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Spectral Decomposition of Path Space in Solvable Lattice Model

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Abstract: We give the *spectral decomposition* of the path space of the $U_q(\widehat{sl}_2)$ vertex model with respect to the local energy functions. The result suggests the hidden Yangian module structure on the $\widehat{sl}_2$ level l integrable modules, which is consistent with the earlier work [1] in the level one case. Also we prove the fermionic character formula of the $\widehat{sl}_2$ level l integrable representations in consequence.

1. Introduction

In the last decade of investigation, various close relations between the solvable lattice model and the conformal field theory have been revealed (for example, [2–5]). The aim of this article is to point out a new interesting relation between the spectrum in the solvable lattice model and the hidden quantum symmetry in the conformal field theory.

Consider the higher spin vertex model associated with the $l+1$ irreducible representation of $U_q(\widehat{sl}_2)$ ([6, 7]). It is well-known that the characters of the $\widehat{sl}_2$ or $U_q(\widehat{sl}_2)$ level l integrable representations $\mathscr{L}(k)$ can be calculated by using its *path space* $\mathscr{P}(k)$ ([2, 8]). The energy of a path $\vec{p}$ is given by the sum of a sequence of numbers $h(\vec{p}) = (h_1(\vec{p}), h_2(\vec{p}), \dots)$ minus the ground state energy which depends on the corresponding boundary condition. Here $h_i(\vec{p})$ is the i^{th} local energy determined from the $i+1^{\text{th}}$ component of $\vec{p}$ and its nearest neighbors by the local energy function. We propose the fact that the local energy functions not only play a combinatorial role, but also can be regarded as the $q \to 0$ limit of the local integrals of motion which commutes with the corner transfer matrix.

At $q = 0$, the energy of a path $\vec{p}$ is essentially the eigenvalue of the logarithm of the corner transfer matrix on the one dimensional configuration space $\sum_{\vec{p}\in\mathscr{P}(k)} \mathbf{C}\vec{p}$. Hence $\vec{p}$ itself is the "eigenvector" of the corner transfer matrix, and at the same time it is a simultaneous "eigenvector" of the mutually commuting infinitely many "local operators" h_i at $q = 0$.

In this paper we studied the *spectral decomposition* of the path space with respect to the local energy functions h_i. That is, we decomposed the path space

$\mathscr{P}(k)$ as

$$\mathscr{P}(k) = \coprod \mathscr{P}(k)_{\vec{h}} ,$$

where $\mathscr{P}(k)_{\vec{h}}$ denotes the "eigenspace" with the spectrum $\vec{h}$. We found that the spectrum can be parameterized by the restricted paths and the Young diagrams. Moreover, it turns out that the spectrum has a rich degeneracy structure. Determining the explicit form of the sl_2 character of $\mathscr{P}(k)_{\vec{h}}$, we see that it equals that of an irreducible Yangian module.

In the level one case, the Yangian structure on the $\widehat{sl}_2$ integrable modules has been described in [1]. Consistent with their work, what our result suggests is that even in the higher level cases, there is a canonical Yangian structure on the integrable module $\mathscr{L}(k)$ such that

$$\mathscr{L}(k) \simeq \bigoplus_{\vec{h}} W_{\vec{h}} ,$$

where each $W_{\vec{h}}$ is an irreducible finite dimensional Yangian module with $\mathrm{ch}\, W_{\vec{h}} = \mathrm{ch}\, \mathscr{P}_{\vec{h}}$. (A part of the Yangian structure is written in [9].)

We believe that this correspondence between the spectral decomposition of the solvable lattice model and the "Yangian multiplets" in conformal field theory is not a mere coincidence, but will play the important role in investigating the hidden quantum symmetry in more general conformal field theories.

In addition, as a byproduct, the spectral decomposition leads to the fermionic character formula conjectured in [9] and proved independently in [10].

This paper is organized as follows. In Sect. 2 we review the basic facts we use in this article about the vertex model. In Sect. 3 we parameterize the spectrum of the path space with the restricted paths and the Young diagrams. In Sect. 4 the degeneracy of the spectrum is calculated. In Sect. 5 we derive several character formulas. In Sect. 6 we discuss the connection of our results in the vertex model with the hidden Yangian symmetry in the conformal field theory.

2. Local Energy Function and Character Formula

In this section we review basic facts about the vertex model. See [2, 6, 7, 8, 11] for details. We consider the higher spin vertex model of $U_q(\widehat{sl}_2)$. Throughout this paper we fix an integer $l \in \mathbf{N}$. Let $S = \{l, l-2, \ldots, -l\}$ and let V be the $l+1$ dimensional irreducible representation of $U_q(sl_2)$ with the standard basis $\{v_s \mid \mathrm{weight}(v_s) = s,\ s \in S\}$. We can extend this to the homogeneous evaluation representation $V(z)$ of $U_q(\widehat{sl}_2)$ with a complex parameter $z \neq 0$. The R-matrix $\check{R}(w)$ is the $U_q(\widehat{sl}_2)$-intertwiner $V(z_1) \otimes V(z_2) \to V(z_2) \otimes V(z_1)$ [6], where $w = \frac{z_1}{z_2}$. We put

$$\check{R}(w) = \sum_{s_1, s_2, s_1', s_2' \in S_l} \check{R}(w)_{s_1 s_2}^{s_1', s_2'} E_{s_1' s_1} \otimes E_{s_2' s_2} , \tag{2.1}$$

where $E_{ss'}$ is an element in $\mathrm{End}_{\mathbf{C}}(V)$ such that $E_{ss'} \cdot v_{s''} = \delta_{s', s''} v_s$ for $s, s', s'' \in S$.

The R-matrix above satisfies several properties, especially the diagonal nature at $q = 0$;

$$\lim_{q \to 0} \check{R}(w) = \sum_{s_1, s_2 \in S_l} w^{-H(s_1, s_2)} E_{s_1 s_1} \otimes E_{s_2 s_2} \tag{2.2}$$

up to a scalar multiplication, where H is a function

$$H : S \times S \to \{0, 1, \ldots, l\}$$

such that

$$H(s,s') = \begin{cases} \frac{1}{2}(s'+l), & \text{if } s+s' \geqq 0\,, \\ \frac{1}{2}(l-s), & \text{if } s+s' < 0\,. \end{cases} \tag{2.3}$$

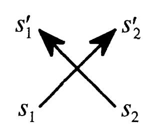
We call this function the *H function* or the *local energy function*. By setting

$$H(s,s') = H_{ss'}\,,$$

we can write the above definition in a matrix form;

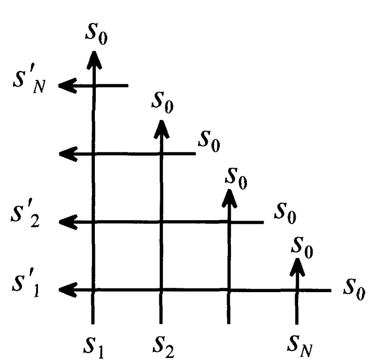
$$(H_{ss'}) = \begin{array}{c} \\ l \\ l-2 \\ \vdots \\ \\ -l+2 \\ -l \end{array} \begin{array}{c} \begin{array}{ccccc} l & l-2 & \ldots & -l+2 & -l \end{array} \\ \begin{pmatrix} l & l-1 & \ldots & 1 & 0 \\ l & l-1 & \ldots & 1 & 1 \\ \ldots & & & & \\ \ldots & & & \vdots & \vdots \\ l & l-1 & \ldots & l-1 & l-1 \\ l & l & \ldots & l & l \end{pmatrix} \end{array}. \tag{2.4}$$

Each matrix element $\check{R}(w)_{s_1,s_2}^{s_1',s_2'}$ defines a Boltzmann weight of the configuration (s_1, s_2, s_1', s_2') round a vertex in a planner square lattice, where s_1, s_2, s_1', and s_2' take values in S. It is often written as

$\check{R}(w)_{s_1,s_2}^{s_1',s_2'} =$  . (2.5)

The corner transfer matrix method reduces the local state probability to the one-dimensional configuration sum [7]. Fix a boundary condition $s_0 \in S$ and let $\Sigma_N = \{\vec{s} = (s_1, s_2, \ldots, s_N) \mid s_i \in S\}$. The corner transfer matrix $A_N^{(s_0)}(w)$ is a matrix whose elements are defined as

$$A_N^{(s_0)}(w)_{\vec{s}}^{\vec{s}'} = \sum_{\text{configuration in the interior edges}} \prod \check{R}(w)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}$$

$=$ 

for $\vec{s} = (s_1, \ldots, s_N), \vec{s}' = (s_1', \ldots, s_N') \in \Sigma_N$.

Consider the limit $q \longrightarrow 0$. By the diagonal nature of the R-matrix, the corner transfer matrix is also diagonalized at $q = 0$, namely,

$$A_N^{(s_0)}(w)_{\vec{s}}^{\vec{s}'} = \delta_{\vec{s}}^{\vec{s}'} w^{-(H(s_1,s_2)+2H(s_2,s_3)+\quad +NH(s_N,s_0))} \tag{2.6}$$

with the normalization of the R-matrix as in (2.2). Hence the value $H(s_1, s_2) + 2H(s_2, s_3) + \cdots + NH(s_N, s_0)$ essentially contributes to the energy of a configuration $\vec{s} \in \Sigma_N$ with this boundary condition s_0.

Let us pass to the case when the lattice size N is infinity.

For $k = 0, \ldots, l$, let $\vec{s}^{(k)}$ denote the k^{th} ground state $(s_1^{(k)}, s_2^{(k)}, \ldots) = (l - 2k, -(l - 2k), l - 2k, -(l - 2k), \ldots)$. Let

$$\Sigma(k) = \{\vec{s} = (s_1, s_2, \ldots) \mid s_i \in S \text{ for all } i,\ \vec{s} \approx \vec{s}^{(k)}\}\,, \tag{2.7}$$

where $s \approx \vec{s}^{(k)}$ denotes the condition that $s_i = s_i^{(k)}$ except for finitely many i's. We call $\Sigma = \bigsqcup_{k=0}^{l} \Sigma(k)$ the space of the spin configurations of the vertex model associated to the $l + 1$ dimensional representation of $U_q(\widehat{sl}_2)$.

Equivalently, we can define the configuration space above using a path walking on the sl_2-weight lattice instead of a spin configuration. Let

$$\mathscr{P}(k) = \{\vec{p} = (p_1, p_2, \ldots) \mid p_i - p_{i+1} \in S \text{ for all } i,\ \vec{p} \approx \vec{p}^{(k)}\}\,, \tag{2.8}$$

where $\vec{p}^{(k)} = (p_1^{(k)}, p_2^{(k)}, \ldots) = (k, l - k, k, l - k, \ldots)$ denotes the ground state path. Then the one-to-one correspondence

$$\begin{gathered} \mathscr{P}(k) \simeq \Sigma(k)\,, \\ (p_1, p_2, \ldots) \leftrightarrow (p_2 - p_1, p_3 - p_2, \ldots) \end{gathered} \tag{2.9}$$

preserves the ground state for each k. We call $\mathscr{P} = \bigsqcup_{k=0}^{l} \mathscr{P}(k)$ the path space of the vertex model associated to the $l + 1$ dimensional irreducible representation of $U_q(\widehat{sl}_2)$. We identify $\Sigma(k)$ with $\mathscr{P}(k)$ hereafter[1].

Define a map

$$\begin{gathered} h : \mathscr{P}(k) \longrightarrow \{0, 1, \ldots, l\}^{\infty} \\ \vec{p} \mapsto h(\vec{p}) = (h_1(\vec{p}), h_2(\vec{p}), \ldots)\,, \end{gathered}$$

by setting

$$h_i(\vec{p}) = H(p_{i+1} - p_i, p_{i+2} - p_{i+1})\,, \tag{2.10}$$

where the function H is defined in (2.3). The value $h_i(\vec{p})$ is called the i^{th} local energy of $\vec{p} \in \mathscr{P}(k)$. Let $\vec{h}^{(k)} = h(\vec{p}^{(k)}) = (k, l - k, k, l - k, \ldots)$.

Now put the (total) energy of a path $\vec{p} \in \mathscr{P}(k)$ as

$$E(\vec{p}) = \sum_{i=1}^{\infty} i(h_i(\vec{p}) - h_i^{(k)}) \tag{2.11}$$

[1] Sometimes a spin configuration $\vec{s}$ is also called a path in the literature through this correspondence

(see (2.6)). Also define the sl_2 weight of a path $\vec{p} \in \mathscr{P}$ as

$$\mathrm{Wt}(\vec{p}) = p_1 \,. \tag{2.12}$$

The following theorem is proved in [2]. (See also [8] for its relation with the crystal basis.)

Theorem 2.1. *For $k = 0, 1, \dots l$, let $\Delta(k) = \frac{k(k+2)}{4(l+2)}$ and let $\mathrm{ch}\,\mathscr{P}(k) = q^{\Delta(k)} \sum_{\vec{p} \in \mathscr{P}(k)} q^{E(\vec{p})} z^{\mathrm{Wt}(\vec{p})}$. Then,*

$$\mathrm{ch}\,\mathscr{P}(k) = \mathrm{ch}_{\mathscr{L}(k)}(q, z)\,, \tag{2.13}$$

where $\mathrm{ch}_{\mathscr{L}(k)}(q, z)$ is the character of the $\widehat{sl}_2$ or $U_q(\widehat{sl}_2)$ level l integrable module of the highest weight $(l-k)\Lambda_0 + k\Lambda_1$.

The image of $\mathscr{P}(k)$ by the map h is denoted by $\mathrm{Sp}(k)$. Then,

Proposition 2.2. *For $k = 0, 1, \dots l$,*

$$\begin{aligned}\mathrm{Sp}(k) = \{\vec{h} = (h_1, h_2, \dots,) \mid\ & (1)\ \text{(Nearest neighbour condition)}\quad h_i + h_{i+1} \geqq l \\ & (2)\ \text{(Boundary condition)}\ \vec{h} \approx \vec{h}^{(k)}\}\,.\end{aligned}$$

Proof. Let $\vec{p} \in \mathscr{P}(k)$. Then $h(\vec{p})$ trivially satisfies the condition (2) by definition. Let us assume that $h_i(\vec{p}) = j$ for some $j \in \{0, \dots, l\}$. Then by the definition of H (formula (2.4)), $p_{i+2} - p_{i+1} \leqq -l + 2j$, which means $h_{i+1}(\vec{p}) \geqq l - j$. Hence $h(\mathscr{P}(k)) \subset \mathrm{Sp}(k)$. The $h(\mathscr{P}(k)) \supset \mathrm{Sp}(k)$ part is similar to the oncoming proof of Theorem 4.2: Put $z = 1$ in it. $\square$

Remark 2.3. If $(h_{i-1}(\vec{p}), h_i(\vec{p}), h_{i+1}(\vec{p})) = (j, l-j, j)$ for some $j \in \{0, 1, \dots, l\}$, then $p_i = p_{i+1} + l - 2j$ and $p_{i+1} = p_{i+2} + 2j - l$, i.e., p_i and p_{i+1} are uniquely determined by p_{i+2}.

We call $\mathrm{Sp} = \bigsqcup_{k=0}^{l} \mathrm{Sp}(k)$ the spectrum of the path space $\mathscr{P}$ with respect to the local energy functions h_i.

Main Problem. *Decompose the path space $\mathscr{P}$ by its spectrum with respect to the local energy functions h_i.*

We carry out the *spectral decomposition* of the path space in the next sections.

3. Decoding Map

In this section we fix $k \in \{0, \dots, l\}$. By the nearest neighbor condition in Proposition 2.2, we can divide each $\vec{h} \in \mathrm{Sp}(k)$ into segments

$$\begin{aligned}\vec{h} &= (g_1 \mid g_2 \mid \cdots \mid g_m \mid g_\infty)\,, \\ g_i &= (h_{\gamma_i}, h_{\gamma_i+1}, \dots, h_{\gamma_{i+1}-1}),\ \ i = 1, \dots, m\,,\end{aligned}$$

$$\begin{aligned} g_\infty &= (h_{\gamma_{m+1}}, h_{\gamma_{m+2}}, \ldots)\,, \\ \gamma_1 &= 1 < \gamma_2 < \gamma_3 < \cdots < \gamma_{m+1} = M+1\,, \end{aligned} \tag{3.1}$$

so that in each segment the sum of two adjacent elements h_i, h_{i+1} is always l, but any sum of the adjacent two lying across different segments is greater than l. We call those segment g_i's elementary blocks of $\vec{h}$. Let $l(g)$ denote the number of the components in an elementary block g when it is finite. Each g_i has the form as follows:

$$g_i = \begin{cases} ([[h_{\gamma_i}]]^{\frac{l(g_i)}{2}}) & \text{if } l(g_i) \text{ is even} \\ (h_{\gamma_i}, [[l - h_{\gamma_i}]]^{\frac{l(g_i)-1}{2}}) & \text{if } l(g_i) \text{ is odd} \end{cases} \quad \text{for } i = 1, \ldots, m\,, \tag{3.2}$$

where $[[i]]^a$ denotes

$$\overbrace{i, l-i, i, l-i, \ldots, i, l-i.}^{a \text{ pairs}} \tag{3.3}$$

And let us write the last block g_∞ as

$$g_\infty = (h_{\gamma_{m+1}}, [[l - h_{\gamma_{m+1}}]]^\infty) = \begin{cases} (k, l-k, k, l-k, \ldots) & \text{if } M \text{ is even} \\ (l-k, k, l-k, k, \ldots) & \text{if } M \text{ is odd} \end{cases} \tag{3.4}$$

for later convenience. Then, supposing there are J elementary blocks of odd length, we can now rewrite the given $\vec{h} \in \mathrm{Sp}(k)$ in the following form:

$$\begin{aligned} \vec{h} = (&[[m_{11}]]^{b_{11}}, [[m_{12}]]^{b_{12}}, \ldots\ldots, [[m_{1n_1}]]^{b_{1n_1}}, l_1, [[l-l_1]]^{c_1}, \\ &[[m_{21}]]^{b_{21}}, [[m_{22}]]^{b_{22}}, \ldots\ldots, [[m_{2n_2}]]^{b_{2n_2}}, l_2, [[l-l_2]]^{c_2}, \\ &[[m_{31}]]^{b_{31}}, [[m_{32}]]^{b_{32}}, \ldots\ldots, [[m_{3n_3}]]^{b_{3n_3}}, l_3, [[l-l_3]]^{c_3}, \\ &\ldots\ldots \qquad\qquad\qquad\qquad \ldots, \\ &[[m_{J1}]]^{b_{J1}}, [[m_{J2}]]^{b_{J2}}, \ldots\ldots, [[m_{Jn_J}]]^{b_{Jn_J}}, l_J, [[l-l_J]]^{c_J}, \\ &[[m_{J+1\,1}]]^{b_{J+1\,1}}, [[m_{J+1\,2}]]^{b_{J+1\,2}}, \ldots, [[m_{J+1\,n_{J+1}}]]^{b_{J+1\,n_{J+1}}}, l_{J+1}, [[l-l_{J+1}]]^\infty)\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \{m_{ij}\} &: \text{the initial elements of the blocks of even length}\,, \\ \{l_i \mid 1 \leqq i \leqq J\} &: \text{the initial elements of the blocks of odd length}\,, \\ l_{J+1} = h_{\gamma_{m+1}} &: \text{the initial elements of the last block } g_\infty\,, \end{aligned} \tag{3.6}$$

and $\{b_{ij}\}$(resp. $\{c_i\}$) are some positive (resp. non-negative) integers determined by the lengths of the corresponding blocks. Notice that $J \equiv M \bmod 2$.

Example 3.1. $(l = 3, k = 1)$

$$\text{(a) } \vec{h} = (1,2,1,2 \mid 2,1 \mid 3 \mid 0,3 \mid 2,1,2 \mid 3,0,3 \mid 1,2 \mid 3,0,3 \mid 1,2,1,2,1,2,\ldots,)$$
$$= [[1]]^2, [[2]]^1, 3, [[0]]^1, 2, [[1]]^1, 3, [[0]]^1, [[1]]^1, 3, [[0]]^1, 1, [[2]]^\infty ,$$

$$\text{(b) } \vec{h} = (0,3 \mid 1,2,1 \mid 3,0,3 \mid 1,2,1,2,1,2 \mid 2,1,2,1 \mid 3,0,3 \mid 2,1,2,1,2,1,2,1,\ldots,)$$
$$= [[0]]^1, 1, [[2]]^1, 3, [[0]]^1, [[1]]^3, [[2]]^2, 3, [[0]]^1, 2, [[1]]^\infty .$$

Lemma 3.2. *For a given* $\vec{h} \in \mathrm{Sp}(k)$, *consider the sequence of the initial elements of elementary blocks*

$$(m_{11}, m_{12}, \ldots, m_{1m_1}, l_1, m_{21}, \ldots, m_{2n_2}, l_2, m_{31}, \ldots, m_{3n_3}, l_3, \ldots, m_{J+1\, n_{J+1}}, l_{J+1}) \tag{3.7}$$

in the notation of (3.5) *and* (3.6). *Then,*

$$\begin{aligned} 0 \leqq m_{11} < m_{12} < \cdots < m_{1n_1} < l_1 \,, \\ l - l_{i-1} < m_{i1} < m_{i2} < \cdots < m_{in_i} < l_i, \quad \text{for } i = 2, \ldots, J+1 \,. \end{aligned} \tag{3.8}$$

Proof. The statement directly follows from the fact that the initials of the adjacent elementary blocks $g_i = (h_{\gamma_i}, \,)$ and $g_{i+1} = (h_{\gamma_{i+1}}, \;)$ satisfy

$$\begin{aligned} h_{\gamma_i} < h_{\gamma_{i+1}} \quad \text{if } l(g_i) \text{ is even} \,, \\ l - h_{\gamma_i} < h_{\gamma_{i+1}} \quad \text{if } l(g_i) \text{ is odd} \end{aligned} \tag{3.9}$$

by Eq. (3.2). $\square$

For the given $\vec{h} \in \mathrm{Sp}(k)$, let

$$\vec{h}^\sharp = (l_1, \ldots, l_J, l_{J+1}) \tag{3.10}$$

be the numbers defined in (3.6). From the lemma above, we see that a sequence of the form

$$(l_i), [[l - l_i]]^{d_1}, [[l - l_i + 1]]^{d_2}, [[l - l_i + 2]]^{d_3}, \ldots, [[l_{i+1} - 1]]^{d_{l_{i+1} - l + l_i}}, (l_{i+1}) \tag{3.11}$$

with some $d_i \in \mathbf{Z}_{\geqq 0}$ is inserted between two initials l_i, l_{i+1} in $\vec{h}$. Set

$$\begin{aligned} N &= N(\vec{h}) = \sum_{j=0}^{J} (l_{j+1} - l + l_j) \,, \\ t_i &= t_i(\vec{h}) = \sum_{j=0}^{i-1} (l_{j+1} - l + l_j) \quad \text{for } i = 1, \ldots, J \,, \end{aligned} \tag{3.12}$$

where $l_0 = l$, and $t_1 < t_2 < \cdots < t_J < N$ by the above lemma. Then we can find a sequence

$$a(\vec{h}) = (a_1, \ldots, a_N) \in (\mathbf{Z}_{\geqq 0})^N \tag{3.13}$$

so that

$$
\begin{aligned}
\vec{h} = (&[[0]]^{a_1}, [[1]]^{a_2}, \ldots, [[l_1 - 1]]^{a_{t_1}}, l_1, \\
&[[l - l_1]]^{a_{t_1+1}}, [[l - l_1 + 1]]^{a_{t_1+2}}, \ldots, [[l_2 - 1]]^{a_{t_2}}, l_2, \\
&\cdots \qquad\qquad\qquad\qquad\qquad\qquad \ldots, \\
&\qquad\qquad\qquad\qquad\qquad \ldots, [[l_i - 1]]^{a_{t_i}}, l_i, \\
[[l - l_i]]^{a_{t_i+1}}&[[l - l_i + 1]]^{a_{t_i+2}}, \ldots, \\
\cdots& \qquad\qquad\qquad\qquad\qquad \ldots, \\
& \qquad\qquad\qquad \ldots, [[l_J - 1]]^{a_{t_J}}, l_J, \\
[[l - l_J]]^{a_{t_J+1}}, \ldots, & \qquad \ldots, [[l_{J+1} - 1]]^{a_N}, l_{J+1}, [[l - l_{J+1}]]^{\infty}) \,.
\end{aligned}
\tag{3.14}
$$

We regard the above defined $a(\vec{h})$ as a Young diagram of depth $N = N(\vec{h})$ by

$$(\mathbf{Z}_{\geqq 0})^N \simeq Y_N = \{(\lambda_1, \lambda_2, \ldots, \lambda_N) \mid \lambda_i \in \mathbf{Z},\ \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_N \geqq 0\}\,,$$

$$\vec{a} = (a_1, \ldots, a_N) \mapsto \left(\sum_{i=1}^{N} a_i, \sum_{i=1}^{N-1} a_i, \ldots, a_1 \right)\,, \tag{3.15}$$

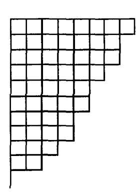
where Y_N denotes the set of the Young diagrams of depth N (see Fig. 1). For later convenience, we set $a(\vec{h}) = \phi$ when $N(\vec{h}) = 0$, and $Y_0 = \{\emptyset\}$.

Example 3.3. (Continued from Example 3.1.)

(a) $h = [[0]]^0, [[1]]^2, [[2]]^1, 3, [[0]]^1, [[1]]^0, 2, [[1]]^1, [[2]]^0, 3, [[0]]^1, [[1]]^1, [[2]]^0, 3, [[0]]^1, 1, [[2]]^{\infty}$.

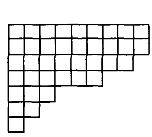
It reads as $N = 11$ and

$$a(\vec{h}) = (0, 2, 1, 1, 0, 1, 0, 1, 1, 0, 1)$$

$=$ 

(b) $\vec{h} = [[0]]^1, 1, [[2]]^1, 3, [[0]]^1, [[1]]^3, [[2]]^2, 3, [[0]]^1, [[1]]^0, 2, [[1]]^{\infty}$
It reads as $N = 7$ and

$$a(\vec{h}) = (1, 1, 1, 3, 2, 1, 0)$$

$=$ 

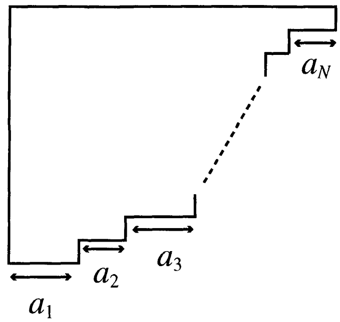


Fig. 1. Young diagram corresponding to $\vec{a} = (a_1, \quad , a_N)$

We have seen so far that once $\vec{h}^{\sharp}$ is known, then $\vec{h} \in \mathrm{Sp}(k)$ is uniquely determined by the index $a(\vec{h}) \in Y_{N(\vec{h})}$. Now let us consider the sequence $\vec{h}^{\sharp} = (l_1, \ldots, l_J, l_{J+1})$ for $\vec{h} \in \mathrm{Sp}(k)$. We shall associate with it a restricted path of length $N = N(\vec{h})$.

A sequence $\vec{r} = (r_0, \ldots, r_N)$ is called a level l restricted path of length N if it satisfies the conditions

$$r_i \in \{0, \ldots, l\}, \; r_i - r_{i+1} = \pm 1 \, .$$

Let $\mathscr{R}_N(k)$ denote the set of the level l restricted paths of length N with the condition

$$r_0 = 0, \qquad r_N = k \, .$$

Note that $\mathscr{R}_N(k) \neq \emptyset$ if and only if $N \equiv k \bmod 2$ and $N \geqq k$. Since the sequence $\vec{h}^{\sharp} = (l_1, \ldots, l_J, l_{J+1})$ satisfies

$$\begin{aligned} &0 < l_1 > l - l_2 < l_3 > \cdots \\ &\qquad \cdots < l_{2i-1} > l - l_{2i} < l_{2i+1} > \cdots \begin{cases} > l - l_J < l_{J+1} = k & \text{if } J \text{ is even} \\ < l_J > l - l_{J+1} = k & \text{if } J \text{ is odd} \end{cases} \end{aligned} \tag{3.16}$$

by Lemma 3.2, we can define $r(\vec{h}) \in \mathscr{R}_N(k)$ for $\vec{h} \in \mathrm{Sp}(k)$ as

$$\begin{aligned} r(\vec{h}) = (&\overset{0\text{th}}{0}, 1, \ldots, l_1 - 1, \overset{t_1\text{th}}{l_1}, l_1 - 1, \ldots, l - l_2 + 1, \overset{t_2\text{th}}{l - l_2}, l - l_2 + 1, \ldots \\ &\ldots, l_{2i-1} - 1, \overset{t_{2i-1}\text{th}}{l_{2i-1}}, l_{2i-1} - 1, \ldots, l - l_{2i} + 1, \overset{t_{2i}\text{th}}{l - l_{2i}}, l - l_{2i} + 1, \ldots, \\ &\ldots \qquad\qquad\qquad\qquad \ldots, \overset{N\text{th}}{k}) \, . \end{aligned} \tag{3.17}$$

(See Fig. 2.) Note that the numbers $(t_1, \ldots, t_J)$ in (3.12) are interpreted as the extremal points of the extremums $(l_1, l - l_2, l_3, \ldots)$ of the path $r(\vec{h})$.

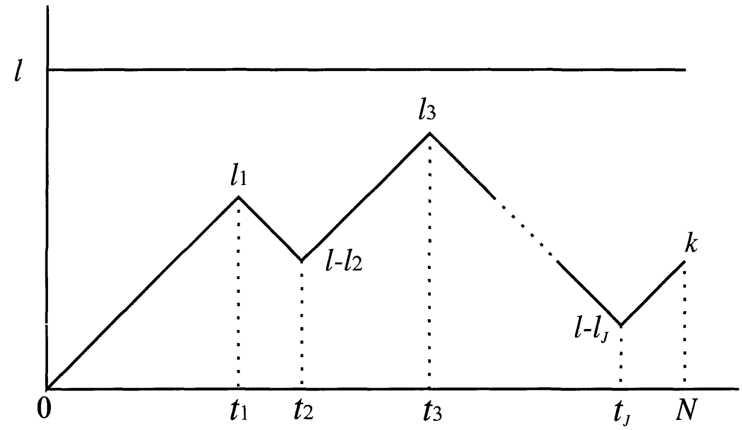
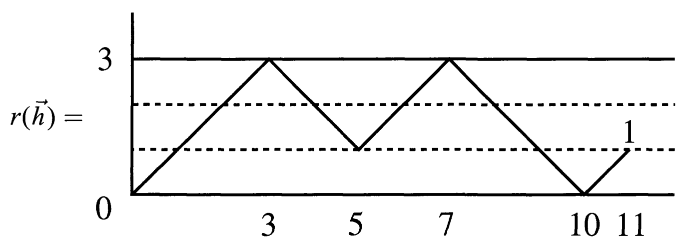


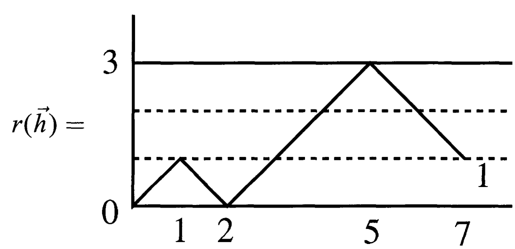
Fig. 2. Restricted path $r(\vec{h})$ (J : even)

Example 3.4. (Continued from Example 3.3.)

(a) $\vec{h}^\sharp = (3,2,3,3,1)$. So



(b) $\vec{h}^\sharp = (1,3,3,2)$. So



Now let us summarize the above argument. We have found the map

$$
\begin{aligned}
\pi_k : \mathrm{Sp}(k) \;&\rightarrow \coprod_{\substack{N=k \\ N\equiv k \mod 2}}^{\infty} (\mathscr{R}_N(k) \times Y_N) \\
\vec{h} \;&\mapsto (r(\vec{h}), a(\vec{h}))
\end{aligned} \tag{3.18}
$$

defined by (3.13) and (3.17). We call the map π_k the decoding map of $\mathrm{Sp}(k)$.

Theorem 3.5. *The decoding map π_k is bijective.*

Proof. It is easy to construct the inverse map. □

By the theorem above, we identify the spectrum $\mathrm{Sp}(k)$ of the path space $\mathscr{P}(k)$ with $\coprod_{\substack{N=k \\ N\equiv k \mod 2}}^{\infty} (\mathscr{R}_N(k) \times Y_N)$. For $\vec{r} \in \mathscr{R}_N(k)$, $\vec{a} \in Y_N$, let

$$\mathscr{P}(k)_{\vec{r},\vec{a}} = \{\vec{p} \in \mathscr{P}(k) \mid h(\vec{p}) = \pi_k^{-1}(\vec{r},\vec{a})\} \tag{3.19}$$

be the path space of the spectrum $(\vec{r},\vec{a})$. And let

$$\mathrm{ch}\,\mathscr{P}(k)_{\vec{r},\vec{a}} = \sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} q^{\Delta(k)+E(\vec{p})} z^{\mathrm{Wt}(\vec{p})} \tag{3.20}$$

be the character of $\mathscr{P}(k)_{\vec{r},\vec{a}}$. Then by definition

$$\mathrm{ch}\,\mathscr{P}(k) = \sum_{(\vec{r},\vec{a})\in\mathrm{Sp}(k)} \mathrm{ch}\,\mathscr{P}(k)_{\vec{r},\vec{a}} \,. \tag{3.21}$$

4. Degeneracy of the Spectrum

In this section we calculate the character $\mathrm{ch}\,\mathscr{P}(k)_{\vec{r},\vec{a}}$, which describes the degeneracy of the spectrum.

For $\vec{r} \in \mathscr{R}_N(k)$, we consider the sequence

$$\vec{n} = (n_1, \dots, n_N), \quad n_1 \leqq \cdots \leqq n_N \tag{4.1}$$

obeying the condition

$$n_1 = 0, \qquad n_i = \begin{cases} n_{i-1}+1, & \text{if } r_{i-2} = r_i < r_{i-1}\,, \\ n_{i-1}, & \text{otherwise}\,. \end{cases} \tag{4.2}$$

We define the *degree* of the restricted path $\vec{r} \in R_N(k)$, denoted by $d(\vec{r})$, as

$$d(\vec{r}) = \sum_{i=1}^{N} n_i \,. \tag{4.3}$$

For an arbitrary $\vec{a} \in Y_N$, we define the *size* of the Young diagram, denoted by $|\vec{a}\,|$, as

$$|\vec{a}\,| = \sum_{i=1}^{N} (N+1-i)a_i \,. \tag{4.4}$$

This is in accordance with the usual definition of the size of the Young diagram through the correspondence (3.15).

Proposition 4.1. *Fix a given $(\vec{r},\vec{a}) \in \mathscr{R}_N(k) \times Y_N$, $(N \geqq k,\ N \equiv k \bmod 2)$. Then for any $\vec{p} \in \mathscr{P}(k)_{\vec{r},\vec{a}}$ the equality*

$$E(\vec{p}) = d(\vec{r}) + |\vec{a}\,| \tag{4.5}$$

holds.

Proof. The proof will be given by the induction for $a_1, \dots, a_N$.

If $\vec{a} = \vec{0}$, it is easy to show both sides of (4.5) are

$$\begin{cases} \sum_{i=1}^{J} il_i + \frac{J}{2}k - \frac{J(J+2)}{4}l, & \text{if } J \text{ is even}, \\ \sum_{i=1}^{J} il_i - \frac{J+1}{2}k - \frac{(J+1)(J-1)}{4}l, & \text{if } J \text{ is odd}. \end{cases} \tag{4.6}$$

Next suppose that the statement is true for some $\vec{a} \in Y_N$. Let $\vec{a'} \in Y_N$ be an element such that $a'_i = a_i + \delta_{i,I}$ for some I $(1 \leqq I \leqq N)$. It is sufficient to prove that for any $\vec{p'} \in \mathscr{P}(k)_{\vec{r},\vec{a'}}$,

$$E(\vec{p'}) = E(\vec{p}) + N + 1 - I, \quad \vec{p} \in \mathscr{P}(k)_{\vec{r},\vec{a}} . \tag{4.7}$$

Let $l_1, \dots, l_J, t_1, \dots, t_J$ be the numbers defined in (3.6) and (3.12). Suppose

$$t_{m-1} < I \leqq t_m \tag{4.8}$$

for some $m, (1 \leqq m \leqq J+1)$, where $t_0 = 0$, $t_{J+1} = N$. We set

$$I = t_{m-1} + t, \quad 1 \leqq t \leqq t_m - t_{m-1} . \tag{4.9}$$

Then $\pi_k^{-1}(\vec{r}, \vec{a'})$ is obtained by inserting a term $[[l - l_{m-1} + t - 1]]$ between the $(m + 2(a_1 + \cdots + a_I) - 1)^{\text{th}}$ component of $\pi_k^{-1}(\vec{r}, \vec{a})$ and the one right next to it. Thus

$$\begin{aligned} E(\vec{p'}) - E(\vec{p}) = & \{(m + 2(a_1 + \cdots + a_I))(l - l_{m-1} + t - 1) \\ & +(m + 2(a_1 + \cdots + a_I) + 1)(l_{m-1} - t + 1)\} \\ & +\{2l(a_{I+1} + \cdots + a_N) + 2(l_m + \cdots + l_J) \\ & -(2(a_1 + \cdots + a_N) + J + 1)l_{J+1} \\ & -(2(a_1 + \cdots + a_N) + J + 2)(l - l_{J+1})\} , \end{aligned}$$

where the first term is the contribution from the inserted sequence $[[l - l_{m-1} + t - 1]]$ and the second term is the one from the shift of the sequence following the inserted sequence. Hence

$$\begin{aligned} E(\vec{p'}) - E(\vec{p}) &= l_{m-1} + 2(l_m + \cdots + l_J) + l_{J+1} - (J + 2 - m)l - t + 1 \\ &= N - t_{m-1} - t + 1 \\ &= N - I + 1 . \quad \square \end{aligned}$$

Let us turn to the calculation of the sl_2 part of the character,

$$\sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} z^{\mathrm{Wt}(\vec{p})} . \tag{4.10}$$

We first introduce a partition of N associated to each Young diagram of depth N. Let B_N be the set of the ordered partitions of N, i.e.,

$$B_N = \left\{ \vec{b} = (b_1,\ldots,b_s) \mid b_1,\ldots,b_s \in \mathbf{N},\ 1 \leqq s \leqq N,\ \sum_{i=1}^{s} b_i = N \right\} . \tag{4.11}$$

Define a map $\beta : Y_N \to B_N$

$$\vec{\alpha} = (a_1,\ldots,a_N) \to \beta(\vec{a}) = \vec{b} = (b_1,\ldots,b_s) \tag{4.12}$$

so that

$$\begin{aligned} a_1 &= a_2 = \cdots = a_{b_1} < a_{b_1+1} = \cdots = a_{b_1+b_2} < \cdots < a_{b_1+\cdots+b_{s-1}+1} \\ &= \cdots = a_N . \end{aligned} \tag{4.13}$$

Note that the map β is a surjection. This definition is more easily seen in Fig. 3.

Let

$$\chi_b(z) = \frac{z^{b+1} - z^{-b-1}}{z - z^{-1}} . \tag{4.14}$$

This is the character of the $(b+1)$-dimensional irreducible sl_2-module V_b.

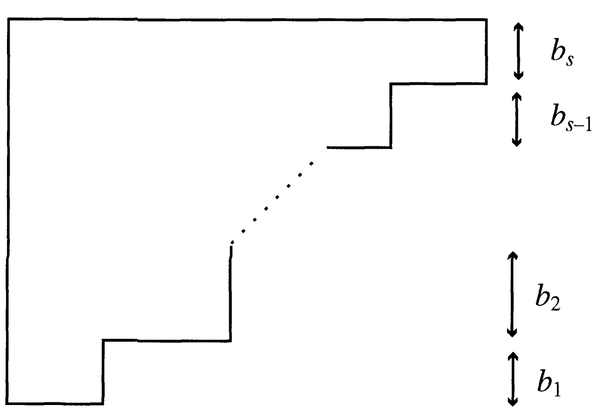


Fig. 3. The partition of N associated to a Young diagram $\vec{a}\in Y_N, \beta(\vec{a}) = \vec{b} = (b_1, b_2, \ldots, b_{s-1}, b_s)$

Theorem 4.2. *Let* $(\vec{r},\vec{a})\in\mathscr{R}_N(k)\times Y_N$, $(N\geqq k,\ N\equiv k \bmod 2)$. *Let* $\vec{b}=(b_1,\dots,b_s)\in B_N$ *be the partition of* N *associated to the Young diagram* $\vec{a}$ *as in* (4.13). *Then*

$$\sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} z^{\mathrm{Wt}(\vec{p})}=\prod_{i=1}^{s}\chi_{b_i}(z)\,. \tag{4.15}$$

Combining (4.15) *with* (4.5), *we have*

$$\mathrm{ch}\,\mathscr{P}(k)_{\vec{r},\vec{a}}=q^{\Delta(k)+d(\vec{r})+|\vec{a}|}\prod_{i=1}^{s}\chi_{b_i}(z)\,. \tag{4.16}$$

The proof of Theorem 4.2 is divided into several steps.

Step 1. In this step we show that the left hand side of (4.15) can be expressed by the product of the *incidence matrix.*

For any $(\vec{r},\vec{a})\in\mathscr{R}_N(k)\times Y_N$, $(n\geqq k,\ N\equiv k \bmod 2)$, let $\vec{h}$ be the corresponding element in the spectrum $\mathrm{Sp}(k)$ (Theorem 3.5). Let M and J be the numbers defined in (3.1) and (3.5) respectively, i.e., J is the number of the elementary blocks of odd length in $\vec{h}$ and

$$M=J+2\sum_{i=1}^{N}a_i\,. \tag{4.17}$$

Let $\Sigma(k)_{\vec{r},\vec{a}}$ be the subset of $\Sigma(k)$ corresponding to $\mathscr{P}(k)_{\vec{r},\vec{a}}$ through the identity (2.9). For $\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}$, let $\vec{s}\in\Sigma(k)_{\vec{r},\vec{a}}$ be the spin configuration corresponding to $\vec{p}$. Then it has the following form:

$$\vec{s}=(s_1,\dots,s_{M+1},-l+2l_{J+1}/l-2l_{J+1}/-l+2l_{J+1},\dots)\,. \tag{4.18}$$

The summation $s_1+\cdots+s_{M+1}$ represents how the path goes up or down from the starting point p_1 to the $(M+2)^{\text{th}}$ point of $\vec{p}$, i.e., $l-l_{J+1}$. Therefore the following equality holds:

$$\sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} z^{\mathrm{Wt}(\vec{p})}=z^{l-l_{J+1}}\sum_{\vec{s}\in\Sigma(k)_{\vec{r},\vec{a}}} z^{-(s_1+\ \ +s_{M+1})}\,. \tag{4.19}$$

We define the function $F(\vec{r},\vec{a};z)$ as

$$F(\vec{r},\vec{a};z)=\sum_{\vec{s}\in\Sigma(k)_{\vec{r},\vec{a}}} z^{s_1+\ \ +s_{M+1}}\,. \tag{4.20}$$

For $a=0,1,\dots,l$, we set

$$M_a=\begin{array}{cc} & \begin{array}{ccccc} \quad & l-a+1 & \quad & \quad & \quad \end{array} \\ a+1 & \left(\begin{array}{ccccc} & z^{l} & & & \\ & z^{l-2} & & & \\ & \vdots & & & \\ & z^{l-2a} & z^{l-2a} & \cdots & z^{l-2a} \\ & & & & \end{array}\right) \end{array}, \tag{4.21}$$

$$H_a = \begin{array}{c c} & l-a+1 \\ a+1 & \begin{pmatrix} & & & & \\ & z^{l-2a} & z^{l-2a} & \cdots & z^{l-2a} \\ & & & & \end{pmatrix} \end{array}, \qquad V_a = \begin{array}{c c} & l-a+1 \\ & \begin{pmatrix} & z^{l} & \\ & z^{l-2} & \\ & \vdots & \\ a+1 & z^{l-2a} & \\ & & \end{pmatrix} \end{array}. \tag{4.22}$$

The matrix M_a is named the *incidence matrix*. This is defined in such a way that

1) $(M_a)_{ij} \neq 0$ if and only if $H_{l+2-2i,l+2-2j} \neq a$ in (2.4).
2) If $(M_a)_{ij} \neq 0$, then $(M_a)_{ij} = z^{l+2-2i}$.

From these properties, the (i,j)-component of the matrix $M_{h_1} \cdots M_{h_m}$ is expressed as

$$(M_{h_1} \cdots M_{h_m})_{ij} = \sum_{\vec{s}=(s_1,\cdots,s_m)} z^{s_1+\cdots+s_m}, \tag{4.23}$$

where the summation is taken over all the spin configurations $\vec{s}$ such that

$$H(s_i, s_{i+1}) = h_i, \qquad s_1 = l+2-2i, \qquad s_{m+1} = l+2-2j. \tag{4.24}$$

Lemma 4.3. 1) *For an arbitrary $\vec{h} = \pi_k{}^{-1}(\vec{r}, \vec{a})$, the function $F(\vec{r}, \vec{a}; z)$ is given by the sum of the matrix elements on the $l - l_{J+1} + 1^{\text{th}}$ column of the matrix*

$$M_{h_1} \cdots M_{h_M} M_{l_{J+1}}. \tag{4.25}$$

2) *Equivalently, the function $F(\vec{r}, \vec{a}; z)$ is also given by the $(l+1, l-l_{J+1}+1)$ component of the matrix*

$$T(\vec{r}, \vec{a}) = z^l H_l M_{h_1} \cdots M_{h_M} V_{l_{J+1}}. \tag{4.26}$$

Proof. 1) For $\vec{s} \in \Sigma(k)_{\vec{r},\vec{a}}$, $s_{M+2} = -l + 2l_{J+1}$ as in (4.18). Applying (4.23), we have the statement.

2) Easily follows from 1). □

Example 4.4. Here we consider the case

$$l = 3,\ N = 3,\ k = 1,$$

$$\vec{r} = (0, 1, 2, 1),$$

$$\vec{a} = (0, 1, 0).$$

In this case $J = 1$, $M = 3$, $l_{J+1} = 2$, and $(l+1, l-l_{J+1}+1) = (4, 2)$. From these data, $\vec{h} = \pi_k{}^{-1}(\vec{r}, \vec{a})$ is

$$\vec{h} = (1, 2, 2, 2, [[1]]^\infty).$$

The spin configurations in $\Sigma(k)_{\vec{r},\vec{a}}$ are

$$(s_1, s_2, s_3, s_4, 1, -1, 1, -1, 1, \ldots),$$

where

$$
\begin{aligned}
(s_1, s_2, s_3, s_4) = \ & (3, -1, 1, 1)\,, \\
& (3, -1, -1, 1)\,, \\
& (3, -1, -1, -1)\,, \\
& (1, -1, 1, 1)\,, \\
& (1, -1, -1, 1)\,, \\
& (1, -1, -1, -1)\,.
\end{aligned}
$$

Thus

$$F(\vec{r}, \vec{a}; z) = z^4 + 2z^2 + 2 + z^{-2}\,.$$

On the other hand,

$$
\begin{aligned}
T(\vec{r}, \vec{a}) &= z^3 H_3 M_1 M_2 M_2 V_2 \\
&= z^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z^{-3} & z^{-3} & z^{-3} & z^{-3} \end{pmatrix} \begin{pmatrix} 0 & 0 & z^3 & 0 \\ 0 & 0 & z & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & z^3 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & z^{-1} & z^{-1} & z^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & z^3 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & z^{-1} & z^{-1} & z^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & z^3 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & z^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & z^4 + 2z^2 + 2 + z^{-2} & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

We see that the $(4, 2)$ component of the matrix $T(\vec{r}, \vec{a})$ is equal to $F(\vec{r}, \vec{a}; z)$.

Let us proceed to the evaluation of $T(\vec{r}, \vec{a})$ in Lemma 4.3.

Step 2. In this step, we represent $T(\vec{r}, \vec{a})$ as a factorized form (4.36).

The next lemma is easily proved and useful for our purpose.

Lemma 4.5. *For $a = 0, 1, \ldots, l$ and $n \in \mathbf{N}$,*

$$(M_a M_{l-a})^n = V_a H_{l-a}\,. \tag{4.27}$$

Lemma 4.6. *For $\vec{a}, \vec{a'} \in Y_N$, if $\beta(\vec{a}) = \beta(\vec{a'})$, then*

$$F(\vec{r}, \vec{a}; z) = F(\vec{r}, \vec{a'}; z)\,. \tag{4.28}$$

Proof. If $a_1 \neq 0$, there is a factor $z^l H_l (M_0 M_l)^{a_1}$ at the left end of $T(\vec{r}, \vec{a})$ (see (3.14) and (4.27)). Since

$$M_0 M_l = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & & & \\ & & & \end{pmatrix}, \tag{4.29}$$

$z^l H_l (M_0 M_l)^{a_1} = z^l H_l$. So it does not affect the function $F(\vec{r}, \vec{a}; z)$ whether a_1 is equal to zero or not. Next we suppose that there is some i $(2 \leqq i \leqq N)$ such that $a_i \geqq 1$, $a'_i \geqq 1$, and $a_i \neq a'_i$. But from Lemma 4.5,

$$(M_a M_{l-a})^{a_i} = (M_a M_{l-a})^{a'_i} = V_a H_{l-a} . \tag{4.30}$$

Thus we have the result. □

From this lemma we could assume that

$$a_1 = 0, \;\; a_2, \ldots, a_N \in \{0, 1\} \tag{4.31}$$

in order to prove Theorem 4.2 without losing generality. Until the end of this proof we set this assumption.

For $\vec{a} \in Y_N$ such that $\beta(\vec{a}) = (b_1, \ldots, b_s)$, let $\beta_0 = 0, \beta_1 = b_1, \ldots, b_i = b_1 + b_2 + \cdots + b_i, \ldots, \beta_s = b_1 + \cdots + b_s = N$, and $0 < t_1 < \cdots < t_J$ be the extremal points of $\vec{r}$. We align β_i's and t_i's as follows:

$$\begin{aligned} &\beta_0 = 0 < t_1 < \cdots < t_{m_1} < \beta_1 \leqq t_{m_1+1} < \cdots < t_{m_2} < \beta_2 \leqq \cdots \\ &\quad \leqq t_{m_{s-2}+1} < \cdots < t_{m_{s-1}} < \beta_{s-1} \leqq t_{m_{s-1}+1} < \cdots < t_J < \beta_s = N . \end{aligned} \tag{4.32}$$

Accordingly $T(\vec{r}, \vec{a})$ has a form

$$\begin{aligned} &z^l H_l M_{l_1} \cdots M_{l_{m_1}} (M_{h'_1} M_{l-h'_1}) M_{l_{m_1+1}} \cdots M_{l_{m_2}} (M_{h'_2} M_{l-h'_2}) M_{l_{m_2+1}} \cdots \\ &\quad M_{l_{m_{s-2}}} (M_{h'_{s-2}} M_{l-h'_{s-2}}) M_{l_{m_{s-2}+1}} \cdots M_{l_{m_{s-1}}} (M_{h'_{s-1}} M_{l-h'_{s-1}}) M_{l_{m_{s-1}+1}} \cdots M_{l_J} V_{l_{J+1}} , \end{aligned} \tag{4.33}$$

where

$$h'_i = \begin{cases} r_{\beta_i} & \text{if } m_i \text{ is even} , \\ l - r_{\beta_i} & \text{if } m_i \text{ is odd} . \end{cases} \tag{4.34}$$

Using Lemma 4.5 one can replace the factors $(M_a M_{l-a})$ by $(V_a H_{l-a})$. Thus we have

$$\begin{aligned} T(\vec{r}, \vec{a}) &= z^l (H_l M_{l_1} \cdots M_{l_{m_1}} V_{h'_1})(H_{l-h'_1} M_{l_{m_1+1}} \cdots M_{l_{m_2}} V_{h'_2}) \cdots \\ &\quad \times (H_{l-h'_{s-2}} M_{l_{m_{s-2}+1}} \cdots M_{l_{m_{s-1}}} V_{h'_{s-1}})(H_{l-h'_{s-1}} M_{l_{m_{s-1}+1}} \cdots M_{l_J} V_{l_{J+1}}) . \end{aligned} \tag{4.35}$$

Therefore $T(\vec{r}, \vec{a})$ is a product of the factors

$$T(\vec{r}, \vec{a}) = z^l S_l S_2 \cdots S_s ,$$

$$S_i = H_{l-h'_{i-1}} M_{l_{m_{i-1}+1}} \cdots M_{l_{m_i}} V_{h'_i} , \tag{4.36}$$

where

$$m_0 = 0, \qquad m_s = J ,$$

Example 4.7. To illustrate the procedures (4.32–35), let us give one more example. Let

$$l = 3,\ N = 9,\ k = 1\,,$$

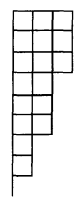
$$\vec{r} = (0,1,2,1,2,3,2,1,0,1) \in \mathscr{R}_9(1)\,,$$

$$\vec{a} = (0,1,0,1,0,0,1,0,0) \in A_9\,.$$

The sequence of the local energies is

$$\pi_1{}^{-1}(\vec{r},\vec{a}) = ([[1]],2,2,[[1]],3,[[1]],3,[[1]]^{\infty})\,.$$

The Young diagram corresponding to $\vec{a}$ is



Then we see that

$$\beta_1 = 1, \qquad \beta_2 = 3, \qquad \beta_3 = 6\,,$$

$$t_1 = 2, \qquad t_2 = 3, \qquad t_3 = 5, \qquad t_4 = 8\,,$$

and the alignment of (4.32) in this case is

$$0 < 1 < 2 < 3 \leqq 3 < 5 < 6 < 8\,.$$

Accordingly

$$T(\vec{r},\vec{a};z) = z^3 H_3 M_1 M_2 M_2 M_2 M_1 M_2 M_3 M_1 M_2 M_3 V_1\,.$$

From Lemma 4.5 this can be factorized as

$$z^3 (H_3 V_1)(H_2 M_2 V_1)(H_1 M_2 M_3 V_1)(H_2 M_3 V_1)\,.$$

Step 3. Finally we evaluate the factor S_i explicitly.

Let $E_{i,j}$ be the matrix whose matrix element is 1 at the (i,j) component and zero otherwise.

Lemma 4.8.

$$S_i = z^{h'_{i-1}-h'_i}\,\chi_{\beta_i-\beta_{i-1}}(z)\,E_{l-h'_{i-1}+1,l-h'_i+1}\,. \tag{4.37}$$

Proof. Let us put $m_{i-1}+1 = n$, $m_i = m$, for simplicity. We need to calculate the product

$$H_{l-h'_{i-1}} M_{l_n} \cdots M_{l_m} V_{h'_i}\,. \tag{4.38}$$

First we consider the product $H_{l-h'_{i-1}}M_{l_n}$.

$$
\begin{aligned}
&H_{l-h'_{i-1}}M_{l_n}\\
&= \begin{array}{c} \\ l-h'_{i-1}+1 \\ \\ \end{array}
\begin{array}{c} h'_{i-1}+1 \\ \left(\begin{array}{ccc} z^{-l+2h'_{i-1}} & \cdots & z^{-l+2h'_{i-1}} \\ & & \\ & & \end{array}\right) \end{array}
\cdot
\begin{array}{c} \\ \\ \\ l_n+1 \end{array}
\begin{array}{c} l-l_n+1 \\ \left(\begin{array}{ccc} z^{l} & & \\ z^{l-2} & & \\ \vdots & & \\ z^{l-2l_n} & \cdots & z^{l-2l_n} \end{array}\right) \end{array}\\
&= \begin{array}{c} \\ l-h'_{i-1}+1 \\ \\ \end{array}
\begin{array}{c} l-l_n+1 \\ \left(\begin{array}{cccc} \varphi_{l_n-h'_{i-1}}(z) & 1 & \cdots & 1 \\ & & & \\ & & & \end{array}\right) \end{array} z^{2(h'_{i-1}-l_n)}\,,
\end{aligned}
\tag{4.39}
$$

where

$$
\varphi_m(z) = z^{2m} + z^{2m-2} + \cdots + 1\,. \tag{4.40}
$$

Since $l_n - h'_{i-1} = t_n - \beta_{i-1}$,

$$
H_{l-h'_{i-1}}M_{l_n} = \begin{array}{c} \\ l-h'_{i-1}+1 \\ \\ \end{array}
\begin{array}{c} l-l_n+1 \\ \left(\begin{array}{cccc} \varphi_{t_n-\beta_{i-1}}(z) & 1 & \cdots & 1 \\ & & & \\ & & & \end{array}\right) \end{array} z^{-t_n+\beta_{i-1}+h'_{i-1}-l_n}\,.
\tag{4.41}
$$

In a similar way

$$
\begin{aligned}
H_{l-h'_{i-1}}M_{l_n}M_{l_{n+1}} &= \begin{array}{c} \\ l-h'_{i-1}+1 \\ \\ \end{array}
\begin{array}{c} l-l_n+1 \\ \left(\begin{array}{cccc} \varphi_{t_n-\beta_{i-1}}(z) & 1 & \cdots & 1 \\ & & & \\ & & & \end{array}\right) \end{array}\\
&\quad\times \begin{array}{c} \\ \\ \\ l_{n+1}+1 \end{array}
\begin{array}{c} l-l_{n+1}+1 \\ \left(\begin{array}{ccc} z^{l} & & \\ z^{l-2} & & \\ \vdots & & \\ z^{l-2l_{n+1}} & \cdots & z^{l-2l_{n+1}} \end{array}\right) \end{array} z^{-t_n+\beta_{i-1}+h'_{i-1}-l-n}\\
&= \begin{array}{c} \\ l-h'_{i-1}+1 \\ \\ \end{array}
\begin{array}{c} l-l_{n+1}+1 \\ \left(\begin{array}{cccc} \varphi_{t_n-\beta_{i-1}+l_{n+1}-(l-l_n)}(z) & 1 & \cdots & 1 \\ & & & \\ & & & \end{array}\right) \end{array}\\
&\quad\times z^{-t_n+\beta_{i-1}+h'_{i-1}-l_n+l-2l_{n+1}}\,.
\end{aligned}
\tag{4.42}
$$

Again, since $l_{n+1}-(l-l_n)=t_{n+1}-t_n$,

$$H_{l-h'_{i-1}}M_{l_n}M_{l_{n+1}}$$

$$= \begin{array}{c c} & l-l_{n+1}+1 \\ l-h'_{i-1}+1 & \begin{pmatrix} & & & & \\ & \varphi_{t_{n+1}-\beta_{i-1}}(z) & 1 & \cdots & 1 \\ & & & & \end{pmatrix} \end{array} z^{-t_{n+1}+\beta_{i-1}+h'_{i-1}-l_{n+1}} . \tag{4.43}$$

Repeating this procedure, the result is shown as

$$H_{l-h'_i}M_{l_n}\cdots M_{l_m}V_{h'_i} = \begin{array}{c c} & l-h'_i+1 \\ l-h'_{i-1}+1 & \begin{pmatrix} & & \\ & \varphi_{\beta_i-\beta_{i-1}}(z) & \\ & & \end{pmatrix} \end{array} z^{-\beta_i+\beta_{i-1}-h'_i+h'_{i-1}}$$

$$= z^{h'_i-h'_{i-1}}\mathrm{ch}_{V_{\beta_i-\beta_{i-1}}}(z)E_{l-h'_{i-1}+1,l-h'_i+1} . \qquad \square \tag{4.44}$$

Now we complete the proof of Theorem 4.2. By (4.20) we see

$$\sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} z^{\mathrm{Wt}(\vec{p})} = z^{l-l_{J+1}}F(\vec{r},\vec{a};z^{-1}) . \tag{4.45}$$

From Lemma 4.8, $F(\vec{r},\vec{a};z)$ is evaluated as

$$F(\vec{r},\vec{a};z) = z^{l-l_{J+1}}\prod_{i=1}^{s}\chi_{\beta_i-\beta_{i-1}}(z) , \tag{4.46}$$

where we have used

$$h'_0-h'_s=-l_{J+1} .$$

Taking into account the definition of $\beta(\vec{a})$, we get the formula

$$\sum_{\vec{p}\in\mathscr{P}(k)_{\vec{r},\vec{a}}} z^{\mathrm{Wt}(\vec{p})} = \prod_{i=1}^{s}\chi_{b_i}(z), \quad \beta(\vec{a})=(b_1,\ldots,b_s) . \qquad \square \tag{4.47}$$

5. Character Formulas

In this section, as an application of our decomposition, we derive some character formulas for the level l integrable $\widehat{sl}_2$-modules.

Owing to Theorems 2.1, 3.5, 4.2 and Proposition 4.1, we immediately obtain the following formula:

Proposition 5.1.

$$\mathrm{ch}_{\mathscr{L}(k)}(q,z) = q^{\Delta(k)}\sum_{\substack{N=k \\ N\equiv k \bmod 2}}^{\infty}\sum_{(\vec{r},\vec{a})\in\mathscr{R}_N(k)\times Y_N} q^{d(\vec{r})+|\vec{a}|}\prod_{i=1}^{s}\chi_{b_i}(z) , \tag{5.1}$$

where $(b_1,\ldots,b_s)=\beta(\vec{a})$.

Since the summation over $\mathscr{R}_N(k)$ and Y_N are independent of each other, (5.1) can be rewritten as the following factorized form:

$$\begin{aligned}
\mathrm{ch}_{\mathscr{L}(k)}(q,z) &= q^{\Delta(k)} \sum_{\substack{N=k \\ N\equiv k \bmod 2}}^{\infty} F_{N,k}(q) G_N(q,z)\,, \\
F_{N,k}(q) &= \sum_{\vec{r}\in\mathscr{R}_N(k)} q^{d(\vec{r})}\,, \\
G_N(q,z) &= \sum_{\vec{a}\in Y_N} q^{|\vec{a}|} \prod_{i=1}^{s} \chi_{b_i}(z)\,.
\end{aligned} \tag{5.2}$$

It is possible to evaluate the functions $F_{N,k}(q)$ and $G_N(q,z)$ more explicitly. Let

$$\begin{aligned}
(q)_n &= \prod_{i=1}^{n} (1-q^i)\,, \\
\begin{bmatrix} N \\ n \end{bmatrix} &= \frac{(q)_N}{(q)_n (q)_{N-n}}, \quad \text{for } 0 \leqq n \leqq N\,.
\end{aligned} \tag{5.3}$$

Proposition 5.2.

$$G_N(q,z) = \frac{1}{(q)_N} \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} z^{N-2n}\,. \tag{5.4}$$

Proof. Let β_i's be the numbers defined in (4.12). Then

$$\begin{aligned}
G_N(q,z) &= \sum_{\vec{a}\in Y_N} q^{|\vec{a}|} \prod_{i=1}^{s} \chi_{b_i}(z) \\
&= \sum_{\substack{b_i \geqq 1 \\ b_1+\quad+b_s=N}} \ \sum_{\substack{a_1\geqq 0 \\ a_{\beta_{m+1}}>0}} q^{\sum_{i=1}^{s}\{\sum_{m=0}^{i-1} a_{\beta_{m+1}}\} b_i} \prod_{i=1}^{i} \chi_{b_i}(z)\,.
\end{aligned} \tag{5.5}$$

Summed up with respect to $a_{\beta_{m+1}}$'s, we see that the above equals

$$\sum_{\substack{b_i\geqq 1 \\ b_1+\quad+b_s=N}} \prod_{i=1}^{s} \chi_{b_i}(z) \frac{q^{b_1}}{1-q^{b_1}} \frac{q^{b_1+b_2}}{1-q^{b_1+b_2}} \cdots \frac{q^{b_1+\quad+b_{s-1}}}{1-q^{b_1+\quad+b_{s-1}}} \frac{1}{1-q^{b_1+\quad+b_s}}\,. \tag{5.6}$$

Considering it particularly as a summation with respect to b_s, again the above equals

$$\sum_{b_s=1}^{N} \frac{q^{N-b_s}}{1-q^N} \chi_{b_s}(z) \sum_{\substack{b_i\geqq 1 \\ b_1+\quad+b_{s-1}=N-b_s}} \prod_{i=1}^{s-1} \chi_{b_i}(z) \frac{q^{b_1}}{1-q^{b_1}} \frac{q^{b_1+b_2}}{1-q^{b_1+b_2}} \cdots \frac{1}{1-q^{b_1+\quad+b_{s-1}}}\,. \tag{5.7}$$

Then we find that $G_N(q,z)$ satisfies the following recursion relation:

$$\begin{aligned}
G_0(q,z) &= 1\,, \\
G_N(q,z) &= \sum_{b=1}^{N} \frac{q^{N-b}}{1-q^N} \chi_b(z) G_{N-b}(q,z)\,.
\end{aligned} \tag{5.8}$$

On the other hand, let

$$H_N(q,z) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} z^{N-2n} . \tag{5.9}$$

The function $H_N(q,z)$ is essentially the Rogers–Szegö polynomial [12], and satisfies the following recursion relation:

$$\begin{aligned} H_0(q,z) &= 1 , \\ H_1(q,z) &= z + z^{-1} , \\ H_N(q,z) &= (z+z^{-1})H_{N-1}(q,z) - (1-q^{N-1})H_{N-2}(q,z) . \end{aligned} \tag{5.10}$$

Then the right hand side of Eq. (5.4) is

$$h_N(q,z) := \frac{1}{(q)_N} H_N(q,z) .$$

Using the recursion relations (5.10) time after time, we see that $h_N(q,z)$ satisfies the same recursion relations as (5.8). □

Next let us evaluate

$$F_{N,k}(q) = \sum_{\vec{r} \in \mathscr{R}_N(k)} q^{d(\vec{r})} .$$

Note that the degree of the restricted path $\vec{r}$ could also be calculated as

$$d(\vec{r}) = \sum_{j=1}^{N-1} j f(r_{N-j} - r_{N+1-j}, r_{N-1-j} - r_{N-j}) , \tag{5.11}$$

where

$$\begin{aligned} &f : \{+,-\} \times \{+,-\} \to \{0,1\} , \\ &\begin{cases} f(+,-) = 1 \\ f(+,+) = f(-,-) = f(-,+) = 0 . \end{cases} \end{aligned} \tag{5.12}$$

By the expression (5.11) of $d(\vec{r})$, we can regard $d(\vec{r})$ as a total energy of the size $N+1$ lattice model which has the local energy function f. This f is essentially equivalent to the local energy function of sl_2 level l RSOS model. Thus we can calculate $F_{N,k}(q)$ in the same way as the one dimensional configuration sum of sl_2 level l RSOS model [13].

Proposition 5.3.

$$F_{N,k}(q) = \sum_{j=-\infty}^{\infty} \left\{ q^{j(k+1)+j^2(l+2)} \begin{bmatrix} N \\ \frac{-k-2j(l+2)+N}{2} \end{bmatrix} - q^{-j(k+1)+j^2(l+2)} \begin{bmatrix} N \\ \frac{k+2-2j(l+2)+N}{2} \end{bmatrix} \right\} . \tag{5.13}$$

Proof. Let

$$g_N^{(\pm,j)}(m,n,n+1) = q^{\pm j(m+1)+j^2(l+2)} \begin{bmatrix} N \\ \frac{n\mp(m+1)+1-2j(l+2)+N}{2} \end{bmatrix},$$

$$g_N^{(\pm,j)}(m,n,n-1) = q^{\pm j(m+1)+j^2(l+2)+\frac{1}{2}\{n\mp(m+1)+1-2j(l+2)+N\}} \begin{bmatrix} N \\ \frac{n\mp(m+1)+1-2j(l+2)+N}{2} \end{bmatrix}, \tag{5.14}$$

and define the function

$$F_N(m,n,n\pm 1) = \sum_{j=-\infty}^{\infty} \{g_N^{(+,j)}(m,n,n\pm 1) - g_N^{(-,j)}(m,n,n\pm 1)\}\,. \tag{5.15}$$

Then we can show the following properties:
(1) Initial condition

$$F_1(m,n,n\pm 1) = g_1^{(+,0)}(m,n,n\pm 1)\,. \tag{5.16}$$

(2) Recursion relation

$$F_{N+1}(m,n,n+1) = F_N(m,n-1,n) + F_N(m,n+1,n)\,,$$
$$F_{N+1}(m,n,n-1) = q^{N+1}F_N(m,n-1,n) + F_N(m,n+1,m)\,, \tag{5.17}$$

(3) Restriction condition

$$F_N(m,-1,0) = F_N(m,l+1,l) = 0\,. \tag{5.18}$$

From these properties it follows that

$$F_{N,k}(q) = F_N(a,0,1)\,. \tag{5.19}$$

Thus we have had the result. □

There is also an alternative (fermionic) expression of $F_{N,k}(q)$ given by Bouwkneght et al. [14]

$$F_{N,k}(q) = q^{-\frac{1}{4}k-\frac{1}{4}N^2} \sum_{m_2,\quad,m_l} q^{\frac{1}{2}(N^2+m_2{}^2+\quad+m_l{}^2-Nm_2-m_2m_3-\quad-m_{l-1}m_l)}$$
$$\times \prod_{i=2}^{l} \begin{bmatrix} \frac{1}{2}(m_{i-1}+m_{i+1}+\delta_{i,k+1}) \\ m_i \end{bmatrix},$$

where the summation is over all odd non-negative integers for $m_{2a}, m_{2a-2}, m_{2a-4}, \ldots$ and over the even non-negative integers for the remaining ones (we set $m_1 = N, m_{l+1} = 0$).

6. Hidden Yangian Symmetry in WZW Models

In this section we point out an intriguing connection between our spectral decomposition of the path space and the hidden quantum group symmetry in the WZW conformal field theory.

The Hilbert space of the $\widehat{sl}_2$ level l WZW model is the direct sum of the integrable representations $\mathscr{L}(k)$ of the $\widehat{sl}_2$ level l.

In the level 1 case the Yangian algebra of sl_2 acts on $\mathscr{L}(k)$ by

$$Q_0^a = J_0^a, \qquad Q_1^a = \frac{1}{2} f^a{}_{bc} \sum_{m>0} J_{-m}^b J_m^c , \tag{6.1}$$

where J_m^a are the Fourier components of the $\widehat{sl}_2$ current and $f^a{}_{bc}$ is the structure constant of sl_2 [1,5]. As a Yangian module, $\mathscr{L}(k)$ decomposes into irreducible finite dimensional representations as

$$\mathscr{L}(k) \cong \bigoplus_{\substack{N=k \\ N\equiv k \bmod 2}}^{\infty} \bigoplus_{\vec{a}\in Y_N} W_{N,\vec{a}} . \tag{6.2}$$

Each irreducible component $W_{N,\vec{a}}$ is an L_0 eigenspace with the eigenvalue

$$\frac{1}{4}(N^2 - k^2) + |\vec{a}| + \Delta(k) , \tag{6.3}$$

and it has the following sl_2-module structure

$$W_{N,\vec{a}} \cong \bigotimes_{i=1}^{s} V_{b_i}, \qquad \beta(\vec{a}) = (b_1, \ldots, b_s) . \tag{6.4}$$

Comparing (6.2)–(6.4) with Theorem 4.2, we see a remarkable coincidence of the Yangian decomposition of $\mathscr{L}(k)$ and the spectral decomposition of the path space $\mathscr{P}(k)$.

Let us consider why this happens. In general the integrability of field theory is synonymous to the existence of an infinite number of the local integrals of motion (IM) commuting with each other. Let us write the abelian algebra generated by these IMs as $\mathscr{I}$. In the spectral decomposition of the path space we regard the local energy operators h_i as the maximal family of the local operators commuting with the energy operator E which is also equal to the Virasoro energy operator L_0. If we identify h_i's with the generators of $\mathscr{I}$, then the degeneracy of the spectrum means the presence of a hidden non-abelian symmetry. In this way we recover the decomposition (6.2) through our spectral decomposition.

Let us turn to the higher level case. We recall that any irreducible finite dimensional representation of the sl_2 Yangian is isomorphic, as an sl_2 module, to a *tensor product* of some irreducible representations of sl_2 [15]. Then, looking at Theorem 4.2, we naturally identify each degeneracy in our spectral decomposition with the character of an irreducible Yangian multiplet. This leads us to the following conjecture:

For $l \geqq 2$, (for $l = 1$, it has been proved in [1]).

Conjecture. 1) *For each integral representation $\mathscr{L}(k)$ of $\widehat{sl}_2$ level l, there is the canonical action of the Yangian and the algebra of the local integrals of motion which are commutant with each other.*

2) *The $\mathscr{L}(k)$ decomposes into a direct sum of irreducible finite dimensional Yangian modules. The set $\bigsqcup_{N=k \atop N\equiv k \bmod 2}^{\infty} \mathscr{R}_N(k) \times Y_N$ parameterizes the Yangian highest weight vectors $v_{\vec{r},\vec{a}}$ in $L(k)$ such that the L_0 eigenvalue of $v_{\vec{r},\vec{a}}$ is $\Delta(k)+d(\vec{r})+|\vec{a}|$, and as an sl_2-module the Yangian multiplet generated by $v_{\vec{r},\vec{a}}$ is isomorphic to $\bigotimes_{i=1}^{s} V_{b_i}$, where $\beta(\vec{a})=(b_1,\dots,b_s)$.*

To some extent the above Yangian module structure has already appeared in [14] through a generalization of the idea of [1].

It has been clarified that the symmetry algebra of a two dimensional integrable *massive* field theory is also the product of the algebra of the integrals of motions $\mathscr{I}$ and some "quantum group" symmetry commuting with each other [16]. Especially, the WZW model allows the integrable massive deformation which has the Yangian symmetry [17]. It is an important problem to understand how the symmetry algebra of the WZW model here is related to the one in the deformed model.

We also comment that in the level 1 case there is a simple correspondence between the spectrum of the vertex model and the one in the Haldane–Shastry model [5]. The natural speculation is that there will be a higher spin analog of the Haldane–Shastry model having a similar correspondence to the one in the vertex model here.

To conclude, we expect that the correspondence between the spectral decomposition of the solvable lattice model and the one of the corresponding conformal field theory (and its massive deformation) will be a universal phenomena. This interplay provides a new and useful way to investigate a hidden quantum symmetry structure of other conformal field theories as well.

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Note added After the submission of this paper, the authors noticed the paper, M. Idzumi, K Iohara, T Jimbo, T Miwa, T Nakashima, and T Tokihiro, *Quantum affine symmetry in vertex models*, Int J. Mod Phys. **A8**, 1479–1511 (1993), in which the path space is described using the combinatorics similar to what we used for the description of the spectrum of the path space in Sect 3

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