partial} - \text{ad} A^*(t)) : T_1 \text{Gau}_{\text{DS}} \rightarrow T_{A^*(t)} \text{SHol}_{\text{DS}}$ with the given Hilbert structures with the zero eigenvalues removed and

$$P(t, \tilde{H})_{i,j} = \langle \partial_\mu A^*(t), p(t, \tilde{H}) \partial_\nu A^*(t) \rangle_{h, \tilde{H}}, \quad i, j = 1, \dots, m, \tag{A.6}$$

where $p(t, \tilde{H})$ is the orthogonal projector on $\text{coker}(\bar{\partial} - \text{ad} A^*(t))$ in $T_{A^*(t)} \text{SHol}_{\text{DS}}$. All determinants are taken on the complex field.

Proof. The calculation of the jacobian requires to begin with the computation of the tangent map of the map (4.4)–(4.5). This is given by

$$\delta H(\tilde{H}, \alpha)H(\tilde{H}, \alpha)^{-1} = \text{Ad } \alpha(\delta \tilde{H} \tilde{H}^{-1} + \alpha^{-1} \delta \alpha + \text{Ad } \tilde{H}(\alpha^{-1} \delta \alpha)^\dagger), \tag{A.7}$$

$$\delta A^*(t, \alpha) = \text{Ad } \alpha((\bar{\partial} - \text{ad} A^*(t))(\alpha^{-1} \delta \alpha) + \delta_t A^*(t)), \tag{A.8}$$

as follows from a simple variational calculation. The Hilbert structure of the tangent bundle of $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s} \cdot)$ may be disentangled by means of the following orthogonal decomposition:

$$\begin{aligned} & T_t \mathcal{M}_{\text{DS}} \oplus ((T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}) \ominus T \text{bit}_{(\tilde{H}, \alpha)}(1; t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)) \\ &= T_t \mathcal{M}_{\text{DS}} \oplus (T_{\tilde{H}} \text{Herm} \ominus T \text{bit}_{\tilde{H}}(1; t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)) \\ & \quad \oplus (T_\alpha \text{Gau}_{\text{DS}} \ominus T \text{bit}_\alpha(1; t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)) \oplus E_t. \end{aligned} \tag{A.9}$$

Here, $\text{bit}_{\tilde{H}}(\cdot; t) : \mathcal{G}_{\text{DS}}(\mathbf{s}_t) \rightarrow \text{Herm}$ is the orbit map associated to the $\mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ action (4.6) on Herm . Its tangent map $T \text{bit}_{\tilde{H}}(1; t)$ maps $\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ into the subspace of $T_{\tilde{H}} \text{Herm}$ spanned by the vectors of the form $\delta \eta + \text{Ad } \tilde{H} \delta \eta^\dagger$ with $\delta \eta \in \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$. Similarly, $\text{bit}_\alpha(\cdot; t) : \mathcal{G}_{\text{DS}}(\mathbf{s}_t) \rightarrow \text{Gau}_{\text{DS}}$ is the orbit map associated to the $\mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ action (4.7) on Gau_{DS} . Its tangent map $T \text{bit}_\alpha(1; t)$ maps $\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ into the subspace of $T_\alpha \text{Gau}_{\text{DS}}$ spanned by the vectors $\delta \eta \in \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$. E_t is the subspace of $T_{\tilde{H}} \text{Herm} \oplus T_\alpha \text{Gau}_{\text{DS}}$ spanned by the vectors of the form $(\delta \eta + \text{Ad } \tilde{H} \delta \eta^\dagger) \oplus \delta \eta$ with $\delta \eta \in \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$, where a sign difference in the second component with respect to the vectors spanning $T \text{bit}_{(\tilde{H}, \alpha)}(1; t) \text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t)$ is to be noticed. Hence, for

any $\delta\tilde{H}\tilde{H}^{-1} \in T_{\tilde{H}}\text{Herm}$ and $\alpha^{-1}\delta\alpha \in T_{\alpha}\text{Gau}_{\text{DS}}$, one has the decompositions

$$\delta\tilde{H}\tilde{H}^{-1} = \delta\tilde{H}\tilde{H}^{-1}_{\perp} + \delta\eta + \text{Ad}\tilde{H}\delta\eta^{\dagger}, \quad (\text{A.10})$$

$$\alpha^{-1}\delta\alpha = \alpha^{-1}\delta\alpha_{\perp} + \delta\eta, \quad (\text{A.11})$$

where $\delta\eta \in \text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$, $\delta\tilde{H}\tilde{H}^{-1}_{\perp} \in T_{\tilde{H}}\text{Herm} \ominus T\text{bit}_{\tilde{H}}(1;t)\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$ and $\alpha^{-1}\delta\alpha_{\perp} \in T_{\alpha}\text{Gau}_{\text{DS}} \ominus T\text{bit}_{\alpha}(1;t)\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$. By substituting (A.10)–(A.11) into (A.7)–(A.8) and the result into (A.1), one finds

$$\begin{aligned} & \|\delta H(\tilde{H}, \alpha)H(\tilde{H}, \alpha)^{-1} \oplus \delta A^*(t, \alpha)\|_{h, H(\tilde{H}, \alpha)|(H(\tilde{H}, \alpha), A^*(t, \alpha))}^2 \\ &= \|\delta\tilde{H}\tilde{H}^{-1}_{\perp} + \alpha^{-1}\delta\alpha_{\perp} + \text{Ad}\tilde{H}(\alpha^{-1}\delta\alpha_{\perp})^{\dagger}\|_{h|\tilde{H}}^2 \\ &+ \|(\bar{\partial} - \text{ad}A^*(t))(\alpha^{-1}\delta\alpha_{\perp} + \alpha^{-1}\delta\alpha_{\perp}(t, \tilde{H}))\|_{\tilde{H}|A^*(t)}^2 \\ &+ \|p(t, \tilde{H})\delta_t A^*(t)\|_{\tilde{H}|A^*(t)}^2 + 2\|(\delta\eta + \text{Ad}\tilde{H}\delta\eta^{\dagger}) \oplus \delta\eta\|_{h, \tilde{H}|(\tilde{H}, 1)}^2. \end{aligned} \quad (\text{A.12})$$

Here, $\alpha^{-1}\delta\alpha_{\perp}(t, \tilde{H})$ is some element of $T_{\alpha}\text{Gau}_{\text{DS}} \ominus T\text{bit}_{\alpha}(1;t)\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$ depending on t and \tilde{H} whose explicit expression will not matter. In deducing (A.12), one exploits the fact that the Cartan Killing form tr_{ad} vanishes on \mathfrak{r} because of the nilpotence of \mathfrak{r} . One also uses the fact that $\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t) \subseteq \text{HECF}_{\text{DSS}_t}^0$ so that, for $\delta\eta \in \text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$, $(\bar{\partial} - \text{ad}A^*(t))\delta\eta = 0$. Using the jacobian relation (A.3), the normalization condition for the measures and (A.12), it is straightforward to obtain (A.4)–(A.6). \square

The jacobian $J(t, h, \tilde{H})$ is a positive (m, m) form on \mathcal{M}_{DS} . It does not depend on α , a consequence of Gau_{DS} gauge invariance. It can be shown that, for any $\eta \in \mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$, $J(t, h, {}^{\eta}\tilde{H}) = J(t, h, \tilde{H})$, i.e. $J(t, h, \tilde{H})$ is invariant under the action (4.6) of $\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$ on Herm . This is expected on general grounds as a consequence of the \mathcal{G}_{DS} symmetry of the parametrization (4.4)–(4.5).

Next, one has to define the Hilbert manifold structure of the isomorphic spaces $\text{Herm} \times \text{Gau}_{\text{DS}}$ and $((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \times \mathcal{G}_{\text{DS}}$.

$\text{Herm} \times \text{Gau}_{\text{DS}}$ has an obvious structure of real Hilbert manifold. For $(H, \omega) \in \text{Herm} \times \text{Gau}_{\text{DS}}$, $T_{(H, \omega)}\text{Herm} \times \text{Gau}_{\text{DS}} \cong T_H\text{Herm} \oplus T_{\omega}\text{Gau}_{\text{DS}}$ equipped with the Hilbert norm

$$\|\delta HH^{-1} \oplus \omega^{-1}\delta\omega\|_{h, H|(H, \omega)}^2 = \|\delta HH^{-1}\|_{h|H}^2 + \|\omega^{-1}\delta\omega\|_{h, H|\omega}^2. \quad (\text{A.13})$$

$((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \times \mathcal{G}_{\text{DS}}$ can also be given a structure of Hilbert manifold. For any $(\tilde{H}, \alpha, g) \in ((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \times \mathcal{G}_{\text{DS}}$, $T_{(\tilde{H}, \alpha, g)}((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \times \mathcal{G}_{\text{DS}} \cong ((T_{\tilde{H}}\text{Herm} \oplus T_{\alpha}\text{Gau}_{\text{DS}})/T\text{bit}_{(\tilde{H}, \alpha)}(1;t)\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \oplus T_g\mathcal{G}_{\text{DS}}$. One equips $T_{\tilde{H}}\text{Herm} \oplus T_{\alpha}\text{Gau}_{\text{DS}} \oplus T_g\mathcal{G}_{\text{DS}}$ with the Hilbert norm

$$\|\delta\tilde{H}\tilde{H}^{-1} \oplus \alpha^{-1}\delta\alpha \oplus \delta g\|_{h, \tilde{H}|(\tilde{H}, \alpha, g)}^2 = \|\delta\tilde{H}\tilde{H}^{-1}\|_{h|\tilde{H}}^2 + \|\alpha^{-1}\delta\alpha\|_{h, \tilde{H}|\alpha}^2 + \|\delta g\|_{g}^2. \quad (\text{A.14})$$

Then, one has the identification $T_{(\tilde{H}, \alpha, g)}((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \times \mathcal{G}_{\text{DS}} \cong ((T_{\tilde{H}}\text{Herm} \oplus T_{\alpha}\text{Gau}_{\text{DS}}) \ominus T\text{bit}_{(\tilde{H}, \alpha)}(1;t)\text{Lie}\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)) \oplus T_g\mathcal{G}_{\text{DS}}$. This above construction is independent up to unitary equivalence from the choice of the representative (\tilde{H}, α) of the corresponding equivalence class modulo the $\mathcal{G}_{\text{DS}}(\mathfrak{S}_t)$ action (4.6)–(4.7).

One has now to compute the jacobian $K(t, g, h, \tilde{H})$ of the map (4.8)–(4.9) relating the functional measures on $\text{Herm} \times \text{Gau}_{\text{DS}}$ and $((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_t)) \times \mathcal{G}_{\text{DS}}$. One has

$$\begin{aligned} & (D\delta H(\tilde{H}, g)H(\tilde{H}, g)^{-1})_{h|H(\tilde{H}, g)} \otimes (D\omega(\alpha, g)^{-1}\delta\omega(\alpha, g))_{h, H(\tilde{H}, g)|\omega(\alpha, g)} \\ &= K(t, g, h, \tilde{H}) \left[(D\delta\tilde{H}\tilde{H}^{-1})_{h|\tilde{H}} \otimes (D\alpha^{-1}\delta\alpha)_{h, \tilde{H}|\alpha} \right. \\ & \quad \left. \otimes (D\delta g)_{|g} \right] \Big|_{(T\text{bit}_{(\tilde{H}, \alpha)}(1;t)\text{Lie } \mathcal{G}_{\text{DS}}(\mathbf{s}_t))^\perp}. \end{aligned} \tag{A.15}$$

The expression obtained is

$$K(t, g, h, \tilde{H}) = 2^q \det Q(t, g, h, \tilde{H}), \tag{A.16}$$

where

$$Q(t, g, h, \tilde{H})_{i,j} = \langle \zeta(g; t)^{-1} \partial_{g^i} \zeta(g; t), \zeta(g; t)^{-1} \partial_{g^j} \zeta(g; t) \rangle_{h, \tilde{H}}, \quad i, j = 1, \dots, q \tag{A.17}$$

and the determinant is taken on the complex field.

Proof. The tangent map of the parametrization (4.8)–(4.9) is given by

$$\begin{aligned} \delta H(\tilde{H}, g)H(\tilde{H}, g)^{-1} &= \text{Ad } \zeta(g; t)(\delta\tilde{H}\tilde{H}^{-1} + \zeta(g; t)^{-1}\delta_g\zeta(g; t) \\ & \quad + \text{Ad } \tilde{H}(\zeta(g; t)^{-1}\delta_g\zeta(g; t))^\dagger), \end{aligned} \tag{A.18}$$

$$\omega(\alpha, g)^{-1}\delta\omega(\alpha, g) = \text{Ad } \zeta(g; t)(\alpha^{-1}\delta\alpha - \zeta(g; t)^{-1}\delta_g\zeta(g; t)). \tag{A.19}$$

By substituting (A.18)–(A.19) into (A.13), one obtains

$$\begin{aligned} & \|\delta H(\tilde{H}, g)H(\tilde{H}, g)^{-1} \oplus \omega(\alpha, g)^{-1}\delta\omega(\alpha, g)\|_{h, H(\tilde{H}, g)|(H(\tilde{H}, g), \omega(\alpha, g))}^2 \\ &= \|\delta\tilde{H}\tilde{H}^{-1}\|_{h|\tilde{H}}^2 + \|\alpha^{-1}\delta\alpha\|_{h, \tilde{H}|\alpha}^2 + 2\|\zeta(g; t)^{-1}\delta_g\zeta(g; t)\|_{h, \tilde{H}|\zeta(g; t)}^2. \end{aligned} \tag{A.20}$$

Using the jacobian relation (A.15), the normalization condition of the measures and (A.20), it is straightforward to obtain (A.16)–(A.17). \square

The jacobian $K(t, g, h, \tilde{H})$ is a (q, q) form on \mathcal{G}_{DS} . Its independence from α is a consequence of the left invariance of the measure on $T_\alpha\text{Gau}_{\text{DS}}$. From (A.17), it is apparent that $L_f^*K(t, g, h, \tilde{H}) = K(t, g, h, \tilde{H})$ for any $f \in \mathcal{G}_{\text{DS}}$, i.e. $K(t, g, h, \tilde{H})$ is left invariant. Under the right action of \mathcal{G}_{DS} , one has instead $R_f^*K(t, g, h, \tilde{H}) = K(t, g, h, \zeta(f; t)^*\tilde{H})$.

Now, all elements required for the implementation of the gauge fixing procedure are available. Consider a Gau_{DS} -invariant functional $\Theta(h, H, A^*)$. Hence, for any $\alpha \in \text{Gau}_{\text{DS}}$, $\Theta(h, \alpha^*H, \alpha^*A^*) = \Theta(h, H, A^*)$. The functional integral

$$\mathcal{I}_\Theta(h) = \int_{\text{Herm} \times \text{SHol}_{\text{DS}}} (DH)_{h|H} \otimes (DA^*)_{H|A^*} \Theta(h, H, A^*) \tag{A.21}$$

is thus divergent because of the Gau_{DS} invariance of the integrand. The problem to solve next is the factorization of the divergent gauge volume.

On account of the isomorphism (4.4)–(4.5) of $\text{Herm} \times \text{SHol}_{\text{DS}}$ and $\mathcal{M}_{\text{DS}} \times (\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{s}_t)$, the jacobian relation (A.3) and the Gau_{DS} invariance

of $\Theta(h, \tilde{H}, A^*)$, one has

$$\begin{aligned} \mathcal{I}_\Theta(h) &= \int_{\mathcal{M}_{\text{DS}}} (Dt)|_t \int_{(\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{S}_t)} (D\tilde{H})_{h|\tilde{H}} \otimes (D\alpha)_{h,\tilde{H}|\alpha} \\ &\quad \times J(t, h, \tilde{H})\Theta(h, \tilde{H}, A^*(t)). \end{aligned} \tag{A.22}$$

Because of the quotient by $\mathcal{G}_{\text{DS}}(\mathbf{S}_t)$, it is not possible to factor out the gauge volume yet. This requires a few extra steps.

Define

$$v(t, h, H) = \int_{\mathcal{G}_{\text{DS}}} (Dg)|_g K(t, g, h, H). \tag{A.23}$$

$v(t, h, H)$ is actually divergent since \mathcal{G}_{DS} is a non-compact group. However, formally, by the form of the right \mathcal{G}_{DS} action on $K(t, g, h, \tilde{H})$, $v(t, h, {}^\eta H) = v(t, h, H)$ for any $\eta \in \mathcal{G}_{\text{DS}}(\mathbf{S}_t)$, i.e. $v(t, h, H)$ is $\mathcal{G}_{\text{DS}}(\mathbf{S}_t)$ invariant. The infinite volume of the gauge group is

$$V(h, H) = \int_{\text{Gau}_{\text{DS}}} (D\omega)_{h,H|\omega}. \tag{A.24}$$

Now, from the isomorphism (4.8)–(4.9) of $\text{Herm} \times \text{Gau}_{\text{DS}}$ and $((\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{S}_t)) \times \mathcal{G}_{\text{DS}}$, using the jacobian relation (A.15) and the $\mathcal{G}_{\text{DS}}(\mathbf{S}_t)$ invariance of $J(t, h, \tilde{H})$, the Gau_{DS} invariance of $\Theta(h, \tilde{H}, A^*)$, (A.23) and the $\mathcal{G}_{\text{DS}}(\mathbf{S}_t)$ invariance of $v(t, h, H)$, one has

$$\begin{aligned} &\int_{\text{Herm}} (DH)_{h|H} \frac{V(h, H)}{v(t, h, H)} J(t, h, H)\Theta(h, H, A^*(t)) \\ &= \int_{\text{Herm} \times \text{Gau}_{\text{DS}}} (DH)_{h|H} \otimes (D\omega)_{h,H|\omega} \frac{1}{v(t, h, H)} J(t, h, H)\Theta(h, H, A^*(t)) \\ &= \int_{(\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{S}_t)} (D\tilde{H})_{h|\tilde{H}} \otimes (D\alpha)_{h,\tilde{H}|\alpha} \int_{\mathcal{G}_{\text{DS}}} (Dg)|_g \\ &\quad \times K(t, g, h, \tilde{H}) \frac{1}{v(t, h, \tilde{H})} J(t, h, \tilde{H})\Theta(h, \tilde{H}, A^*(t)) \\ &= \int_{(\text{Herm} \times \text{Gau}_{\text{DS}})/\mathcal{G}_{\text{DS}}(\mathbf{S}_t)} (D\tilde{H})_{h|\tilde{H}} \otimes (D\alpha)_{h,\tilde{H}|\alpha} J(t, h, \tilde{H})\Theta(h, \tilde{H}, A^*(t)). \end{aligned} \tag{A.25}$$

Combining (A.22) and (A.25), one has

$$\mathcal{I}_\Theta(h) = \int_{\mathcal{M}_{\text{DS}}} (Dt)|_t \int_{\text{Herm}} (DH)_{h|H} \frac{V(h, H)}{v(t, h, H)} J(t, h, H)\Theta(h, H, A^*(t)). \tag{A.26}$$

Gauge fixing is now easy. One simply deletes the infinite gauge volume $V(h, H)$ in the above expression. The gauge fixed functional integral is then

$$\mathcal{I}_\Theta^{\text{gf}}(h) = \int_{\mathcal{M}_{\text{DS}}} (Dt)|_t \int_{\text{Herm}} (DH)_{h|H} \frac{1}{v(t, h, H)} J(t, h, H)\Theta(h, H, A^*(t)). \tag{A.27}$$

$v(t, h, H)$ depends on H and this is inconvenient. One can separate the H dependence from the group volume by the following method. Let $\nu(g)$ be a left invariant positive (q, q) form on \mathcal{G}_{DS} . So $L_f^* \nu(g) = \nu(g)$, for any $f \in \mathcal{G}_{\text{DS}}$. Using $\nu(g)$, one

can define the group volume $v_v = \int_{\mathcal{G}_{\text{DS}}} (Dg)|_g v(g)$ of \mathcal{G}_{DS} . From the left \mathcal{G}_{DS} invariance of $K(t, g, h, \tilde{H})$, (A.23) and the left \mathcal{G}_{DS} invariance of v , it is easy to show the formal relation

$$v(t, h, H) = v_v v(1)^{-1} K(t, 1, h, H). \quad (\text{A.28})$$

Using (A.28), (A.27) can be cast

$$\mathcal{I}_{\Theta}^{\text{gf}}(h) = \int_{\mathcal{M}_{\text{DS}}} (Dt)|_t \frac{v(1)}{v_v} \int_{\text{Herm}} (DH)_{h|H} \frac{J(t, h, H)}{K(t, 1, h, H)} \Theta(h, H, A^*(t)). \quad (\text{A.29})$$

This is the final form of the expression of $\mathcal{I}_{\Theta}^{\text{gf}}(h)$. Now, (4.20) follows from (A.29) by a straightforward calculation.

Using the isomorphisms $(\text{Lie Gau}_{\text{DS}})^{\vee} \cong \text{ECF}_{\text{DS}}^{\vee, 1, 0}$ and $\text{Lie Gau}_{\text{DS}} \cong \text{ECF}_{\text{DS}}^{0, 0}$, one can define real Hilbert structures on $(\text{Lie Gau}_{\text{DS}})^{\vee}$ and $\text{Lie Gau}_{\text{DS}}$. One simply views $(\text{Lie Gau}_{\text{DS}})^{\vee}$ and $\text{Lie Gau}_{\text{DS}}$ as the real Hilbert manifolds $\text{ECF}_{\text{DS}}^{\vee, 1, 0r}$ and $\text{ECF}_{\text{DS}}^{0, 0r}$ with the Hilbert structure $\langle \cdot, \cdot \rangle_{\text{DS}h, H}^{\vee r}$ and $\langle \cdot, \cdot \rangle_{\text{DS}h, H}^r$, respectively. This yields the ghost functional measures appearing in (4.22).

Appendix B

In this appendix, I shall provide some detail about the derivation of (6.7)–(6.8). To lighten the notation, I shall not indicate the \mathfrak{s} dependence of the various objects. I also identify $\varphi^{-1}(n)$ and $\varphi^{-1}(n)H_{\text{cl}}(n)^{-1}$.

Let $n \in \mathcal{N}$. For any $\Phi \in \varphi^{-1}(n)$, the tangent space $T_{\Phi}\varphi^{-1}(n)$ is the subspace of $\text{ECF}^{0, 0r}$ spanned by the $H_{\text{cl}}(n)$ -hermitian elements $\exp(-\Phi/2)\delta \exp \Phi \exp(-\Phi/2)$ and is equipped with the Hilbert structure $\frac{1}{2}\langle \cdot, \cdot \rangle_{h, H_{\text{cl}}(n)}^r$. Hence, one has

$$\|\exp(-\Phi/2)\delta \exp \Phi \exp(-\Phi/2)\|_{h, H_{\text{cl}}(n)|\Phi}^2 = \|\exp(-\Phi/2)\delta \exp \Phi \exp(-\Phi/2)\|_{h, H_{\text{cl}}(n)}^2,$$

where in the right-hand side the norm is that of $\text{ECF}^{0, 0}$. In this way, $\varphi^{-1}(n)$ becomes a real Hilbert manifold.

Consider now \mathcal{N} . For any $n \in \mathcal{N}$, $T_n\mathcal{N}$ is just \mathbb{R}^r , where $r = \dim \mathcal{N}$, with the standard euclidean inner product $\langle \cdot, \cdot \rangle$. So, for $\delta n \in T_n\mathcal{N}$, $\|\delta n\|_n^2 = |\delta n|^2$.

$\mathcal{N} \times \varphi^{-1}(\cdot)$ can be given the structure of Hilbert manifold as follows. For any $(n, \Phi) \in \mathcal{N} \times \varphi^{-1}(\cdot)$, $T_{(n, \Phi)}\mathcal{N} \times \varphi^{-1}(\cdot) = T_n\mathcal{N} \oplus T_{\Phi}\varphi^{-1}(n)$. The tangent vectors are of the form $\delta n \oplus \exp(-\Phi/2)\delta_n \exp \Phi \exp(-\Phi/2)$, where the notation δ_n means variation at fixed n . The norm is given by

$$\begin{aligned} & \|\delta n \oplus \exp(-\Phi/2)\delta_n \exp \Phi \exp(-\Phi/2)\|_{h, H_{\text{cl}}(n)|(n, \Phi)}^2 \\ &= \|\delta n\|_n^2 + \|\exp(-\Phi/2)\delta_n \exp \Phi \exp(-\Phi/2)\|_{h, H_{\text{cl}}(n)|\Phi}^2. \end{aligned} \quad (\text{B.1})$$

The jacobian $M(h; n)$ of the map (6.6) relating the measures on Herm and $\mathcal{N} \times \varphi^{-1}(\cdot)$ is defined by

$$\begin{aligned} & (D\delta H(\Phi; n)H(\Phi; n)^{-1})_{h|H(\Phi; n)} \\ &= M(h; n) \left[(D\delta n)|_n \otimes (D \exp(-\Phi/2)\delta_n \exp \Phi \exp(-\Phi/2))_{h, H_{\text{cl}}(n)|\Phi} \right]. \end{aligned} \quad (\text{B.2})$$

By explicit calculations one finds

$$M(h; n) = [\det J(h; n)]^{\frac{1}{2}}, \quad (\text{B.3})$$

where $J(h; n)$ is given by (6.9).

Proof. The tangent map of the parametrization (6.6) is given by

$$\begin{aligned} \delta H(\Phi; n)H(\Phi; n)^{-1} &= \exp(\text{ad } \Phi/2)[\exp(-\Phi/2)\delta'_n \exp \Phi \exp(-\Phi/2) \\ &\quad + \delta' H_{\text{cl}}(n)H_{\text{cl}}(n)^{-1}]. \end{aligned} \quad (\text{B.4})$$

The two terms in the right-hand side are the components of $\delta H(\Phi; n)H(\Phi; n)^{-1}$ on $T_{H(\Phi; n)}\varphi^{-1}(n)$ and $\mathcal{H}_{H(\Phi; n)}(n)$, respectively. The notation δ' is used instead of δ since the decomposition does not follow by a straightforward variation of the relation (6.6). Then, by the orthogonality in $T_{H(\Phi; n)}\text{Herm}$ of the two terms in the right-hand side of (B.4), one has

$$\begin{aligned} \|\delta H(\Phi; n)H(\Phi; n)^{-1}\|_{h|H(\Phi; n)}^2 &= \|\exp(-\Phi/2)\delta'_n \exp \Phi \exp(-\Phi/2)\|_{h, H_{\text{cl}}(n)|\Phi}^2 \\ &\quad + \|\delta' H_{\text{cl}}(n)H_{\text{cl}}(n)^{-1}\|_{h|H_{\text{cl}}(n)}^2. \end{aligned} \quad (\text{B.5})$$

Using the jacobian relation (B.2), the normalization condition of the measures and (B.5), it is straightforward to obtain (B.3). \square

From (B.2)–(B.3), (6.7)–(6.8) follows readily.

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