Commun. Math. Phys. 179, 185–214 (1996)

Communications in Mathematical Physics


# Target-Space Duality between Simple Compact Lie Groups and Lie Algebras under the Hamiltonian Formalism: I. Remnants of Duality at the Classical Level

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Received: 20 April 1995 / Accepted: 23 October 1995

**Abstract:** It has been suggested that a possible classical remnant of the phenomenon of target-space duality (T-duality) would be the equivalence of the classical string Hamiltonian systems. Given a simple compact Lie group $G$ with a bi-invariant metric and a generating function $\Gamma$ suggested in the physics literature, we follow the above line of thought and work out the canonical transformation $\Phi$ generated by $\Gamma$ together with an Ad-invariant metric and a B-field on the associated Lie algebra $\mathfrak{g}$ of $G$ so that $G$ and $\mathfrak{g}$ form a string target-space dual pair at the classical level under the Hamiltonian formalism. In this article, some general features of this Hamiltonian setting are discussed. We study properties of the canonical transformation $\Phi$ including a careful analysis of its domain and image. The geometry of the T-dual structure on $\mathfrak{g}$ is lightly touched. We leave the task of tracing back the Hamiltonian formalism at the quantum level to the sequel of this paper.

## 0. Introduction and Outline

*0.1. Introduction.* Target space duality (T-duality) is a very surprising phenomenon in string theory[1]. In essence, two target-spaces are dual to each other if both lead to the same string theory. The usual technical definition involves using path-integrals to sum over the space of all smooth maps from surfaces (string world-sheets) to target manifolds [B1, B2, F-J, R-V, G-R1, M-V]. In this aspect, it is a quantum mechanical phenomenon. Nevertheless, it is natural to ask:

"*Q*: *Are there classical aspects of the phenomenon of target space duality*?

As already pointed out in the literature (e.g. [A-AG-B-L, A-AG-L2, C-Z, G-P-R, G-R3, G-R-V]), one possible answer may be the equivalence of the associated string Hamiltonian systems.

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[1] See the review [G-P-R] for a comprehensive set of references.

As will be discussed in Sect. 1.3, for the simplest known example, the $(R \leftrightarrow \frac{1}{R})$-duality for $S^1$, the above naive picture after appropriate modification captures many features of target-space duality. Backed by this example and some lessons learned from it, we next turn our attention to another known example in the physics literature [A-AG-B-L, A-AG-L1, C-Z, dlO-Q, E-G-R-S-V, G-K, G-R2, G-R-V]. Recall that the simple compact Lie group $SU(2)$ and its associated Lie algebra form a target-space dual pair when $SU(2)$ is endowed with a bi-invariant metric and its associated Lie algebra $\mathfrak{su}(2)$ is endowed with a metric and a $B$-field which, up to a constant multiple, are written in linear coordinates respectively as [C-Z]

$$\tilde{g}_{ij} = \frac{(\delta_{ij} + 4v^i v^j)}{1 + 4v^2} \quad \text{and} \quad \widetilde{B}_{ij} = \frac{\varepsilon_{ijk} v^k}{1 + 4v^2} .$$

In terms of Hamiltonian systems, the duality of this pair comes from a formal canonical transformation from the loop space $LT^*SU(2)$ to the loop space $LT^*\mathfrak{su}(2)$. This canonical transformation is generated by

$$\Gamma(g, v) = \int_{S^1} \mathrm{Tr}(v\, g^{-1} dg)$$

in coordinate-free, fundamental matrix form. The latter expression is immediately applicable for general Lie groups and their associated Lie algebras. This observation leads us to this present work.

Recently, a geometrical picture of duality [K-S1, K-S2] has emerged which allows one to write down the general duality transformation when there is a group action on a manifold. In the present paper, we use the formalism of [C-Z] to look more closely at the example of the target being a simple compact Lie group.

In brief, given a simple compact Lie group $G$ with a bi-invariant metric, let $\mathfrak{g}$ be its associated Lie algebra. We take the generating function $\Gamma$ as the foundation of our approach and work out the canonical transformation $\Phi$ it generates from $LT^*G$ to $LT^*\mathfrak{g}$. We obtain also an Ad-invariant metric and an Ad-invariant $B$-field (a 2-form) on the associated Lie algebra $\mathfrak{g}$ so that they form a T-dual pair at the classical level under the Hamiltonian formalism. This could possibly be an exact dual pair in terms of path-integrals at the quantum level. In this first paper, we focus on properties of the canonical transformation $\Phi$ and the T-dual geometry on $\mathfrak{g}$ and leave the important issue of how exactly $G$ and $\mathfrak{g}$ form a dual pair at the quantum level to the sequel.

Recently there appeared an article [Lo] by Y. Lozano on the same subject. Interested readers may compare our setting here with hers.

## Contents

## 1. Target-Space Duality in Hamiltonian Formalism

*1.1. Hamiltonian Formalism for String Theory.* In string theory a particle is assumed to be a one-dimensional extended object. There are two kinds of them, open and closed strings. In this article we shall restrict ourselves only to closed strings, given by smooth maps from $S^1$ into a smooth target manifold.

Neglecting the dilaton and other fields, the target-space data for a string theory consists of a Riemannian manifold with a 2-form (usually called a $B$-field by physicists) $(M, ds^2, B)$. We shall denote it collectively by $M$ when both the Riemannian metric and the $B$-field are understood from the text. The configuration space consists of all possible positions of the particle and hence is given by the loop space

$$LM = \{\phi : S^1 \to M \mid \phi \text{ is } C^\infty\} .$$

The phase space requires however some choices. Since we are only interested in objects describable as smooth objects along a circle, we choose the phase space to be $LT^*M$ instead of the much larger $T^*LM$. There is a canonical symplectic structure $\omega$ on $LT^*M$ induced from the canonical symplectic structure $\omega$ on $T^*M$ given by

$$\omega_\gamma(\eta, \xi) = \int_{S^1} d\sigma\, \omega(\eta_{\gamma(\sigma)}, \xi_{\gamma(\sigma)}) ,$$

where $\gamma$ is in $LT^*M$ and $\eta$, $\xi$ are two tangent vectors at $\gamma$. They are simply two vectors fields in $T^*M$ along the loop $\gamma$.

The Lagrangian density from the (1+1)-dimensional $\sigma$-model over a cylinder can be thought of as an energy function for paths in the configuration space. It can be rephrased as a Lagrangian $\mathscr{L}$ defined on the tangent bundle $T_*LM = LT_*M$ of $LM$. Denote a point in $T_*LM$ by $(\phi, X)$, where $\phi \in LM$ and $X$ is a smooth vector field in $M$ along $\phi$, then the Lagrangian can be written as

$$\mathscr{L}(\phi, X) = \int_{S^1} d\sigma\, \mathscr{L}(\phi, X; \sigma)$$

with

$$\mathscr{L}(\phi, X; \sigma) = \tfrac{1}{2}(\langle X(\sigma), X(\sigma)\rangle - \langle \phi_*\partial_\sigma, \phi_*\partial_\sigma\rangle) + B(X(\sigma), \phi_*\partial_\sigma) ,$$

where $\langle\, ,\rangle$ stands for the metric on $M$, and $\partial_\sigma$ is the coordinate vector field along $S^1$.

The canonical momentum density $\pi$ associated to $\mathscr{L}$ for $(\phi, X)$ is given by

$$\pi(\sigma) = \frac{\delta\mathscr{L}}{\delta X}(\sigma) = \langle\, \cdot\, , X(\sigma)\rangle + B(\,\cdot\, , \phi_*\partial_\sigma) .$$

The Legendre transformation now takes functions on $LT_*M$ to functions on $LT^*M$. The image of the Lagrangian $\mathscr{L}$ becomes the string Hamiltonian function $\mathscr{H}$ on $LT^*M$. Its density function along $S^1$ is given by

$$\mathscr{H}(\phi,\pi;\sigma) = \tfrac{1}{2}\langle \pi(\sigma) - B(\,\cdot\,,\phi_*\partial_\sigma), \pi(\sigma) - B(\,\cdot\,,\phi_*\partial_\sigma)\rangle^{\sim} + \tfrac{1}{2}\langle \phi_*\partial_\sigma, \phi_*\partial_\sigma\rangle\,,$$

where $\langle\,,\,\rangle^{\sim}$ stands for the induced metric on the fiber of $T^*M$ from $\langle\,,\,\rangle$.

Basically all the information about the classical physics for a closed string is contained in this Hamiltonian system.

*1.2. Target-Space Duality.* Though the term "*target-space duality*" has become more or less official in the literature, a better name for it would be "*string-equivalence between target-spaces*" [A-G-M]. The latter says exactly the meaning hidden under the former. Technically, this means that there exists a correspondence $\Phi$ that takes the states and observables in the string theory associated to one target-space $(M, ds^2, B, \ldots)$ to those of the string theory associated to another target-space $(\widetilde{M}, \widetilde{ds^2}, \widetilde{B}, \ldots)$ such that the related correlation functions are all identical. Thus, as long as physics is concerned, one cannot tell whether the particle is moving about in one or another target-space in the same equivalence class. Since these correlation functions are all defined formally via Feynman's path-integral, the definition indicates that target-space duality is actually a quantum level phenomenon. One would like to know *if this phenomenon manifests itself at the classical level.*

Since all the information of the classical physics for a string theory is completely contained in the string Hamiltonian system described in Sect. 1.1, a naive guess for the classical remnant of target-space duality is the equivalence of string Hamiltonian systems. This equivalence would be given by canonical transformations between string phase spaces that take the string Hamiltonian function on one phase space to that on another. In the next section we shall do a redemonstration of a known example where the target space is a circle. The classical remnant of target space duality will be a classical Hamiltonian equivalence except that the respective phase spaces have to be restricted. In this example "classical duality" only exists between *reduced Hamiltonian systems*. This may be a general feature of target space duality. Namely, its classical remnant is a reduced Hamiltonian system though a general rule for this classical reduction is not known yet.

In this paper we will explore these issues for the case of a $G$-$\mathfrak{g}$ pair.

*1.3. A Lesson from the $(R \leftrightarrow \frac{1}{R})$-Duality of $S^1$.* The $S^1$-target case is the simplest and best known example of target-space duality. It indicates a new relation between physics in the small scale and physics in the large scale. Such a relationship may be applicable to the removal of initial singularities of a space-time in general relativity.

The target-space in this example is $S^1_R$, a circle of radius $R$. The phase space is $LT^*S^1_R$. Let $(\theta, \pi)$ be a canonical coordinate system for $T^*S^1_R$, where $\theta$ runs over the interval $[0, 2\pi]$ and is proportional to the length of the circle. Let $\gamma(\sigma) = (\theta(\sigma), \pi(\sigma))$ be a loop in $T^*S^1_R$. Then the value of the Hamiltonian function at $\gamma$ is given by

$$\mathscr{H}(\gamma) = \int_{S^1} d\sigma\, \mathscr{H}(\gamma;\sigma)$$

with

$$\mathscr{H}(\gamma;\sigma) = \frac{1}{2R^2}\pi(\sigma)^2 + \frac{R^2}{2}\left(\frac{d\theta}{d\sigma}\right)^2 .$$

The Hamiltonian vector field $X_{\mathscr{H}}$ on $LT^*S^1_R$ associated to $\mathscr{H}$ can be computed straightforwardly. Its value at a $\gamma$ in $LT^*S^1_R$ is a vector field along $\gamma$ in $T^*S^1_R$ and is given by

$$X_{\mathscr{H}}|_{\gamma}(\sigma) = \left(R^2\frac{d^2\theta}{d\sigma^2}\right)\partial_\pi|_{\gamma(\sigma)} + \left(\frac{1}{R^2}\pi(\sigma)\right)\partial_\theta|_{\gamma(\sigma)} .$$

The form of the Hamiltonian function suggests immediately a transformation $\Phi$ from $LT^*S^1_R$ to $LT^*S^1_{\frac{1}{R}}$ that leaves the form invariant:

$$\Phi(\gamma)(\sigma) = \left(\int_0^\sigma d\varsigma\pi(\varsigma), \frac{d\theta}{d\sigma}(\sigma)\right).$$

Let $\tilde{\gamma} = \Phi(\gamma)$, then $\widetilde{\mathscr{H}}$, the pushed-forward of $\mathscr{H}$ to $LT^*S^1_{\frac{1}{R}}$, has density

$$\widetilde{\mathscr{H}}(\tilde{\gamma};\sigma) = \frac{1}{2R^2}\left(\frac{d\tilde{\theta}}{d\sigma}\right)^2 + \frac{R^2}{2}\tilde{\pi}^2 ,$$

which is exactly the string Hamiltonian function with target space $S^1_{\frac{1}{R}}$. Unfortunately, as we shall see, this natural candidate for the sought canonical transformation is not extendable to the whole phase space. Nevertheless, a natural "*quantization condition*" comes in to select the correct reduced phase space on which everything works.

**Proposition 1.1.** *Let $\gamma_\theta$ and $\gamma_\pi$ be respectively the $\theta$- and $\pi$-component of a loop $\gamma$ in $T^*S^1_R$. Let*

$$L_R^{(m,n)} = \left\{\gamma \,|\, \deg\gamma_\theta = m, \int_{S^1} d\sigma\,\pi(\sigma) = 2\pi n\right\} .$$

*Then*

1. *Each $L_R^{(m,n)}$ is a sub-Hamiltonian system in $(LT^*S^1_R, \omega, \mathscr{H})$;*
2. *$\Phi$ is a canonical transformation from $L_R^{(m,n)}$ onto $L_{\frac{1}{R}}^{(n,m)}$.*

*Schematic representation of $\Phi$ may is depicted in (Fig. 1).*

*Proof.* Let $t$ be the parameter for the string Hamiltonian flow. Continuity of the flow implies that winding number of $\gamma_\theta$ is invariant. Using the explicit expression for $X_{\mathscr{H}}$, the first claim then follows from the fact that along the flow

$$\begin{aligned}\frac{d}{dt}\int_{S^1} d\sigma\,\pi(\sigma,t) &= \int_{S^1} d\sigma\,\frac{\partial}{\partial t}\pi(\sigma,t)\\ &= R^2\int_{S^1} d\sigma\frac{\partial^2}{\partial\sigma^2}\theta(\sigma,t) = R^2\int_{S^1} d\sigma\frac{\partial}{\partial\sigma}\left(\frac{\partial}{\partial\sigma}\theta(\sigma,t)\right),\end{aligned}$$

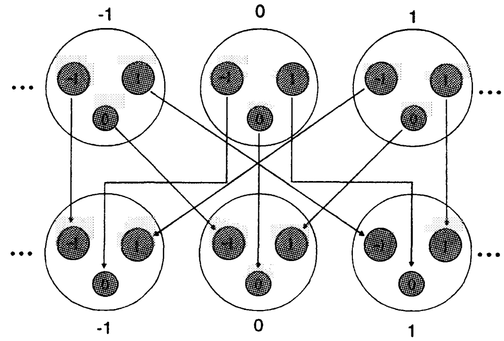


**Fig. 1.** T-duality between $S_R^1$ and $S_{\frac{1}{R}}^1$. The outer lables denote the winding "quantum" number; the inner labels denote the momentum "quantum" number.

which vanishes since $T_*S^1$ is trivial and, thus, $\frac{\partial}{\partial\sigma}\theta(\sigma,t)$ can be regarded as a map from $S^1$ to $\mathbb{R}$.

That $\Phi$ maps $L_R^{(m,n)}$ onto $L_{\frac{1}{R}}^{(n,m)}$ is clear. Its inverse is given by

$$\Phi^{-1}(\tilde{\gamma})(\sigma) = \left( \int_0^\sigma ds\, \tilde{\pi}, \frac{d}{d\sigma}\tilde{\theta}(\sigma) \right).$$

One can check that $\Phi^* \widetilde{\mathscr{H}} = \mathscr{H}$.

Now let $Y$ be in $T_\gamma L_R^{(m,n)}$. In the canonical coordinates, as a vector field along $\gamma$ in $T^*S^1$, one may write

$$Y(\sigma) = A(\sigma) \left.\frac{\partial}{\partial\theta}\right|_{\gamma(\sigma)} + B(\sigma) \left.\frac{\partial}{\partial\pi}\right|_{\gamma(\sigma)} .$$

Then

$$\Phi_* : T_\gamma L_R^{(m,n)} \longrightarrow T_{\Phi(\gamma)} L_{\frac{1}{R}}^{(n,m)}$$

$$A(\sigma) \left.\frac{\partial}{\partial\theta}\right|_{\gamma(\sigma)} + B(\sigma) \left.\frac{\partial}{\partial\pi}\right|_{\gamma(\sigma)} \mapsto \left( \int_0^\sigma ds\, B(s) \right) \left.\frac{\partial}{\partial\tilde{\theta}}\right|_{\Phi(\gamma)(\sigma)} + \left( \frac{d}{d\sigma} A(\sigma) \right) \left.\frac{\partial}{\partial\tilde{\pi}}\right|_{\Phi(\gamma)(\sigma)} .$$

From this one can check straightforwardly that

$$\tilde{\omega}(\Phi_* Y_1, \Phi_* Y_2) = \omega(Y_1, Y_2) ;$$

and hence $\Phi$ is a symplectomorphism from $L_R^{(m,n)}$ onto $L_{\frac{1}{R}}^{(n,m)}$. This concludes the proof. □

## 2. The T-Duality Transformation and T-Dual Structures

With the preparation in Sect. 1 we shall focus for the rest of this article on the case of simple compact Lie groups and their associated Lie algebras. To avoid confusion with other duals in the discussion, we will write "T-dual" for "target-space dual."

### 2.1. A Generating Function and the Induced Canonical Transformation

*2.1.1. A Natural Generating Function $\Gamma : L\mathfrak{g} \times LG \to \mathbb{R}$.* Let $G$ be a simple compact Lie group and $\mathfrak{g}$ be its associated Lie algebra. We shall identify $\mathfrak{g}$ constantly with $T_eG$, the tangent space of $G$ at the identity $e$ or occasionally with the space of all left-invariant vector fields on $G$ whenever necessary. $G$ admits a bi-invariant positive-definite metric which is unique up to a constant multiple. This metric is proportional to the Killing form of $G$. Its restriction to $\mathfrak{g} = T_eG$ provides an Ad-invariant inner product in the Lie algebra. For simplicity of notation, we shall denote both of them by $\langle\, , \rangle$.

Let $\Omega$ be the left invariant Maurer–Cartan 1-form of $G$. Recall that, for $X \in T_gG$, it is defined by

$$\Omega(X) = (l_{g^{-1}})_*(X) \in T_eG = \mathfrak{g} ,$$

where $l_g : G \to G$ is left multiplication by $g$.

Then we choose a generating function $\Gamma$ defined as follows.

**Definition 2.1 (Generating function).** *Let $(\psi : S^1 \to \mathfrak{g}) \in L\mathfrak{g}$ and $(\varphi : S^1 \to G) \in LG$. With $S^1$ parameterized by $\sigma$, we define*

$$\Gamma(\psi, \varphi; \sigma) = \langle \psi(\sigma), \Omega(\varphi_* \partial_\sigma) \rangle ,$$

*where $\partial_\sigma$ is the coordinate vector field along $S^1$. We choose the generating function $\Gamma : L\mathfrak{g} \times LG \to \mathbb{R}$ to be*

$$\Gamma(\psi, \varphi) = \int_{S^1} d\sigma \Gamma(\psi, \varphi; \sigma) .$$

*Remark.* Notice that when $G$ is identified with a classical matrix group, the above expression for $\Gamma$ is exactly

$$\Gamma(\psi, \varphi) = \text{constant} \cdot \int_{S^1} \mathrm{Tr} \left( \psi(\sigma)\varphi(\sigma)^{-1} \frac{d}{d\sigma}\varphi(\sigma) \right) ,$$

which appears already in the literature for constructing dual models of the chiral $SU(2)$-model.

*2.1.2. The Induced Canonical Transformations.* In the following arguments we shall denote points in $LT^*G$ by $(\varphi, \varpi)$, where $\varphi$ is a smooth map from $S^1$ into $G$ and $\varpi$ is a 1-form along $\varphi$. Similarly we shall denote points in $LT^*\mathfrak{g}$ by $(\psi, \pi)$, where $\psi$ is a smooth map from $S^1$ into $\mathfrak{g}$ and $\pi$ is a 1-form along $\psi$. Our first task is to work out the functional derivative

$$\varpi = \frac{\delta \Gamma(\psi, \varphi)}{\delta \varphi}, \qquad \pi = -\frac{\delta \Gamma(\psi, \varphi)}{\delta \psi} ;$$

and then to solve $(\psi, \pi)$ and $(\varphi, \varpi)$ in terms of each other.

**Proposition 2.1.** *The functional derivatives of the generating function $\Gamma$ with respect to its arguments are respectively*

$$\varpi = \frac{\delta\Gamma(\psi,\varphi)}{\delta\varphi} = -\left\langle \left(\frac{d}{d\sigma} + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\right)\psi, \Omega(\,\cdot\,)\right\rangle ,$$

$$\pi = -\frac{\delta\Gamma(\psi,\varphi)}{\delta\psi} = -\langle \Omega(\varphi_*\partial_\sigma),\ \cdot\ \rangle ,$$

*where* $\mathrm{ad}_{(\,\cdot\,)}$ *is the* ad-*representation of* $(\,\cdot\,)$ *on* $\mathfrak{g}$.

*Remark.* One may notice that, in the expression, the part,

$$\langle \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\psi, \Omega(\,\cdot\,)\rangle ,$$

up to a constant, is exactly the canonical 3-form $\Xi$ on a simple Lie group defined by

$$\Xi(X,Y,Z) = K([\Omega(X),\Omega(Y)],\Omega(Z)) ,$$

where $X,Y,Z$ are some tangent vectors at some point in $G$, $[\ ,\ ]$ is the Lie bracket for the associated Lie algebra $\mathfrak{g}$ and $K$ is the Killing form of $G$.

*Proof.* Since $\Gamma(\psi,\varphi)$ is linear with respect to $\psi$, one has immediately

$$-\frac{\delta\Gamma(\psi,\varphi)}{\delta\psi} = -\langle \Omega(\varphi_*\partial_\sigma),\ \cdot\ \rangle .$$

Thus for the rest of the proof we shall focus on the computation of $\frac{\delta\Gamma(\psi,\varphi)}{\delta\varphi}$.

(i) Let $X$ be a vector field along $\varphi$. Let

$$\begin{aligned} \Upsilon : S^1 \times (-\varepsilon,\varepsilon) &\to G \\ (\sigma,\tau) & \end{aligned}$$

such that

$$\Upsilon(\,\cdot\,,0) = \varphi, \qquad \Upsilon_*(\partial_\tau|_{\tau=0}) = X .$$

Let $\varphi_\tau = \Upsilon(\,\cdot\,,\tau)$. One has

$$\begin{aligned} \int_{S_1} \frac{\delta\Gamma(\psi,\varphi)}{\delta\varphi}(\sigma)(X(\sigma))d\sigma &= \left.\frac{d}{d\tau}\right|_{\tau=0} \Gamma(\psi,\varphi_\tau) \\ &= \int_{S_1} \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \langle \psi(\sigma), \Omega(\varphi_{\tau*}\partial_\sigma)\rangle d\sigma \\ &= \int_{S_1} \left\langle \psi(\sigma), \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \Omega(\varphi_{\tau*}\partial_\sigma)\right\rangle d\sigma . \end{aligned}$$

(ii) We shall show next that

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \Omega(\varphi_{\tau*}\partial_\sigma) = \frac{d}{d\sigma}\Omega(X) + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\Omega(X) .$$

Let $S = \Upsilon_* \partial_\sigma$, $T = \Upsilon_* \partial_\tau$. Let $e_i$ be a basis for $\mathfrak{g}$ and $\omega^i$ be the 1-forms on $G$ obtained by left-translating the dual basis of $e_i$ around $G$. Then $\Omega = e_i \otimes \omega^i$; and

$$\begin{aligned}\frac{\partial}{\partial\tau}\bigg|_{\tau=0} \Omega(\varphi_{\tau *}\partial_\sigma) &= X(e_i \otimes \omega^i(S)) = e_i \cdot T\omega^i(S)|_{\tau=0} \\ &= e_i \cdot \{S\omega^i(T) + \omega([T,S]) + 2\,d\omega^i(T,S)\}_{\tau=0} \\ &= S\Omega(X) + 2\,d\Omega(T,S), \quad \text{since } T_{\tau=0} = X \text{ and } [T,S] = 0; \\ &= \frac{d}{d\sigma}\Omega(X) + 2\,d\Omega(T,S)\,.\end{aligned}$$

By the Maurer–Cartan equation, i.e.

$$d\Omega + \tfrac{1}{2}[\Omega,\Omega] = 0\,,$$

where $[\ ,\ ]$ means Lie bracket for the Lie algebra part and wedge product for the 1-form part, the second term in the last equation can be rewritten as

$$\begin{aligned}2\,d\Omega(T,S) &= -[\Omega,\Omega](T,S) \\ &= [\Omega(S),\Omega(T)] = \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\Omega(X), \text{ at } \tau = 0\,.\end{aligned}$$

(iii) Finally from the fact that ad is skew-symmetric with respect to $\langle\ ,\ \rangle$, we have

$$\begin{aligned}&\int_{S^1} d\sigma \left\langle \psi(\sigma), \frac{d}{d\sigma}\Omega(X) + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\Omega(X)\right\rangle \\ &= \int_{S^1} d\sigma \frac{d}{d\sigma}\langle\psi(\sigma),\Omega(X)\rangle - \int_{S^1} d\sigma \left\langle \frac{d}{d\sigma}\psi(\sigma),\Omega(X)\right\rangle - \int_{S^1} d\sigma \langle \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\psi(\sigma),\Omega(X)\rangle\,.\end{aligned}$$

The first term is a total derivative of a function on $S^1$; hence vanishes. In conclusion,

$$\int_{S^1} d\sigma \frac{\delta\Gamma(\psi,\varphi)}{\delta\varphi}(\sigma)(X(\sigma)) = -\int_{S^1} d\sigma \left\langle \frac{d}{d\sigma}\psi(\sigma) + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\psi(\sigma), \Omega(X)\right\rangle\,.$$

Since we are in the smooth category and the above is true for all smooth $X$ along $\varphi$, one must have

$$\frac{\delta\Gamma(\psi,\varphi)}{\delta\varphi} = -\left\langle \frac{d}{d\sigma}\psi + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\psi, \Omega(\,\cdot\,)\right\rangle\,.$$

This completes the proof. □

From the previous proposition, one can now obtain the sought out canonical transformations formally. Before doing so, we introduce the following correspondence for necessity.

Let $\psi \in L\mathfrak{g}$. Denote by $E(\psi)$ a path in $G$ such that

$$\psi = \Omega(E(\psi)_*(\partial_\sigma))\,.$$

From the theory of ordinary differential equations, $E(\psi)$ always exists but in general doesn't close up to form a loop. Given $\psi, E(\psi)$ is unique up to a left translation in $G$.

**Theorem 2.1 (Canonical Transformations).** *The formal canonical transformation $\Phi$ from $LT^*G$ to $LT^*\mathfrak{g}$ induced from the generating function $\Gamma$ is*

$$\psi = -(\nabla^{\varphi}_{\partial_\sigma})^{-1}(\Omega(\varpi^{\sim}))\,,$$

$$\pi = -\Omega(\varphi_*\partial_\sigma)^{\sim}\,,$$

*where* $\nabla^{\varphi}_{\partial_\sigma} = \frac{d}{d\sigma} + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}$ *and "$\sim$" represents the metric dual with respect to* $\langle,\rangle$ *(Fig. 2).*

*Its formal inverse $\Phi^{-1}$ is given by*

$$\varphi = E(-\pi^{\sim})\,,$$

$$\varpi = \left((\Omega|_{\varphi})^{-1} \circ \left(-\frac{d}{d\sigma} + \mathrm{ad}_{\pi^{\sim}}\right)\psi\right)^{\sim}\,.$$

*Remark.* Requiring that $\psi, \varphi$ be loops puts constraints on $\varpi$ and $\pi$ respectively. Thus, like in the case of $S^1$, $\Phi$ and its inverse are defined only on a reduced phase space. One may think of this as part of certain "*quantization conditions.*" (Or one may consider the more general space of "*twisted loops*" [P–S], which we will not discuss here.)

*Proof of Theorem 2.1.* The transformations are read off straightforwardly from the previous proposition. These transformations are only formal due to the multivaluedness of operators $(\nabla^{\varphi}_{\partial_\sigma})^{-1}$ and $E$. However it turns out that the multi-valuedness can be completely understood, as will be explained in the next sub-subsection. $\square$

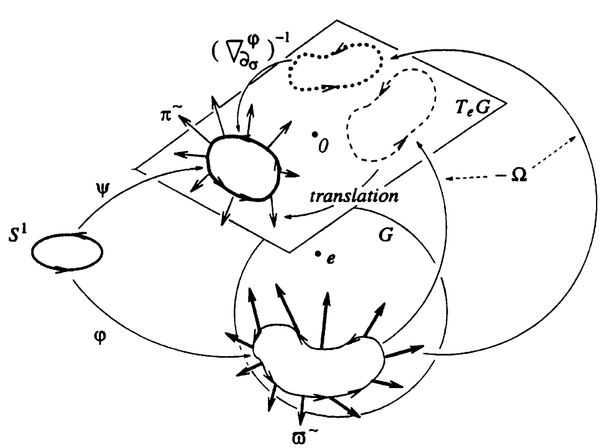


**Fig. 2.** The formal canonical transformation between $LT^*G$ and $LT^*\mathfrak{g}$. In the picture the metric dual with respect to $\langle,\rangle$ is used to represent a 1-form along a loop. Notice that the tangent vector field to a loop and the momentrum vector field along it are exchanged under the formal canonical transformation.

*2.1.3. Multi-Valuedness of the Formal Canonical Transformations.* We shall show that, under our choice of generating function, the multi-valuedness for the induced canonical transformation in either direction is exactly what is expected.

Recall that the multi-valuedness of the map $E : L\mathfrak{g} \to \operatorname{Path} G$ is parameterized by $G$ itself. Hence we only need to take care of the multi-valuedness of the inverse, $(\nabla^{\varphi}_{\partial_\sigma})^{-1}$.

From the expression

$$\nabla^{\varphi}_{\partial_\sigma} = \frac{d}{d\sigma} + \mathrm{ad}_{\Omega(\varphi_* \partial_\sigma)} ,$$

if we regard $\psi : S^1 \to \mathfrak{g}$ as a section in the trivial bundle

$$\begin{array}{ccc} S^1 & \times & \mathfrak{g} \\ & \downarrow & \\ & S^1 , & \end{array}$$

then, given $\varphi$, the differential operator $\nabla^{\varphi}_{\partial_\sigma}$ defines a connection on this bundle. Due to the linearity of this differential operator, for any fixed $\varphi$, the multi-valuedness of $(\nabla^{\varphi}_{\partial_\sigma})^{-1}$ is parameterized by the kernel $\ker(\nabla^{\varphi}_{\partial_\sigma})$. From the horizontal lifting property of paths in the base $S^1$, it must be isomorphic to a subspace of $\mathfrak{g}$.

**Lemma 2.1.** *For any $\varphi : S^1 \to G$, the induced connection $\nabla^{\varphi}_{\partial_\sigma}$ on the bundle $S^1 \times \mathfrak{g}$ is trivial.*

*Proof.* Let $s$ be a section in our trivial bundle $S^1 \times \mathfrak{g}$. Observe that the following three statements are equivalent:

$$\begin{array}{ll} (1) & \nabla^{\varphi}_{\partial_\sigma} s = 0 ; \\ (2) & \dfrac{d}{d\sigma} s(\sigma) = -\mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)} s(\sigma) ; \\ (3) & s(\sigma) = \mathrm{Ad}_{\varphi(\sigma)^{-1}} X_0 \quad \text{for some } X_0 \in \mathfrak{g} . \end{array}$$

That (1) and (2) are equivalent follows from definition.

For (2) and (3), one can check that the section defined in (3) satisfies the first-order ordinary differential equation given in (2). Conversely, given a section $s$ that satisfies the differential equation in (2) with $s(\sigma_0)$ specified for some $\sigma_0 \in S^1$, there exists some $X_0 \in \mathfrak{g}$ such that $s(\sigma_0) = \mathrm{Ad}_{\varphi(\sigma_0)^{-1}} X_0$ since $\mathrm{Ad}_g$ is an automorphism of $\mathfrak{g}$ for any $g \in G$. One can then define a new section as in (3) that coincides with $s$ at $\sigma = \sigma_0$. Uniqueness of solutions for ordinary differential equations implies these two sections must coincide everywhere. Thus (2) and (3) are equivalent.

Altogether, this shows that our bundle with connection $\nabla^{\varphi}_{\partial_\sigma}$ admits a combing by globally well-defined flat sections parameterized by the Lie algebra $\mathfrak{g}$.

This concludes the proof. $\square$

**Corollary 2.1.** *The kernel* $\ker(\nabla^{\varphi}_{\partial_\sigma})$ *is isomorphic to* $\mathfrak{g}$.

*Proof.* Use the trivial connection $\nabla^{\varphi}_{\partial_\sigma}$ to retrivialize the bundle $S^1 \times \mathfrak{g}$. The kernel consists of exactly the constant sections with respect to the new trivialization and the space of all such sections is isomorphic to $\mathfrak{g}$. $\square$

*Remark.* Recall that our generating function is given by

$$\Gamma(\psi, \varphi) = \int_{S^1} d\sigma \langle \psi(\sigma), \Omega(\varphi_* \partial_\sigma) \rangle .$$

Let $l_g$ be the left-translation by $g$. Since $\Gamma(\psi, \varphi) = \Gamma(\psi, l_g \circ \varphi)$ for any $g \in G$, it is expected that $\Gamma$ would determine a canonical transformation from $LT^*G$ to $LT^*\mathfrak{g}$ only up to a freedom parameterized by $G$. The corollary now shows that there is another part of freedom parameterized by the Lie algebra. This puts the group and its associated Lie algebra on an equal footing, which is a nice feature as far as duality is concerned.

*2.2. The Dual Structures on the Associated Lie Algebra.* Continuing the previous arguments, we shall show that

**Theorem 2.2.** *Let $G$ be a simple compact Lie group with a bi-invariant metric $\langle \, , \rangle$. Let $\mathfrak{g}$ be its associated Lie algebra identified with $T_eG$. In order to make the canonical transformations worked out in the previous section be T-duality transformations, the metric $\langle\!\langle \, , \rangle\!\rangle$ and the B-field (a 2-form $B$) on $\mathfrak{g}$ are uniquely determined. They are given by*

$$\begin{aligned} \langle\!\langle X, Y \rangle\!\rangle &= \langle (\mathrm{Id} - \mathrm{ad}_v)^{-1} X, (\mathrm{Id} - \mathrm{ad}_v)^{-1} Y \rangle , \\ B(X, Y) &= \langle\!\langle X, \mathrm{ad}_v Y \rangle\!\rangle , \end{aligned}$$

*where $v \in \mathfrak{g}$, $X, Y \in T_v\mathfrak{g}$ and* ad *is the* ad*-representation of $\mathfrak{g}$ on itself.*

Notice that in the above expressions we implicitly identify $T_v\mathfrak{g}$, for any $v$ in $\mathfrak{g}$, with $\mathfrak{g}$ itself by the vector space structure of $\mathfrak{g}$.

*Proof.* We sketch first the basic ideas in the proof and then present the details of the manipulations.

(i) *Basic ideas.* The inverse formal canonical transformation $\Phi^{-1}$ from $LT^*\mathfrak{g}$ to $LT^*G$ pulls back the string Hamiltonian function $\mathscr{H}$ on $LT^*G$ to some function $\widetilde{\mathscr{H}}$ on $LT^*\mathfrak{g}$. It turns out that this is also a string Hamiltonian function, from which one reads off the dual metric $\langle\!\langle \, , \rangle\!\rangle$ and the dual $B$-field $B$ on $\mathfrak{g}$.

(ii) *Details*: Recall that the Hamiltonian $\mathscr{H}$ on $LT^*G$ is given by

$$\mathscr{H} = \int_{S^1} d\sigma \mathscr{H}(\varphi, \varpi; \sigma) ,$$

where

$$\mathscr{H}(\varphi, \varpi; \sigma) = \tfrac{1}{2}\{\langle \varphi_* \partial_\sigma, \varphi_* \partial_\sigma \rangle + \langle \varpi(\sigma), \varpi(\sigma) \rangle^\sim\} .$$

To get the pulled back Hamiltonian $\widetilde{\mathscr{H}}$ on $LT^*\mathfrak{g}$, one simply rewrites $\mathscr{H}$ in terms of $(\psi, \pi)$ by using

$$\varphi = E(-\pi^\sim), \qquad \varpi = \left( (\Omega|_\varphi)^{-1} \circ \left( -\frac{d}{d\sigma} + \mathrm{ad}_{\pi^\sim} \right) \psi \right)^\sim .$$

Now

$$\langle \varphi_* \partial_\sigma, \varphi_* \partial_\sigma \rangle = \langle \Omega(\varphi_* \partial_\sigma), \Omega(\varphi_* \partial_\sigma) \rangle = \langle -\pi^\sim, -\pi^\sim \rangle = \langle \pi^\sim, \pi^\sim \rangle = \langle \pi, \pi \rangle^\sim ;$$

and

$$
\begin{aligned}
\langle \varpi, \varpi \rangle^{\sim} &= \left\langle \Omega|_{\varphi}^{-1} \circ \left( -\frac{d}{d\sigma} + \mathrm{ad}_{\pi^{\sim}} \right) \psi, \Omega|_{\varphi}^{-1} \circ \left( -\frac{d}{d\sigma} + \mathrm{ad}_{\pi^{\sim}} \right) \psi \right\rangle \\
&= \left\langle \left( -\frac{d}{d\sigma} + \mathrm{ad}_{\pi^{\sim}} \right) \psi, \left( -\frac{d}{d\sigma} + \mathrm{ad}_{\pi^{\sim}} \right) \psi \right\rangle \\
&= \langle \psi_* \partial_\sigma, \psi_* \partial_\sigma \rangle - 2\langle \psi_* \partial_\sigma, \mathrm{ad}_{\pi^{\sim}} \psi \rangle + \langle \mathrm{ad}_{\pi^{\sim}} \psi, \mathrm{ad}_{\pi^{\sim}} \psi \rangle \, .
\end{aligned}
$$

Thus

$$
\begin{aligned}
2\widetilde{\mathscr{H}} &= \langle \pi, \pi \rangle^{\sim} + \langle \psi_* \partial_\sigma, \psi_* \partial_\sigma \rangle - 2\langle \psi_* \partial_\sigma, \mathrm{ad}_{\pi^{\sim}} \psi \rangle + \langle \mathrm{ad}_{\pi^{\sim}} \psi, \mathrm{ad}_{\pi^{\sim}} \psi \rangle \\
&= (\langle \pi^{\sim}, \pi^{\sim} \rangle + \langle \mathrm{ad}_\psi \pi^{\sim}, \mathrm{ad}_\psi \pi^{\sim} \rangle) + 2\langle \mathrm{ad}_\psi \pi^{\sim}, \psi_* \partial_\sigma \rangle + \langle \psi_* \partial_\sigma, \psi_* \partial_\sigma \rangle ,
\end{aligned}
$$

since $\mathrm{ad}_{\pi^{\sim}} \psi = -\mathrm{ad}_\psi \pi^{\sim}$ .

Next we try to put it into the string Hamiltonian form

$$
\langle\!\langle \pi - B(\,\cdot\,, \psi_* \partial_\sigma), \pi - B(\,\cdot\,, \psi_* \partial_\sigma) \rangle\!\rangle^{\wedge} + \langle\!\langle \psi_* \partial_\sigma, \psi_* \partial_\sigma \rangle\!\rangle \, ,
$$

where $\langle\!\langle\ ,\ \rangle\!\rangle$ and $B$ are respectively the sought-for metric and 2-form on $\mathfrak{g}$ and $\langle\!\langle\ ,\ \rangle\!\rangle^{\wedge}$ is the induced metric on $T^*\mathfrak{g}$ from $\langle\!\langle\ ,\ \rangle\!\rangle$.

Since $\langle \pi^{\sim}, \pi^{\sim} \rangle + \langle \mathrm{ad}_\psi \pi^{\sim}, \mathrm{ad}_\psi \pi^{\sim} \rangle$ contains all the quadratic terms in $\pi$, by comparison, we must have

$$
\begin{aligned}
\langle\!\langle \pi, \pi \rangle\!\rangle^{\wedge} &= \langle \pi^{\sim}, \pi^{\sim} \rangle + \langle \mathrm{ad}_\psi \pi^{\sim}, \mathrm{ad}_\psi \pi^{\sim} \rangle \\
&= \langle (\mathrm{Id} + \mathrm{ad}_\psi) \pi^{\sim}, (\mathrm{Id} + \mathrm{ad}_\psi) \pi^{\sim} \rangle, \quad \text{since } \langle \pi^{\sim}, \mathrm{ad}_\psi \pi^{\sim} \rangle = 0 \, .
\end{aligned}
$$

Notice that the argument also implies that $\langle\!\langle\ ,\ \rangle\!\rangle$ is positive definite and that, for each $\sigma$, $\mathrm{Id} + \mathrm{ad}_{\psi(\sigma)}$ is an invertible linear transformation from $T_{\psi(\sigma)}\mathfrak{g} = \mathfrak{g}$ to itself.

To get $\langle\!\langle\ ,\ \rangle\!\rangle$ itself, fix an orthonormal basis and its dual basis for $(\mathfrak{g}, \langle\ ,\ \rangle)$. We may then regard elements in $\mathfrak{g}$ as a column vector and elements in $\mathfrak{g}^*$ as a row vector. Then, with respect to such bases, $\pi^{\sim} = \pi^t$, where "$t$" stands for transpose; and

$$
\langle\!\langle \pi, \pi \rangle\!\rangle^{\wedge} = \pi (\mathrm{Id} + \mathrm{ad}_\psi)^t (\mathrm{Id} + \mathrm{ad}_\psi) \pi^t \, .
$$

Consequently, for $X, Y \in T_{\psi(\sigma)}\mathfrak{g} = \mathfrak{g}$,

$$
\begin{aligned}
\langle\!\langle X, Y \rangle\!\rangle &= X^t ((\mathrm{Id} + \mathrm{ad}_\psi)^t (\mathrm{Id} + \mathrm{ad}_\psi))^{-1} Y \\
&= \langle ((\mathrm{Id} + \mathrm{ad}_\psi)^t)^{-1} X, ((\mathrm{Id} + \mathrm{ad}_\psi)^t)^{-1} Y \rangle \\
&= \langle (\mathrm{Id} - \mathrm{ad}_\psi)^{-1} X, (\mathrm{Id} - \mathrm{ad}_\psi)^{-1} Y \rangle \, ,
\end{aligned}
$$

where we use the fact that $(\mathrm{Id} + \mathrm{ad}_\psi)^t = \mathrm{Id} - \mathrm{ad}_\psi$ since

$$
\langle (\mathrm{Id} + \mathrm{ad}_\psi) X, Y \rangle = \langle X, (\mathrm{Id} - \mathrm{ad}_\psi) Y \rangle \, .
$$

The mixed term

$$
\begin{aligned}
\langle\!\langle \pi, B(\,\cdot\,,\psi_*\partial_\sigma)\rangle\!\rangle^\wedge &= -\langle \mathrm{ad}_\psi \pi^\sim, \psi_*\partial_\sigma\rangle = \langle \pi^\sim, \mathrm{ad}_\psi \psi_*\partial_\sigma\rangle \\
&= \langle (\mathrm{Id}+\mathrm{ad}_\psi)\pi^\sim, (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi\psi_*\partial_\sigma\rangle \\
&= \langle (\mathrm{Id}+\mathrm{ad}_\psi)\pi^\sim, (\mathrm{Id}+\mathrm{ad}_\psi)((\mathrm{Id}+\mathrm{ad}_\psi)^{-1}(\mathrm{Id}-\mathrm{ad}_\psi)^{-1})\mathrm{ad}_\psi\psi_*\partial_\sigma\rangle \\
&= \langle (\mathrm{Id}+\mathrm{ad}_\psi)\pi^\sim, (\mathrm{Id}+\mathrm{ad}_\psi)B(\,\cdot\,,\psi_*\partial_\sigma)^\sim\rangle\,.
\end{aligned}
$$

Thus

$$B(\,\cdot\,,\psi_*\partial_\sigma)^\sim = (\mathrm{Id}+\mathrm{ad}_\psi)^{-1}(\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi\psi_*\partial_\sigma\,;$$

and

$$
\begin{aligned}
B(X,\psi_*\partial_\sigma) &= \langle (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}X, (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi\psi_*\partial_\sigma\rangle \\
&= \langle\!\langle X, \mathrm{ad}_\psi\psi_*\partial_\sigma\rangle\!\rangle\,.
\end{aligned}
$$

By choosing $\psi$ appropriately, we could make it pass through any point in $\mathfrak{g}$ along any tangent vector at that point. Thus the above implies that

$$B(X,Y) = \langle\!\langle X, \mathrm{ad}_v Y\rangle\!\rangle, \qquad \text{for } X, Y \in T_v\mathfrak{g}\,.$$

That $B$ is a 2-form follows from the commutativity of $(\mathrm{Id}+\mathrm{ad}_v)^{-1}$, $(\mathrm{Id}-\mathrm{ad}_v)^{-1}$, $\mathrm{ad}_v$ and that $\langle X, \mathrm{ad}_v Y\rangle = -\langle \mathrm{ad}_v X, Y\rangle$.

Completing the square, we then obtain

$$2\widetilde{\mathscr{H}} = \langle\!\langle \pi - B(\,\cdot\,,\psi_*\partial_\sigma), \pi - B(\,\cdot\,,\psi_*\partial_\sigma)\rangle\!\rangle^\wedge + \langle\!\langle \psi_*\partial_\sigma, \psi_*\partial_\sigma\rangle\!\rangle + \mathit{Remainder}\,,$$

where

$$\mathit{Remainder} = \langle \psi_*\partial_\sigma, \psi_*\partial_\sigma\rangle - \langle\!\langle \psi_*\partial_\sigma, \psi_*\partial_\sigma\rangle\!\rangle - \langle\!\langle B(\,\cdot\,,\psi_*\partial_\sigma), B(\,\cdot\,,\psi_*\partial_\sigma)\rangle\!\rangle^\wedge\,.$$

We shall now show that *Remainder* $= 0$.

$$
\begin{aligned}
&\mathit{Remainder} \\
&= \langle \psi_*\partial_\sigma, \psi_*\partial_\sigma\rangle - \langle (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\psi_*\partial_\sigma, (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\psi_*\partial_\sigma\rangle \\
&\quad - \langle (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi\psi_*\partial_\sigma, (\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi\psi_*\partial_\sigma\rangle \\
&= \langle \psi_*\partial_\sigma, [\mathrm{Id} - (\mathrm{Id}+\mathrm{ad}_\psi)^{-1}(\mathrm{Id}-\mathrm{ad}_\psi)^{-1} + (\mathrm{Id}+\mathrm{ad}_\psi)^{-1}(\mathrm{Id}-\mathrm{ad}_\psi)^{-1}\mathrm{ad}_\psi^2]\psi_*\partial_\sigma\rangle \\
&= \langle \psi_*\partial_\sigma, (\mathrm{Id}-\mathrm{ad}_\psi^2)^{-1}[(\mathrm{Id}-\mathrm{ad}_\psi^2) - \mathrm{Id} + \mathrm{ad}_\psi^2]\psi_*\partial_\sigma\rangle \\
&= 0 \text{ as claimed.}
\end{aligned}
$$

From the argument it is clear that the $\langle\!\langle\ ,\ \rangle\!\rangle$ and $B$ are uniquely determined by the formal canonical transformation.

This completes the proof. □

*2.3. A Second Glance at $\Phi$.* As already pointed out in a remark following Theorem 2.1, the formal canonical transformation $\Phi$ that we constructed from the generating function $\Gamma$ is not a map from the whole $LT^*G$ to the whole $LT^*\mathfrak{g}$. Instead, it singles out a reduced phase-space, $\operatorname{Dom}\Phi$, the domain of $\Phi$ in $LT^*G$ and a reduced phase-space, $\operatorname{Im}\Phi$, the image of $\Phi$ in $LT^*\mathfrak{g}$. Both are of codimension $\dim G$ in the original phase-spaces they reside. One last question concerning the target-space duality between $(G, \langle\, ,\, \rangle)$ and $(\mathfrak{g}, \langle\!\langle\, ,\, \rangle\!\rangle, B)$ at the classical level under the Hamiltonian formalism based on $\Gamma$ is then:

*Q. Are both* $\operatorname{Dom}\Phi$ *and* $\operatorname{Im}\Phi$ *invariant under the string Hamiltonian flows?*

Naively, one would expect that if $\operatorname{Dom}\Phi$ is invariant under the string Hamiltonian flow in $LT^*G$, then so is $\operatorname{Im}\Phi$ in $LT^*\mathfrak{g}$ due to the way the string Hamiltonian $\widetilde{\mathscr{H}}$ on $LT^*\mathfrak{g}$ is constructed. Also notice that, a priori, the domain and image of a canonical transformation that a generating function generates do not necessarily have to do with Hamiltonian flows. However, as a thumb rule that whatever is natural tends to work, the answer to the above question is affirmative. This certainly gives another backup of the Hamiltonian setting presented here and in the literature.

**Theorem 2.3 (Invariance under String Hamiltonian Flow).** *Both* $\operatorname{Dom}\Phi$ *and* $\operatorname{Im}\Phi$ *are invariant under the related string Hamiltonian flow. Thus they do form sub-string Hamiltonian systems.*

*Proof.* Recall from Theorem 2.1 the map $E$ from $L\mathfrak{g}$ to Path $G$ and the connection $\nabla^{\varphi}_{\partial_\sigma}$. These together with the proof of Lemma 2.1 leads to

$$\begin{aligned}\operatorname{Dom}\Phi &= \{(\varphi, \varpi) \,|\, \varphi \in LG, \text{ and } \Omega(\varpi^{\sim}) \in \operatorname{Im}\nabla^{\varphi}_{\partial_\sigma}\} \\ &= \left\{(\varphi, \varpi) \middle|\, \varphi \in LG, \text{ and } \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)}\, \Omega(\varpi^{\sim})(\sigma) = 0\right\};\end{aligned}$$

and

$$\operatorname{Im}\Phi = \{(\psi, \pi) \,|\, \psi \in L\mathfrak{g}, \text{ and } E(-\pi^{\sim}) \text{ is a loop in } G\}.$$

We shall first show that $\operatorname{Dom}\Phi$ is invariant under the flow generated by the string Hamiltonian vector field $X_{\mathscr{H}}$. We begin with a condition that characterizes $T_*(\operatorname{Dom}\varphi)$ and then show that $X_{\mathscr{H}}$, when restricted to $\operatorname{Dom}\varphi$, satisfies this condition and hence has to be tangent to $\operatorname{Dom}\varphi$. The flow generated therefore leaves $\operatorname{Dom}\varphi$ invariant. The invariance of $\operatorname{Im}\Phi$ under the flow generated by $X_{\widetilde{\mathscr{H}}}$ is also demonstrated by a similar approach.

Let us introduce the following trivializations of bundles in the discussion:

$$T_*G \overset{\Omega}{\simeq} G \times \mathfrak{g}, \quad T^*G \overset{\Omega}{\simeq} G \times \mathfrak{g}^*, \quad LT^*G \overset{\Omega}{\simeq} LG \times L\mathfrak{g}^*,$$

$$T_*(LT^*G) \overset{\Omega}{\simeq} T_*LG \times T_*L\mathfrak{g}^*, \quad T^*(LT^*G) \overset{\Omega}{\simeq} T^*LG \times T^*L\mathfrak{g}^*.$$

We denote all these bundle isomorphisms by $\Omega$ since they all arise from the first isomorphism which defines the Maurer–Cartan form $\Omega$.

Let $(\varphi_t, \varpi_t)$ be a path in $LT^*G$ that lies in $\operatorname{Dom}\Phi$ with

$$\Omega\left(\left.\frac{d}{dt}\right|_{t=0}(\varphi_t, \varpi_t)\right) = (Y, Z)\,.$$

Then

$$\int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi_t(\sigma)}\, \Omega(\varpi_t^{\sim}(\sigma)) = 0 \quad \text{for all } t\,.$$

Thus

$$0 = \left.\frac{d}{dt}\right|_{t=0} \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi_t(\sigma)}\, \Omega(\varpi_t^{\sim}(\sigma))$$

$$= \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi_0(\sigma)} \left.\frac{\partial}{\partial t}\right|_{t=0} (\mathrm{Ad}_{\varphi_0(\sigma)^{-1}\varphi_t(\sigma)}\, \Omega(\varpi_t^{\sim}(\sigma)))$$

$$= \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi_0(\sigma)} \{\mathrm{ad}_{Y(\sigma)}\, \Omega(\varpi_0^{\sim}(\sigma)) + Z(\sigma)^{\sim}\}\,.$$

And we lead to a criterion for a tangent vector $(Y, Z)$ at $(\varphi, \varpi)$ in $\operatorname{Dom}\Phi$ to be in $T_*(\operatorname{Dom}\Phi)$:

$$\int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)} \{\mathrm{ad}_{Y(\sigma)}\, \Omega(\varpi^{\sim}(\sigma)) + Z(\sigma)^{\sim}\} = 0\,.$$

(One should think of $Y$ as an arbitrary smooth vector field along $\varphi$ in $G$; and then $Z$ is subject to the above constraint.)

Next recall that the string Hamiltonian $\mathscr{H}$ on $LT^*G$ has density

$$\mathscr{H}(\varphi, \varpi; \sigma) = \tfrac{1}{2}\langle\varpi(\sigma), \varpi(\sigma)\rangle^{\sim} + \tfrac{1}{2}\langle\varphi_*\partial_\sigma, \varphi_*\partial_\sigma\rangle\,,$$

from which one has

$$d\mathscr{H}(\varphi, \varpi; \sigma) = \langle\varpi(\sigma),\ \cdot\ \rangle^{\sim} - \langle\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma,\ \cdot\ \rangle$$

$$= \varpi^{\sim}(\sigma) - (\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma)^{\sim}\,.$$

Consequently,

$$\Omega(d\mathscr{H}(\varphi, \varpi))(\sigma) = (-\Omega((\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma)^{\sim}),\, \Omega(\varpi^{\sim})(\sigma))\,;$$

and

$$\Omega(X_{\mathscr{H}}|_{(\varphi,\varpi)})(\sigma) = (\Omega(\varpi^{\sim}(\sigma)),\, \Omega((\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma)^{\sim}))\,.$$

Now we only need to check that $\Omega(X_{\mathscr{H}}|_{(\varphi,\varpi)})$ satisfies the above criterion for $(\varphi, \varpi)$ in $\operatorname{Dom}\Phi$,

$$\int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)} \{\mathrm{ad}_{\Omega(\varpi^{\sim}(\sigma))}\Omega(\varpi^{\sim}(\sigma)) + \Omega(\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma)\}$$

$$= \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)} \frac{d}{d\sigma}\Omega(\varphi_*\partial_\sigma)$$

since $\mathrm{ad}_Y Y = 0$ and $\Omega(\nabla_{\varphi_*\partial_\sigma}\varphi_*\partial_\sigma)$;

$$
\begin{aligned}
&= \frac{d}{d\sigma}\Omega(\varphi_*\partial_\sigma);\\
&= -\int_{S^1} d\sigma \left(\frac{d}{d\sigma}\mathrm{Ad}_{\varphi(\sigma)}\right)\Omega(\varphi_*\partial_\sigma)\\
&= -\int_{S^1} d\sigma\,\mathrm{Ad}_{\varphi(\sigma)}\,\mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\Omega(\varphi_*\partial_\sigma) = 0\,.
\end{aligned}
$$

This shows the invariance of $\mathrm{Dom}\,\Phi$ under the flow generated by $X_{\mathscr{H}}$.

Likewise for the image $\mathrm{Im}\,\Phi$ we shall introduce the trivialization of the respective bundles induced by the trivialization

$$T_*\mathfrak{g} = \mathfrak{g}\times\mathfrak{g}\,,$$

arising from the vector space structure of $\mathfrak{g}$. A similar argument as in the first part, using the identity

$$\left.\frac{\partial}{\partial t}\right|_{t=0}\Omega(\varphi_{t*}\partial_\sigma) = \frac{d}{d\sigma}\Omega(T_0) + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\Omega(T_0)$$

in the proof of Proposition 2.1 with $T_0(\sigma)$ being $\left.\frac{\partial}{\partial t}\right|_{t=0}\varphi_t(\sigma)$, gives the following criterion for $T_*\mathrm{Im}\,\Phi$:

$$(Y,Z)\in T_{(\psi,\pi)}\,\mathrm{Im}\,\Phi \quad\text{iff}\quad Z^{\sim}\in \mathrm{Im}\left(-\frac{d}{d\sigma}+\mathrm{ad}_{\pi^{\sim}}\right).$$

It remains to show that $X_{\widetilde{\mathscr{H}}}$, when restricted to $\mathrm{Im}\,\Phi$ satisfies this criterion. Let

$$X_{\widetilde{\mathscr{H}}} = (Y_{\widetilde{\mathscr{H}}}, Z_{\widetilde{\mathscr{H}}})\,.$$

Then in terms of a parallel orthonormal frame in $\mathfrak{g}$ with respect to $\langle\;,\;\rangle$, one has

$$X_{\widetilde{\mathscr{H}}} = (Y_{\widetilde{\mathscr{H}}}, Z_{\widetilde{\mathscr{H}}}) = \left(\frac{\delta\widetilde{\mathscr{H}}}{\delta\pi}, -\frac{\delta\widetilde{\mathscr{H}}}{\delta\psi}\right).$$

Since only the $Z$-component matters, we shall work the latter functional derivative out.

Recall from the proof of Theorem 2.2 that

$$
\begin{aligned}
\widetilde{\mathscr{H}}(\psi,\pi) = \frac{1}{2}\int_{S^1} d\sigma\,\{&\langle\pi^{\sim},\pi^{\sim}\rangle + \langle\mathrm{ad}_\psi\pi^{\sim},\mathrm{ad}_\psi\pi^{\sim}\rangle\\
&+2\,\langle\mathrm{ad}_\psi\pi^{\sim},\psi_*(\partial_\sigma)\rangle + \langle\psi_*\partial_\sigma,\psi_*\partial_\sigma\rangle\}\,.
\end{aligned}
$$

Let $\psi_t$ be a path in $L\mathfrak{g}$ with $\psi_0 = \psi$ and $\left.\frac{d}{dt}\right|_{t=0}\psi_t = T_0$. Then

$$
\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0}\widetilde{\mathscr{H}}(\psi_t,\pi)\\
&\qquad = \int_{S^1} d\sigma\,\{\langle\mathrm{ad}_{T_0}\pi^{\sim},\mathrm{ad}_\psi\pi^{\sim}\rangle + \langle\mathrm{ad}_{T_0}\pi^{\sim},\psi_*\partial_\sigma\rangle\\
&\qquad\quad + \langle\mathrm{ad}_\psi\pi^{\sim},\nabla_{T_0}\psi_{t*}\partial_\sigma\rangle + \langle\nabla_{T_0}\psi_{t*}\partial_\sigma,\psi_{t*}\partial_\sigma\rangle\}
\end{aligned}
$$

(where $\nabla$ is the connection associated to the flat metric $\langle\, , \,\rangle$ on $\mathfrak{g}$)

$$= \int_{S^1} d\sigma \left\langle T_0\,,\ \mathrm{ad}_{\pi^\sim}\mathrm{ad}_{\psi}\pi^\sim + \mathrm{ad}_{\pi^\sim}\psi_*\partial_\sigma - \frac{d}{d\sigma}(\mathrm{ad}_{\psi}\pi^\sim) - \frac{d^2}{dt^2}\psi \right\rangle .$$

Thus, with respect to the same orthonormal parallel frame,

$$\left(Z_{\widetilde{\mathscr{H}}}\right)^\sim = (\mathrm{ad}_{\pi^\sim})^2\psi - \mathrm{ad}_{\pi^\sim}\psi_*\partial_\sigma - \frac{d}{d\sigma}(\mathrm{ad}_{\pi^\sim}\psi) + \frac{d^2}{d\sigma^2}\psi$$
$$+ \left(-\frac{d}{d\sigma} + \mathrm{ad}_{\pi^\sim}\right)(\mathrm{ad}_{\pi^\sim}\psi - \psi_*\partial_\sigma)\ ,$$

which satisfies the criterion for tangency to Im $\Phi$.

This completes the proof. □

*Remark.* From the conditions that characterizes Dom $\Phi$ and Im $\Phi$, one can see that:

1. Dom $\Phi$ is a codimension dim $G$ vector subbundle in $LT^*G$ over $LG$; it has one component over each component of $LG$ and these components are labeled exactly by $\pi_1(G)$ because $\pi_0(LG) = \pi_1(G)$.

2. Im $\Phi$ is also a codimension dim $G$ subspace in $LT^*\mathfrak{g}$; it is a bundle over $L\mathfrak{g}$ with components of the fiber parameterized again by $\pi_1(G)$.

3. $\Phi$ then takes a component of Dom $\Phi$ onto a component of Im $\Phi$ (Fig. 3).

This indicates that analogously to the $S^1$ case, $\Phi$ cannot be extended to a bijection between the two unreduced phase spaces after factoring out the redundancy.

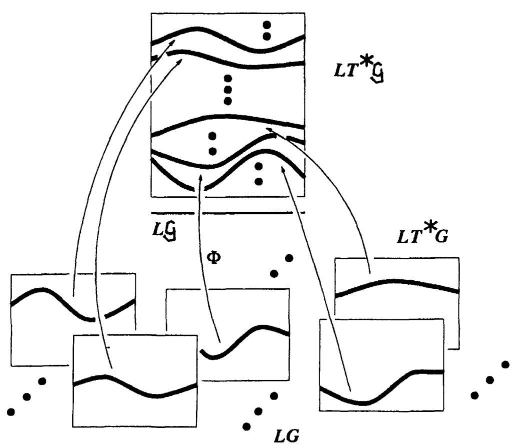


**Fig. 3.** The rectangles on the bottom represent the components of $LT^*G$ and the thick curves are the components of Dom $\Phi$. On the top rectangle, the thick curves represent the components of Im $\Phi$.

*Remark.* In comparison with the $S^1$ (i.e. $U(1)$) case, some features are similar and other features are missing. Actually the $S^1_R$–$S^1_{\frac{1}{R}}$ T-dual pair can be obtained from the general setting with the additional introduction of an appropriate compactification of the associated Lie algebra $\mathfrak{g}$. Naively, when this is done, say by a lattice in $\mathfrak{g}$, one may then extend the scope from the loop space to the space of paths with the difference of end-points lying in the lattice. In the case of $\mathfrak{u}(1)$ there is no difficulty, however, for general simple Lie group – Lie algebra pairs, it is not clear how such a compactification can be introduced that lives compatibly with all other properties.

So far, we have rarely touched upon the symmetries in the theory. A reason for this is that many of the concepts and discussions used here are quite general. Actually, one may see that all the arguments seem to be applicable even to cases without symmetries as long as one is able to tell what is the generating function. We hope that using arguments which avoid relying heavily on symmetries may shed some light on the more challenging situations. Nevertheless, it is worthwhile to see if symmetry provides any conceptual reason why things should work in the present case.

*2.4. Symmetries in the Theory.* Conceptually and naively, the following may be related:

- symmetries of the generating function $\Gamma$;
- the redundancy (i.e. the non-injectiveness and multi-valuedness) of $\Phi$;
- symmetries of the Hamiltonian systems for the $G$-part and the $\mathfrak{g}$-part respectively.

We shall try to clarify their relationships with each other.

*2.4.1. Symmetries of $\Gamma$ and the Redundancy of $\Phi$.* A more symmetric way to think of $\Phi$ is to regard it as a *symplectic relation.* The section $d\Gamma$ in $T^*(L\mathfrak{g} \times LG)$ over $L\mathfrak{g} \times LG$ gives rise to an embedded Lagrangian submanifold in the product space $LT^*\mathfrak{g} \times LT^*G$ with the symplectic structure $\tilde{\omega} \ominus \omega$; this then leads to a relation from $LT^*G$ to $LT^*\mathfrak{g}$, which is exactly $\Phi$. Let $\mathrm{Sym}_\Gamma$ be a group acting on $L\mathfrak{g} \times LG$ that leaves $\Gamma$ invariant. Then its induced action on $LT^*\mathfrak{g} \times LT^*G$ is symplectic and leaves $d\Gamma$ invariant. The intersection of $\mathrm{Sym}_\Gamma$-orbits in $LT^*\mathfrak{g} \times LT^*G$ with the vertical leaves of the product space then contributes to the non-injectiveness of $\Phi$, while that with the horizontal leaves contributes to the multi-valuedness of $\Phi$. This gives a general picture how the symmetry of $\Gamma$ and the redundancy of $\Phi$ are related.

In the present case, there are at least two groups of symmetries for $\Gamma$:

$$(\mathrm{Sym}_\Gamma)_1 \;=\; L\mathfrak{g} \quad \text{with the pointwise addition operation from } \mathfrak{g}\,,$$

whose action on $L\mathfrak{g} \times LG$ is defined by

$$\begin{aligned}(\mathrm{Sym}_\Gamma)_1 \times (L\mathfrak{g} \times LG) &\to L\mathfrak{g} \times LG \\ (\eta, \psi, \varphi) &\mapsto (\psi + \mathrm{ad}_{\Omega(\varphi_*\partial_\sigma)}\eta\,,\ \varphi)\,;\end{aligned}$$

and

$$(\mathrm{Sym}_\Gamma)_2 \;=\; G_L \times G_R \quad \text{with the componentwise multiplication from that of } G\,,$$

whose action on $L\mathfrak{g} \times LG$ is defined by

$$\begin{aligned}(\mathrm{Sym}_\Gamma)_2 \times (L\mathfrak{g} \times LG) &\to L\mathfrak{g} \times LG \\ ((g_1, g_2), \psi, \varphi) &\mapsto (\mathrm{Ad}_{g_2^{-1}}\psi\,, l_{g_1} r_{g_2}\varphi)\,.\end{aligned}$$

Direct computation shows that the intersection of an $L\mathfrak{g}$-orbit in $LT^*\mathfrak{g} \times LT^*G$ with either a vertical or a horizontal leaf is in general just a point; hence this huge symmetry of $\Gamma$ actually won't contribute to the redundancy of $\Phi$ in a major way. However the $G_L \times G_R$-orbit of a point in $LT^*\mathfrak{g} \times LT^*G$ intersects the vertical leaf through that point by the $G_L \times e_R$-suborbit, which is homeomorphic to $G$. Thus the $G_L \times G_R$-symmetry of $\Gamma$ accounts for the non-injectiveness of $\Phi$ completely. On the other hand, the same orbit intersects the horizontal leaf through that point at only one point and, hence, this action doesn't contribute to the multi-valuedness of $\Phi$.

### *2.4.2. Symmetries of the String Hamiltonian Systems*

*(a) The G-Part.* Since the metric $\langle\, ,\, \rangle$ on $G$ is bi-invariant, the Hamiltonian system $(LT^*G, \mathscr{H})$ admits a $G_L \times G_R$ action induced by the left- and right-multiplication in $G$. Moreover, since this action preserves the canonical symplectic potential $\theta$ on $LT^*G$, there is a *moment map* [A-M]

$$\mu = (\mu_L, \mu_R) : LT^*G \;\to\; \mathfrak{g}_L^* \oplus \mathfrak{g}_R^*$$

defined by

$$\begin{aligned}\mu(\varphi, \varpi)(v_1, v_2) &= (\mu_L(\varphi,\varpi)(v_1)\,,\, \mu_R(\varphi,\varpi)(v_2)) \\ &= \left( \int_{S^1} d\sigma\, \varpi(\sigma)(\xi^L_{v_1}|_{\varphi(\sigma)}),\, \int_{S^1} d\sigma\, \varpi(\sigma)(\xi^R_{v_2}|_{\varphi(\sigma)}) \right) ,\end{aligned}$$

where $\xi^L_{v_1}$ (resp. $\xi^R_{v_2}$) is the left (resp. right) invariant vector field on $G$ generated by $v_1$ (resp. $v_2$).

**Proposition 2.2.** $\mathrm{Dom}\,\Phi \;=\; \mu_R^{-1}(0)$.

*Proof.* This follows from the computation:

$$\begin{aligned}\mu_R(\varphi,\varpi)(v) &= \int_{S^1} d\sigma\, \varpi(\sigma)(\xi^R_v|_{\varphi(\sigma)}) \\ &= \int_{S^1} d\sigma\, \langle \varpi(\sigma)^\sim\,,\, \xi^R_v|_{\varphi(\sigma)} \rangle \;=\; \int_{S^1} d\sigma\, \langle \Omega(\varpi(\sigma)^\sim)\,,\, \mathrm{Ad}_{\varphi(\sigma)^{-1}} v \rangle \\ &= \int_{S^1} d\sigma\, \langle \mathrm{Ad}_{\varphi(\sigma)} \Omega(\varpi(\sigma)^\sim)\,,\, v \rangle \;=\; \left\langle \int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)} \Omega(\varpi(\sigma)^\sim)\,,\, v \right\rangle .\end{aligned}$$

This vanishes for all $v$ iff

$$\int_{S^1} d\sigma\, \mathrm{Ad}_{\varphi(\sigma)} \Omega(\varpi(\sigma)^\sim) \;=\; 0\,,$$

which is exactly the condition that characterizes $\mathrm{Dom}\,\Phi$. $\square$

This proposition suggests that one may apply the *Marsden–Weinstein reduction* to $(LT^*G, \mathscr{H})$ and consider the quotient space $\mathrm{Dom}\,\Phi / G_R$ as the true classical physical phase space. It also provides a "*true*" reason for the invariance of $\mathrm{Dom}\,\Phi$ under the flow generated by $X_{\mathscr{H}}$.

*(b) The* $\mathfrak{g}$*-Part.* The identity

$$\Gamma(\psi, l_{g_1} r_{g_2} \varphi) = \Gamma(\mathrm{Ad}_{g_2}\psi, \varphi)$$

suggests that $\Phi$ transforms the $G_L \times G_R$ action on $LT^*G$ into the $\mathrm{Ad}\, G_R$ action on $LT^*\mathfrak{g}$. Indeed as will be shown in Sect. 3.3, the T-dual Hamiltonian system $(LT^*\mathfrak{g}, \widetilde{\mathscr{H}})$ does admit the $\mathrm{Ad}\, G$ action. Analogous to the $G$-part, this also leads to a moment map

$$\tilde{\mu} : LT^*\mathfrak{g} \to \mathfrak{g}^*$$

defined by

$$\tilde{\mu}(\psi, \pi)(v) = \int_{S^1} d\sigma\, \pi(\sigma)(\eta_v|_{\psi(\sigma)}),$$

where $\eta_v$ is the vector field on $\mathfrak{g}$ associated to $v$ via the Ad-action. Explicitly, $\eta_v|_{\psi(\sigma)}$ is just $\mathrm{ad}_v \psi(\sigma)$. Direct computation then gives

$$\tilde{\mu}(\psi, \pi)(v) = \left\langle \int_{S^1} d\sigma\, \mathrm{ad}_{\psi(\sigma)} \pi^{\sim}(\sigma), v \right\rangle .$$

Unfortunately, we are not able to see if $\mathrm{Im}\, \Phi$ is of the form $\tilde{\mu}^{-1}(A)$ for some subset $A$ in $\mathfrak{g}$ by using the above formula. The condition that characterizes $\mathrm{Im}\, \Phi$ is more related to the following *holonomy map*:

$$\begin{aligned} \mathrm{Hol} : LT^*\mathfrak{g} &\to G \\ (\psi, \pi) &\mapsto E(-\pi^{\sim})_0^{-1} E(-\pi^{\sim})_{2\pi} . \end{aligned}$$

Notice that this is well-defined regardless of the initial point $E(-\pi^{\sim})_0$ chosen. It is not clear to us how to translate the fact that $\mathrm{Im}\, \Phi = \mathrm{Hol}^{-1}(e)$ into the language of $\tilde{\mu}$. There might be other symmetries that would give the correct moment map for such a translation. And we shall conclude our discussion of symmetries with this open end.

The last issue that we shall touch upon in this article is about the T-dual structures on $\mathfrak{g}$. There are surely many more properties worth studying, in particular, the curvature properties, the asymptotic behavior of geodesics of the T-dual Riemannian manifold and the existence of symplectic leaves of $B$. However we shall be contented here only to give a light feel of the T-dual geometry on $\mathfrak{g}$.

## 3. The Geometry of $(\mathfrak{g}, \langle\langle\ ,\ \rangle\rangle\, B)$

*3.1. Preliminaries to Study the Dual Geometry.* The dual structures on $\mathfrak{g}$ worked out in the previous section links closely to the ad-representation of $\mathfrak{g}$ on itself. Thus in this sub-section, we shall digress to prepare ourselves necessary facts about real simple Lie algebras and their ad-representation for studying the dual geometry. These facts either are contained in [Sa] or can be derived from material therein.

*3.1.1. The Characteristic Polynomials and the Characteristic Variety.* For any $v \in \mathfrak{g}$, the characteristic polynomial is defined to be

$$\det(\mathrm{ad}_v - t\,\mathrm{Id}) = (-1)^n (t^n - D_1(v) t^{n-1} + D_2(v) t^{n-2} - \cdots + (-1)^{n-r} D_{n-r}(v) t^r),$$

where $n = \dim \mathfrak{g}$ and $r = \mathrm{rank}\, \mathfrak{g}$. Notice that the coefficients $D_i$ are homogeneous polynomials in $v$ of degree $i$.

Since $\mathfrak{g}$ is simple, one has that $n-r$ is even and that the characteristic polynomials can be written in the form

$$\det(\mathrm{ad}_v - t\,\mathrm{Id}) = (-1)^n t^r \prod_{i=1}^{\frac{n-r}{2}} (t^2 + a_i(v)^2)\,,$$

where $a_i$ are some functions in $v$. This implies that

$$D_i \equiv 0, \qquad \text{for } i \text{ odd}\,.$$

The characteristic variety $V_0$ is defined to be the zero set of the homogeneous polynomial $D_{n-r}$. It is naturally stratified by the following "tower":

$$V_0 \supset V_1 \supset \cdots \supset V_k \supset \cdots \supset V_{\frac{n-r}{2}-1} (= \{0\})\,,$$

where

$$V_k = \{v \,|\, D_{n-r}(v) = D_{n-r-2}(v) = \cdots = D_{n-r-2k}(v) = 0\}\,.$$

*3.1.2. Cartan Subalgebras.*

- Let $v \in \mathfrak{g}$. Then there exists exactly one Cartan subalgebra that contains $v$ if and only if $v \in \mathfrak{g} - V_0$.
- If $v \in V_k$ for $k > 0$, then there exists at least a $2(k+1)$-dimensional family of Cartan subalgebras and each contains $v$.
- Fix a Cartan subalgebra $\mathfrak{h}$ and an Ad-invariant inner product $\langle\ ,\ \rangle$ in $\mathfrak{g}$. Let $\Delta$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with a fixed order. Then $\mathfrak{g}$ decomposes orthogonally into

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta^+} \Pi_\alpha \right),$$

where $\Delta^+$ is the set of all positive roots and each $\Pi_\alpha$ is a 2-dimensional subspace invariant under $\mathrm{ad}\,\mathfrak{h}$.
- For any $v \in \mathfrak{g}$, the kernel of the endomorphism $\mathrm{ad}_v : \mathfrak{g} \to \mathfrak{g}$ contains all the Cartan subalgebras that contain $v$.

*3.1.3. The* Ad-*Action on* $\mathfrak{g}$ *and the Weyl Group.*

- The Ad-action of $G$ on $\mathfrak{g}$ induces a $G$-action on the space of all Cartan subalgebras (with the subset topology from an appropriate Grassmann manifold). This induced action is transitive.
- Since $\langle\ ,\ \rangle$ is Ad-invariant and $\mathfrak{g}$ is compact simple, one has a group homomorphism

$$\mathrm{Ad} : G \to SO(n) \subset \mathrm{Isom}(\mathfrak{g}, \langle\ ,\ \rangle)\,.$$

- Let $T$ be the maximal torus in $G$ associated to $\mathfrak{h}$, i.e. $\mathfrak{h} = T_e T$. The restricted action $\mathrm{Ad}T$ on $\mathfrak{g}$ leaves $\mathfrak{h}$ fixed and are rotations on each $\Pi_\alpha$.
- Let $h_\alpha$ be root vectors associated to roots $\alpha \in \Delta$. Recall that the Weyl group action $\mathscr{W}$ on $\mathfrak{h}$ associated to $\Delta$ is generated by the reflections with respect to $h_\alpha^\perp$, the orthogonal complement of $h_\alpha$ in $\mathfrak{h}$ with respect to $\langle\ ,\ \rangle$. Then every element in $\mathscr{W}$ comes from an $\mathrm{Ad}_g$, for some $g \in G$, that leaves $\mathfrak{h}$ invariant.

Conversely, if some $\mathrm{Ad}_g$, $g \in G$, leaves $\mathfrak{h}$ invariant, then the restriction $\mathrm{Ad}_g|_{\mathfrak{h}}$ is in $\mathscr{W}$.

*3.1.4. The Weyl Chamber.* For our purposes, the *Weyl chamber* associated to $(\mathfrak{h}, \Delta)$ with a fixed order shall mean any of the following:

- The closed Weyl chamber is the quotient space $\mathfrak{h}/\mathscr{W}$. It is a convex cone with boundary. The interior is called the open Weyl chamber.
- Let $F$ be a fixed fundamental system. Then the closed Weyl chamber is the cone

$$C = C_F = \{v \in \mathfrak{h} | \langle h_\alpha, v \rangle \geqq 0\} .$$

  Its interior is the open Weyl chamber.
- Let $\Sigma_\alpha = \{v \in \mathfrak{h} | \langle h_\alpha, v \rangle = 0\}$, for $\alpha \in \Delta$. Then an open Weyl chamber is any of the connected components of $\mathfrak{h} - \bigcup_{\alpha \in \Delta} \Sigma_\alpha$. Its closure is a closed Weyl chamber.

Notice that the closed Weyl chamber is linear isomorphic to the orthant

$$\boldsymbol{R}^r_+ = \{(a_1, \dots, a_r) | a_r \geqq 0\} ,$$

where $r = \dim \mathfrak{h}$.

*3.2. An* Ad-*Invariant Polarization in* $\mathfrak{g} - V_0$. There is a collection of integrable distributions (i.e. a *polarization*) in $\mathfrak{g}$ that arises from the ad-representation of $\mathfrak{g}$ on itself. It plays an important role in understanding the T-dual geometry on $\mathfrak{g}$ and we shall explain it in some detail.

Let $C$ be a closed Weyl chamber in $\mathfrak{g}$ and $\mathrm{Int}\, C$ be its interior. Then, from Sect. 3.1.3, one has

$$\mathfrak{g} - V_0 = \mathrm{Ad}\, G \cdot \mathrm{Int}\, C .$$

Let

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta^+} \Pi_\alpha \right)$$

be as in Sect. 3.1.2 with $C$ lying in $\mathfrak{h}$. Let $\widehat{\mathfrak{h}}$, $\widehat{\Pi_\alpha}$ be the distributions along $\mathrm{Int}\, C$ obtained by translating respectively $\mathfrak{h}$, $\Pi_\alpha$ over $\mathrm{Int}\, C$ using the vector space structure of $\mathfrak{g}$. Applying the Ad-action to move them around, one then obtains a collection of distributions on $\mathfrak{g} - V_0$. Denote the one associated to $\widehat{\mathfrak{h}}$ by $\mathscr{D}_0$ and the one associated to $\widehat{\Pi_\alpha}$ by $\mathscr{D}_\alpha$. The whole collection is independent of the choice of Weyl chamber and one has

$$T_*(\mathfrak{g} - V_0) = \mathscr{D}_0 \oplus \left( \bigoplus_{\alpha \in \Delta^+} \mathscr{D}_\alpha \right) .$$

This decomposition is orthogonal with respect to both $\langle\, ,\, \rangle$ and $\langle\!\langle\, ,\, \rangle\!\rangle$.

**Proposition 3.1 (Integrability).** *The distributions $\mathscr{D}_0$ and $\mathscr{D}_\alpha$'s on $\mathfrak{g} - V_0$ are integrable.*

*Proof.* From the setting, it is clear that $\mathscr{D}_0$ is integrable. An integral submanifold of $\mathscr{D}_0$ is the intersection of some Cartan subalgebra with $\mathfrak{g} - V_0$. In other words, it is an open Weyl chamber.

As for $\mathscr{D}_\alpha$, let $v \in \mathfrak{g} - V_0$; its stabilizer $\mathrm{Stab}(v)$ under the Ad-action is the maximal torus $T$ that gives the unique Cartan subalgebra $\mathfrak{h}$ containing $v$. The Ad-orbit $Q$ through $v$ is diffeomorphic to $G/T$ and one has

$$T_* Q = \bigoplus_\alpha (\mathscr{D}_\alpha|_Q)\,.$$

Let

$$\mathrm{proj} : G \to Q$$

be the quotient map and $\overline{\mathfrak{h} + \Pi_\alpha}$ be the left-invariant distribution on $G$ whose restriction at the identity is $\mathfrak{h} + \Pi_\alpha$. Since $\mathfrak{h} + \Pi_\alpha$ is a subalgebra in $\mathfrak{g}$, $\overline{\mathfrak{h} + \Pi_\alpha}$ is integrable. From the fact that

$$\mathscr{D}_\alpha|_Q = \mathrm{proj}_*(\overline{\mathfrak{h} + \Pi_\alpha})\,,$$

one concludes that $\mathscr{D}_\alpha$ is integrable when restricted to $Q$ and hence it is integrable in $\mathfrak{g}$. Its integral submanifolds are the projection of those for $\overline{\mathfrak{h} + \Pi_\alpha}$ in $G$. This completes the proof. □

We shall call either $\{\mathscr{D}_\alpha\}$ or the family of their integral leaves *the polarization of* $\mathfrak{g}$ *indexed by a root system.*

*3.3. Basic Properties of the T-Dual Geometry.*

**Proposition 3.2.** *The dual structures* $\langle\!\langle\, ,\, \rangle\!\rangle$ *and* $B$ *on* $\mathfrak{g}$ *are both* Ad-*invariant.*

*Proof.* After identifying the tangent space at any $v \in \mathfrak{g}$ with $\mathfrak{g}$ itself using the vector space structure, one may write $(\mathrm{Ad}_g)_*$ for $g \in G$ simply as $\mathrm{Ad}_g$. With this convention, for $X, Y \in T_v\mathfrak{g}$, one has

$$\begin{aligned}
&\langle\!\langle \mathrm{Ad}_g X, \mathrm{Ad}_g Y\rangle\!\rangle_{\mathrm{Ad}_g v}\\
&\quad = \langle (\mathrm{Id} - \mathrm{ad}_{\mathrm{Ad}_g v})^{-1}\mathrm{Ad}_g X, (\mathrm{Id} - \mathrm{ad}_{\mathrm{Ad}_g v})^{-1}\mathrm{Ad}_g Y\rangle\\
&\quad = \langle \mathrm{Ad}_g(\mathrm{Id} - \mathrm{ad}_v)^{-1}X, \mathrm{Ad}_g(\mathrm{Id} - \mathrm{ad}_v)^{-1}Y\rangle, \ \text{ since } \mathrm{ad}_{\mathrm{Ad}_g v} = \mathrm{Ad}_g \mathrm{ad}_v \mathrm{Ad}_g^{-1};\\
&\quad = \langle (\mathrm{Id} - \mathrm{ad}_v)^{-1}X, (\mathrm{Id} - \mathrm{ad}_v)^{-1}Y\rangle, \ \text{ since } \langle\, ,\, \rangle \text{ is Ad-invariant;}\\
&\quad = \langle\!\langle X, Y\rangle\!\rangle_v\,.
\end{aligned}$$

And similarly,

$$\begin{aligned}
B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y)_{\mathrm{Ad}_g v} &= \langle\!\langle \mathrm{Ad}_g X, \mathrm{ad}_{\mathrm{Ad}_g v}\mathrm{Ad}_g Y\rangle\!\rangle_{\mathrm{Ad}_g v}\\
&= \langle\!\langle \mathrm{Ad}_g X, \mathrm{Ad}_g \mathrm{ad}_v Y\rangle\!\rangle_{\mathrm{Ad}_g v} = \langle\!\langle X, \mathrm{ad}_v Y\rangle\!\rangle_v\\
&= B(X, Y)_v\,. \qquad \square
\end{aligned}$$

**Corollary 3.1.** *As a Riemannian submanifold in* $(\mathfrak{g}, \langle\!\langle\, ,\, \rangle\!\rangle)$, *every Cartan subalgebra is totally geodesic.*

*Proof.* Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$ and $v \in \mathfrak{h} - V_0$. Then, for $\varepsilon > 0$ but small enough, $\mathrm{Ad}_{\exp(\varepsilon v)}$ is an isometry of $(\mathfrak{g}, \langle\!\langle\, ,\, \rangle\!\rangle)$ whose set of fixed points is exactly $\mathfrak{h}$. This shows that $\mathfrak{h}$ is totally geodesic. □

Let $v \in \mathfrak{g}$ then $\langle\langle X,X\rangle\rangle|_v = \langle X,X\rangle_v$ for $X \in \ker(\mathrm{ad}_v)$. From Sect. 3.2.1, $\ker(\mathrm{ad}_v)$ is $\mathscr{D}_0|_v$ for $v \in \mathfrak{g} - V_0$. Thus for any tangent vector to $\mathfrak{g}$ that lies in $\mathscr{D}_0$, its norms with respect to $\langle\ ,\ \rangle$ and $\langle\langle\ ,\ \rangle\rangle$ are the same. Consequently, any path that lies in some Cartan subalgebra has the same length with respect to either $\langle\ ,\ \rangle$ or $\langle\langle\ ,\ \rangle\rangle$. Together with the previous corollary then implies that all the affine lines in $\mathfrak{g}$ that lie in a Cartan subalgebra are bi-infinite geodesics with respect to $\langle\langle\ ,\ \rangle\rangle$. Particularly, all the half lines from the origin are infinite geodesic rays with respect to $\langle\langle\ ,\ \rangle\rangle$. Thus the exponential map at the origin with respect to $\langle\langle\ ,\ \rangle\rangle$

$$\mathrm{Exp}_O : T_O\mathfrak{g} \to \mathfrak{g}$$

is well-defined on the whole $T_O\mathfrak{g}$. It actually coincides with the exponential map with respect to $\langle\ ,\ \rangle$. By Hopf–Rinow theorem [C–E] this shows that

**Corollary 3.2.** $(\mathfrak{g}, \langle\langle\ ,\ \rangle\rangle)$ *is a complete metric space.*

Since $G$ is compact connected, for any $v \in \mathfrak{g}$, its stabilizer $\mathrm{Stab}(v)$ under the Ad-action is a connected closed subgroup in $G$ [He] with

$$\ker(\mathrm{ad}_v) \subset T_e\mathrm{Stab}(v)\,.$$

Since the jump of the dimension of $\ker(\mathrm{ad}_v)$ when varying $v$ is always even,

$$\dim\,\mathrm{Stab}(v)\begin{cases} = r(\text{i.e. rank } G) & \text{if } v \in \mathfrak{g} - V_0 \\ \geqq r+2 & \text{if } v \in V_0\,. \end{cases}$$

On the other hand, for any closed Weyl chamber $C$ in a fixed Cartan subalgebra,

$$\mathfrak{g} - V_0 = \mathrm{Ad}\,G \cdot \mathrm{Int}\,C \quad \text{and}$$
$$V_0 = \mathrm{Ad}\,G \cdot \partial C\,.$$

This implies that $V_0$ is a homogeneous variety of codimension $\geqq 2$ and hence the isometric embedding

$$\mathfrak{g} - V_0 \hookrightarrow (\mathfrak{g}, \langle\langle\ ,\ \rangle\rangle)$$

is distance-preserving. In other words, $(\mathfrak{g}, \langle\langle\ ,\ \rangle\rangle)$ is the metric completion of $(\mathfrak{g} - V_0, \langle\langle\ ,\ \rangle\rangle|_{\mathfrak{g}-V_0})$ by a subset of codimension $\geqq 2$ in $\mathfrak{g}$. Consequently, the *generic part* $\mathfrak{g} - V_0$ itself captures nearly all the metric properties of the whole $(\mathfrak{g}, \langle\langle\ ,\ \rangle\rangle)$.

*3.4. Riemannian Geometry of the T-Dual Metric.* Due to the fact that the Ad-action of $G$ on $\mathfrak{g} - V_0$ has stabilizers isomorphic to a maximal torus $T$ of $G$ and that the quotient space is the interior of a Weyl chamber $C$ which is contractible, one has the following trivial fibration:

$$\begin{array}{ccc} G/T & \to & \mathfrak{g} - V_0 \\ & & \downarrow \mathrm{pr} \\ & & \mathrm{Int}\,C\,. \end{array}$$

Also recall the common Ad-invariant orthogonal decomposition

$$T_*(\mathfrak{g} - V_0) = \mathscr{D}_0 \oplus \left( \bigoplus_{\alpha\in\Delta^+} \mathscr{D}_\alpha \right)$$

with respect to both $\langle\ ,\ \rangle$ and $\langle\langle\ ,\ \rangle\rangle$. The following proposition shows that the T-dual metric on $\mathfrak{g}$ is a *polarized conformal deformation* of the flat Killing metric and the generic part is a *polarized warped-product* of the flat cone $\mathrm{Int}\,C$ with $G/T$.

**Proposition 3.3.** *Let*

$$\langle\ ,\ \rangle = ds_0^2 + \sum_{\alpha\in\Delta^+} ds_\alpha^2$$

*with respect to the above decomposition. Then, for any* $v \in \mathfrak{g} - V_0$,

$$\langle\!\langle\ ,\ \rangle\!\rangle_v = ds_0^2|_v + \sum_{\alpha\in\Delta^+} \frac{1}{1-\alpha(\bar{v})^2}\, ds_\alpha^2|_v\,,$$

*where* $\bar{v} = \mathrm{pr}(v) \in \mathrm{Int}\, C$.

*Proof.* Since $\mathscr{D}_0$, $\mathscr{D}_\alpha$'s are invariant under the Ad-action, the decomposition of $\langle\ ,\ \rangle$ is also invariant under the Ad-action. Without loss of generality, we may assume that the open Weyl chamber $\mathrm{Int}\, C$ is embedded in $\mathfrak{g} - V_0$ and contains $v$. Now for $X \in \mathscr{D}_\alpha$, $Y \in \mathscr{D}_\beta$ ($\alpha$, $\beta$ could be 0), one has

$$\begin{aligned}\langle\!\langle X, Y\rangle\!\rangle &= \langle(\mathrm{Id}-\mathrm{ad}_v)^{-1}X, (\mathrm{Id}-\mathrm{ad}_v)^{-1}Y\rangle \\ &= \langle X, (\mathrm{Id}+\mathrm{ad}_v)^{-1}(\mathrm{Id}-\mathrm{ad}_v)^{-1}Y\rangle \\ &= \langle X, (\mathrm{Id}-(\mathrm{ad}_v)^2)^{-1}Y\rangle\,.\end{aligned}$$

Notice that $(\mathrm{ad}_v)^2$ is symmetric with respect to $\langle\ ,\ \rangle$. The eigenspace decomposition of $T_v\mathfrak{g}$ for $(\mathrm{ad}_v)^2$ coincides with $(\mathscr{D}_0 \oplus (\bigoplus_{\alpha\in\Delta^+}\mathscr{D}_\alpha))|_v$. For $Z \in \mathscr{D}_\alpha$,

$$(\mathrm{ad}_v)^2(Z) = \alpha(v)^2 Z\,.$$

Thus

$$\langle\!\langle X, Y\rangle\!\rangle = \langle X, (1-\beta(v)^2)^{-1}Y\rangle = \frac{\delta_{\alpha\beta}}{1-\alpha(\bar{v})^2}\langle X, Y\rangle\,.$$

This concludes the proof. □

Notice that $\alpha(\bar{v})$ is purely imaginary, thus

$$0 < \frac{1}{1-\alpha(\bar{v})^2} \leqq 1\,.$$

Since $\mathfrak{g} - V_0$ is open and dense in $\mathfrak{g}$, we have

**Corollary 3.3.** *For any* $X \in T_*\mathfrak{g}$, $\langle\!\langle X, X\rangle\!\rangle \leqq \langle X, X\rangle$.

As $v$ approaches the characteristic variety $V_0$, some of the $\alpha(\bar{v})$'s get closer and closer to 0. In the limit, their corresponding $\mathscr{D}_\alpha$'s are absorbed into the undistorted flat directions at the limit point in $V_0$.

The explicit expression in the set of the polarized conformal factor $\{\frac{1}{1-\alpha(\bar{v})^2}\}$ together with the polarized warped-product structure on $\mathfrak{g} - V_0$ gives a clear picture of what the T-dual $\mathfrak{g}$ looks like as a Riemannian manifold (Fig. 4). We summarize them partially as

**Proposition 3.4 (Asymptotic stability).** *Let* $\bar{\gamma}(t)$ *be a ray in* $\mathrm{Int}\, C$ *from the origin parameterized by arc-length. Let* $(G/T)_t$ *be the fiber over* $\bar{\gamma}(t)$ *in the polarized*

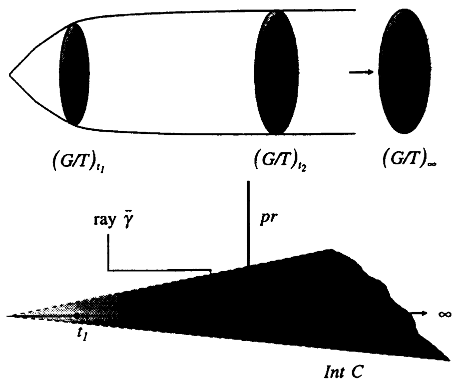


**Fig. 4.** Asymptotic stability of the T-dual Riemannian geometry. With a neighborhood of $V_0$ deleted, the rest of the metric space looks like a cone on a sufficiently large scale.

*warped-product. Then $(G/T)_t$ is a polarized conformal deformation of $(G/T)_1$ with polarization $\{\mathcal{D}_\alpha\}$ and family of factors*

$$\left\{\frac{t^2(1-\alpha(\bar{\gamma}(1))^2)}{1-t^2\alpha(\bar{\gamma}(1))^2}\right\}_\alpha .$$

*Consequently, $(G/T)_\infty$ is a polarized conformal deformation of $(G/T)_1$ with factors $\{-\frac{1}{\alpha(\bar{\gamma}(1))^2}\}_\alpha$ and hence is compact. With a neighborhood of $V_0$ deleted from $\mathfrak{g}$, the rest is quasi-isometric [Gr] to the base cone with a neighborhood of boundary deleted.*

*Proof.* All this follows from the fact that the collection of Riemannian manifolds $\{(G/T)_t\}$ forms a radial infinite cone at the origin with base $(G/T)_1$. Using the radial projection from $(G/T)_t$ to $(G/T)_1$ and the invariance of $\mathcal{D}_\alpha$ under radial scaling maps, one immediately justifies all the claims. This concludes the proof. $\square$

*Remark.* When applied to the special pair, $SU(2)$ and $\mathfrak{su}(2)$,

$$V_0 = \text{the origin, } \operatorname{Int} C = \text{ a half line } L_+, \text{ and } G/T = S^2 .$$

The proposition says that $\mathfrak{su}(2) - \{\text{origin}\}$ with the T-dual metric is a warped-product of $L_+$ with $S^2$ with factor $\frac{t^2}{1+4t^2}$, whose limit is $\frac{1}{4}$ as $t \to \infty$. One can check that this coincides with the known results from the literature.

*3.5. The B-Field.* Recall that, with respect to the Ad-invariant metric $\langle\ ,\ \rangle$, $\mathfrak{g}$ has a Poisson structure given by a closed 2-form $\zeta$ with

$$\zeta_v(X,Y) = \langle X, \mathrm{ad}_v Y\rangle ,$$

for $X$, $Y$ in $T_v\mathfrak{g}$. Its symplectic leaves are the Ad-orbits. Analogous to the T-dual metric, $B$ can be written as a polarized conformal deformation of $\zeta$. Explicitly, assuming that $v$ is in $\mathfrak{g} - V_0$, let $X, Y \in T_v\mathfrak{g}$ and decompose

$$X = \sum_\alpha X_\alpha, \qquad Y = \sum_\alpha Y_\alpha ,$$

where $X_\alpha, Y_\alpha \in \mathscr{D}_\alpha$ ($\alpha$ could be zero here). Straightforward computation then gives

$$B(X,Y) = \sum_\alpha \frac{1}{1-\alpha(v)^2}\zeta(X_\alpha, Y_\alpha)\,,$$

which, for the special case of $SU(2)$–$\mathfrak{su}(2)$ pair, again gives the known $B$-field. However, direct checking shows that, after this polarized distortion, $B$ is no longer closed for general simple Lie algebras; nor do the Ad-orbits remain symplectic in general. It's not clear to us at the moment what kind of geometry this $B$-field provides on $\mathfrak{g}$ in general.

*Acknowledgements.* We would like to thank Hung-Wen Chang for very helpful discussions, Thom Curtright for sharing with us his joint work with C. Zachos, and Marco Monti for help with Xfig.

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Communicated by R.H. Dijkgraaf