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Integral Representations of the Macdonald Symmetric Polynomials

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Abstract: Multiple-integral representations of the (skew-)Macdonald symmetric polynomials are obtained. Some bosonization schemes for the integral representations are also constructed.

1. Introduction

The Calogero–Sutherland model [1] and its various generalizations [2, 3] have been extensively studied and these $1/r^2$ type models are known to describe systems with the generalized exclusion principle in $1+1$ dimension [4]. The Calogero–Sutherland model describes a system of non-relativistic particles on a circle under the inverse square potential. Its Hamiltonian and momentum are

$$H_{CS} = \sum_{j=1}^{N_0} \frac{1}{2}\left(\frac{1}{i}\frac{\partial}{\partial q_j}\right)^2 + \left(\frac{\pi}{L}\right)^2 \sum_{\substack{i,j=1 \\ i<j}}^{N_0} \frac{\beta(\beta-1)}{\sin^2 \frac{\pi}{L}(q_i - q_j)}, \qquad P_{CS} = \sum_{j=1}^{N_0} \frac{1}{i}\frac{\partial}{\partial q_j}, \tag{1.1}$$

where β is a coupling constant. This Calogero–Sutherland model is related to many branches of low-dimensional physics and mathematics: quantum Hall effect [5], 2D Yang–Mills theory [6, 7], matrix model [8, 9], Yangian symmetry [10, 11], Virasoro symmetry [21–23], $W_{1+\infty}$ symmetry [12], Laplace–Beltrami operator [13, 14], etc. One of the recent remarkable developments was the evaluation of some dynamical correlation functions [9, 15–18]. In these calculations the Jack symmetric polynomials [19, 20] play a central role, because they describe the excited states of the Calogero–Sutherland model. In the previous works [21–24], the free field realization of the wave functions, in other words, the integral representations of the Jack symmetric polynomials is discussed.

Several years ago, Ruijsenaars constructed a relativistic (or lattice regularized) version of the Calogero system [25]. That model is integrable, since it has mutually

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commuting hermitian operators $\hat{S}_k$ $(k = 1, 2, \ldots, N_0)$:

$$\hat{S}_k = \sum_{\substack{I \subset \{1,\ldots,N_0\} \\ |I|=k}} \prod_{\substack{i \in I \\ j \notin I}} h(q_j - q_i)^{\frac{1}{2}} \cdot \exp\left(\rho \sum_{j \in I} \frac{1}{i} \frac{\partial}{\partial q_j}\right) \prod_{\substack{i \in I \\ j \notin I}} h(q_i - q_j)^{\frac{1}{2}}, \tag{1.2}$$

where $h(q) = \sigma(q + \mu)/\sigma(q)$ and $\rho \in \mathbf{R}$, $\mu \in \mathbf{C}$. Here $\sigma(z)$ denotes the Weierstrass σ-function defined by

$$\sigma(z) = z \prod_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq (0,0)}} \left(1 - \frac{z}{\Omega_{m,n}}\right) \exp\left(\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}}\right),$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$ and $2\omega_1$ and $2\omega_3$ denote primitive periods. The relation to the Calogero–Sutherland model is the following:

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \left(\frac{\hat{S}_1 + \hat{S}_{N_0}^{-1} \hat{S}_{N_0-1}}{2} - N_0\right) = \sum_{j=1}^{N_0} \frac{1}{2} \left(\frac{1}{i} \frac{\partial}{\partial q_j}\right)^2 + \beta(\beta - 1) \sum_{\substack{i,j=1 \\ i<j}}^{N_0} \wp(q_i - q_j), \tag{1.3}$$

$$\lim_{\rho \to 0} \frac{1}{\rho} \left(\frac{\hat{S}_1 - \hat{S}_{N_0}^{-1} \hat{S}_{N_0-1}}{2}\right) = \sum_{j=1}^{N_0} \frac{1}{i} \frac{\partial}{\partial q_j}, \tag{1.4}$$

where we set $\mu = i\beta\rho$ and $\wp(z)$ is the Weierstrass $\wp$-function given by $\wp(z) = -\frac{d}{dz}\zeta(z)$, $\zeta(z) = \frac{d}{dz}\sigma(z)/\sigma(z)$. If we consider the case $2\omega_1 = L$, $\omega_3 = i\infty$, we have

$$\wp(z) = \left(\frac{\pi}{L}\right)^2 \left(\frac{1}{\sin^2 \frac{\pi}{L} z} - \frac{1}{3}\right).$$

Thus the system reduces to the model defined by (1.1).

To solve Ruijsenaars' system, we need an explicit formula for the simultaneous eigenfunctions of $\hat{S}_k$'s. When the $\wp$-function degenerates to the trigonometric function, the commuting operators $\hat{S}_k$'s essentially degenerate Macdonald's operators [19]. Therefore, the eigenfunctions are given by the Macdonald symmetric polynomials. (As for definitions of the Macdonald symmetric polynomials, see the following sections.) In this article, we construct integral representations of the Macdonald symmetric polynomials and construct some boson realization schemes of the integral formula. These results are considered as natural deformation theories of the previous works on the Jack symmetric polynomials [21–24]. We hope that the general elliptic case may be treated in a similar manner.

This paper is organized as follows. In Sect. 2 we give a short summary of the Macdonald symmetric functions. In Sect. 3 we derive multiple integral representation formulas for the Macdonald symmetric polynomials by using two types of maps. Moreover using the isomorphism between the ring of symmetric functions and the boson Fock space, we derive the integral representations of the skew Macdonald polynomials and the Kostka matrix. In Sect. 4 we construct two bosonization formulas for the integral representation of the Macdonald symmetric polynomials, following the idea of the work by Jing [27]. Two cases ($\beta \in \mathbf{Z}_{>0}$ and $\beta \in \mathbf{C}$) are discussed separately. The A-type structure and finite temperature calculation method are used respectively. Section 5 is devoted to discussions.

2. Brief Review of Macdonald Symmetric Functions

In this section we review some basic facts about the Macdonald symmetric functions [19].

2.1. Notations and the Scalar Product $\langle\,,\,\rangle_{q,t}$. Let q, t be independent indeterminates and $\Lambda_{n;\mathbf{Q}(q,t)}$ be the ring of symmetric functions in n variables $(x_1,\ldots,x_n)$ over the field of rational functions in q and t. We sometimes write $t=q^\beta$. In the limit of $q \to 1$ this β is understood as the coupling constant of the Calogero–Sutherland model. The ring $\Lambda_{n;\mathbf{Q}(q,t)}$ is a graded ring $\Lambda_{n;\mathbf{Q}(q,t)} = \bigoplus_{k\geqq 0} \Lambda^k_{n;\mathbf{Q}(q,t)}$, where $\Lambda^k_{n;\mathbf{Q}(q,t)}$ consists of the homogeneous symmetric polynomials of degree k, together with the zero polynomial. Introduce the homomorphism

$$\begin{aligned} \rho_{m,n} : \Lambda_{m;\mathbf{Q}(q,t)} &\longrightarrow \Lambda_{n;\mathbf{Q}(q,t)} \quad (m \geqq n)\,, \\ f(x_1,\ldots,x_m) &\mapsto f(x_1,\ldots,x_n,0,\ldots,0)\,. \end{aligned}$$

and restricting it, we have

$$\rho^k_{m,n} : \Lambda^k_{m;\mathbf{Q}(q,t)} \longrightarrow \Lambda^k_{n;\mathbf{Q}(q,t)}\,,$$

for $k \geqq 0$, $m \geqq n$, which is always surjective, and are bijective for $m \geqq n \geqq k$. Consider the inverse limit

$$\Lambda^k_{\mathbf{Q}(q,t)} = \varprojlim_n \Lambda^k_{n,\mathbf{Q}(q,t)}\,,$$

relative to the homomorphism $\rho^k_{m,n}$ and set $\Lambda_{\mathbf{Q}(q,t)} = \bigoplus_{k\geqq 0} \Lambda^k_{\mathbf{Q}(q,t)}$. We call the graded ring $\Lambda_{\mathbf{Q}(q,t)}$ as the ring of the symmetric functions.

There are various bases of the ring $\Lambda_{\mathbf{Q}(q,t)}$. They are indexed by partitions. A partition λ is a sequence $\lambda = (\lambda_1,\lambda_2,\lambda_3,\ldots)$ of non-negative integers, such that $\lambda_1 \geqq \lambda_2 \geqq \cdots$ and $|\lambda| = \sum_i \lambda_i < \infty$. The nonzero λ_i's are called the parts of λ, and the number of parts is the length $l(\lambda)$ of λ. For two partitions λ,μ, we define $\lambda+\mu = (\lambda_1+\mu_1,\lambda_2+\mu_2,\ldots)$. The natural partial ordering is defined as follows:

$$\lambda \geqq \mu \Leftrightarrow |\lambda| = |\mu| \quad \text{and} \quad \lambda_1+\cdots+\lambda_r \geqq \mu_1+\cdots+\mu_r \quad \text{for all } r \geqq 1\,. \tag{2.1}$$

A partition is identified with the Young diagram. The conjugate partition of λ, whose diagram is obtained by interchanging rows and columns, is denoted by λ'. x^λ stands for the monomial $x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\cdots$. We give some bases of $\Lambda_{\mathbf{Q}(q,t)}$:

(i) $m_\lambda = \sum_{\substack{\alpha:\,\text{distinct}\\ \text{permutation of }\lambda}} x^\alpha$, (monomial symmetric function).

(ii) $p_\lambda = p_{\lambda_1}p_{\lambda_2}\ldots$, (the r^{th} power sum $p_r = \sum_i x_i^r$).

(iii) $e_\lambda = e_{\lambda_1}e_{\lambda_2}\ldots$, (the r^{th} elementary symmetric function $e_r = \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}$).

We endow $\Lambda_{\mathbf{Q}(q,t)}$ with the following scalar product:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda(q,t)\,, \tag{2.2}$$

where

$$z_\lambda(q,t) = \prod_{r \geqq 1} r^{m_r} m_r! \cdot \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \quad \lambda = (1^{m_1} 2^{m_2} \cdots)\,, \tag{2.3}$$

with $m_r \equiv \#\{i \mid \lambda_i = r\}$.

The Macdonald symmetric function is characterized by the following existence theorem:

Theorem 2.1 [19]. *For each partition λ there is a unique symmetric function $P_\lambda = P_\lambda(x;q,t) \in \Lambda_{\mathbf{Q}(q,t)}$ such that*

$$\text{(A)}\ P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu, \quad u_{\lambda\mu} \in \mathbf{Q}(q,t)\,, \tag{2.4}$$

$$\text{(B)}\ \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu\,. \tag{2.5}$$

Even though this definition is concise, it is more useful to define the Macdonald symmetric function by introducing an operator which can be regarded as a natural deformation of the Calogero–Sutherland Hamiltonian. The following operator is called the Macdonald operator:

$$D_1 = \sum_{i=1}^{n} \left(\prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \right) T_{q,x_i}\,, \tag{2.6}$$

where T_{q,x_i} is the q-shift operator defined by $(T_{q,x_i} f)(x_1,\ldots,x_n) = f(x_1,\ldots, qx_i,\ldots,x_n)$. The other way to define the Macdonald symmetric functions is the following:

Theorem 2.2 [19]. *For each partition λ (of length $\leqq n$), there is a unique symmetric function $P_\lambda(x;q,t) \in \Lambda_{n,\mathbf{Q}(q,t)}$ satisfying the two conditions*:

$$\text{(A)}\ P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu, \quad u_{\lambda\mu} \in \mathbf{Q}(q,t)\,, \tag{2.7}$$

$$\text{(C)}\ D_1 P_\lambda = \sum_{i=1}^{n} t^{n-i} q^{\lambda_i} \cdot P_\lambda\,. \tag{2.8}$$

It was shown by Macdonald that D_1 can be included in a family of mutually commuting operators $\{D_r | r = 1,\ldots,n\}$. The operator D_r is defined by

$$D_r = \sum_{\substack{I \subset \{1,2,\ldots,n\} \\ |I|=r}} t^{r(r-1)/2} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \cdot \prod_{i \in I} T_{q,x_i}\,. \tag{2.9}$$

Theorem 2.3 [19]. *The operators D_r $(r = 1,\ldots,n)$ commute with each other and the Macdonald symmetric function P_λ is the simultaneous eigenvector of these operators with the eigenvalues given by the coefficient of X^{n-r} in $\prod_{i=1}^{n}(X + t^{n-i}q^{\lambda_i})$.*

One can notice that Macdonald's D_r and Ruijsenaars' $\hat{S}_k$ have a similar structure.

Here we list some particular cases of the Macdonald symmetric function $P_\lambda(q,t)$:

(i) When $t=q$, $P_\lambda(q,q)$ is the Schur function s_λ,
(ii) When $q=0$, $P_\lambda(0,t)$ is the Hall–Littlewood function $P_\lambda(t)$,
(iii) $\lim_{q\to 1} P_\lambda(q,q^\beta)$ is the Jack symmetric function $J_\lambda(\beta)$,
(iv) When $t=1$, $P_\lambda(q,1)=m_\lambda$,
(v) When $q=1$, $P_\lambda(1,t)=e_{\lambda'}$,
(vi) $P_\lambda(q^{-1},t^{-1})=P_\lambda(q,t)$.

Let Q_λ's be the dual basis of P_λ's, that is

$$\langle Q_\lambda(q,t),P_\mu(q,t)\rangle_{q,t}=\delta_{\lambda,\mu}\,. \tag{2.10}$$

The following proposition is easily proved:

Proposition 2.4. *We have the following equation*:

$$\sum_\lambda P_\lambda(x;q,t)Q_\lambda(y;q,t)=\Pi(x,y;q,t)\,, \tag{2.11}$$

where

$$\Pi(x,y)=\Pi(x,y;q,t)=\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\,. \tag{2.12}$$

Here we have used the following notation:

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\quad\text{for } a\in\mathbf{C}\,. \tag{2.13}$$

We remark that $\Pi(x,y)=\Pi(y,x)$ is a Taylor series in x_i and y_j in the region $|x_iy_j|<1$.

If we write

$$Q_\lambda(x;q,t)=b_\lambda(q,t)P_\lambda(x;q,t)\,, \tag{2.14}$$

then we have the explicit formula for $b_\lambda(q,t)$. To state it we need the following notation: For any $s=(i,j)\in\lambda$ (i^{th} row, j^{th} column in the Young diagram λ), let us define arm-length $a(s)$, leg-length $l(s)$, arm-colength $a'(s)$ and leg-colength $l'(s)$ as follows:

$$\begin{cases} a(s)=\lambda_i-j, & a'(s)=j-1\,,\\ l(s)=\lambda'_j-i, & l'(s)=i-1\,. \end{cases} \tag{2.15}$$

Theorem 2.5 [19]. *The explicit formula for the coefficient $b_\lambda(q,t)$ is*

$$b_\lambda(q,t)=\frac{1}{\langle P_\lambda,P_\lambda\rangle_{q,t}}=\prod_{s\in\lambda}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}\,. \tag{2.16}$$

2.2. The Dual Transformation. Let us define an automorphism

$$\omega_{q,t}:\Lambda_{\mathbf{Q}(q,t)}\to\Lambda_{\mathbf{Q}(q,t)} \tag{2.17}$$

by fixing the action on p_r as

$$\omega_{q,t}(p_r)=(-1)^{r-1}\frac{1-q^r}{1-t^r}p_r \tag{2.18}$$

and extending it naturally. We have the following theorem which describes the duality transformation of the Macdonald symmetric functions.

Theorem 2.6 [19]. *For any partition λ, we have*

$$\omega_{q,t}P_\lambda(q,t) = Q_{\lambda'}(t,q) \quad (\text{or equivalently } \omega_{q,t}Q_\lambda(q,t) = P_{\lambda'}(t,q)) . \tag{2.19}$$

It is easy to show

$$\omega^y_{q,t}\Pi(x,y;q,t) = \prod_{i,j}(1+x_iy_j) \equiv \tilde{\Pi}(x,y) , \tag{2.20}$$

where $\omega^y_{q,t}$ acts on the variable y. Hence we have

$$\sum_\lambda P_\lambda(x;q,t)P_{\lambda'}(y;t,q) = \tilde{\Pi}(x,y) . \tag{2.21}$$

2.3. The Scalar Product $\langle\,,\rangle'_{n;q,t}$. Next we consider the properties of another scalar product $\langle\,,\rangle'_{n;q,t}$ that will be defined below. We shall work with a finite number of indeterminates $x = (x_1,\ldots,x_n)$. We set the parameters q and t as $0 < q < 1$ and $0 < t < 1$. Define

$$\Delta(x) = \Delta(x;q,t) = \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} . \tag{2.22}$$

In the region $t < |x_i/x_j| < t^{-1}$ $(i\neq j)$, Δ is a Laurent series in x_i's. For $f,g \in \Lambda_{n,\mathbf{Q}(q,t)}$, we define[1]

$$\begin{aligned}\langle f,g\rangle'_{n;q,t} &= \frac{1}{n!}\oint\prod_{j=1}^{n}\frac{dx_j}{2\pi i x_j}\cdot f(\bar{x})g(x)\Delta(x;q,t)\\ &= \frac{1}{n!}(\text{constant term in } f(\bar{x})g(x)\Delta(x)) .\end{aligned} \tag{2.23}$$

The following proposition is the most fundamental one.

Proposition 2.7 [19]. *The operator D_1 is self-adjoint with respect to this scalar product, namely,*

$$\langle D_1 f,g\rangle'_{n;q,t} = \langle f,D_1 g\rangle'_{n;q,t} , \tag{2.24}$$

for all $f,g \in \Lambda_{n,\mathbf{Q}(q,t)}$.

From this proposition we have

Proposition 2.8 [19].

$$\langle P_\lambda(q,t),P_\mu(q,t)\rangle'_{n;q,t} = 0 \quad \textit{if } \lambda\neq\mu . \tag{2.25}$$

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[1] For $f(x) = f(x_1,x_2,\ldots)$, we define $f(\bar{x}) = f(1/x_1,1/x_2,\ldots)$.

Furthermore we have

Proposition 2.9 [19].

$$\begin{aligned}
&\langle P_\lambda(q,t), P_\lambda(q,t)\rangle'_{n;q,t} \\
&= \prod_{1\leqq i<j\leqq n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty (q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty} \\
&= \prod_{i=1}^{n} \frac{\Gamma_q(i\beta)}{\Gamma_q(\beta)\Gamma_q((i-1)\beta+1)} \cdot \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{n-l'(s)}}{1-q^{a'(s)+1}t^{n-l'(s)-1}} \cdot b_\lambda^{-1}(q,t)\,, \qquad (2.26)
\end{aligned}$$

where $\Gamma_q(x)$ is the q-gamma function defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}\,.$$

3. Integral Representation of the Macdonald Symmetric Polynomials

In this section, we construct integral representation formulas for the Macdonald symmetric polynomials. We adopt the same idea as that of the case of Jack symmetric polynomials [21–24].

3.1. Maps $\mathscr{G}_s$, $\mathscr{N}_{n,m}$ and an Integral Formula for $P_\lambda(x;q,t)$. Let us define the map $\mathscr{G}_s$ and $\mathscr{N}_{n,m}$ as follows:

$$\begin{aligned}
&\mathscr{G}_s : \Lambda_{r,\mathbf{Q}(q,t)} \to \Lambda_{r,\mathbf{Q}(q,t)} \\
&\qquad f(x) \mapsto (\mathscr{G}_s f)(x) = \prod_{i=1}^{r} (x_i)^s \cdot f(x)\,, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
&\mathscr{N}_{n,m} : \Lambda_{m,\mathbf{Q}(q,t)} \to \Lambda_{n,\mathbf{Q}(q,t)} \\
&\qquad f(x) \mapsto (\mathscr{N}_{n,m} f)(x) = \oint \prod_{j=1}^{m} \frac{dy_j}{2\pi i y_j} \cdot \Pi(x,\bar{y};q,t)\Delta(y;q,t)f(y)\,. \qquad (3.2)
\end{aligned}$$

Here $r, m < \infty$ and n can be equal to ∞.

Proposition 3.1. *The actions of $\mathscr{G}_s$ and $\mathscr{N}_{n,m}$ on the Macdonald symmetric polynomial P_λ are as follows:*

$$\text{(i)}\quad P_{(s^r)+\lambda}(x_1,\ldots,x_r;q,t) = \mathscr{G}_s P_\lambda(x_1,\ldots,x_r;q,t)\,, \qquad (3.3)$$

$$\text{(ii)}\quad P_\lambda(x_1,\ldots,x_n;q,t) = \frac{\langle P_\lambda, P_\lambda\rangle_{q,t}}{m!\langle P_\lambda, P_\lambda\rangle'_{m;q,t}} \mathscr{N}_{n,m} P_\lambda(x_1,\ldots,x_m;q,t)\,. \qquad (3.4)$$

Proof. As for (i), we can easily check the conditions (A) and (C) in Theorem 2.2. The statement (ii) can be proved as follows:

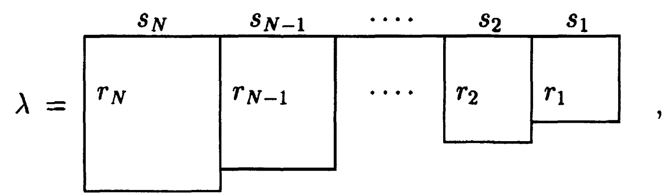
$$\frac{\langle P_\lambda, P_\lambda\rangle_{q,t}}{m!\langle P_\lambda, P_\lambda\rangle'_{m;q,t}} \oint \prod_{j=1}^{m} \frac{dy_j}{2\pi i y_j} \cdot \Pi(x, \bar{y}; q, t)\Delta(y; q, t)P_\lambda(y; q, t)$$

$$= \frac{\langle P_\lambda, P_\lambda\rangle_{q,t}}{m!\langle P_\lambda, P_\lambda\rangle'_{m;q,t}} \oint \prod_{j=1}^{m} \frac{dy_j}{2\pi i y_j} \cdot \sum_\mu Q_\mu(x; q, t)P_\mu(\bar{y}; q, t)\Delta(y; q, t)P_\lambda(y; q, t)$$

$$= \langle P_\lambda, P_\lambda\rangle_{q,t} Q_\lambda(x; q, t)$$

$$= P_\lambda(x; q, t)\,. \qquad \text{Q.E.D.}$$

Any Young diagram λ can be uniquely decomposed into rectangles:



where $r_N > \cdots > r_2 > r_1$. Therefore the partition λ is parametrized as follows:

$$\lambda = (s_N^{r_N}) + \cdots + (s_2^{r_2}) + (s_1^{r_1})\,, \tag{3.5}$$

where $(s^r) = (\overbrace{s, s, \ldots, s}^{r})$. For the partition λ, we assign a set of partitions $\lambda^{(a)}$ $(a = 1, \ldots, N)$ as follows:

$$\lambda^{(a)} = (s_a^{r_a}) + \cdots + (s_2^{r_2}) + (s_1^{r_1})\,. \tag{3.6}$$

Here we state our main theorem:

Theorem 3.2. *Let λ be the partition given by* (3.5). *We have the following multiple integral representation of the Macdonald symmetric function $P_\lambda(x; q, t) \in \Lambda_{\mathbf{Q}(q,t)}$:*

$$\begin{aligned} P_\lambda(x; q, t) &= C_\lambda^+ \mathcal{N}_{r_{N+1}, r_N} \mathcal{G}_{s_N} \mathcal{N}_{r_N, r_{N-1}} \cdots \mathcal{G}_{s_2} \mathcal{N}_{r_2, r_1} \mathcal{G}_{s_1} \cdot 1 \\ &= C_\lambda^+ \oint \prod_{a=1}^{N} \prod_{j=1}^{r_a} \frac{dx_j^a}{2\pi i x_j^a} (x_j^a)^{s_a} \cdot \prod_{a=1}^{N} \Pi(x^{a+1}, \overline{x^a}; q, t)\Delta(x^a; q, t)\,, \end{aligned} \tag{3.7}$$

where $x_i = x_i^{N+1}$, $r_{N+1} = \infty$ and

$$C_\lambda^+ = C_\lambda^+(q, t) = \prod_{a=1}^{N} \frac{\langle P_{\lambda^{(a)}}, P_{\lambda^{(a)}}\rangle_{q,t}}{r_a!\langle P_{\lambda^{(a)}}, P_{\lambda^{(a)}}\rangle'_{r_a;q,t}}\,. \tag{3.8}$$

Proof. Use Proposition 3.1 iteratively. $\mathcal{G}_s$ adds a rectangle and $\mathcal{N}_{n,m}$ increases the number of variables. Q.E.D.

Notice that the variable x's have both lower and upper suffices and the upper suffices should not be understood as the powers of x. Hereafter, we will frequently use this convention.

3.2. Another Integral Formula for $P_{\lambda'}(x;t,q)$. Next we consider another integral representation of the Macdonald symmetric polynomial $P_{\lambda'}(x;t,q)$ that is obtained from $Q_\lambda(x;q,t)$ by applying the automorphism $\omega_{q,t}$. Let us introduce one more map defined by

$$\tilde{\mathscr{N}}_{n,m} \,:\, \Lambda_{m,\mathbf{Q}(q,t)} \to \Lambda_{n,\mathbf{Q}(q,t)}$$

$$f(x) \mapsto (\tilde{\mathscr{N}}_{n,m} f)(x) = \oint \prod_{j=1}^{m} \frac{dy_j}{2\pi i y_j} \cdot \tilde{\Pi}(x,\bar{y})\Delta(y;q,t)f(y)\,. \tag{3.9}$$

We can prove the following Proposition 3.3 and Theorem 3.4 in the same way as Proposition 3.1 and Theorem 3.2, respectively.

Proposition 3.3. *The following equality holds*:

$$P_{\lambda'}(x_1,\ldots,x_n;t,q) = \frac{1}{m!\langle P_\lambda, P_\lambda\rangle'_{m;q,t}} \tilde{\mathscr{N}}_{n,m} P_\lambda(x_1,\ldots,x_m;q,t)\,. \tag{3.10}$$

Theorem 3.4. *Let λ be the partition given by* (3.5). *We have the following multiple integral representation of the Macdonald symmetric polynomial $P_{\lambda'}(x;t,q) \in \Lambda_{\mathbf{Q}(q,t)}$*:

$$\begin{aligned} P_{\lambda'}(x;t,q) &= C_\lambda^- \tilde{\mathscr{N}}_{r_{N+1},r_N} \mathscr{G}_{s_N} \mathscr{N}_{r_N,r_{N-1}} \cdots \mathscr{G}_{s_2} \mathscr{N}_{r_2,r_1} \mathscr{G}_{s_1} \cdot 1 \\ &= C_\lambda^- \oint \prod_{a=1}^{N} \prod_{j=1}^{r_a} \frac{dx_j^a}{2\pi i x_j^a} (x_j^a)^{s_a} \\ &\quad \times \tilde{\Pi}(x^{N+1}, \overline{x^N}; q,t) \prod_{a=1}^{N-1} \Pi(x^{a+1}, \overline{x^a}; q,t) \prod_{a=1}^{N} \Delta(x^a;q,t)\,, \end{aligned} \tag{3.11}$$

where $x_i = x_i^{N+1}$, $r_{N+1} = \infty$ and

$$C_\lambda^- = C_\lambda^-(q,t) = \frac{C_\lambda^+(q,t)}{\langle P_\lambda, P_\lambda\rangle_{q,t}}\,. \tag{3.12}$$

3.3. An Integral Formula for the Skew Macdonald Polynomials. Now let us proceed to discuss how to obtain an integral representation of the skew Macdonald polynomials. To this end, let us start with introducing a boson Fock space $\mathscr{F}$ which is isomorphic as the vector space to $\Lambda_{\mathbf{Q}(q,t)}$ [26, 21]. Define the commutation relations of the bosonic oscillators a_n $(n \in \mathbf{Z}_{\neq 0})$ as follows:

$$[a_n, a_m] = n \frac{1-q^{|n|}}{1-t^{|n|}} \delta_{n+m,0}\,. \tag{3.13}$$

Let $|0\rangle$ be the vacuum vector such that $a_n|0\rangle = 0$ for $n < 0$ and $\mathscr{F}$ be the Fock space defined by $\mathscr{F} = \mathbf{Q}(q,t)[a_{-1}, a_{-2}, \ldots]|0\rangle$. Let $\langle 0|$ be the dual of $|0\rangle$ i.e., $\langle 0|0\rangle = 1$. Define $\mathscr{F}^* = \langle 0|\mathbf{Q}(q,t)[a_1, a_2, \ldots]$.

We can construct an isomorphism $\imath$ between $\mathscr{F}$ and $\Lambda_{\mathbf{Q}(q,t)}$ as follows:

$$\imath : \mathscr{F} \longrightarrow \Lambda_{\mathbf{Q}(q,t)} \tag{3.14}$$

$$|f\rangle \mapsto f(x) = \langle 0|C(x)|f\rangle\,, \tag{3.15}$$

and an isomorphism $\imath^*$ between $\mathscr{F}^*$ and $\Lambda_{\mathbf{Q}(q,t)}$ by

$$\imath^* : \mathscr{F}^* \longrightarrow \Lambda_{\mathbf{Q}(q,t)} \tag{3.16}$$

$$\langle f| \mapsto f(x) = \langle f|C^*(x)|0\rangle\,, \tag{3.17}$$

where

$$C(x) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n}\frac{a_n}{n} p_n\right), \tag{3.18}$$

$$C^*(x) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n}\frac{a_{-n}}{n} p_n\right), \tag{3.19}$$

where p_n is the power sum $p_n = x_1^n + x_2^n + \cdots$. For example, we have $\imath : a_{-n}|0\rangle \mapsto p_n$.

We will use the following notation. For any symmetric polynomial $f \in \Lambda_{\mathbf{Q}(q,t)}$, we assign an operator $\hat{f} \in \mathbf{Q}(q,t)[a_{-1}, a_{-2}, \ldots]$ and a vector $|f\rangle \in \mathscr{F}$ such that $\imath(\hat{f}|0\rangle) = \imath(|f\rangle) = f(x)$. In the same way, we assign an operator $\hat{f}^* \in \mathbf{Q}(q,t)$ $[a_1, a_2, \ldots]$ and a vector $\langle f| \in \mathscr{F}^*$ such that $\imath^*(\langle 0|\hat{f}^*) = \imath^*(\langle f|) = f(x)$. For example, $\imath(\hat{P}_\lambda(q,t)|0\rangle) = \imath(|P_\lambda(q,t)\rangle) = P_\lambda(x;q,t)$ and $\imath^*(\langle 0|\hat{P}^*_\lambda(q,t)) = \imath^*(\langle P_\lambda(q,t)|) = P_\lambda(x;q,t)$. For a product $f(x)g(x)$, the corresponding state is $\hat{f}\hat{g}|0\rangle = \hat{f}|g\rangle = |f{\cdot}g\rangle$. We have the following proposition:

Proposition 3.5.

(i) *Let* $\langle f| \in \mathscr{F}^*$ *and* $|g\rangle \in \mathscr{F}$. *We have*

$$\langle f|g\rangle = \langle f(x), g(x)\rangle_{q,t}\,. \tag{3.20}$$

(ii) *Let* $\langle f| \in \mathscr{F}^*$ *and* $|g{\cdot}h\rangle \in \mathscr{F}$. *We have*

$$\langle f|g{\cdot}h\rangle = \langle \langle f|C^*(x)|g\rangle, \langle 0|C(x)|h\rangle\rangle_{q,t}\,. \tag{3.21}$$

Proof. We defined the commutation relations of a_n so that (i) holds.
(ii) is proved as follows:

$$\begin{aligned}\langle f|g\cdot h\rangle &= \langle f|\hat{g}|h\rangle = \langle i|h\rangle \\ &= \langle\langle i|C^*(x)|0\rangle, \langle 0|C(x)|h\rangle\rangle_{q,t} = \langle\langle f|\hat{g}C^*(x)|0\rangle, \langle 0|C(x)|h\rangle\rangle_{q,t} \\ &= \langle\langle f|C^*(x)|g\rangle, \langle 0|C(x)|h\rangle\rangle_{q,t}\,,\end{aligned}$$

where we have set $\langle i| = \langle f|\hat{g} \in \mathscr{F}^*$ (which may be 0), and used (i) and $\hat{g}C^*(x) = C^*(x)\hat{g}$. Q.E.D.

We remark that in this boson language, for example, Proposition 2.4 is a consequence of the completeness condition $\sum_\lambda |P_\lambda\rangle\langle Q_\lambda| = \mathbf{1}$.

By Theorems 3.2 and 3.4, we have the following bosonization formulas for the Macdonald symmetric polynomial $P_\lambda(x;q,t)$:

Proposition 3.6. *Let* λ *be the partition given by* (3.5). *We have*

$$\hat{P}_\lambda(q,t) = \oint \prod_{j=1}^{r_N} \frac{dx_j}{2\pi i x_j} \cdot F_\lambda^+(x;q,t) \prod_{j=1}^{r_N} \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n} \frac{a_{-n}}{n} (x_j)^{-n}\right) \tag{3.22}$$

$$= \oint \prod_{j=1}^{r_N} \frac{dx_j}{2\pi i x_j} \cdot F_\lambda^-(x;q,t) \prod_{j=1}^{r_N} \exp\left(-\sum_{n=1}^{\infty} \frac{a_{-n}}{n} (-x_j)^{-n}\right), \tag{3.23}$$

$$\hat{P}^*_\lambda(q,t) = \oint \prod_{j=1}^{r_N} \frac{dx_j}{2\pi i x_j} \cdot F_\lambda^+(x;q,t) \prod_{j=1}^{r_N} \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n} \frac{a_{n}}{n} (x_j)^{-n}\right) \tag{3.24}$$

$$= \oint \prod_{j=1}^{r_N} \frac{dx_j}{2\pi i x_j} \cdot F_\lambda^-(x;q,t) \prod_{j=1}^{r_N} \exp\left(-\sum_{n=1}^{\infty} \frac{a_{n}}{n} (-x_j)^{-n}\right). \tag{3.25}$$

Namely, $\langle 0|C(x)\hat{P}_\lambda(q,t)|0\rangle = \langle 0|\hat{P}^*_\lambda(q,t)C^*(x)|0\rangle = P_\lambda(x;q,t)$. *Here* $F_\lambda^\pm$ *is defined by*

$$F_\lambda^+(x^N;q,t) = C_\lambda^+(q,t)\Delta(x^N;q,t)\prod_{j=1}^{r_N}(x_j^N)^{s_N}$$
$$\times \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dx_j^a}{2\pi i x_j^a}(x_j^a)^{s_a} \cdot \prod_{a=1}^{N-1} \Pi(x^{a+1}, \overline{x^a};q,t)\Delta(x^a;q,t), \tag{3.26}$$

$$F_{\lambda'}^-(x^N;q,t) = C_\lambda^-(t,q)\Delta(x^N;t,q)\prod_{j=1}^{r_N}(x_j^N)^{s_N}$$
$$\times \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dx_j^a}{2\pi i x_j^a}(x_j^a)^{s_a} \cdot \prod_{a=1}^{N-1} \Pi(x^{a+1}, \overline{x^a};t,q)\Delta(x^a;t,q). \tag{3.27}$$

Let μ and ν be two partitions. We define the structure constants $f_{\mu\nu}^\lambda$ of the ring $\Lambda_{\mathbf{Q}(q,t)}$ by

$$P_\mu(x;q,t)P_\nu(x;q,t) = \sum_\lambda f_{\mu\nu}^\lambda(q,t)P_\lambda(x;q,t), \tag{3.28}$$

or equivalently,

$$f_{\mu\nu}^\lambda = f_{\mu\nu}^\lambda(q,t) = \langle Q_\lambda, P_\mu P_\nu\rangle_{q,t} \in \mathbf{Q}(q,t). \tag{3.29}$$

The skew Q-function is defined by

$$Q_{\lambda/\mu}(x;q,t) = \sum_\nu f_{\mu\nu}^\lambda(q,t)Q_\nu(x;q,t). \tag{3.30}$$

This is equivalent to the following condition:

$$\langle Q_{\lambda/\mu}, P_\nu\rangle_{q,t} = \langle Q_\lambda, P_\mu P_\nu\rangle_{q,t}. \tag{3.31}$$

The skew P-function is given by $P_{\lambda/\mu} = b_\lambda^{-1} b_\mu Q_{\lambda/\mu}$.

Now we are ready to state the boson representation of the skew Q-polynomial.

Theorem 3.7. *We have the following boson realization of the skew Q-polynomial:*

$$Q_{\lambda/\mu}(x;q,t) = \langle Q_\lambda|C^*(x)|P_\mu\rangle = \langle P_\mu|C(x)|Q_\lambda\rangle\ . \tag{3.32}$$

Proof. From Propositions 3.5 and 3.6 we have the following:

$$\begin{aligned}\langle\langle Q_\lambda|C^*(x)|P_\mu\rangle, P_\nu\rangle_{q,t} &= \langle\langle Q_\lambda|C^*(x)|P_\mu\rangle, \langle 0|C(x)|P_\nu\rangle\rangle_{q,t}\\ &= \langle Q_\lambda, P_\mu P_\nu\rangle_{q,t}\ .\end{aligned}$$

This proves the first equality. The second equality can be proved in the same way. Q.E.D.

As a corollary of this theorem and Proposition 3.6, we obtain integral representation formulas for the skew Q-polynomial.

Corollary 3.8. *Let* λ *be the partition given by* (3.5) *and* μ *be another partition* $\mu = (\sigma_M^{\rho_M}) + \cdots + (\sigma_1^{\rho_1})$. *We have the integral representation formulas for* $Q_{\lambda/\mu}(x;q,t) \in \Lambda_{\mathbf{Q}(q,t)}$ *as follows*:

$$b(q,t)_\lambda^{-1} Q_{\lambda/\mu}(x;q,t)$$

$$= \oint \prod_{j=1}^{r_N} \frac{dz_j}{2\pi i z_j} \cdot \prod_{j=1}^{\rho_M} \frac{dw_j}{2\pi i w_j} \cdot F_\lambda^+(z;q,t) F_\mu^+(w;q,t) \Pi(\overline{z},\overline{w};q,t) \times \begin{cases} \Pi(x,\overline{z};q,t) \\ \Pi(x,\overline{w};q,t) \end{cases} \tag{3.33}$$

$$= \oint \prod_{j=1}^{r_N} \frac{dz_j}{2\pi i z_j} \cdot \prod_{j=1}^{\rho_M} \frac{dw_j}{2\pi i w_j} \cdot F_\lambda^-(z;q,t) F_\mu^+(w;q,t) \tilde{\Pi}(\overline{z},\overline{w}) \times \begin{cases} \tilde{\Pi}(x,\overline{z}) \\ \Pi(x,\overline{w};q,t) \end{cases} \tag{3.34}$$

$$= \oint \prod_{j=1}^{r_N} \frac{dz_j}{2\pi i z_j} \cdot \prod_{j=1}^{\rho_M} \frac{dw_j}{2\pi i w_j} \cdot F_\lambda^+(z;q,t) F_\mu^-(w;q,t) \tilde{\Pi}(\overline{z},\overline{w}) \times \begin{cases} \Pi(x,\overline{z};q,t) \\ \tilde{\Pi}(x,\overline{w}) \end{cases} \tag{3.35}$$

$$= \oint \prod_{j=1}^{r_N} \frac{dz_j}{2\pi i z_j} \cdot \prod_{j=1}^{\rho_M} \frac{dw_j}{2\pi i w_j} \cdot F_\lambda^-(z;q,t) F_\mu^-(w;q,t) \Pi(\overline{z},\overline{w};t,q) \times \begin{cases} \tilde{\Pi}(x,\overline{z}) \\ \tilde{\Pi}(x,\overline{w})\ . \end{cases} \tag{3.36}$$

Remark. More generally, one can directly prove that the skew Macdonald polynomial can be written in the integral transformation $\mathcal{N}_{n,m}$ of Eq. (3.2) or in the power-sums p_n as follows:

$$\begin{aligned} Q_{\lambda/\mu}(x;q,t) &= \frac{\langle Q_\lambda, Q_\lambda\rangle_{q,t}}{m!\langle Q_\lambda, Q_\lambda\rangle'_{m;q,t}} (\mathcal{N}_{n,m} Q_\lambda \overline{P}_\mu)(x;q,t)\ ,\\ Q_{\lambda/\mu}(p;q,t) &= P_\mu(\overline{p};q,t)\ Q_\lambda(p;q,t)\cdot 1\ , \end{aligned} \tag{3.37}$$

for all $m \geqq l(\lambda)$. Here $\overline{P_\mu(x)} \equiv P_\mu\left(\frac{1}{x}\right)$ and $\overline{p}_n \equiv n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n}$.

3.4. The Kostka Matrix. As another application of the bosonization constructed in the last subsection, we will give integral representations of the Kostka matrix $K_{\lambda\mu}(q,t)$. Let

$$h_\lambda(q,t) = \prod_{s\in\lambda} (1 - q^{a(s)}t^{l(s)+1}) , \tag{3.38}$$

$$h'_\lambda(q,t) = \prod_{s\in\lambda} (1 - q^{a(s)+1}t^{l(s)}) = h_\lambda(t,q) . \tag{3.39}$$

So, we have

$$b_\lambda(q,t) = h_\lambda(q,t)/h'_\lambda(q,t) . \tag{3.40}$$

Let us define

$$M_\lambda(x;q,t) = h_\lambda(q,t)P_\lambda(x;q,t) = h'_\lambda(q,t)Q_\lambda(x;q,t) . \tag{3.41}$$

The q-analogue of the Kostka–Foulks polynomial $K_{\lambda\mu}(q,t)$ introduced by Macdonald is defined by

$$M_\mu(x;q,t) = \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(x;t) , \tag{3.42}$$

where $S_\lambda(x;t)$ is the dual base of the Schur function $s_\lambda(x)$ with respect to the scalar product $\langle\, , \rangle_{0,t}$. Further let us define the dual base of $S_\lambda(x;t)$ with respect to the scalar product $\langle\, , \rangle_{q,t}$ by $S_\lambda(x;q,t)$.

We have the following:

Proposition 3.9. *For a partition λ, we have*

$$|s_\lambda\rangle = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{\lambda_j} \cdot \prod_{i<j} (1 - x_i/x_j) \cdot \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{a_{-n}}{n} x_j^{-n}\right) |0\rangle , \tag{3.43}$$

$$|S_\lambda(t)\rangle = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{\lambda_j} \cdot \prod_{i<j} (1 - x_i/x_j) \cdot \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{1-t^n}{1} \frac{a_{-n}}{n} x_j^{-n}\right) |0\rangle , \tag{3.44}$$

$$|S_\lambda(q,t)\rangle = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{\lambda_j} \cdot \prod_{i<j} (1 - x_i/x_j) \cdot \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{1}{1-q^n} \frac{a_{-n}}{n} x_j^{-n}\right) |0\rangle , \tag{3.45}$$

$$\langle s_\lambda| = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{i<j} (1 - x_j/x_i) \cdot \langle 0| \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{a_n}{n} x_j^{n}\right) , \tag{3.46}$$

$$\langle S_\lambda(t)| = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{i<j} (1 - x_j/x_i) \cdot \langle 0| \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{1-t^n}{1} \frac{a_n}{n} x_j^{n}\right) , \tag{3.47}$$

$$\langle S_\lambda(q,t)| = \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{i<j} (1 - x_j/x_i) \cdot \langle 0| \prod_{j=1}^{l(\lambda)} \exp\left(\sum_{n>0} \frac{1}{1-q^n} \frac{a_n}{n} x_j^{n}\right) . \tag{3.48}$$

Proof. An integral representation of the Schur polynomial is well known [26]:

$$s_\lambda(x) = \oint \prod_{j=1}^{l(\lambda)} \frac{dy_j}{2\pi i y_j} y_j^{\lambda_j} \cdot \prod_{i<j} (1 - y_i/y_j) \cdot \prod_{i,j} (1 - x_i/y_j)^{-1} . \tag{3.49}$$

Therefore (3.43) and (3.46) are correct states. Since $\langle s_\lambda | s_\mu \rangle|_{t=q} = \delta_{\lambda,\mu}$, we have

$$\oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{j=1}^{l(\mu)} \frac{dy_j}{2\pi i y_j} y_j^{\mu_j} \cdot \prod_{i<j} (1 - x_j/x_i) \cdot \prod_{i,j} \frac{1}{1 - x_i/y_j} \cdot \prod_{i<j} (1 - y_i/y_j) = \delta_{\lambda,\mu} . \tag{3.50}$$

Note that $S_\lambda(x;t)$ is independent of q. Using the above identity and Proposition 3.5, we can show the following:

$$\langle S_\lambda(t), s_\mu \rangle_{0,t} = \langle S_\lambda(t) | s_\mu \rangle \Big|_{q=0} = \delta_{\lambda,\mu} ,$$

$$\langle S_\lambda(q,t), S_\mu(t) \rangle_{q,t} = \langle S_\lambda(q,t) | S_\mu(t) \rangle = \delta_{\lambda,\mu} . \qquad \text{Q.E.D.}$$

We remark that we obtain another expression of these states by using another integral representation of the Schur polynomial [26],

$$s_{\lambda'}(x) = (-1)^{|\lambda|} \oint \prod_{j=1}^{l(\lambda)} \frac{dy_j}{2\pi i y_j} y_j^{\lambda_j} \cdot \prod_{i<j} (1 - y_i/y_j) \cdot \prod_{i,j} (1 - x_i/y_j) . \tag{3.51}$$

Since $K_{\lambda\mu}(q,t) = \langle S_\lambda(q,t), M_\mu(q,t) \rangle_{q,t}$, by using Propositions 3.5, 3.6 and 3.9, we can show the following theorem:

Theorem 3.10. *Let λ, μ be partitions; $\mu = (\sigma_M^{\rho_M}) + \cdots + (\sigma_1^{\rho_1})$. Kostka matrix $K_{\lambda\mu}(q,t)$ is represented as follows:*

$$\begin{aligned} K_{\lambda\mu}(q,t) &= \langle S_\lambda(q,t) | M_\mu(q,t) \rangle \\ &= h_\mu(q,t) \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{j=1}^{\rho_M} \frac{dy_j}{2\pi i y_j} \cdot \prod_{i<j} (1 - x_j/x_i) \\ &\quad \cdot \prod_{i,j} \frac{1}{(x_i/y_j;q)_\infty} \cdot F_\mu^+(y;q,t) \\ &= h_\mu(q,t) \oint \prod_{j=1}^{l(\lambda)} \frac{dx_j}{2\pi i x_j} x_j^{-\lambda_j} \cdot \prod_{j=1}^{\rho_M} \frac{dy_j}{2\pi i y_j} \cdot \prod_{i<j} (1 - x_j/x_i) \\ &\quad \cdot \prod_{i,j} (-x_i/y_j;t)_\infty \cdot F_\mu^-(y;q,t) . \end{aligned} \tag{3.52}$$

4. Bosonizations of the Integral Formula

In the last section, we introduced the Fock space of the boson field to discuss how to obtain integral formulas for the skew Macdonald polynomials. One may notice, however, the bosonization is something *partial* compared with the case of the Schur polynomials [26] and the Hall–Littlewood polynomials [27], because we only bosonized the variable $x = x^{N+1}$. We consider some *total* bosonization schemes of the integral representation formula for the Macdonald symmetric polynomials which was obtained in the last section.

4.1. Firstly we treat the case of $\beta \in \mathbf{Z}_{>0}$. In this case, we can bosonize the integral formula by using a similar method to the Jack symmetric polynomial's case discussed in our previous paper [24]. Let us introduce the following bosonic oscillators having A-type like symmetry:

$$[a_n^a, a_m^b] = \begin{cases} 0 & \text{for } A^{ab} = 0\,, \\ -n\dfrac{1-t^{|n|}}{1-q^{|n|}}\delta_{n+m,0} & \text{for } A^{ab} = -1\,, \\ n\left(\dfrac{1-t^n}{1-q^n} + \dfrac{1-t^{-n}}{1-q^{-n}}\right)\delta_{n+m,0} & \text{for } A^{ab} = 2\,, \end{cases} \tag{4.1}$$

and $[a_0^a, Q^b] = \beta A^{ab}$, where $n, m \in \mathbf{Z}$ and $a, b \in \{1, \dots, N+1\}$. Here, $A^{ab} = 2\delta^{a,b} - \delta^{a,b+1} - \delta^{a,b-1}$ is the Cartan matrix of A_{N+1} type. Let us define A-type boson fields as follows:

$$\phi^a(z) = \phi^a_{\leqq 0}(z) + \phi^a_{\geqq 0}(z), \qquad \begin{cases} \phi^a_{\leqq 0}(z) = \sum_{n>0} \dfrac{a^a_{-n}}{n} z^n + Q^a\,, \\ \phi^a_{\geqq 0}(z) = -\sum_{n>0} \dfrac{a^a_n}{n} z^{-n} + a^a_0 \log z\,. \end{cases} \tag{4.2}$$

The normal ordering $:\ :$ is defined as moving $\phi_{\geqq 0}$ to the right of $\phi_{\leqq 0}$. The operator product expansion (in the region $|w/z| < q^{\beta-1}$) is given as follows:

$$\phi^a(z)\phi^b(w) \sim \begin{cases} 0 & \text{for } A^{ab} = 0\,, \\ \log\left(z^{-\beta} \prod_{k=0}^{\beta-1} \left(1-q^k w/z\right)^{-1}\right) & \text{for } A^{ab} = -1\,, \\ \log\left((-zw)^{\beta} q^{-\frac{1}{2}\beta(\beta-1)} \prod_{k=0}^{\beta-1} \left(1-q^k w/z\right)\left(1-q^k z/w\right)\right) & \text{for } A^{ab} = 2\,. \end{cases} \tag{4.3}$$

For $\alpha = (\alpha^1, \dots, \alpha^{N+1})$, let $|\alpha\rangle = \exp(\frac{1}{\beta}\sum_{a,b=1}^{N+1} \alpha^a (A^{-1})_{ab} Q^b)|0\rangle$, where $|0\rangle$ satisfies $a_n^a|0\rangle = 0$ ($a = 1, \dots, N+1$ and $n \geqq 0$). This $|\alpha\rangle$ satisfies $a_n^a|\alpha\rangle = 0$ ($a = 1, \dots, N+1$ and $n > 0$) and $a_0^a|\alpha\rangle = \alpha^a|\alpha\rangle$. We also introduce $\langle\alpha|$ as the dual of $|\alpha\rangle$, i.e. $\langle\alpha|\alpha'\rangle = \delta_{\alpha,\alpha'}$.

We can state our result as follows:[2]

Proposition 4.1. *Let λ be defined by* (3.5). *We have the following A-type bosonic realization of the Macdonald symmetric function for $\beta \in \mathbf{Z}_{>0}$:*

$$P_\lambda(x;q,t) = C_\lambda^+(q,t)(-q^{\frac{1}{2}(\beta-1)})^{\frac{1}{2}\beta\sum_{a=1}^N r_a(r_a-1)}$$

$$\times \oint \prod_{a=1}^N \prod_{j=1}^{r_a} \frac{dx_j^a}{2\pi i x_j^a} \cdot \langle\tilde{\alpha}| \prod_{a=1}^N \prod_{j=1}^{r_a} :e^{\phi^a(x_j^a)}: \cdot \prod_{i=1}^{r_{N+1}} e^{\phi_{\leqq 0}^{N+1}(x_i^{N+1})}|\alpha\rangle\,, \tag{4.4}$$

where $\alpha^a = \beta(r_{a+1} - r_a + 1) + s_a$, $(r_{N+2} = 0)$, $\tilde{\alpha}^a = \alpha^a + \beta\sum_{b=1}^{N+1} A^{ab} r_b$, *and* $x_i = x_i^{N+1}$, $r_{N+1} = \infty$ *(after calculation)*.

Proof. First we remark that for $\beta \in \mathbf{Z}_{>0}$ $(t = q^\beta)$,

$$\Delta(x;q,t) = \prod_{i \neq j} \prod_{k=0}^{\beta-1} (1 - q^k x_i/x_j), \qquad \Pi(x,y;q,t) = \prod_{i,j} \prod_{k=0}^{\beta-1} \frac{1}{1 - q^k x_i y_j}\,. \tag{4.5}$$

A straightforward calculation of the operator product expansion shows that this integrand agrees with that of Theorem 3.1. Q.E.D.

4.2. Next we construct another bosonization scheme which is applicable for the case of $\beta \in \mathbf{C}$. We utilize Jing's boson field which was introduced to consider the Hall–Littlewood symmetric function $P(x;t)$ having one parameter t [27]. Notice that in this case we will not utilize an A_N structure but derive a bosonization formula for $P(x;q,t)$ using finite temperature calculation regarding the parameter q as playing the role of temperature. Let us introduce an N copy of boson oscillators a_n^a $(a = 1, 2, \ldots, N)$, whose commutation relations are given as follows:

$$[a_n^a, a_m^b] = n\frac{1}{1 - t^{|n|}}\delta_{n+m,0}\delta^{ab}\,. \tag{4.6}$$

Let $\mathscr{F}$ be the Fock space of these boson fields: $\mathscr{F} = \mathbf{Q}(q,t)[a_{-1}^a, a_{-2}^a, \ldots]|0\rangle$. Normal ordering $:\ :$ is defined as moving a_n^a to the right of a_{-n}^a $(n > 0)$. We define the grading operator L_0 as

$$L_0 = \sum_{a=1}^N \sum_{n=1}^\infty (1 - t^n) a_{-n}^a a_n^a\,, \tag{4.7}$$

which satisfies $[L_0, a_n^a] = -n a_n^a$. We introduce boson fields as follows:

$$\phi^a(z) = -\sum_{n \in \mathbf{Z}_{\neq 0}} (1 - t^{|n|})\frac{a_n^a}{n} z^{-n}, \qquad \phi_-^a(z) = \sum_{n>0} (1 - t^n)\frac{a_{-n}^a}{n} z^n\,. \tag{4.8}$$

Here we state another bosonization formula:

[2] We use the convention $\prod_{i=1}^n \mathcal{O}_i = \mathcal{O}_1\mathcal{O}_2 \cdots \mathcal{O}_n$ for non-commuting $\mathcal{O}_i$'s.

Proposition 4.2. *Let λ be defined by* (3.5), *and let $\beta \in \mathbf{C}$. We have the following bosonization formula for the Macdonald symmetric function*:

$$P_\lambda(x;q,t) = C_\lambda^+(q,t)(q;q)_\infty^{N-\sum_{a=1}^N r_a}(tq;q)_\infty^{\sum_{a=1}^N r_a}\prod_{a=1}^N r_a!\prod_{k=1}^{r_a}\frac{1-t}{1-t^k}\oint\prod_{a=1}^N\prod_{j=1}^{r_a}\frac{dx_j^a}{2\pi i x_j^a}(x_j^a)^{s_a}$$
$$\cdot \mathrm{Tr}_{\mathscr{F}}\left(q^{L_0}\prod_{a=1}^N\prod_{j=1}^{r_a}:e^{\phi^a(x_j^a)}:\cdot\prod_{a=1}^N\prod_{j=1}^{r_{a+1}}e^{-\phi_-^a(x_j^{a+1})}\right), \tag{4.9}$$

where $x_i = x_i^{N+1}$ and $r_{N+1} = \infty$.

Proof. To calculate the trace, we apply the Clavelli–Shapiro trace technique [28]. We introduce the boson oscillators b_n^a, which satisfy the same commutation relation as a_n^a and commutes with a_m^a, and take the following combinations ($n > 0$):

$$\tilde{a}_n^a = \frac{a_n^a}{1-q^n} + b_{-n}^a, \qquad \tilde{a}_{-n}^a = a_{-n}^a + \frac{q^n b_n^a}{1-q^n}\,. \tag{4.10}$$

Clavelli and Shapiro's argument tells us that

$$\mathrm{Tr}_{\mathscr{F}}(q^{L_0}\mathcal{O}) = \frac{\langle 0|\tilde{\mathcal{O}}|0\rangle}{\prod_{k=1}^\infty(1-q^k)^N}\,, \tag{4.11}$$

where $\mathcal{O}$ is an operator in a_n^a, and $\tilde{\mathcal{O}}$ is defined as the operator obtained from $\mathcal{O}$ by replacing a_n^a with $\tilde{a}_n^a$. Then we obtain

$$:e^{\tilde{\phi}^a(z)}: = \frac{(q;q)_\infty}{(qt;q)_\infty}\exp\left(\sum_{n>0}(1-t^n)\frac{a_{-n}^a}{n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{1-q^n}\frac{a_n^a}{n}z^{-n}\right)$$
$$\times\exp\left(-\sum_{n>0}(1-t^n)\frac{b_{-n}^a}{n}z^{-n}\right)\exp\left(\sum_{n>0}\frac{(1-t^n)q^n}{1-q^n}\frac{b_n^a}{n}z^n\right), \tag{4.12}$$

and

$$:e^{-\tilde{\phi}_-^a(z)}: = \exp\left(-\sum_{n>0}(1-t^n)\frac{a_{-n}^a}{n}z^n\right)\exp\left(-\sum_{n>0}\frac{(1-t^n)q^n}{1-q^n}\frac{b_n^a}{n}z^n\right). \tag{4.13}$$

We have the following OPE's: in the region $q < |z_2/z_1| < 1$,

$$:e^{\tilde{\phi}^a(z_1)}::e^{\tilde{\phi}^b(z_2)}: = \begin{cases} :e^{\tilde{\phi}^a(z_1)+\tilde{\phi}^a(z_2)}:\dfrac{(z_2/z_1;q)_\infty}{(tz_2/z_1;q)_\infty}\dfrac{(qz_1/z_2;q)_\infty}{(qtz_1/z_2;q)_\infty} & \text{for } a=b\,, \\ :e^{\tilde{\phi}^a(z_1)+\tilde{\phi}^b(z_2)}: & \text{for } a\neq b\,, \end{cases} \tag{4.14}$$

and in the region $|w/z| < 1$,

$$:e^{\tilde{\phi}^a(z)}::e^{-\tilde{\phi}^b_-(w)}: = \begin{cases} :e^{\tilde{\phi}^a(z)-\tilde{\phi}^a_-(w)}: \dfrac{(tw/z;q)_\infty}{(w/z;q)_\infty} & \text{for } a=b\,, \\ :e^{\tilde{\phi}^a(z)-\tilde{\phi}^b_-(w)}: & \text{for } a \neq b\,. \end{cases} \tag{4.15}$$

By using these equations, $(q;q)_\infty^{N-\sum_{a=1}^N r_a}(tq;q)_\infty^{\sum_{a=1}^N r_a}\mathrm{Tr}_{\mathscr{F}}(\cdots)$ in (4.9) becomes

$$\prod_{a=1}^N \prod_{\substack{i,j=1\\ i<j}}^{r_a} \frac{(x_j^a/x_i^a;q)_\infty}{(tx_j^a/x_i^a;q)_\infty}\frac{(qx_i^a/x_j^a;q)_\infty}{(qtx_i^a/x_j^a;q)_\infty}\cdot\prod_{a=1}^N\prod_{i=1}^{r_a}\prod_{j=1}^{r_{a+1}}\frac{(tx_i^{a+1}/x_j^a;q)_\infty}{(x_i^{a+1}/x_j^a;q)_\infty}$$

$$=\prod_{a=1}^N \Delta(x^a;q,t)\Pi(x^{a+1},\overline{x^a};q,t)\cdot\prod_{a=1}^N\prod_{\substack{i,j=1\\ i<j}}^{r_a}\frac{(1-tx_i^a/x_j^a)}{(1-x_i^a/x_j^a)}\,. \tag{4.16}$$

For each a and a permutation σ, we change the integration variables $x_i^a \to x_{\sigma(i)}^a$. Then by using the identity [19]

$$\sum_{\sigma\in S_n}\prod_{1\leqq i<j\leqq n}\frac{x_{\sigma(i)}-tx_{\sigma(j)}}{x_{\sigma(i)}-x_{\sigma(j)}}=\prod_{k=1}^n\frac{1-t^k}{1-t}\,, \tag{4.17}$$

where S_n is the n^{th} symmetric group, the integrand agrees with that of Theorem 3.1. Q.E.D.

5. Discussion

In this paper we have obtained integral representations of the (skew-)Macdonald symmetric polynomials (Theorems 3.2, 3.4 and Corollary 3.8) and their boson realizations (Propositions 4.1, 4.2 and Theorem 3.7). The two maps in Proposition 3.1 have played an essential role in our derivation.

Our first physical motivation for this study is the calculation of the correlation functions of the Calogero–Sutherland model. The results obtained in this paper and ref. [21–24] will help us to do it. In particular skew Jack symmetric functions will be useful for higher point correlation functions. Of course, concerning the analysis for the Calogero–Sutherland model, the Macdonald symmetric polynomials are unnecessary, but sometimes calculation for q-deformed quantities is more transparent than the original ones. We have also constructed free boson realizations for the integral representations. These realizations will also help us in calculation for correlation functions. However, in comparison with the case of the Jack symmetric polynomials, these free field expressions are ad hoc in the sense that they merely give the desired integrands of the integral representations (see the next paragraph). Another motivation is to solve the Ruijsenaars model, i.e. model with elliptic potential. At present this problem still seems to be difficult.

For mathematical interest, we would like to mention the relation between free field realizations and symmetry algebras. In the case of the Jack symmetric polynomial [21–24], this polynomial is realized on the boson Fock space as the state obtained by the action of screening currents $:e^{\alpha_\pm\phi^a(z)}:$ on the vacuum. This state is the singular vector of the W_N algebra. On the other hand, in the free boson realization, the W_N algebra is the commutant of these screening currents $:e^{\alpha_\pm \phi^a(z)}:$. So we have the following natural question: in the case of the Macdonald symmetric polynomials, what algebra appears as the commutant of the vertex operators $:e^{\phi^a(x)}:$ used in Sect. 4?

After finishing this work, we learned that Frenkel and Reshetikhin constructed certain q-deformations of the Virasoro and W-algebras [29] by utilizing the free boson realization of the quantum affine algebra $U_q(\widehat{sl}_N)$ studied in ref. [30]. It seems interesting to clarify the connection between our vertex operators introduced in Sect. 4 and the q-deformed algebras.

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