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On Finite 4D Quantum Field Theory in Non-Commutative Geometry

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Abstract: The truncated 4-dimensional sphere S^4 and the action of the self-interacting scalar field on it are constructed The path integral quantization is performed while simultaneously keeping the $SO(5)$ symmetry and the finite number of degrees of freedom The usual field theory UV-divergences are manifestly absent

1. Introduction

The basic ideas of non-commutative geometry were developed in [1, 2], and in the form of the matrix geometry in [3, 4] The applications to physical models were presented in [2, 5], where the non-commutativity was in some sense minimal the Minkowski space was not extended by some standard Kaluza–Klein manifold describing internal degrees of freedom, but just by two discrete points. The algebra of functions on this manifold remains commutative, but the complex of the differential forms does not This led to a new insight on the $SU(2)_L \otimes U(1)_R$ symmetry of the standard model of electroweak interactions The consideration of gravity was included in [6] Such models, of course, do not lead to UV-regularization, since they do not introduce any modification of the space-time short-distance behaviour

To achieve the UV-regularization one should introduce a non-commutative deformation of the algebra of functions on a space-time manifold in the Minkowski case, or on the space manifold in the Euclidean version One of the simplest locally Euclidean manifolds is the sphere S^2 Its non-commutative (fuzzy) deformation was described by [7, 8] in the framework of the matrix geometry. A more general construction of some non-commutative homogeneous spaces was described in [9] using coherent-states technique

The first attempts to construct fields on a truncated sphere were presented in [8,10] within the matrix formulation Using a more general approach, the fields on truncated S^2 were investigated in detail in [11–13] In particular, in [11] it was the

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quantum scalar field on the truncated S^2 and it was explicitly demonstrated that the UV-regularization automatically takes place upon the non-commutative deformation of the algebra of functions

In this article we extend this approach from the 2-dimensional sphere S^2 to the 4-dimensional one Since S^4 is not a (co)-adjoint orbit, this extension has some new nontrivial features We shall introduce only the necessary notions of the non-commutative geometry we needed for our approach

In Sect. 2 we describe briefly the standard (commutative) sphere S^4 as the Hopf fibration $S^7 \to S^4$ and the scalar self-interacting field on it Section 3 is devoted to the generalization of the model to the non-commutative truncated sphere S^4 introducing the non-commutative analogue of the Hopf fibration Then, using Feynman (path) integrals, we perform the quantization of the model in question Last, Sect 4 contains a brief discussion and concluding remarks

2. Scalar Field on the Commutative S^4

Here we describe the standard sphere S^4 in the form that will be suitable for the non-commutative generalization Our basic tools are the real quaternions

$$\varphi = \varphi_{(a)} e_a \in \mathbf{H} \tag{1}$$

with $\varphi_{(a)}$ real and the quaternionic units

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

satisfying the relations

$$e_i e_j = -\delta_{ij} - \varepsilon_{ijk} e_k, \qquad e_4 e_i = e_i e_4 \tag{3}$$

We shall usually write 1 instead of e_4 The coefficient $\varphi_{(0)} = \frac{1}{2}\mathrm{tr}\,\phi$ is called the real part of the quaternion, and $\varphi_{(i)}$, $i = 1, 2, 3$, are pure quaternionic components The explicit realization (2) of the quaternionic units allows us to identify the space of quaternions with $\mathbf{C}^2$· any quaternion we represent by 2×2 complex matrix

$$\varphi = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} \tag{4}$$

The quaternionic conjugation $\varphi \to \varphi^*$ defined by

$$e_i \to e_i^* = -e_i, \quad i = 1, 2, 3, \qquad e_4 \to e_4^* = e_4\,,$$

then corresponds to the Hermitian conjugation of complex matrices We shall frequently use both descriptions without an explicit specification. Further, the quaternionic length $|\varphi|$ is defined by

$$|\varphi|^2 = \varphi^* \varphi = \varphi_{(a)}^2 = \det \varphi \tag{5}$$

If $|\varphi| = 1$, φ is called a unit quaternion. The set of unit quaternions is isomorphic to the group $SU(2)$ (and as a topological space to S^3)

The group $Sp(4)$ we identify with the group of 2×2 quaternionic matrices of the form

$$A = \begin{pmatrix} \cos\frac{\theta}{2}\alpha & \sin\frac{\theta}{2}\gamma\beta^* \\ -\sin\frac{\theta}{2}\gamma^*\alpha & \cos\frac{\theta}{2}\beta^* \end{pmatrix}, \tag{6}$$

where α, β, γ are unit quaternions, and $\theta \in [0, \pi]$ is a real angle

The Lie algebra $sp(4) = so(5)$ is spanned by 10 anti-Hermitian matrices $\xi_{AB} = -\xi_{BA}$, $A, B = 1, \quad, 5$, given as

$$\xi_{a5} = \begin{pmatrix} 0 & e_a \\ -e_a^* & 0 \end{pmatrix} = \xi_a, \qquad \xi_{ab} = \xi_a\xi_b, \tag{7}$$

where $a, b = 1, \quad, 4$, $a \neq b$. The matrices ξ_{ab} span the Lie algebra $so(4) = so(3) \oplus so(3)$ Supplementing (7) by five matrices

$$\tilde{\xi}_a = \begin{pmatrix} 0 & e_a \\ e_a^* & 0 \end{pmatrix}, \qquad \tilde{\xi}_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a = 1, \ldots, 4, \tag{8}$$

we recover the basis of the Lie algebra $su^*(4) = so(5, 1)$ It is closely related to the Clifford algebra $C^{4,0}$ with the basis ξ_a, $a = 1, \ldots, 4$·

$$C^{4,0} = \begin{pmatrix} \xi_1\xi_2\xi_3\xi_4 \\ \xi_a\xi_b\xi_c, \ 1 \leqq a < b < c \leqq 4 \\ \xi_a\xi_b, \ 1 \leqq a < b \leqq 4 \\ \xi_a, \ 1 \leqq a \leqq 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_5 \\ \tilde{\xi}_a, \ 1 \leqq a \leqq 4 \\ \xi_{ab}, \ 1 \leqq a < b \leqq 4 \\ \xi_a, \ 1 \leqq a \leqq 4 \\ 1 \end{pmatrix}, \tag{9}$$

where the matrices ξ_a, ξ_{ab} are anti-Hermitian whereas the matrices $\tilde{\xi}_A$, $A = 1, \quad, 5$, are Hermitian and transform as an $SO(5)$ vector

The matrices $A \in Sp(4)$ act in a natural way in the space $\mathbf{H}^2$

$$z = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \in \mathbf{H}^2 \to Az = \begin{pmatrix} \tilde{a}\varphi + \tilde{b}\chi \\ \tilde{c}\varphi + \tilde{d}\chi \end{pmatrix} \in \mathbf{H}^2 \tag{10}$$

The sphere S^7, given by the equation

$$z^+ z = |\varphi|^2 + |\chi|^2 = 1, \tag{11}$$

is transitively invariant under this action Introducing the equivalence relation

$$z \sim z' = z\alpha, \quad \alpha - \text{unit quaternion}, \tag{12}$$

we recover the sphere S^4 as the Hopf fibration $S^7 \to S^4$ To any equivalence class (13) we assign the $SO(5)$ vector given by the Cartesian coordinates in $\mathbf{R}^5$:

$$x_A = \frac{1}{2}\mathrm{tr}(z^+ \tilde{\xi}_A z) = \frac{1}{2}\mathrm{tr}(z'^+ \tilde{\xi}_A z') \tag{13}$$

These are just the Cartesian coordinates of the sphere S^4 embedded into $\mathbf{R}^5$ (similar objects were used in [8] within a relativistic context)

As $\mathscr{A}_\infty$ we denote the commutative algebra of analytic functions (polynomials) in the variables x_A, $A = 1, \ldots, 5$

$$\Phi(x) = \sum A_M x^M, \quad A_M - \text{complex}, \tag{14}$$

with the usual point-wise multiplication Here we used the multi-index notation· $M = (M_1, \quad, M_5)$, $x^M = x_1^{M_1} \cdot x_5^{M_5}$ In $\mathscr{A}_\infty$ we introduce the scalar product

$$(\Phi_1, \Phi_2)_\infty = I_\infty[\Phi_1^* \Phi_2], \tag{15}$$

where $I_\infty[\quad]$ denotes the usual $SO(5)$-invariant integral on S^4.

$$I_\infty[\ \ldots] = \frac{3}{4\pi^2} \int d^5x\ \delta(x_A^2 - 1)\ [\quad], \tag{16}$$

where the normalization guarantees that $I_\infty[1] = 1$

The $Sp(4)$ action (10) in the algebra $\mathscr{A}_\infty$ generates $\mathbf{R}^5$ rotations, leaving the quantity $x_A^2 = 1$ invariant The generators of this action (anti-Hermitian with respect to the scalar product given above) are given as

$$\hat{J}_{AB}\Phi = \frac{1}{2}(\psi_\alpha^*\ \xi_{AB}^{\alpha\beta}\ \partial_{\psi_\beta^*} + \psi_\beta\ \xi_{AB}^{\alpha\beta}\ \partial_{\psi_\alpha})\Phi \tag{17}$$

Here $\xi_{AB}^{\alpha\beta}$ are elements of the 4×4 complex matrix assigned to the 2×2 quaternionic matrix ξ_{AB}, and ψ_γ, ψ_α^*, $\alpha = 1, \ldots, 4$, are complex variables identified with the elements of complex matrices assigned to the quaternions φ and χ in the following way

$$\begin{gathered} \psi_1 = \varphi_1,\ \psi_2 = \varphi_2,\ \psi_3 = \chi_1,\ \psi_4 = \chi_2, \\ \psi_1^* = \varphi_1^*,\ \psi_2^* = \varphi_2^*,\ \psi_3^* = \chi_1^*,\ \psi_4^* = \chi_2^* \end{gathered} \tag{18}$$

It follows from (17) that the quantities ψ_α, ψ_α^*, $\alpha = 1, \ldots, 4$, transform as S^4 spinors,

$$\hat{J}_{AB}\,\psi_\alpha = \frac{1}{2}\xi_{AB}^{\alpha\beta}\,\psi_\beta, \qquad \hat{J}_{AB}\,\psi_\beta^* = \frac{1}{2}\xi_{AB}^{\alpha\beta}\,\psi_\gamma^* \tag{19}$$

Consequently, the quantities x_A, $A = 1, \ldots, 5$ given as

$$x_A = \psi^+ \tilde{\xi}_A \psi = \psi_\alpha^*\ \tilde{\xi}_A^{\alpha\beta}\ \psi_\beta, \tag{20}$$

where $\tilde{\xi}_A^{\alpha\beta}$ are elements of the complex matrix assigned to $\tilde{\xi}_A$, transform as a vector in $\mathbf{R}^5$ Moreover, the function $C(x) = x_A^2$ satisfies

$$\hat{J}_{AB}C(x) = 0, \quad A, B = 1, \quad, 5, \tag{21}$$

i e $C(x)$ is an invariant function as expected

The $Sp(4)$ action (17) in the algebra $\mathscr{A}_\infty$ is reducible and we have the following expansion:

$$\mathscr{A}_\infty = \bigoplus_{p=0}^{\infty} \mathscr{A}_\infty^p, \tag{22}$$

where $\mathscr{A}_\infty^p$ is the carrier space of the irreducible representation of the $Sp(4)$ group spanned by the harmonic polynomials Ψ_μ^p of degree p in the variables $x_A, A =$

1, ..., 5 The polynomials Ψ^p_μ are orthonormal with respect to the scalar product (15). The dimension of the space $\mathscr{A}^p_\infty$ is

$$d_p = \dim \mathscr{A}^p_\infty = \frac{1}{6}(p+1)(p+2)(2p+3)\,,$$

which means that any field $\Phi \in \mathscr{A}_\infty$ can be expanded as

$$\Phi(x) = \sum_{p=0}^{\infty} \sum_{\mu=0}^{d_p} a^p_\mu \Psi^p_\mu \tag{23}$$

The field action corresponding to the real scalar field Φ is given as

$$S[\Phi] = I_\infty \left[\frac{1}{2}(\hat{J}_{AB}\Phi)^2 + V(\Phi) \right], \tag{24}$$

where $V(\cdot)$ is a polynomial bounded from below

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral over fields from $\Phi \in \mathscr{A}^R_\infty$ by

$$\langle F[\Phi] \rangle = \frac{\int D\Phi e^{-S[\Phi]} F[\Phi]}{\int D\Phi e^{-S[\Phi]}}\,, \tag{25}$$

where $D\Phi = \prod_x d\Phi(x) = \prod_{p,\mu} da^p_\mu$ (eventually, with the reality conditions for a^p_μ included).

Since here $p = 0, 1, \ldots, \infty$, the formula for the measure is only formal We shall not discuss the complicated (and not completely solved) problems related to its rigorous definition. As we shall see below, such problems do not appear in the framework of the non-commutative version of the model.

3. Scalar Field on the Non-Commutative S^4

In this section we shall use various unitary irreducible representations of the group $Sp(4)$ Any such representation is characterized by its signature (p,k) with integer $p \geqq k \geqq 0$ and can be expressed as the Young product

$$(p,k) = \pi_1^{p-k}\ \pi_2^k\,, \tag{26}$$

of $Sp(4)$ fundamental representations $\pi_1 = (1,0)$ 4-dimensional quaternionic and $\pi_2 = (1,1)$ 5-dimensional real (see e g. [14]) The dimension of the representation (p,k) is

$$d_{pk} = \frac{1}{6}(p+2)(k+1)(p-k+1)(p+k+3)\,. \tag{27}$$

In the non-commutative (fuzzy) case we replace the commuting parameters (18) by the non-commutative ones Namely, we shall express the parameters ψ_α, ψ^*_α, $\alpha = 1, \ldots, 4$ in terms of annihilation and creation operators as

$$\psi_\alpha = A_\alpha R^{-1/2}, \qquad \psi^*_\alpha = R^{-1/2} A^*_\alpha\,, \tag{28}$$

where

$$R = A^*_\alpha A_\alpha\,, \tag{29}$$

so that the condition $\psi_\alpha^*\psi_\alpha = 1$ is satisfied (the operators ψ_α are well defined everywhere, except in the vacuum, we complete the definition by postulating that they annihilate the vacuum). The operators A_α and A_α^* (* denotes the Hermitian conjugation) act in the Fock space $\mathscr{F}$ spanned by the orthonormal vectors $|n\rangle = |n_1, \ldots, n_4\rangle$ labelled by the occupation numbers n_α, $\alpha = 1, \ldots, 4$ They satisfy in $\mathscr{F}$ the commutation relations

$$[A_\alpha, A_\beta] = [A_\alpha^*, A_\beta^*] = 0, \qquad [A_\alpha, A_\beta^*] = \delta_{\alpha\beta} \tag{30}$$

The operators

$$J_{AB} = \frac{1}{2} A_\alpha^* \, \xi_{AB}^{\alpha\beta} A_\beta, \quad A, B = 1, \ldots, 5 \tag{31}$$

satisfy in the Fock space $\mathscr{F}$ the $sp(4) = so(5)$ Lie algebra commutation relations The subspace $\mathscr{F}_N$ with the fixed total occupation number

$$\mathscr{F}_N = \{|n\rangle, |n| = N\}, \quad N = 0, 1, 2, \ldots \tag{32}$$

has the dimension

$$d_{N0} = \binom{N+3}{3}, \tag{33}$$

and is the carrier space of the $Sp(4)$ unitary irreducible representation $(N, 0)$

As the $\mathscr{A}_N$ we denote the non-commutative algebra of operators $\mathscr{F}_N \to \mathscr{F}_N$, which can be expressed as polynomials

$$\Phi(x) = \sum A_M x^M, \quad A_M \text{ complex}, \tag{34}$$

in operators

$$x_A = \psi_\alpha^* \, \tilde{\xi}_A^{\alpha\beta} \, \psi_\beta = \psi^+ \tilde{\xi}_A \psi, \quad A = 1, \ldots, 5 \tag{35}$$

restricted to the space $\mathscr{F}_N$ The operators x_A, $A = 1, \ldots, 5$, form a vector in $\mathbf{R}^5$

In $\mathscr{A}_N$ we introduce the scalar product

$$(\Phi_1, \Phi_2)_N = I_N[\Phi_1^* \Phi_2], \tag{36}$$

where $I_N[\ \ ]$ is the analogue of the $SO(5)$-invariant integral on S^4:

$$I_N[\ldots] = \frac{1}{d_{N0}} \mathrm{Tr}_N[\ldots] \tag{37}$$

Here $\mathrm{Tr}_N[\ldots]$ denotes the trace in the algebra $\mathscr{A}_N$, and the normalization guarantees that $I_N[1] = 1$

As a non-commutative analogue of (18) we have a commutator action of the $sp(4)$ algebra in $\mathscr{A}_N$:

$$\hat{J}_{AB} \Phi(x) = [J_{AB}, \Phi(x)], \tag{38}$$

with J_{AB} defined in (31). This is a reducible representation with the following decomposition to $Sp(4)$ irreducible components

$$(N, 0) \otimes (N, 0) = \bigoplus_{p=0}^{N} \bigoplus_{k=0}^{p} (p + k, p - k). \tag{39}$$

This decomposition induces the decomposition of the algebra $\mathscr{A}_N$.

$$\mathscr{A}_N = \bigoplus_{p=0}^{N} \bigoplus_{k=0}^{p} \mathscr{A}_N^{p+k,p-k}, \tag{40}$$

where $\mathscr{A}_N^{p'k'}$ is the carrier space of the $Sp(4)$ representation (p',k') This means that any $\Phi \in \mathscr{A}_N$ can be expanded as

$$\Phi(x) = \sum_{p=0}^{N} \sum_{k=0}^{p} \sum_{\mu=1}^{d'_{pk}} a_\mu^{p+k,p-k} \Psi_\mu^{p+k,p-k}, \tag{41}$$

where $d'_{pk} = d_{p+k,p-k}$ and $\Psi_\mu^{p'k'}$, $\mu = 1, \ldots, d_{p'k'}$, span the space $\mathscr{A}_N^{p'k'}$.

In the commutative case, the decomposition (22) of the algebra $\mathscr{A}_\infty$ contains only representations $(p,p) = \pi_2^p$ corresponding to terms with $k = 0$ in the decomposition (40)

Note We would like to stress that it is not essential that the generators x_A, $A = 1, \ldots, 5$, given in (35) do not close to some Lie algebra (they close to a Lie algebra only after supplementing them by the operators (31)) The following point is important, however the decomposition (40) of the basic algebra $\mathscr{A}_N$ under the symmetry transformation in question (this aspect was less transparent for the truncated sphere S^2, since in this case the generators closed to a Lie algebra, see [11]) The detailed information contained in Eq. (40) is necessary for realistic numerical or symbolical calculations

We identify the space of the configurations of a real scalar field with the subspace

$$\mathscr{A}_N^R = \bigoplus_{p=0}^{N} \mathscr{A}_N^{pp}, \tag{42}$$

of symmetric polynomials in x_A, $A = 1, \ldots, 5$, with real coefficients

Such fields can be expanded as

$$\Phi(x) = \sum_{p=0}^{N} \sum_{\mu=1}^{d'_{pk}} a_\mu^p \Psi_\mu^{pp}, \tag{43}$$

where the coefficient a_μ^p are real provided that Ψ_μ^{pp} are chosen to be Hermitian (if this is not the case the coefficients a_μ^p satisfy some relations that guarantee that the field in question is a Hermitian operator in $\mathscr{F}_N$) This guarantees that in the commutative limit $N \to \infty$ we recover from (43) only fields that have the proper form (23)

In the non-commutative case the field action corresponding to the real scalar field Φ is given as

$$S[\Phi] = I_N \left[\frac{1}{2} (\hat{J}_{AB}\Phi)^2 + V(\Phi) \right], \tag{44}$$

where $V(\cdot)$ is a polynomial bounded from below Obviously, this action has the following basic properties:

1) it has the full $SO(5)$ symmetry corresponding to S^4 rotations, and

2) it describes a model with a finite number of modes since, in fact, it corresponds to a particular matrix model

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral

$$\langle F[\Phi]\rangle = \frac{\int D\Phi e^{-S[\Phi]}F[\Phi]}{\int D\Phi e^{-S[\Phi]}} \,. \tag{45}$$

However, here $D\Phi = \prod_{p,\mu} da^p_\mu$ (eventually with the reality conditions included) is the usual Lebesgue measure, since now the product is finite ($p = 0, 1, \ldots, N$, and $\mu = 1, \ldots, d'_{pp}$) The quantum mean values are well defined for any polynomial functional $F[\Phi]$.

4. Concluding Remarks

We have demonstrated that the interacting scalar field on the noncommutative sphere S^4 represents a quantum system that has the following properties

1) The model has the full $SO(5)$ space symmetry under the rotations of the sphere S^4. This is exactly the same symmetry as the interacting scalar field on the standard sphere S^4 possesses

2) The field has only a finite number of modes Then the number of degrees of freedom is finite, which leads to the non-perturbative UV-regularization, i.e all quantum mean values of polynomial field functionals are well defined and finite

In our approach the UV cut-off in the number of modes is supplemented with a highly non-trivial vertex modification due to non-trivial products of fields Our UV-regularization is non-perturbative and is completely determined by the algebra $\mathscr{A}_N$ It is originated by the short-distance structure of the space, and does not depend on the field action of the model in question.

Moreover, it can be shown that the Schwinger functions

$$S_n(F) = \langle F_n[\Phi]\rangle \,, \tag{46}$$

where $F_n[\Phi] = \sum \alpha^{p_1 \ldots p_n}_{\mu_1 \ldots \mu_n} (\Psi^{p_1}_{\mu_1}, \Phi)_N \ldots (\Psi^{p_n}_{\mu_n}, \Phi)_N$ satisfy the Osterwalder–Schrader axioms.

(OS1) *Hermiticity*

$$S^*_n(F) = S_n(\Theta F) \,,$$

where ΘF is the involution defined by $\Theta F_n[\Phi] = (F_n[\Phi])^*$.

(OS2) *Covariance*

$$S_n(F) = S_n(\mathscr{R}F) \,,$$

where $\mathscr{R}F$ is a mapping of functionals induced by $SO(5)$ rotations

(OS3) *Reflection positivity*

$$\sum_{n,m\in\mathscr{I}} S_{n+m}(\Theta F_n \otimes F_m) \geqq 0 \,.$$

(OS4) *Symmetry*

$$S_n(F) = S_n(\pi F),$$

where πF is a functional obtained from F by arbitrary permutation of indices of $\alpha^{p_1 \ldots p_n}_{\mu_1 \ldots \mu_n}$

Note We do not include the last Osterwalder–Schrader axiom, the cluster property, since the compact manifold requires a special treatment (however, it can be recovered in the limit where the radius of the sphere grows to infinity, but these considerations go beyond the presented scheme). Qualitatively, the properties of the Schwinger functions are the same as those valid for the truncated sphere S^2, see [11] We would like to stress that the properties of standard Schwinger functions not included above (e g support, or singularity and growth, specification) are essential again in the commutative limit $N \to \infty$.

The usual divergences will appear only in the commutative limit $N \to \infty$ It would be very interesting to isolate the large-N behaviour non-perturbatively By this we mean the Wilson-like approach in which the renormalization group flow in the space of Lagrangians is studied. In this context a connection may be found with similar recent works [15]

Combining the results of this paper with those of [11–13] we obtain a set of UV-regularized Euclidean quantum field models on S^2 and S^4.

a) the scalar field on the truncated S^2, which is super-renormalizable,

b) the Gross–Neveu model on the truncated S^2, which is renormalizable,

c) the scalar field on the truncated S^4 with Φ^4 interaction which is renormalizable too

Analogous models formulated on standard Euclidean planes ($\mathbf{R}^2$ or $\mathbf{R}^4$ instead of spheres) served as important examples in the proof of the existence of quantum fields in continuum Euclidean spaces in the framework of the Wilson approach (see [16, 17] for the super-renormalizable case, and [18, 19] for the renormalizable one)

We have an alternative approach. the regularization procedure is non-perturbative and preserves all space symmetries of the models in question. The UV-regularization in our scheme can be interpreted as a direct consequence of the short-distance structure induced by the non-commutative geometry of the underlying space This can lead to a better understanding of the origin and properties of divergences in quantum field theory

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References

1 Connes, A : Publ IHES **62**, 257 (1986)
2 Connes, A : Non-Commutative Geometry London: Academic Press, 1994
3 Dubois-Violette, M : C R Acad Sci Paris **307**, Ser I, 403 (1988)
4 Dubois-Violette, M , Kerner, R and Madore, J : J Math Phys **31**, 316 (1990)
5 Coquereaux, R , Esposito-Farese, G and Vaillant, G : Nucl Phys **B353**, 689 (1991)
6 Chamsedine, A H , Felder, G and Fröhlich, J : Commun Math Phys **155**, 205 (1993)
7 Madore, J : J Math Phys **32**, 332 (1991)

8 Madore, J : Non-Commutative Geometry and Applications Cambridge: Cambridge University Press, 1995
9 Grosse, H and Prešnajder, P : Lett Math Phys **33**, 171 (1995)
10 Grosse, H and Madore, J : Phys Lett **B283**, 218 (1992)
11 Grosse, H , Klimčík, C and Prešnajder, P : Int J Theor Phys **35** (1996) 231
12 Grosse, H , Klimčík, C and Prešnajder, P : Field theory on a supersymmetric lattice Preprint CERN-TH/95-195, UWThPh-20-1995, hep-th/9507074
13 Grosse, H , Klimčík, C and Prešnajder, P : Commun Math Phys **178** (1996) 507
14 Zhelobenko, D P and Stern, A I : Group representations Moscow: Nauka, 1983, in Russian
15 Kempf, A : On Quantum Field Theory with Non-Zero Minimal Uncertainties in Positions and Momenta hep-th/9602085 and references therein
16 Simon, B : The $P(\Phi)_2$ Euclidean (quantum) field theory Princeton, NJ: Princeton Univ Press, 1974
17 Glimm, J and Jaffe, A : Quantum physics A functional integral point of view New York: Springer-Verlag, 1981
18 Gawędzki, K and Kupiainen, A : Commun Math Phys **99**, 197 (1985)
19 Gawędzki, K and Kupiainen, A : Commun Math Phys **102**, 1 (1985)

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