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The Higher Order Hamiltonian Structures for the Modified Classical Yang–Baxter Equation

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Abstract: We consider constructing the higher order Hamiltonian structures on the dual of the Lie algebra from the first Hamiltonian structure of the coadjoint orbit method. For this purpose we show that the structure of the Lie algebra g is inherited to the algebra of vector fields on g^* through the solution of the Modified Classical Yang–Baxter equation (Classical r matrix). We study the algebra that generates the compatible Poisson brackets.

Introduction

Let D be a ring of differential operators and E be a ring of pseudo-differential operators. We have a direct sum decomposition such as

$$E = D \oplus E_{-1},$$

where E_{-1} is a subring of E consisted of pseudo-differential operators whose orders are at most -1. For $P \in E$, we abbreviate $Proje_D P$ and $Proje_{E_{-1}} P$ as P_+ and P_- respectively. Let L be a monic p^{th} order differential operator, $L = \partial^p + a_{p-1}(x)\partial^{p-1} + \cdots + a_0(x)$, where $\partial = \frac{\partial}{\partial x}$. We define the space of δ functions K such as

$$K = \left\{ \sum_{i_1, \cdots, i_m} a_{i_1 \cdots i_m} \delta^{(i_1)}(x_{i_1}) \cdots \delta^{(i_m)}(x_{i_m}) | a_{i_1 \cdots i_m} \in \mathbf{C} \right\}.$$

We regard that

$$K = \bigoplus_{n \geqq 0} \otimes^n C^{-\infty}(\mathbf{R}),$$

where $C^{-\infty}(\mathbf{R})$ is distribution of $\mathbf{R}$. Let M be a space of functional of L such as

$$M = \left\{ F(L) = \sum_{i_1, \cdots, i_m, j_1, \cdots, j_m} f^{j_1, \cdots, j_m}_{i_1, \cdots, i_m} a^{(j_1)}_{i_1}(x_{i_1}) \cdots a^{(j_m)}_{i_m}(x_{i_m}) | f^{j_1, \cdots, j_m}_{i_1, \cdots, i_m} \in K \right\}.$$

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We call M as phase space. The phase space M is generated by

$$a_{p-1}(x_{p-1}),\ldots,a_0(x_0), \quad x_{p-1},\ldots,x_0 \in \mathbf{R}$$

in the following sense

$$f^{j_1, \ ,j_m}_{i_1, \ ,i_m} a^{(j_1)}_{i_1}(x_{i_1})\cdots a^{(j_m)}_{i_m}(x_{i_m})$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f^{j_1, \ ,j_m}_{i_1, \ ,i_m}\delta(y_{i_1}-x_{i_1})\cdots\delta(y_{i_m}-x_{i_m})\times a^{(j_1)}_{i_1}(y_{i_1})$$

$$\cdots a^{(j_m)}_{i_m}(y_{i_m})dy_{i_1}\cdots dy_{i_m}$$

$$= f^{j_1, \ ,j_m}_{i_1, \ ,i_m}\int_{-\infty}^{\infty}\delta(y_{i_1}-x_{i_1})a^{(j_1)}_{i_1}(y_{i_1})dy_{i_1}\cdots\int_{-\infty}^{\infty}\delta(y_{i_m}-x_{i_m})a^{(j_m)}_{i_1}(y_{i_m})dy_{i_m}$$

$$= f^{j_1, \ ,j_m}_{i_1, \ ,i_m}\int_{-\infty}^{\infty}(-)^{j_1}\delta^{(j_1)}(y_{i_1}-x_{i_1})a_{i_1}(y_{i_1})dy_{i_1}$$

$$\cdots\int_{-\infty}^{\infty}(-)^{j_m}\delta^{(j_m)}(y_{i_m}-x_{i_m})a_{i_m}(y_{i_m})dy_{i_m}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f^{j_1, \ ,j_m}_{i_1, \ ,i_m}(-)^{j_1+ \ +j_m}\delta^{(j_1)}(y_{i_1}-x_{i_1})$$

$$\cdots\delta^{(j_m)}(y_{i_m}-x_{i_m})a_{i_1}(y_{i_1})\cdots a_{i_m}(y_{i_m})dy_{i_1}\cdots dy_{i_m}.$$

Then we only have to consider the functional such as

$$F(L) = \sum_{i_1, \ ,i_m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{i_1, \ ,i_m}(x_{i_1},\ldots,x_{i_m})a_{i_1}(x_{i_1})\cdots a_{i_m}(x_{i_m})dx_{i_1}\cdots dx_{i_m},$$

where $f_{i_1, \ ,i_m}(x_{i_1},\ldots,x_{i_m}) \in K$. Thus we can regard the functions $a_{p-1}(x_{p-1}),\ldots,$ $a_0(x_0)$ $x_{p-1},\ldots,x_0 \in \mathbf{R}$ as generators of M. If $F \in M$ has the parameter x, we call F as function of x and sometime; denote $F(x)$. We define the functional derivative $\frac{\delta}{\delta_{a_i}(x)}$ such as $\frac{\delta_{a_j}(y)}{\delta_{a_i}(x)} = \delta_{i,j}\delta(x-y)$ and

$$\frac{\delta F(L)}{\delta a_i(x)} = \sum_{i_1, \ ,i_m}\sum_{\mu=1}^{m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{i_1, \ ,i_m}a_{i_1}(x_{i_1})\cdots a_{i_{\mu-1}}(x_{i_{\mu-1}})a_{i_{\mu+1}}(x_{i_{\mu+1}})$$

$$\cdots a_{i_m}(x_{i_m})\times\delta_{i_\mu,i}\delta(x-x_{i_\mu})dx_{i_1}\cdots dx_{i_m}.$$

From the above definition $\frac{\delta F(L)}{\delta a_i(x)}$ has parameter x. Then it is legitimate to write $\frac{\delta F(L)}{\delta a_i(x)}$ as $\frac{\delta F(L)}{\delta a_i}(x)$. For $P \in E$, we write P_{-1} as coefficient of ∂^{-1} of P. The inner product of E is defined as follows:

$$\langle P,Q\rangle = \int_{-\infty}^{\infty}(PQ)_{-1}dx, \quad P,Q \in E.$$

Put $Z = z_{p-1}(x)\partial^{p-1} + \cdots z_0(x)$. We define the gradient $\nabla F(L)$ by

$$\left.\frac{d}{dt}\right|_{t=0} F(L+tZ) = \langle Z, \nabla F(L)\rangle.$$

It is easy to see that $\nabla F(L) = \sum_{i=0}^{p-1} \partial^{-i-1} \frac{\delta F(L)}{\delta a_i}(x)$. For $F(L), G(L) \in M$, we define the Poisson bracket by [1],

$$\{F, G\} = \langle L, [\nabla F, \nabla G]\rangle.$$

By the following property of the bracket we only have to calculate on generators,

$$\{F(L), G(L)\} = \sum_{0\leqq i,j\leqq p-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta F}{\delta a_i}(x)\frac{\delta G}{\delta a_j}(y)\{a_i(x), a_j(y)\}dx\,dy. \tag{0.1}$$

In other words, we can see the Poisson bracket as a contravariant skew symmetric 2-tensor

$$\omega^1(dF, dG) = \sum_{0\leqq i,j\leqq p-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega_{ij}^1(x,y)\frac{\delta F}{\delta a_i}(x)\frac{\delta G}{\delta a_j}(y)dx\,dy,$$

where $\omega^1(x,y) = \{a_i(x), a_j(y)\}$ and $dF = \sum_{0\leqq i\leqq p-1}\int_{-\infty}^{\infty} \frac{\delta F}{\delta a_i}(x)da_i(x)$. By definition, $\nabla a_i(x) = \partial_z^{-i-1}\delta(x-z)$. Then we have

$$\begin{aligned}\{a_i(x), a_j(y)\} &= \langle L, [\partial_z^{-i-1}\delta(x-z), \partial_z^{-j-1}\delta(y-z)]\rangle \\ &= \sum_{k-\mu=i+j+1} \binom{k-i-1}{\mu} a_k(y)\delta^{(\mu)}(x-y) \\ &\quad - \binom{k-j-1}{\mu} a_k(x)\delta^{(\mu)}(y-x).\end{aligned}$$

Notice that the resulting Poisson structure is linear with respect to the coefficients of L. A vector field v on M is defined as follows:

$$v(F(L)) = \sum_{i=0}^{p-1}\int_{-\infty}^{\infty} v_i(x)\frac{\delta F}{\delta a_i}(x)dx.$$

We mean that $v(L) = \sum_{j=0}^{p-1} v(a_j(x))\partial^j$. Furthermore we see that $v(a_j(x)) = \int_{-\infty}^{\infty} v_j(y)\delta(x-y)dy = v_j(x)$. Then we have $v(L) = \sum_{j=0}^{p-1} v_j(x)\partial^j$ and $v(F(L)) = \langle v(L), \nabla F(L)\rangle$. We define the Hamiltonian vector field $X_H^{\omega^1}$ for $H \in M$ by

$$X_H^{\omega^1}(G) = \{H, G\}, \quad G \in M.$$

Notice that

$$X_H^{\omega^1}(G) = \langle L, [\nabla H, \nabla G]\rangle = \langle [L, \nabla H]_+, \nabla G\rangle.$$

On the other hand $X_H^{\omega^1}(G)$ is equal to $\langle X_H^{\omega^1}(L), \nabla G\rangle$. This leads us to

$$X_H^{\omega^1}(L) = [L, \nabla H]_+. \qquad (0.2)$$

In general, on the manifold X, the Schouten bracket $[\omega, \eta]$ is defined as follows, where ω, η are contravariant skew symmetric k and l tensors respectively.

$$i([\omega, \eta])t = ((-)^{kl+l} i(\omega) d i(\eta) + (-)^k i(\eta) d i(\omega))t,$$

for any covariant skew symmetric $k+l-1$ tensor t, where i is inner derivative and d is exterior derivative. The reader can refer precise definition of i and d in the next section. In particular ω is 1-form, that is, ω is a vector field v on X, the Schouten bracket $[v, \eta]$ is called Lie derivative of η with respect to v. By easy calculation, we see that the contravariant skew symmetric ω defines a Poisson structure on X if and only if $[\omega, \omega] = 0$. Since L is a p^{th} monic differential operator, one can construct $B_n \in E$ satisfying

$$[B_n, L] = -L^{n+1}, \quad n \geqq -1,$$

where the coefficients of B_n are differential polynomials of that of L. It is easy to see that $v_n(L) = [-B_{n-}, L]$, $n \geqq -1$ define the vector fields on M. Adler and Moerbeke shows the following facts.

Theorem 0.1. [3].

(1) $$[v_n, v_m] = (m-n)v_{m+n},$$

(2) $$X_H^{[v_k, \omega^1]}(L) = -(k+1)(L(\nabla H L^k)_- - (L^k \tilde{\nabla} H)_- L) \quad k \geqq -1,$$

where $\tilde{\nabla} H \in E$ is defined by $[L, \tilde{\nabla} H] = [L, \nabla H]_+$.

In particular they show that $X_H^{[v_1, \omega^1]}$ is a vector field of second Hamiltonian structure of KdV equation defined by Gel'fand–Dikki [4–6]. Put $\omega^k = \frac{-1}{k}[v_{k-1}, \omega^1], k \geqq 1$. They show that $\omega^1, \omega^2, \ldots$ define the compatible Poisson structures.

Theorem 0.2. [3]. *Put $\omega = \lambda_1 \omega^1 + \cdots + \lambda_k \omega^k$, where $\lambda_1, \ldots, \lambda_k \in \mathbf{C}$. It holds that*

$$[\omega, \omega] = 0.$$

Roughly speaking, Theorem 0.2 is induced from (1) of Theorem 0.1.

Let $L = \partial + a_1(x)\partial^{-1} + a_2(x)\partial^{-2} + \cdots$ be a Lax operator of the KP hierarchy. We can define the phase space M as in the previous case. By Watanabe, the first Hamiltonian structure is defined on the Lax operator of the KP hierarchy [16]. To get the second Hamiltonian structure of the KP hierarchy systematically, it is natural to consider to apply the method of Adler and Moerbeke. To apply this method to the Lax operator of the KP hierarchy, there is an obstacle. In the case of $L \in D$, L satisfies $L_+ = L$ and $L_- = 0$. These properties are necessary to prove Theorem 0.1. Although the Lax operator of the KP hierarchy does not have these properties. One can easily show that $R = Proj_D - Proj_{E_{-1}}$ satisfies the Modified Classical Yang

Baxter Equation (MCYBE)

$$[RX, RY] - R([RX, Y] + [X, RY]) = -[X, Y], \quad X, Y \in E. \tag{0.3}$$

The motivation of this paper is to find suitable $R \in \operatorname{End} E$ satisfying (0.3) that the operator $\frac{1}{2}(R+1)P$ and $\frac{-1}{2}(R-1)P$ taking the place of P_+ and P_- for $P \in E$ to avoid the obstacle mentioned above. To this purpose we study what relation the vector fields should satisfy to generate the compatible Poisson structures like Theorem 0.2 in the general Lie algebra.

Let g be an infinite dimensional Lie algebra and $R \in \operatorname{End} g$ is the classical r matrix, that is, R satisfies (0.3). If one assumes $R^2 = 1$, then g is decomposed into the eigenspaces of g_+ and g_- of R, where $g_+ = \{x \in g | Rx = x\}$ and $g_- = \{x \in g | Rx = -x\}$. Since R is the classical r matrix, $g_\pm$ are Lie subalgebras. In this case $\frac{1}{2}(R+1)$ and $\frac{-1}{2}(R-1)$ are projection to g_+ and g_- respectively. From $R = \frac{1}{2}(R+1) - \frac{-1}{2}(R-1)$, R is the difference of the projection. In this paper we study a little more complicated case. We assume that R has three eigenvalues $1, 0, -1$ and g is decomposed into the corresponding eigenspace, $g = g_+ \oplus g_0 \oplus g_-$. Since R is a classical r matrix, $g_\pm$ and g_0 are Lie subalgebras, especially g_0 is abelian. Moreover we assume that the invariant and nondegenerate inner product $\langle\, ,\rangle$ is defined in g. Since R satisfies (0.3), g_+ and g_- are isotropic and g_0 is orthogonal to $g_\pm$ with respect to $\langle\, ,\rangle$. We can choose the generators of $g_+, e_1, e_2, \ldots$, that of $g_-, f_1, f_2, \ldots$, and that of $g_0, h_1, h_2, \ldots$ satisfying

$$\langle e_i, f_j\rangle = \delta_{ij}, \qquad \langle h_i, h_j\rangle = \delta_{ij}.$$

Put $L = L_1 e_1 + L_2 e_2 + \cdots \in g_+$. We denote the commutative algebra over $\mathbf{C}$ generated by $L_1, L_2, \ldots$ as A. For $F(L) \in A$, we define $\nabla F(L)$ by $\frac{dF(L+tZ)}{dt}|_{t=0} = \langle Z, \nabla F(L)\rangle$, for $Z \in g_+$. In this case $\nabla F(L) = \sum_{i \geqq 1} \frac{\delta F(L)}{\delta L_i} f_i$. The Poisson bracket on A is defined by

$$\{F, G\} = -\langle L, [\nabla F(L), \nabla G(L)]\rangle. \tag{0.4}$$

Let ω^1 be a contravariant skew symmetric 2-tensor which corresponds to $\{\, ,\}$. Furthermore we define the Hamiltonian vector field for $H \in A$, $X_H^{\omega^1}(F) = \{H, F\}$. By (0.4) and invariance of $\langle\, ,\rangle$, we have

$$X_H^{\omega^1}(L) = -R_+([L, \nabla H]) \mod g_0,$$

where $R_+ = \frac{1}{2}(R+1)$. If there is $B \in A \otimes_{\mathbf{C}} g$ such as $[B, L] \in g_+$. One can see that $v(L) = [R_-(B), L]$ is a vector field on A, where $R_- = \frac{1}{2}(R-1)$. Let $[v, \omega^1]$ be the Lie derivative of ω^1 with respect to v. We have

$$X_H^{[v,\omega^1]}(L) = R_+\left(\left[R_-\left(\frac{dB}{dL} - \left(\frac{dB}{dL}\right)^T\right)(R_+([L, \nabla H])), L\right]\right) \mod g_0.$$

In this paper we do not treat the associative algebra but the Lie algebra. Thus we can not define L^n. For this reason we consider $B_{-1}, B_0, B_1, \ldots \in g$ such as

$$[B_n, L] = K_n(L) \in g_+. \tag{0.5}$$

Instead of considering L^n, we impose the following 2-conditions on K_n.

(i) The invariance of vector fields on L with respect to dK_n, $n \geqq -1$, that is, if $v(L)$ is vector fields on L then $dK_n(v) = v$.

It is easy to see that $[B_n, K_m(L)] = dK_m(K_n(L)) \in g_+$. Then we assume

(ii) $$dK_m(K_n(L)) = \sum_{i=-1}^{m+n} b^i_{mn} K_i(L), \quad b^i_{mn} \in \mathbf{C}.$$

Put $B_n = \sum_{i\geqq 1} B^i_n e_i + \sum_{i \geqq 1} \tilde{B}^i_n f_i$. Under the situation $[B_n, L] = K_n(L) \in g_+$, we can determine the coefficients of $B^i_n, \tilde{B}^i_n, i = 1, 2, \ldots$ From assumption (ii), the commutation relations of B_n's are obtained such as

$$[B_m, B_n] = \sum_{k=1}^{m+n} a^k_{mn} B_k,$$

where $a^k_{mn} = b^k_{nm} - b^k_{mn}$. We define the vector fields $v_n(L)$ by $v_n(L) = [R_-(B_n), L]$, $n \geqq -1$. With some technical conditions, we have the following results.

The commutation relations of $B_n, n \geqq -1$ are inherited to v_n, $n \geqq -1$,

(I) $$[v_m, v_n] = -\sum_{k=-1}^{m+n} a^k_{mn} v_k.$$

Put $\omega^{k+1} = [v_k, \omega^1]$, $k \geqq 0$. Then ω^k, $k \geqq 1$ define the compatible Poisson structures, that is, for any linear combinations of $\omega = \lambda_1 \omega^1 + \cdots + \lambda_k \omega^k$ it holds that

(II) $$[\omega, \omega] = 0.$$

Section 1. Let g be an infinite dimensional Lie algebra and R be an element of $\mathrm{End}\, g$ satisfying the Modified Classical Yang–Baxter Equation (MCYBE),

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -[x, y], \quad x, y \in g. \tag{1.1}$$

We suppose that g is decomposed into the eigenspace of R such as

$$g = g_+ \oplus g_0 \oplus g_-\ ,$$

where

$$g_\pm = \{x \in g | Rx = \pm x\}, \qquad g_0 = \{x \in g | Rx = 0\}.$$

Let $\langle\, , \rangle$ be an invariant nondegenerate inner product on g. We also assume that R is skew symmetric with respect to $\langle\, , \rangle$, i.e. $\langle Rx, y\rangle = -\langle x, Ry\rangle$, $x, y \in g$. We denote R_+ and R_- as $R_\pm = \frac{(R\pm 1)}{2}$. Notice that $R_+ x = 0$, $x \in g_-$, $R_- x = 0$, $x \in g_+$ and $R_\pm x = \pm\frac{1}{2}x$, $x \in g_0$. We also notice that $R_+ x = x$ (resp. $R_- x = -x$) implies $x \in g_+$ (resp. $x \in g_-$).

Proposition 1. *The eigenspaces g_+ and g_- are subalgebras of g. Moreover g_0 is abelian.*

Proof. It is easy to see that

$$[R_\pm x, R_\pm y] = R_\pm [x,y]_R, \quad x,y \in g,$$

where $[x,y]_R = \frac{1}{2}([Rx,y]+[x,Ry])$. If $x,y \in g_+, [x,y] = [R_+x, R_+y] = R_+[x,y]_R$. Notice that

$$\frac{1}{2}([Rx,y]+[x,Ry]) = \frac{1}{2}([x,y]+[x,y]) = [x,y].$$

Then we see that $R_+[x,y]=[x,y]$ for $x,y \in g_+$. It implies $[x,y] \in g_+$. We can show g_- to be a subalgebra in the same way. Suppose $x,y \in g_0$, then

$$[x,y] = 4[R_+x, R_+y] = 2R_+([Rx,y]+[x,Ry]) = 0. \qquad \text{Q.E.D.}$$

Proposition 2. $[g_\pm, g_0] \subset g_\pm$.

Proof. Suppose $x \in g_+$ and $y \in g_0$. Then we have

$$R_+[x,y] = 2R_+\frac{1}{2}([Rx,y]+[x,Ry]) = 2[R_+x, R_+y] = 2\left[x, \frac{y}{2}\right] = [x,y]\,.$$

We can show $[g_-, g_0] \subset g_-$ in the same way. Q.E.D.

Proposition 3. *Each g_+ and g_- are isotropic with respect to $\langle\,,\rangle$. Moreover g_0 is orthogonal to $g_\pm$.*

Proof. Assume that $x,y \in g_+$. From skew symmetry of R, we see that $\langle x,y\rangle = \langle R_+x, y\rangle = -\langle x,\ R_-y\rangle$. Since $y \in g_+$, $R_-y=0$, then $\langle x,y\rangle = 0$. We can show the case of g_- in the same way. Suppose $x \in g_+$ and $y \in g_0$. Thus.

$$\langle x,y\rangle = \langle Rx,y\rangle = \langle x,-Ry\rangle = \langle x,0\rangle = 0.$$

We can show $\langle x,y\rangle = 0$, where $x \in g_-$, $y \in g_0$, in the same way. Q.E.D.

Proposition 4. *We can choose the basis of $g, \{e_n\}_{n=1}^\infty \subset g_+$, $\{f_n\}_{n=1}^\infty \subset g_-$ and $\{h_n\}_{n=1}^\infty \subset g_0$ such as $\langle e_i, f_j\rangle = \delta_{ij}$ and $\langle h_i, h_j\rangle = \delta_{ij}$.*

Proof. At first we take $e_1 \neq 0$. From the assumption of nondegeneracy of $\langle\,,\rangle$, we can take $f_1 \in g_-$, such that $\langle e_1, f_1\rangle \neq 0$. We normalize f_1 to be $\langle e_1, f_1\rangle = 1$. We take e_2 according to the following two cases. Let us write g_+ as $g_+ = M \oplus \mathbf{C}e_1$. If f_1 is orthogonal to every element of M, we take an arbitrary element from M as e_2. Thus $\langle f_1, e_2\rangle = 0$. If there exists the element of $M, \tilde{e}_2$, such as $\langle f_1, \tilde{e}_2\rangle \neq 0$. Put $e_2 = \tilde{e}_2 - \langle f_1, \tilde{e}_2\rangle e_1$. Then $\langle e_2, f_1\rangle = 0$. From nondegeneracy of $\langle\,,\rangle$, we can take the element of $g_-, \tilde{f}_2$, such as $\langle \tilde{f}_2, e_2\rangle \neq 0$. Put $f_2 = \tilde{f}_2 - \langle \tilde{f}_2, e_1\rangle f_1$. Then it follows that $\langle e_1, f_2\rangle = 0$ and $\langle e_2, f_2\rangle = \langle e_2, \tilde{f}_2\rangle \neq 0$. We normalize f_2 to be $\langle e_2, f_2\rangle = 1$. To choose e_3 and f_3, we again consider according to the following two cases.

Let us write $g_+ = N \oplus \mathbf{C}e_2 \oplus \mathbf{C}e_1$. If f_2 is orthogonal to N, we take an arbitrary element from N as $\tilde{e}_3$. Put $e_3 = \tilde{e}_3 - \langle f_1, \tilde{e}_3\rangle e_1$. Then $\langle e_3, f_1\rangle = 0$. Moreover $e_3 \in N \oplus \mathbf{C}e_1$, then $\langle e_3, f_2\rangle = 0$. If there exists an element of $N, \tilde{e}_3$, such as $\langle \tilde{e}_3, f_2\rangle \neq 0$, put $e_3 = \tilde{e}_3 - \langle \tilde{e}_3, f_2\rangle e_2 - \langle \tilde{e}_3, f_1\rangle e_1$. Then it holds that $\langle e_3, f_2\rangle = \langle e_3, f_1\rangle = 0$. By the non-degeneracy of $\langle\,,\rangle$, we can take f_3 such that $\langle e_3, f_3\rangle \neq 0$. We normalize f_3 to be $\langle e_3, f_3\rangle = 1$. We can choose $e_4, e_5, \ldots$ and $f_4, f_5, \ldots$ in the same manner. Let $\tilde{h}_1, \tilde{h}_2, \ldots$ be the basis of g_0. If $\langle \tilde{h}_1, \tilde{h}_1\rangle \neq 0$, put $h_1 = \tilde{h}_1/\langle \tilde{h}_1, \tilde{h}_1\rangle^{\frac{1}{2}}$. In the case of $\langle \tilde{h}_1, \tilde{h}_1\rangle = 0$, we can choose $\tilde{h}_i$ such that $\langle \tilde{h}_1, \tilde{h}_i\rangle \neq 0$ by virtue of non-degeneracy of $\langle\,,\rangle$. We exchange $\tilde{h}_2$ and such $\tilde{h}_i$ whose index is smallest. If $\langle \tilde{h}_2, \tilde{h}_2\rangle \neq 0$, we exchange $\tilde{h}_1$ and $\tilde{h}_2$. We consider the case of $\langle \tilde{h}_2, \tilde{h}_2\rangle = 0$. Put

$$h_1 = \tilde{h}_1 + \frac{1}{2\langle \tilde{h}_1, \tilde{h}_2\rangle}\tilde{h}_2,$$

then we have $\langle h_1, h_1\rangle = 1$. We project $\tilde{h}_i, i \geqq 2$ to the orthogonal complement of h_1 such as $\tilde{h}_i - \langle \tilde{h}_i, h_1\rangle h_1$. Then $\langle h_1, \tilde{h}_i\rangle = 0$, $i \geqq 2$. If $\langle \tilde{h}_2, \tilde{h}_2\rangle \neq 0$, we put $h_2 = \tilde{h}_2/\langle \tilde{h}_2, \tilde{h}_2\rangle^{\frac{1}{2}}$. We consider the case of $\langle \tilde{h}_2, \tilde{h}_2\rangle = 0$. By the non-degeneracy of $\langle\,,\rangle$, there exists $\tilde{h}_i, i > 2$ such that $\langle \tilde{h}_2, \tilde{h}_i\rangle \neq 0$. We exchange $\tilde{h}_3$ and such $\tilde{h}_i$ whose index is smallest, that is, $\langle \tilde{h}_2, \tilde{h}_3\rangle \neq 0$. If $\langle \tilde{h}_3, \tilde{h}_3\rangle \neq 0$, we change $\tilde{h}_2$ and $\tilde{h}_3$. We consider the case of $\langle \tilde{h}_3, \tilde{h}_3\rangle = 0$. Put $h_2 = \tilde{h}_2 + \frac{1}{2\langle \tilde{h}_2, \tilde{h}_3\rangle}\tilde{h}_3$. We see that $\langle h_1, h_2\rangle = 0$ and $\langle h_2, h_2\rangle = 1$. We can define $h_3, h_4, \ldots$, in the same way. Q.E.D.

Put $[e_i, e_j] = \sum_{k \geqq 1} c_{ij}^k e_k$ and $[f_i, f_j] = \sum_{k \geqq 1} \tilde{c}_{ij}^k f_k$.

Proposition 5. *It holds that*

$$[e_i, f_j] = \sum_{k \geqq 1} \tilde{c}_{jk}^i\, e_k - c_{ik}^j f_k \quad \mathrm{mod}\, g_0.$$

Proof. Put $[e_i, f_j] = \sum_{k \geqq 1} d_{ij}^k e_k + \tilde{d}_{ij}^k f_k + a$, where $a \in g_0$. One sees that

$$\langle [e_i, e_j], f_k\rangle = \sum_{l \geqq 1} c_{ij}^l \langle e_l, f_k\rangle = c_{ij}^k.$$

On the other hand, from the invariance of $\langle\,,\rangle$, one sees that

$$\langle [e_i, e_j], f_k\rangle = \langle e_i, [e_j, f_k]\rangle = \left\langle e_i, \sum_{l \geqq 1} d_{jk}^l e_l + \tilde{d}_{jk}^l f_l\right\rangle = \tilde{d}_{jk}^i.$$

Thus we see that $c_{ij}^k = \tilde{d}_{jk}^i$. We can show $\tilde{c}_{ij}^k = -d_{kj}^i$ in the same way. Q.E.D.

Put $L = L_1 e_1 + L_2 e_2 + \cdots \in g_+$. We consider the commutative algebra $A = \mathbf{C}[[L_1, L_2, \ldots]]$. For the element $F \in A$, we define $\nabla F(L) \in A \otimes_{\mathbf{C}} g_-$, such as $\frac{d}{dt}\big|_{t=0} F(L + tZ) = \langle Z, \nabla F(L)\rangle$, where $Z = Z_1 e_1 + Z_2 e_2 + \cdots$. Notice that $\nabla F(L) = \sum_i \frac{\partial F}{\partial L_i} f_i$. We introduce the Poisson structure as follows. For $F, G \in A$, the Poisson bracket is defined by $\{F, G\} = \frac{1}{2}\langle L, [R\nabla F, \nabla G] + [\nabla F, R\nabla G]\rangle = -\langle L, [\nabla F, \nabla G]\rangle$. From the calculation, $\{F, G\} = \sum_{i,j} \frac{\partial F}{\partial L_i}\frac{\partial G}{\partial L_j}\{L_i, L_j\}$, we can regard the Poisson bracket as a contravariant skew symmetric 2-tensor. We identify the Poisson bracket defined above with $\omega^1 = \sum_{i,j} \omega_1^{ij} \frac{\partial}{\partial L_i} \wedge \frac{\partial}{\partial L_j}$. The Hamiltonian vector fields associated with

$H \in A$ defined by $X_H^{\omega^1}(F) = \{H, F\}$ satisfy

$$X_H^{\omega^1}(L) = -R_+([L, \nabla H]) \mod g_0.$$

We consider the complex of contravariant alternating forms with the coefficient A. Let a_1 be a space of vector fields on A. We consider the de Rham complex over a_1. Let Ω^q be the space of covariant alternating q-forms over A. The exterior derivative $d : \Omega^q \to \Omega^{q+1}$ is defined as follows:

$$d\omega(X_1, \dots, X_{q+1}) = \sum_{i=1}^{q+1} (-)^{i+1} X_i \omega(X_1, \dots, \hat{X}_i, \dots, X_{q+1})$$
$$+ \sum_{i<j} (-)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}),$$

where $X_1, \dots, X_{q+1}$ are elements of a_1. Note that $d^2 = 0$. For $X \in a_1$, the inner derivative $i_X : \Omega^q \to \Omega^{q-1}$ is defined as follows:

$$i_X \omega(X_1, \dots, X_{q-1}) = \omega(X, X_1, \dots, X_{q-1}).$$

Put $\Omega = \oplus_{q \geqq 0} \Omega^q$. We call (Ω, d) the de Rham complex. We denote $\wedge^q a_1$ as a space of skew symmetric q-tensors of a_1. Moreover we denote $\wedge a_1 = \oplus_{q \geqq 0} \wedge^q a_1$. In order to introduce the bracket product in $\wedge a_1$, we use some new notations. The operator defined below is a generalization of the inner product. For $\omega \in \wedge^q a_1$, the operator $i_\omega : \Omega^p \to \Omega^{p-q}$ is defined as follows:

$$i_\omega t(X_1, \dots, X_{p-q}) = t(\omega, X_1, \dots, X_{p-q}),$$

where $X_1, \dots, X_{p-q} \in a_1$ and $t \in \Omega^p$. For $\omega \in \wedge^p a_1$ and $\eta \in \wedge^q a_1$, the Schouten bracket $[\omega, \eta] \in \wedge^{p+q-1}$ is defined as follows [12, 13]. For any $t \in \Omega^{p+q-1}$

$$i_{[\omega,\eta]} t = (-)^{pq+q} i_\omega d i_\eta t + (-)^p i_\eta d i_\omega t.$$

This definition is well defined because of the following lemma.

Lemma 6. *The operator i_ω, $\omega \in \wedge^q a_1$ is non-degenerate, that is, $i_\omega t = 0$ for any $t \in \Omega^q$ implies $\omega = 0$.*

Proof. Put $t_{i_1, \ ,i_q} = dL_{i_1} \wedge \cdots \wedge dL_{i_q}$. Then it is easy to see that

$$t_{i_1, \ ,i_q}\left(\frac{\partial}{\partial L_{j_1}}, \dots, \frac{\partial}{\partial L_{j_q}}\right) = \pm \delta_{\{i_1, \ ,i_q\},\{j_1, \ ,j_q\}},$$

where $\delta_{I,J}$ is Kronecker's delta with respect to the finite set I and J. Thus $i_\omega t_{i_1, \ ,i_q} = \pm \omega^{i_1, \ ,i_q}$. Then $i_\omega t = 0$ for any $t \in \Omega^q$ implies $\omega = 0$. Q.E.D.

It is easy to see that the Schouten bracket satisfies the following relation:

$$[\omega, \eta] = (-)^{pq} [\eta, \omega],$$

$$(-)^{pr} [[\omega, \eta], \xi] + (-)^{pq} [[\eta, \xi], \omega] + (-)^{qr} [[\xi, \omega], \eta] = 0,$$

where $\omega \in \wedge^p a_1$, $\eta \in \wedge^q a_1$ and $\xi \in \wedge^r a_1$. We call the second formula a Jacobi identity of the Schouten bracket. Suppose that $\omega \in \wedge^2 a_1$ satisfies $[\omega, \omega] = 0$, then

ω defines the Poisson bracket. For $v \in a_1$ and $\omega \in \wedge^2 a_1$, the Schouten bracket $[v, \omega]$ is called the Lie derivative of ω with respect to v. By easy calculation, we see that

$$[v,\omega]^{ij} = v\omega^{ij} - \sum_k \omega^{kj} \frac{\partial v^i}{\partial L_k} - \sum_k \omega^{ik} \frac{\partial v^j}{\partial L_k}.$$

In line with [3], we calculate the Lie derivative $[v, \omega^1]$, where the vector field v is defined such as $v(L) = [R_-(B), L]$, where $[B, L] \in g_+$, $B \in A \otimes_{\mathbf{C}} g$.

Lemma 7. *It holds that*

$$X_H^{[v,\omega]} = [v, X_H^{\omega}] - X_{vH}^{\omega}.$$

Proof. Put $v = \sum_k v^k \frac{\partial}{\partial L_k}$ and $\omega = \sum_{i,j} \omega^{ij} \frac{\partial}{\partial L_i} \wedge \frac{\partial}{\partial L_j}$. We see that

$$\begin{aligned} vX_H^{\omega} &- X_H^{\omega} v - X_{vH}^{\omega} \\ &= \sum_k \sum_{i,j} v^k \frac{\partial}{\partial L_k} \omega^{ij} \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_j} - \omega^{ij} \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_j} v^k \frac{\partial}{\partial L_k} - \omega^{ij} \frac{\partial}{\partial L_i} \left(v^k \frac{\partial H}{\partial L_k} \right) \frac{\partial}{\partial L_j} \\ &= \sum_{i,j,k} v^k \frac{\partial \omega^{ij}}{\partial L_k} \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_j} - \omega^{ij} \frac{\partial v^k}{\partial L_j} \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_k} - \omega^{ij} \frac{\partial v^k}{\partial L_i} \frac{\partial H}{\partial L_k} \frac{\partial}{\partial L_j} \\ &= \sum_{i,j,k} \left(v^k \frac{\partial \omega^{ij}}{\partial L_k} - \omega^{ik} \frac{\partial v^j}{\partial L_k} - \omega^{kj} \frac{\partial v^i}{\partial L_k} \right) \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_j} = \sum_{i,j} [v,\omega]^{ij} \frac{\partial H}{\partial L_i} \frac{\partial}{\partial L_j}. \qquad \text{Q.E.D.} \end{aligned}$$

Recall that $v(L) = [R_-(B), L]$ and $[B, L] \in g_+$.

Proposition 8. *It holds that*

$$X_H^{[v,\omega^1]}(L) = R_+ \left(\left[R_- \left(\frac{dB}{dL} - \left(\frac{dB}{dL} \right)^T \right) (R_+[L, \nabla H]), L \right] \right) \mod g_0,$$

where $(\frac{dB}{dL})^T$ is the adjoint operator of $\frac{dB}{dL}$ with respect to $\langle , \rangle$.

Proof. We first show that $v(L) = [R_-(B), L]$ defines a vector field on A.

Lemma 9. *It holds that*

$$[R_-(B), L] \in g_+.$$

Proof. We may show $R_-[R_-(B), L] = 0$. We see that

$$R_-[R_-(B), L] = \frac{1}{2} R_-([R(B), L] - [B, L]),$$

since $[B, L] = [B, RL] \in g_+$,

$$= \frac{1}{2} R_-[R(B), L] = \frac{1}{2} R_-([R(B), L] + [B, RL]),$$

since R satisfies MCYBE,

$$= [R_-(B), R_-L] = [R_-(B), 0] = 0. \qquad \text{Q.E.D.}$$

By Lemma 7,

$$\begin{aligned} X_H^{[v,\omega^1]}(L) &= vX_H^{\omega^1}(L) - X_H^{\omega_1}v(L) - X_{vH}^{\omega^1}(L) \\ &= -vR_+[L, \nabla H] - X_H^{\omega^1}[R_-(B), L] + R_+[L, \nabla vH] \quad \mathrm{mod}_{g_0} \\ &= -R_+[vL, \nabla H] - R_+[L, v\nabla H] - [R_-(X_H^{\omega_1}B), L] - [R_-(B), X_H^{\omega_1}(L)] \\ &\quad +R_+[L, \nabla vH] \quad \mathrm{mod}\, g_0, \end{aligned}$$

since $X_H^{[v,\omega^1]}(L)$ is vector field,

$$\begin{aligned} &= -R_+[[R_-(B), L], \nabla H] - R_+[L, v\nabla H] - [R_-(X_H^{\omega_1}B), L] \\ &\quad +R_+[R_-B, R_+[L, \nabla H]] + R_+[L, \nabla vH] \quad \mathrm{mod}\, g_0. \end{aligned} \tag{1.2}$$

Notice that

$$R_+[R_-(B), R_+[L, \nabla H]] = R_+[R_-(B), [L, \nabla H]] - R_+[R_-(B), R_-[L, \nabla H]].$$

For any two $p, q \in g$, we can decompose such as $p = p_+ + p_0 + p_-$ and $q = q_+ + q_0 + q_-$, where $p_+, q_+ \in g_+$, $p_0, q_0 \in g_0$ and $p_-, q_- \in g_-$. We see that

$$\begin{aligned} [R_- p, R_- q] &= \left[-\frac{1}{2}p_0 - p_-, -\frac{1}{2}q_0 - q_-\right] \\ &= \frac{1}{2}[p_0, q_-] + \frac{1}{2}[q_-, p_0] + [q_-, p_-] \in g_-. \end{aligned}$$

Then we have $R_+[R_-(B),\ R_+[L, \nabla H]] = R_+[R_-(B), [L, \nabla H]]$. We proceed with the calculation

$$\begin{aligned} (1.2) &= -R_+([[R_-(B), L], \nabla H] + [[L, \nabla H], R_-(B)]) \\ &\quad -R_+[L, v\nabla H - \nabla vH] - [R_-(X_H^{\omega_1}B), L] \quad \mathrm{mod}\, g_0 \\ &= R_+([[\nabla H, R_-B] + v\nabla H - \nabla vH, L]) - [R_-(X_H^{\omega_1}B), L] \quad \mathrm{mod}\, g_0. \end{aligned} \tag{1.3}$$

We calculate $v\nabla H - \nabla vH$ independently of [3]. Notice that

$$v\nabla H - \nabla vH = \sum_{i,j} v^i \frac{\partial}{\partial L_i}\frac{\partial H}{\partial L_j} f_j - \frac{\partial}{\partial L_j} v^i \frac{\partial H}{\partial L_i} f_j = -\sum_{i,j} \frac{\partial v^i}{\partial L_j}\frac{\partial H}{\partial L_i} f_j.$$

By definition,

$$v(L) = [R_-(B), L] = \sum_i [R_-(B), L]_i e_i = \sum_i v^i e_i.$$

Thus we have

$$\frac{\partial v^i}{\partial L_j} = \frac{\partial}{\partial L_j}[R_-(B),L]_i = \left(\frac{\partial}{\partial L_j}[R_-(B),L]\right)_i = \left(\left[\frac{\partial}{\partial L_j}R_-(B),L\right]\right.$$

$$\left.+\left[R_-(B),\frac{\partial}{\partial L_j}L\right]\right)_i = \left\langle f_i,\left[\frac{\partial}{\partial L_j}R_-(B),L\right]+[R_-(B),e_j]\right\rangle.$$

Then we see that

$$v\nabla H - \nabla vH = \sum_{i,j} - \left\langle f_i,\left[\frac{\partial R_-(B)}{\partial L_j},L\right]\right\rangle \frac{\partial H}{\partial L_i} f_j - \langle f_i,[R_-(B),e_j]\rangle \frac{\partial H}{\partial L_i} f_j.$$

By calculation, we see that

$$\sum_{i,j} - \langle f_i,[R_-(B),e_j]\rangle \frac{\partial H}{\partial L_i} f_j = -\sum_{i,j}\left\langle f_i\frac{\partial H}{\partial L_i},[R_-(B),e_j]\right\rangle f_j$$

$$= -\sum_j \langle \nabla H,[R_-(B),e_j]\rangle f_j = -\sum_j \langle [\nabla H,R_-(B)],e_j\rangle f_j$$

$$= -\sum_j [\nabla H,R_-(B),]_j f_j = -[\nabla H,R_-(B)].$$

Moreover we have

$$-\sum_{i,j}\left\langle f_i,\left[\frac{\partial R_-(B)}{\partial L_j}\right]\right\rangle \frac{\partial H}{\partial L_i} f_j = -\sum_{i,j}\left\langle f_i\frac{\partial H}{\partial L_i},\left[\frac{\partial R_-(B)}{\partial L_j},L\right]\right\rangle f_j$$

$$= -\sum_j\left\langle \nabla H,\left[\frac{\partial R_-(B)}{\partial L_j},L\right]\right\rangle f_j = -\sum_j\left\langle \frac{\partial R_-B}{\partial L_j},[L,\nabla H]\right\rangle f_j$$

$$= -\sum_{i,j} R_+([L,\nabla H])_i\left(\frac{\partial R_-B}{\partial L_j}\right)_i f_j.$$

Furthermore we see that

$$\frac{dR_-(B)}{dL}(e_i) = \sum_j\left(\frac{dR_-(B)}{dL}\right)_j (e_i) f_j = \sum_j \langle \nabla(R_-(B)_j),e_i\rangle f_j$$

$$= \sum_{k,j}\left\langle \frac{\partial R_-(B)_j}{\partial L_k} f_k,e_i\right\rangle f_j = \sum_j \frac{\partial R_-(B)_j}{\partial L_i} f_j = \frac{\partial R_-(B)}{\partial L_i}.$$

Then we have

$$-\sum_{i,j} R_+([L,\nabla H])_i\left(\frac{\partial R_-(B)}{\partial L_j}\right)_i f_j = -\sum_{i,j} R_+([L,\nabla H])_i \frac{dR_-(B)}{dL}(e_j)_i f_j$$

$$= -\sum_j\left\langle \frac{dR_-(B)}{dL}(e_j),R_+([L,\nabla H])\right\rangle f_j$$

$$= -\sum_j\left\langle e_j,\left(\frac{dR_-(B)}{dL}\right)^T(R_+([L,\nabla H]))\right\rangle f_j.$$

Notice that

$$\frac{dR_-(B)}{dL}(Z) = \lim_{\varepsilon\to 0} \frac{R_-(B)(L+\varepsilon Z) - R_-(B)(L)}{\varepsilon}$$
$$= R_-\left(\lim_{\varepsilon\to 0} \frac{B(L+\varepsilon Z) - B(L)}{\varepsilon}\right) = R_-\left(\frac{dB}{dL}\right)(Z).$$

This fact yields

$$-\sum_j \left\langle e_j, \left(\frac{dR_-(B)}{dL}\right)^T (R_+[L,\nabla H])\right\rangle f_j = -\sum \left\langle e_j, R_-\left(\frac{dB}{dL}\right)^T (R_+[L,\nabla H])\right\rangle f_j.$$

Since T and R_- commute, we have

$$-\sum_j \left\langle e_j, R_-\left(\frac{dB}{dL}\right)^T (R_+[L,\nabla H])\right\rangle f_j$$
$$= -\sum_j \left\langle e_j, R_-\left(\left(\frac{dB}{dL}\right)^T\right) (R_+[L,\nabla H])\right\rangle f_j$$
$$= -R_-\left(\left(\frac{dB}{dL}\right)^T\right)(R_+[L,\nabla H]).$$

Then we have

$$v\nabla H - \nabla vH = -[\nabla H, R_-(B)] - R_-\left(\left(\frac{dB}{dL}\right)^T\right)(R_+[L,\nabla H]).$$

Finally we get

$$(1.2) = R_+\left(\left[[\nabla H, R_-(B)] - [\nabla H, R_-(B)] - R_-\left(\left(\frac{dB}{dL}\right)^T\right)(R_+[L,\nabla H]), L\right]\right)$$
$$-[R_-(X_H^{\omega_1}(B)), L] \mod g_0,$$

$$= -R_+\left(\left[R_-\left(\left(\frac{dB}{dL}\right)\right)^T (R_+[L,\nabla H]), L\right]\right) - R_+[R_-(X_H^{\omega^1}(B)), L]$$
$$-R_-[R_-(X_H^{\omega^1}(B)), L] \mod g_0. \qquad (*)$$

Note that

$$R_-(X_H^{\omega^1}(B)) = \sum_i X_H^{\omega^1}(B_i) f_i = \sum_i \left\langle X_H^{\omega^1}(L), \nabla B_i\right\rangle f_i.$$

Although equality $X_H^{\omega^1}(L) = -R_+[L,\nabla H]$ has ambiguity of modulo g_0, g_0 is orthogonal to ∇B_i. Then we have $R_-(X_H^{\omega^1}(B)) = -R_-(\frac{dB}{dL}(R_+[L,\nabla H]))$. Thus we see

$$(*) = R_+\left(\left[R_-\left(\frac{dB}{dL}-\left(\frac{dB}{dL}\right)^T\right)(R_+[L,\nabla H]),L\right]\right)$$

$$+R_-[R_-(X_H^{\omega^1}(B)),L] \mod g_0.$$

Since $X_H^{[v,\omega_1]}$ is a vector field on g_+, we have

$$X_H^{[v,\omega_1]}(L) = R_+\left(\left[R_-\left(\frac{dB}{dL}-\left(\frac{dB}{dL}\right)^T\right)(R_+[L,\nabla H]),L\right]\right) \mod g_0. \qquad \text{Q.E.D.}$$

We consider the vector fields which preserve the Poisson structure. Let $\{B_n\}_{n\geqq -1}$ be elements of g such that

$$[B_n,L] = K_n(L) \in g_+ \quad n \geqq -1.$$

We imagine $K_n(L)$ like L^{n+1}. Since the algebra g is not associative but a Lie algebra, we cannot define L^{n+1}. Instead of explicit realization of $K_n(L)$, we assume the following two conditions: (i) If $v = v(L)$ is a vector field on L, then v is also a vector field on $K_n(L)$ and $dK_n(v) = v$, $n \geqq -1$. Before stating the second assumption for $K_n(L)$, $n \geqq -1$, we show the fact $[B_n, K_m(L)] \in g_+$. We define the vector fields $v_n(L)$, $n \geqq -1$ such as

$$v_n(L) = [R_-(B_n),L].$$

From Lemma 9, v_n, $n \geqq -1$ are vector fields on L. We decompose $[B_n, K_m(L)]$ into 2 parts as follows:

$$[B_n,K_m(L)] = [R_+(B_n),K_m(L)] - [R_-(B_n),K_m(L)].$$

It is clear that $[R_+(B_n),K_m(L)] \in g_+$. Furthermore we see that

$$[R_-(B_n),K_m(L)] = dK_m(v_n) = v_n(K_m(L)) \in g_+.$$

On the other hand we see that

$$[B_n,K_m(L)] = \sum_{i\geqq 1} K_m^i(L)[B_n,e_i] = dK_m([B_n,L]) = dK_m(K_n(L)).$$

The second assumption is

$$\text{(ii)} \qquad dK_m(K_n(L)) = \sum_{i=-1}^{m+n} b_{mn}^i K_i(L),\ b_{mn}^i \in \mathbf{C} \quad i = -1,\dots m+n.$$

Put $B_n = \sum_{i\geqq 1} B_n^i e_i + \sum_{i\geqq 1} \tilde{B}_n^i f_i$. Under the condition $[B_n,L] = K_n(L) \in g_+$, we determine the coefficients B_n^i and $\tilde{B}_n^i$ in the localization of $A = \mathbf{C}[[L_1,L_2,\dots]]$ at $(0,0,\dots)$. We see that

$$0 = R_-[B_n,L] = R_-[R_+B_n,R_+L] - R_-[R_-B_n,L] = R_-R_+([B_n,L]_R) - R_-[R_-B_n,L].$$

From this one can see $R_-[R_-B_n, L] = 0 \mod g_0$. Expanding B_n and L with respect to the basis of g such as

$$-R_-[R_-B_n, L] = R_-\left[\sum_{j\geqq 1}\tilde{B}_n^j f_j, \sum_{i\geqq 1} L_i e_i\right] = \sum_{i,j\geqq 1}\tilde{B}_n^i L_j R_-[f_j, e_i]$$

$$= \sum_{k\geqq 1}\sum_{i,j\geqq 1}\tilde{B}_n^j L_i c_{ik}^j f_k \mod g_0.$$

Put $A_{ij} = \sum_{\mu\geqq 1} L_\mu c_{\mu j}^i$. Furthermore we assign $\tilde{B}_n^1$ the role of moduli. Then we have the system of the equation,

$$(\tilde{B}_n^2, \tilde{B}_n^3, \dots)\begin{pmatrix} A_{21} & A_{22} & \cdots \\ A_{31} & A_{32} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = -\tilde{B}_n^1(A_{11}, A_{12}, \dots).$$

By Cramer's formula, we have

$$\tilde{B}_n^i = -\tilde{B}_n^1 \det\begin{pmatrix} A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \ddots \\ A_{i-1,1} & A_{i-1,2} & \cdots \\ A_{11} & A_{12} & \cdots \\ A_{i+1,1} & A_{i+1,2} & \cdots \\ \vdots & & \ddots \end{pmatrix} / \det(A_{\mu\nu})_{\mu\geqq 2, \nu\geqq 1} \quad i \geqq 2.$$

Moreover we have

$$K_n(L) = [B_n, L] = \sum_{\mu\geqq 1}\left(\sum_{j\geqq 1, i\geqq 1} B_n^j L_i c_{ji}^\mu\right) e_\mu - \sum_{\mu\geqq 1}\left(\sum_{j\geqq 1, i\geqq 1}\tilde{B}_n^j L_i \tilde{c}_{j\mu}^i\right) e_\mu.$$

Put $A'_{ij} = \sum_{\mu\geqq 1} L_\mu c_{i\mu}^j$ and $D_l = \sum_{j\geqq 1, i\geqq 1}\tilde{B}_n^j L_i \tilde{c}_{jl}^i + K_n^l(L)$. Then we have

$$(B_n^1, B_n^2, \dots)\begin{pmatrix} A'_{11} & A'_{12} & \cdots \\ A'_{12} & A'_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = (D_1, D_2, \dots).$$

By Cramer's formula we have

$$B_n^j = \det\begin{pmatrix} A'_{11} & A'_{12} & \cdots \\ \vdots & \vdots & \ddots \\ A'_{j-1,1} & A'_{j-1,2} & \cdots \\ D_1 & D_2 & \cdots \\ A'_{j+1,1} & A'_{j+1,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} / \det(A'_{\mu\nu})_{\mu,\nu\geqq 1} \quad j \geqq 1.$$

From condition (ii) of $K_n(L)$, we can calculate the commutation relations for B_n's as follows:

$$\begin{aligned} [[B_m,B_n],L] &= -[B_n,[B_m,L]] + [B_m,[B_n,L]] = -[B_n,K_m(L)] + [B_m,K_n(L)] \\ &= -dK_m(K_n(L)) + dK_n(K_m(L)) = \sum_{i=-1}^{m+n} (b_{nm}^i - b_{mn}^j)K_i(L) \\ &= \left[\sum_{i=-1}^{m+n} (b_{nm}^i - b_{mn}^i)B_i, L\right]. \end{aligned}$$

Then we have $[B_m,B_n] = \sum_{i=-1}^{m+n} a_{mn}^i B_i$, where $a_{mn}^i = b_{nm}^i - b_{mn}^i$. We show the following rather general theorem.

Theorem 10. *Suppose that* $R \in \operatorname{End} g$ *satisfies* MCYBE (1.1). *Then it holds that*

$$[v_i, v_j] = -\sum_{k\geqq -1} a_{ij}^k v_k.$$

Proof. We first show the following lemma.

Lemma 11. *It holds that*

$$v_n(B_m) = [R_-(B_n), B_m].$$

Proof. It is easy to see that

$$v_n(B_m) = \frac{dB_m}{dL}(v_n(L)) = \frac{dB_m}{dL}([R_-(B_n),L]).$$

Taking the differentials of $[B_m,L] = K_m(L)$, we have

$$[dB_m,L] + [B_m,dL] = dK_m(dL).$$

Then we have

$$dB_m = ad_L^{-1} ad_{B_m} dL - ad_L^{-1} dK_m(dL).$$

From the fact that $dB_m = \frac{dB_m}{dL}(dL)$, it holds that

$$\begin{aligned}
&\frac{dB_m}{dL}([R_-(B_n),L]) \\
&= ad_L^{-1}[B_m,[R_-(B_n),L]] - ad_L^{-1}dK_m([R_-(B_n),L]) \\
&= -ad_L^{-1}[L,[B_m,R_-(B_n)]] - ad_L^{-1}[R_-(B_n),[L,B_m]] - ad_L^{-1}K_m([R_-(B_n),L]) \\
&= [R_-(B_n),B_m] + ad_L^{-1}[R_-(B_n),K_m(L)] - ad_L^{-1}dK_m([R_-(B_n),L]). \qquad (1.4)
\end{aligned}$$

By definition, the vector field $dK_m(v_n)$ acts $K_m(L)$ such as

$$dK_m(v_n)K_m(L) = \sum_{i,j} v_n^i \frac{\partial K_m^j}{\partial L_i}(L)e_j = \sum_j \langle \nabla K_m^j(L), v_n(L)\rangle e_j.$$

On the other hand, we have

$$[R_-(B_n),K_m(L)] = \sum_j [R_-(B_n),K_m^j(L)e_j] = \sum_j K_m^j(L)[R_-(B_n),e_j] = v_n(K_m(L)).$$

Since $v_n = dK_m(v_n)$, we have

$$[R_-(B_n),K_m(L)] = dK_m([R_-(B_n),L]).$$

Thus we have $v_n(B_m) = [R_-(B_n),B_m]$. Q.E.D.

We proceed with the proof of Theorem 10. From Lemma 11, we have

$$\begin{aligned}
&[v_m,v_n](L) \\
&= v_m(v_n(L)) - v_n(v_m(L)) = v_m([R_-(B_n),L]) - v_n([R_-(B_m),L]) \\
&= [R_-(v_m(B_n)),L] + [R_-(B_n),v_m(L)] - [R_-(v_n(B_m)),L] - [R_-(B_m),v_n(L)] \\
&= [R_-([R_-(B_m),B_n]),L] + [R_-(B_n),[R_-(B_m),L]] \\
&\quad -[R_-([R_-(B_n),B_m]),L] - [R_-(B_m),[R_-(B_n),L]] \\
&= [R_-([R_-(B_m),B_n]) + [R_-(B_n),R_-(B_m)] + R_-([B_m,R_-(B_n)]),L].
\end{aligned}$$

Furthermore we see that

$$\begin{aligned}
&R_-([R_-(B_m),B_n]) + [R_-(B_n),R_-(B_m)] + R_-[B_m,R_-(B_n)] \\
&= \frac{1}{4}\{-2R([B_m,B_n]) + [RB_n,RB_m] - R[B_n,RB_m] - R[RB_n,B_m] + [B_m,B_n]\},
\end{aligned}$$

by MCYBE

$$= \frac{1}{4}\{-2R([B_m,B_n]) - [B_n,B_m] + [B_m,B_n]\} = -R_-[B_m,B_n]$$

$$= -\sum_{k\geqq -1} a^k_{mn} R_-(B_k).$$

Then we have $[v_m, v_n] = -\sum_{k\geqq -1} a^k_{mn} v_k$. Q.E.D.

In [3], they show that the vector fields on the differential operator which satisfy the Virasoro relations preserve the Poisson structure. We also introduce the vector fields to preserve the Poisson bracket whose commutation relations are a generalization of Virasoro. In [3], they construct the pseudo-differential operators B_n, $n \geqq -1$, satisfying

$$[B_n, L] = -L^{n+1} \quad n \geqq -1.$$

Furthermore they construct the vector fields satisfying the Virasoro relation such as

$$v_n(L) = [-B_{n_-}, L].$$

However we show that the algebra of vector fields $v_n(L) = [R_-(B_n), L]$, $n \geqq -1$ generate the compatible Poisson structures.

We exchange a^l_{mn} for $-a^l_{mn}$ in the assumption (ii) of $K_n(L)$. Then the commutation relations are

$$[B_m, B_n] = -\sum_{i=-1}^{m+n} a^i_{mn} B_i \tag{1.5}$$

and

$$[v_m, v_n] = \sum_{l=-1}^{m+n} a^l_{mn} v_l. \tag{1.6}$$

In the commutation relations for B_n's, we assume non-degeneracy, that is, $a^{m+n}_{mn} \neq 0$. We define the contravariant 2-tensor ω^k, $k \geqq 2$ and assume some properties like [3] such as

$$\omega_{k+1} = [v_k, \omega^1], \quad k \geqq 1$$

and

$$[v_{-1}, \omega] = 0, \qquad \omega \in \operatorname{span}\{\omega^k, k \geqq 2, \quad [v_i, \omega^j], \quad i+j \geqq 2\}$$

implies $\omega = 0$ while $[v_{-1}, \omega^1] = 0$.

Theorem 12. *The Lie derivative of ω^n with respect to v_m is equal to the linear combination of $\omega^1, \ldots, \omega^{m+n}$, that is,*

$$[v_m, \omega^n] = A_{m+n}\omega^{m+n} + \cdots + A_1\omega^1.$$

Before we prove Theorem 12, we apply this theorem to show that ω^k, $k \geqq 1$ define the compatible Poisson brackets.

Proposition 13. *It holds that $[\omega^i, \omega^j] = 0$, $i, j \geqq 1$.*

Proof. From the definition and Jacobi identity of Schouten bracket, we see that

$$[\omega^n, \omega^1] = [[v_{n-1}, \omega^1], \omega^1] = -[[\omega^1, \omega^1], v_{n-1}] - [[\omega^1, v_{n-1}], \omega^1]. \tag{1.7}$$

Since ω^1 defines the Poisson structure, $[\omega^1, \omega^1] = 0$, then we have

$$[\omega^n, \omega^1] = -[[\omega^1, v_{n-1}], \omega^1] = -[\omega^n, \omega^1].$$

This implies $[\omega^n, \omega^1] = 0$. Next, we calculate the general case,

$$[\omega^m, \omega^n] = [[v_{m-1}, \omega^1], \omega^n] = -[[\omega^1, \omega^n], v_{m-1}] - [[\omega^n, v_{m-1}], \omega^1].$$

From the previous calculation, $[\omega^1, \omega^n] = 0$, then the first term vanishes. By Theorem 12, $[\omega^n, v_{m-1}]$ is equal to a linear combination of $\omega^1, \dots, \omega^{m+n}$. Then the second term also vanishes. Q.E.D.

Proof of Theorem 12. We show this theorem by 3 steps.

Step 1. We show at first

$$[v_{-1}, \omega^{k+1}] = a_{-1,k}^{k-1}\omega^k + \cdots + a_{-1,k}^{0}\omega^1.$$

We see that

$$[v_{-1}, \omega^{k+1}] = [v_{-1}, [v_k, \omega^1]],$$

by the Jacobi identity,

$$= [v_k, [\omega^1, v_{-1}]] + [\omega^1, [v_{-1}, v_k]]$$

$$= [\omega^1, a_{-1,k}^{k-1} v_{k-1} + \cdots + a_{-1,k}^{0} v_0] = \sum_{i=1}^{k} a_{-1,k}^{i-1}\omega^i.$$

Step 2. The two assumptions,

$$[v_j, \omega^n] = A_{j+n}\omega^{j+n} + \cdots + A_1\omega^1, \quad -1 \leqq j \leqq m-1,$$

$$[v_m, \omega^k] = B_{m+k}\omega^{m+k} + \cdots + B_1\omega^1, \quad 1 \leqq k \leqq n-1$$

imply

$$[v_m, \omega^n] = C_{m+n}\omega^{m+n} + \cdots + C_1\omega^1,$$

where A_i, B_i and $C_i \in \mathbf{C}$.

By the Jacobi identity and Step 1, we have

$$[[v_m, \omega^n], v_{-1}]$$

$$= [[\omega^n, v_{-1}], v_m] + [[v_{-1}, v_m], \omega^n]$$

$$= [v_m, a_{-1,n-1}^{n-2}\omega^{n-1} + \cdots + a_{-1,n-1}^{0}\omega^1] + [a_{-1,m}^{m-1} v_{m-1} + \cdots + a_{-1,m}^{-1} v_{-1}, \omega^n]$$

$$= a_{-1,n-1}^{n-2}[v_m, \omega_{n-1}] + \cdots + a_{-1,n-1}^{0}[v_m, \omega_1] + a_{-1,m}^{m-1}[v_{m-1}, \omega^n]$$

$$+ \cdots + a_{-1,m}^{-1}[v_{-1}, \omega^n],$$

by assumption of induction,

$$= C_{m+n-1}\omega^{m+n-1} + \cdots + C_1\omega^1. \tag{1.8}$$

From Step 1, we see that

$$(1.8) = \frac{C_{m+n-1}}{a^{m+n-2}_{-1,m+n-1}}[v_{-1},\omega^{m+n}] + \tilde{C}_{m+n-2}\omega^{m+n-2} + \cdots + \tilde{C}_1\omega^1.$$

Using Step 1 again and again we have

$$\begin{aligned}(1.8) &= \frac{C_{m+n-1}}{a^{m+n-2}_{-1,m+n-1}}[v_{-1},\omega^{m+n}] + \frac{\tilde{C}_{m+n-2}}{a^{m+n-3}_{-1,m+n-2}}[v_{-1},\omega^{m+n-1}] \\ &\quad + \cdots + \tilde{\tilde{C}}_2[v_{-1},\omega^2] + \tilde{\tilde{C}}_1\omega^1 \\ &= \frac{C_{m+n-1}}{a^{m+n-2}_{-1,m+n-1}}[v_{-1},\omega^{m+n}] + \frac{\tilde{C}_{m+n-2}}{a^{m+n-3}_{-1,m+n-2}}[v_{-1},\omega^{m+n-1}] \\ &\quad + \cdots + \tilde{\tilde{C}}_2[v_{-1},\omega^3] + \frac{\tilde{\tilde{C}}_1}{a^0_{-1,1}}[v_{-1},\omega^2].\end{aligned}$$

By the assumption for the kernel of $[v_{-1},\cdot\,]$, we have

$$[v_m,\omega^n] = \frac{C_{m+n-1}}{a^{m+n-2}_{-1,m+n-1}}\omega_{m+n} + \frac{\tilde{C}_{m+n-2}}{a^{m+n-3}_{-1,m+n-2}}\omega^{m+n-1} + \cdots + \tilde{\tilde{C}}_2\omega^3 + \frac{\tilde{\tilde{C}}_1}{a^0_{-1,1}}\omega^2 \quad \mathrm{mod}\,\omega^1.$$

Step 3. By Step 2 and

$$[v_0,\omega^1] = \omega^1, \qquad [v_{-1},\omega^2] = a^0_{-1,1}\omega^1,$$

we have

$$[v_0,\omega^2] = A_2\omega^2 + A_1\omega^1, \quad A_1, A_2 \in \mathbf{C}.$$

Moreover $[v_1,\omega^1] = \omega^2$ and with Step 2, we have

$$[v_1,\omega^2] = B_3\omega^3 + B_2\omega^2 + B_1\omega^1, \quad B_1, B_2, B_3 \in \mathbf{C}.$$

By the same process, we can show

$$[v_j,\omega^2] = A_{j+2}\omega^{j+2} + \cdots + A_1\omega^1, \quad A_1,\ldots,A_{j+2} \in \mathbf{C}, \quad j \geqq -1. \tag{1.9}$$

Furthermore by Step 2 and

$$[v_{-1},\omega^3] = a^1_{-1,2}\omega^2 + a^0_{-1,2}\omega^1$$

$$[v_0,\omega^2] = A_2\omega^2 + A_1\omega^1,$$

we have

$$[v_0,\omega^3] = A_3\omega^3 + A_2\omega^2 + A_1\omega^1.$$

In the same way as the previous case, we can show

$$[v_j, \omega^3] = A_{j+3}\omega^{j+3} + \cdots + A_1\omega^1, \quad A_1,\ldots,A_{j+3} \in \mathbf{C}, \quad j \geqq -1.$$

Thus we can show

$$[v_m, \omega^n] = A_{m+n}\omega^{m+n} + \cdots + A_1\omega^1. \qquad \text{Q.E.D.}$$

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