Commun. Math. Phys. 179, 61 – 120 (1996)

Communications in Mathematical Physics

Conformal Field Theories, Representations and Lattice Constructions

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Received: 15 September 1994/Accepted: 23 October 1995

Abstract: An account is given of the structure and representations of chiral bosonic meromorphic conformal field theories (CFT's), and, in particular, the conditions under which such a CFT may be extended by a representation to form a new theory. This general approach is illustrated by considering the untwisted and $\mathbf{Z}_2$-twisted theories, $\mathscr{H}(\Lambda)$ and $\widetilde{\mathscr{H}}(\Lambda)$ respectively, which may be constructed from a suitable even Euclidean lattice Λ. Similarly, one may construct lattices $\Lambda_{\mathscr{C}}$ and $\widetilde{\Lambda}_{\mathscr{C}}$ by analogous constructions from a doubly-even binary code $\mathscr{C}$. In the case when $\mathscr{C}$ is self-dual, the corresponding lattices are also. Similarly, $\mathscr{H}(\Lambda)$ and $\widetilde{\mathscr{H}}(\Lambda)$ are self-dual if and only if Λ is. We show that $\mathscr{H}(\Lambda_{\mathscr{C}})$ has a natural "triality" structure, which induces an isomorphism $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}}) \equiv \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ and also a triality structure on $\widetilde{\mathscr{H}}(\widetilde{\Lambda}_{\mathscr{C}})$. For $\mathscr{C}$ the Golay code, $\widetilde{\Lambda}_{\mathscr{C}}$ is the Leech lattice, and the triality on $\widetilde{\mathscr{H}}(\widetilde{\Lambda}_{\mathscr{C}})$ is the symmetry which extends the natural action of (an extension of) Conway's group on this theory to the Monster, so setting triality and Frenkel, Lepowsky and Meurman's construction of the natural Monster module in a more general context. The results also serve to shed some light on the classification of self-dual CFT's. We find that of the 48 theories $\mathscr{H}(\Lambda)$ and $\widetilde{\mathscr{H}}(\Lambda)$ with central charge 24 that there are 39 distinct ones, and further that all 9 coincidences are accounted for by the isomorphism detailed above, induced by the existence of a doubly-even self-dual binary code.

1. Introduction

In this paper we shall provide the details omitted from the summary of our results given in [1].

The principal result of the paper will be to show how a study of binary linear codes leads to an understanding of some of the symmetries of conformal field theories (CFT's). We shall restrict ourselves of self-dual chiral bosonic theories, which are regarded as trivial by approaches to the CFT classification problem which rely upon a study of the fusion rules for the representations of some chiral algebras [2]. (For general reviews of CFT see [3,4].) Hence, a complete understanding of these "trivial" theories would seem to be essential to obtain, and further our results show that such theories are not necessarily without an interesting structure.

Indeed, one such theory, constructed initially by Frenkel, Lepowsky and Meurman (FLM) [5] from the Leech lattice, possesses only discrete automorphisms, which close to form the largest of the sporadic simple groups, the Monster group [8–10]. Building on previous work generalising the construction of this Monster module to other lattices [11], we show that a certain subgroup of discrete symmetries, known as triality, which is the key to the construction of the action of the Monster in the FLM theory, can be seen in this more general context. Triality is seen to occur in some theories as an obvious consequence of the existence of a corresponding binary code, and can be lifted to provide an isomorphism between otherwise potentially distinct CFT's and further to the triality structure of the form exhibited by FLM, though in a more general setting. Hence, we see that triality and binary codes provide insight into the classification of bosonic self-dual theories, and a more general framework in which to view the hitherto mysterious Monster group.

In addition to these investigations of lattice constructions, we provide a general treatment of the representations of bosonic CFT's. We discuss the notions of a subconformal field theory and of a hermitian structure on a CFT, and demonstrate that, under certain conditions, we may extend a CFT by a representation to form a new CFT. A particular example of this is provided by the twisted lattice construction, which gives the Monster module in the case of the Leech lattice. Our treatment is based on the approach of [12], which was inspired by the work of FLM and Borcherds' general approach to "vertex operator algebras" [13]. Results in a similar direction have also been independently described in [14]. This, and references therein to the mathematical literature on vertex operator algebras, contain material on the calculus of formal variables which provides an alternative method of rendering our results mathematically rigorous.

The layout of the paper is as follows. Sections 2–4 cover the general aspects of conformal field theories and their representations. In Sect. 5, we sketch the straight and $\mathbf{Z}_2$-twisted lattice constructions of CFT's and the analogous constructions of lattices from binary codes. Further details may be found in [11]. Section 6 gives the results of these constructions, and discusses the connection with the Monster provided by the work of FLM, while in Sects. 7 and 8 we exhibit the triality structure in this general framework. Our conclusions are presented in Sect. 9.

2. Definitions and Elementary Properties

In this section, we define what we shall mean by a meromorphic bosonic conformal field theory (frequently referred to in this paper simply as a "conformal field theory" for brevity), and review some of the properties which follow from this definition. [For physicists familiar with conformal field theory, note that we shall consider only bosonic meromorphic chiral conformal field theories defined on the Riemann sphere, i.e., they are holomorphic, in the sense that there is only a dependence on the complex variable z and not its conjugate z^*, with meromorphic matrix elements and "commuting" vertex operators in the sense of (2.4)].

Definition 2.1. *A* **meromorphic bosonic conformal field theory** $(\mathscr{H}, \mathscr{F}, \mathbf{V}, |0\rangle, \psi_L)$ *consists of a Hilbert space of states* $\mathscr{H}$, *a dense subspace* $\mathscr{F}$ [*typically the Fock space of states of finite occupation number for some set of harmonic* *oscillators*] *and a set* **V** *of linear operators called* **vertex operators** $V(\psi,z)$, *which are linear maps* $\mathscr{H} \to \mathscr{H}$ *parameterised by a complex variable* z, *in one-to-one correspondence with the states* $\psi \in \mathscr{F}$. [*We shall use the Dirac notation* $|\psi\rangle$, *but will write this simply as* ψ, *where it is notationally convenient.*] *There are two special states in* $\mathscr{F}$, *the* **vacuum** $|0\rangle$ *and a* **conformal state** ψ_L. *We shall take the product* $V(\psi_1,z_1)V(\psi_2,z_2)\cdots$ *to be well-defined only for* $|z_1| > |z_2| > \cdots$ (*see* [28]). *The theory must satisfy the properties* P1–6 *detailed below, and is said to be a* **hermitian conformal field theory** *if it satisfies in addition property* P7.

P1. We define the moments of the vertex operator of ψ_L to be given by

$$V(\psi_L,z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}\,, \tag{2.1}$$

and demand that they provide a representation of the centrally extended Virasoro algebra

$$[L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}\,, \tag{2.2}$$

with $L_n^\dagger = L_{-n}$ and $L_n|0\rangle = 0$, $n \geqq -1$. [Note that we shall see later that the requirement (2.2) may be weakened slightly and still hold, in the presence of the remaining axioms.]

P2. The vertex operators satisfy

$$V(\psi,z)|0\rangle = e^{zL_{-1}}\psi\,, \tag{2.3}$$

and also

P3. The bosonic "locality" relation

$$V(\psi,z)V(\phi,\zeta) = V(\phi,\zeta)V(\psi,z)\,. \tag{2.4}$$

More precisely, we require that the matrix elements of the product $V(\psi,z)V(\phi,\zeta)$ between states in $\mathscr{F}$ should be defined for $|z| > |\zeta|$ and that the function this defines by analytic continuation be regular except for possible poles at z, $\zeta = 0,\infty$ and $z=\zeta$. Then (2.4) should be interpreted to mean that the functions obtained from either side in such a manner are equal. (Note that any extension of the definition of the vertex operators $V(\psi,z)$ from $\mathscr{F}$ to the space of generalised coherent states $V(\psi_1,z_1)V(\psi_2,z_2)\cdots V(\psi_N,z_N)\phi$ for $|z|>|z_1|>|z_2|>\cdots|z_N|$ and $\phi\in\mathscr{F}$, if it exists, is unique because of the fact that $\mathscr{F}$ is dense in $\mathscr{H}$.) (Note also that we could allow a relative minus sign between the two sides of (2.4), corresponding to fermionic fields.)

The property P2 is equivalent to the conditions

$$[L_{-1},V(\psi,z)] = \frac{d}{dz}V(\psi,z) \tag{2.5}$$

and

$$\lim_{z\to 0} V(\psi,z)|0\rangle = \psi\,. \tag{2.6}$$

(In particular $\psi_L = L_{-2}|0\rangle$.) Equivalently (2.5) can be expressed in the form of the "translation property"

$$e^{wL_{-1}}V(\psi,z)e^{-wL_{-1}} = V(\psi,z+w)\,. \tag{2.7}$$

From the locality condition (2.4) we can establish a uniqueness property of the vertex operators.

Proposition 2.2. *If* $U(z)$ *satisfies*

$$U(z)|0\rangle = e^{zL_{-1}}\phi\,, \tag{2.8}$$

for some $\phi \in \mathscr{F}$, *and is local with respect to the system of vertex operators, then* $U(z) = V(\phi,z)$.

Proof. The proof is straightforward, since

$$\begin{aligned} U(z)e^{\zeta L_{-1}}\psi &= U(z)V(\psi,\zeta)|0\rangle = V(\psi,\zeta)U(z)|0\rangle \\ &= V(\psi,\zeta)e^{zL_{-1}}\phi = V(\psi,\zeta)V(\phi,z)|0\rangle \\ &= V(\phi,z)V(\psi,\zeta)|0\rangle = V(\phi,z)e^{\zeta L_{-1}}\psi\,. \end{aligned} \tag{2.9}$$

Thus, taking $\zeta \to 0$, we deduce $U(z) = V(\phi,z)$. □

Thus, to demonstrate a given operator to be a vertex for a particular state all we have to do is show that it is local with respect to **V** and has the appropriate action on the vacuum.

We may apply this uniqueness property to (2.5) to deduce that, since $\frac{d}{dz}V(\psi,z)$ is local with respect to **V**,

$$\frac{d}{dz}V(\psi,z) = V(L_{-1}\psi,z)\,, \tag{2.10}$$

since both sides are local with respect to **V** and have the same action on the vacuum, from (2.3).

Similarly, uniqueness immediately implies that $V(\psi,z)$ is linear in ψ and that $V(|0\rangle,z) \equiv 1$, again using (2.3) (and the fact that $L_{-1}|0\rangle = 0$).

In addition, we have

Proposition 2.3. *The "duality" relation*:

$$V(\psi,z)V(\phi,\zeta) = V(V(\psi,z-\zeta)\phi,\zeta)\,. \tag{2.11}$$

Proof. Again this is a consequence of the uniqueness argument (note that the product on the left-hand side of (2.11) is local with respect to **V**, because each of the factors is). We use (2.3) and the translation property, i.e.,

$$\begin{aligned} V(\psi,z)V(\phi,\zeta)|0\rangle &= V(\psi,z)e^{\zeta L_{-1}}\phi = e^{\zeta L_{-1}}V(\psi,z-\zeta)\phi \\ &= V(V(\psi,z-\zeta)\phi,\zeta)|0\rangle\,. \quad \square \end{aligned} \tag{2.12}$$

[Note that, as discussed following property P3, these equalities are to be interpreted in the sense of analytic continuation of the functions one obtains by taking matrix elements of the given operators in $\mathscr{F}$.]

These results serve to demonstrate the powerful role played by locality in conformal field theory.

Proposition 2.4. *Skew-symmetry*:

$$V(\psi,z)\phi = e^{zL_{-1}}V(\phi,-z)\psi\,. \tag{2.13}$$

Proof. Using (2.11) together with (2.3) we obtain

$$V(\psi,z)V(\phi,\zeta)|0\rangle = V(V(\psi,z-\zeta)\phi,\zeta)|0\rangle = e^{\zeta L_{-1}}V(\psi,z-\zeta)\phi\,. \tag{2.14}$$

But using (2.4) first gives

$$V(\psi,z)V(\phi,\zeta)|0\rangle = e^{zL_{-1}}V(\phi,\zeta-z)\psi\,. \tag{2.15}$$

Thus, comparing (2.14) and (2.15) one obtains (2.13). □

This result will be of use in later chapters when we come to defining what are called the intertwining operators. Note that it also immediately implies linearity of $V(\psi,z)$ in the state ψ.

Let us also assume

P4. x^{L_0} acts locally with respect to $\mathbf{V}$, i.e., $x^{L_0}V(\psi,z)x^{-L_0}$ is local with respect to $\mathbf{V}$.

[Note: we could alternatively *assume* that the spectrum of L_0 is integral (which we deduce in Proposition 2.9 in our present treatment), and then (2.16), and thus the locality of the action of L_0, would follow as a consequence of (2.59) and its implications for L_{-1} descendent states.]

Then using $[L_0,L_{-1}]=L_{-1}$, from the Virasoro algebra (2.2), and the uniqueness argument (together with $L_0|0\rangle=0$) we deduce that

$$x^{L_0}V(\psi,z)x^{-L_0} = x^{h_\psi}V(\psi,xz)\,, \tag{2.16}$$

when $L_0\psi = h_\psi\psi$, or equivalently

$$[L_0, V(\psi,z)] = \left(z\frac{d}{dz}+h_\psi\right)V(\psi,z)\,. \tag{2.17}$$

For a general state, i.e., not necessarily an L_0 eigenstate, we can write (2.16) by linearity as

$$x^{L_0}V(\psi,z)x^{-L_0} = V(x^{L_0}\psi,xz)\,. \tag{2.18}$$

Later, by imposing the requirement P7 that the conformal field theory has a hermitian structure, we will see that the conformal weights h_ψ for the states ψ must always be non-negative integers.

We begin by writing $\mathscr{F}$ as a direct sum of eigenspaces of L_0, i.e.,

$$\mathscr{F} = \bigoplus_h \mathscr{F}_h\,, \tag{2.19}$$

where $L_0\psi = h\psi$ for $\psi\in\mathscr{F}_h$.

Definition 2.5. *A state ψ is said to be an $su(1,1)$* **highest weight state** *or a* **quasi-primary state** *if $L_1\psi=0$. The corresponding vertex operator is said to be a* **quasi-primary field**.

(The elements $L_{\pm1}$, L_0 of the Virasoro algebra generate a subalgebra isomorphic to $su(1,1)$ [note that for m, $n=0,\pm1$ in (2.2) the central term vanishes].)

Let us also assume

P5. The spectrum of L_0 is bounded below.

Note that this assumption is physically reasonable, since in a conformally invariant quantum field theory we have both a holomorphic and an anti-holomorphic conformal structure, with Virasoro generators L_n and $\overline{L}_n$ respectively. The Hamiltonian H of the theory is given by $L_0 + \overline{L}_0$, and in any sensible quantum field theory the Hamiltonian should be bounded below. Since the holomorphic and anti-holomorphic sectors are independent, then L_0 and $\overline{L}_0$ should be separately bounded from below.

Proposition 2.6. *The eigenvalues of L_0 are non-negative.*

Proof. For $\psi \in \mathscr{F}_h$ a quasi-primary state, using $L_{-1}^\dagger = L_1$ and the relation $[L_1, L_{-1}] = 2L_0$, we obtain

$$\|L_{-1}\psi\|^2 = 2h\|\psi\|^2 , \tag{2.20}$$

so that, by positive definiteness of the norm on the Hilbert space of states, $h \geqq 0$. If ϕ is an arbitrary non-zero (not necessarily quasi-primary) state with negative conformal weight Δ, then the sequence of states $L_1^N\phi$ for $N = 0, 1, 2, \ldots$ have conformal weights $\Delta - N$. If any of these states vanishes, let N_0 be the smallest such value of N. Then $L_1^{N_0-1}\phi$ is a quasi-primary state ψ with conformal weight $h = \Delta + 1 - N_0 < 0$. The left-hand side of (2.20) is non-negative, but the right-hand side (since $\psi \neq 0$ as we chose N_0 to be as small as possible) is negative. This contradiction implies that the sequence of states $L_1^N\phi$ are all non-vanishing. Hence, if any state has negative weight, the spectrum of L_0 is unbounded below. This contradicts P5 and hence establishes the result. □

If a state ϕ has conformal weight zero then $L_1\phi = 0$, otherwise we would have a state with negative conformal weight. Thus, we may apply (2.20) with $\phi = \psi$ to see that $L_{-1}\phi = 0$ also, i.e., a state has zero conformal weight if and only if it is $su(1,1)$ invariant. We shall assume

P6. The vacuum is the only $su(1,1)$ invariant state in the theory.

From the fact that the conformal weights are bounded below, we see that $L_1^N\phi = 0$ for N sufficiently large, where ϕ is an arbitrary state of some definite conformal weight. So we have

Proposition 2.7. *$\mathscr{F}$ splits up into a direct sum of $su(1,1)$ highest weight representations.*

Each is generated by repeated action of L_{-1} on an $su(1,1)$ highest weight state (a quasi-primary state). This fact will be of use later in proving certain locality relations, since, by (2.10), we only have to consider quasi-primary states, and for these the hermitian structure takes a particularly simple form. Let us now define this hermitian structure.

Proposition 2.8. *If $V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger$ is local with respect to $\mathbf{V}$ then*

$$V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger = V(\overline{\psi}, z) , \tag{2.24}$$

where

$$\overline{\psi} = \lim_{z\to 0} V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger|0\rangle , \tag{2.25}$$

and the conjugation map $\psi \mapsto \overline{\psi}$ is antilinear.

Proof. All we need do is demonstrate that the left-hand side of (2.24) satisfies the obvious analogue of (2.5), or equivalently the translation property (2.7). Then, being local with respect to $\mathbf{V}$, from Proposition 2.2 we see that it must be the vertex operator with argument z for a particular state $\overline{\psi}$, which must, using (2.6), be given by (2.25). (Note that the limit is seen to exist from the translation property.) The map $\psi \mapsto \overline{\psi}$ is clearly antilinear, from the linearity of the map from states to the corresponding vertex operators noted above as a simple consequence of the uniqueness theorem. We now establish that the translation property is satisfied. $e^{\varepsilon L_1} V(e^{L_1/\zeta}\psi,\zeta)e^{-\varepsilon L_1}$ is a local operator (see (2.59)) and so we calculate its effect on the vacuum and then use the uniqueness theorem. Now

$$e^{\varepsilon L_1} V(e^{L_1/\zeta}\psi,\zeta)e^{-\varepsilon L_1}|0\rangle = e^{\varepsilon L_1}e^{\zeta L_{-1}}e^{L_1/\zeta}\psi \tag{2.26}$$

$$= \exp\left\{\frac{\zeta}{1-\varepsilon\zeta}L_{-1}\right\}(1-\varepsilon\zeta)^{-2L_0}\exp\left\{\frac{1}{\zeta(1-\varepsilon\zeta)}L_1\right\}\psi \tag{2.27}$$

$$= (1-\varepsilon\zeta)^{-2h}\exp\left\{\frac{\zeta}{1-\varepsilon\zeta}L_{-1}\right\}\exp\left\{\frac{1-\varepsilon\zeta}{\zeta}L_1\right\}\psi\,, \tag{2.28}$$

where to get from (2.26) to (2.27) we have used

$$e^{\varepsilon L_1}e^{\zeta L_{-1}} = \exp\left\{\frac{\zeta}{1-\varepsilon\zeta}L_{-1}\right\}(1-\varepsilon\zeta)^{-2L_0}\exp\left\{\frac{\varepsilon}{1-\varepsilon\zeta}L_1\right\} \tag{2.29}$$

and, to get from (2.27) to (2.28), $[L_0,L_{-1}] = L_{-1}$, and we have taken $L_0\psi = h\psi$. Thus it follows that

$$e^{\varepsilon L_1} V(e^{L_1/\zeta}\psi,\zeta)e^{-\varepsilon L_1} = (1-\varepsilon\zeta)^{-2h} V\left(\exp\left\{\frac{1-\varepsilon\zeta}{\zeta}L_1\right\}\psi, \frac{\zeta}{1-\varepsilon\zeta}\right)\,. \tag{2.30}$$

Taking ε small gives

$$[L_1, V(e^{L_1/\zeta}\psi,\zeta)] = 2h\zeta V(e^{L_1/\zeta}\psi,\zeta) - \frac{d}{d(1/\zeta)}V(e^{L_1/\zeta}\psi,\zeta)\,. \tag{2.31}$$

Therefore

$$[L_{-1}, z^{-2h}V(e^{L_1 z^*}\psi, 1/z^*)^\dagger] = \frac{d}{dz}\{z^{-2h}V(e^{L_1 z^*}\psi, 1/z^*)^\dagger\}\,, \tag{2.32}$$

so that the translation property is satisfied by $z^{-2h}V(e^{z^*L_1}\psi, 1/z^*)^\dagger$. The required result follows by linearity of the vertex operators. □

So, let us assume

P7. For all $\psi \in \mathscr{F}$ $V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger$ is local with respect to $\mathbf{V}$.

The conclusions of Proposition 2.8 then follow. The following result details some of the useful properties which follow from this hermitian structure.

Proposition 2.9. *For* $\psi, \phi \in \mathscr{F}$,

(i) *if* $L_0\psi = h\psi$ *then* $L_0\overline{\psi} = h\overline{\psi}$, *i.e., the conjugation operation preserves conformal weights,*
(ii) $\overline{L_{-1}\psi} = -L_{-1}\overline{\psi}$,
(iii) $\overline{\overline{\psi}} = \psi$,
(iv) $(f_{\phi_1\phi_2\phi_3})^* = (-1)^{h_1+h_2+h_3} f_{\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3}$,

where $\phi_1, \phi_2, \phi_3 \in \mathscr{F}$ *with* $L_0\phi_j = h_j\phi_j$ *for* $1 \leqq j \leqq 3$ *and*

$$f_{\phi_1\phi_2\phi_3} = \langle\overline{\phi}_1|V(\phi_2, 1)|\phi_3\rangle \,, \tag{2.33}$$

(v) $\overline{L_1\psi} = -L_1\overline{\psi}$,

(vi) *the spectrum of* L_0 *is integral, i.e., all states have integral conformal weight,*

(vii) $\langle\overline{\psi}|\phi\rangle = \langle\overline{\phi}|\psi\rangle$.

Proof.

(i) Consider

$$e^{\varepsilon L_0}V(e^{L_1/\zeta}\psi, \zeta)e^{-\varepsilon L_0}|0\rangle = e^{\varepsilon L_0}e^{\zeta L_{-1}}e^{L_1/\zeta}\psi \tag{2.34}$$

$$= e^{h\varepsilon}\exp\{e^{\varepsilon}\zeta L_{-1}\}\exp\{e^{-\varepsilon}L_1/\zeta\}\psi \,. \tag{2.35}$$

Therefore

$$e^{\varepsilon L_0}V(e^{L_1/\zeta}\psi, \zeta)e^{-\varepsilon L_0} = e^{h\varepsilon}V(\exp\{e^{-\varepsilon}L_1/\zeta\}\psi, e^{\varepsilon}\zeta) \,, \tag{2.36}$$

and by taking ε small we see

$$[L_0, V(e^{L_1/\zeta}\psi, \zeta)] = \left(\zeta\frac{d}{d\zeta} + h\right)V(e^{L_1/\zeta}\psi, \zeta) \,, \tag{2.37}$$

$$[L_0, V(\overline{\psi}, z)] = \left(z\frac{d}{dz} + h\right)V(\overline{\psi}, z) \,, \tag{2.38}$$

showing that $L_0\overline{\psi} = h\overline{\psi}$ as required.

(ii) If $\phi = L_{-1}\psi$,

$$V(\overline{\phi}, z) = z^{-2h-2}V(e^{L_1 z^*}L_{-1}\psi, 1/z^*)^\dagger \,. \tag{2.39}$$

Now,

$$e^{L_1 z^*}L_{-1}\psi = (L_{-1} + 2z^*L_0 + (z^*)^2L_1)e^{L_1 z^*}\psi$$

$$= L_{-1}e^{L_1 z^*}\psi + e^{L_1 z^*}(2z^*L_0 - (z^*)^2L_1)\psi \,, \tag{2.40}$$

using the algebra (2.2). Thus,

$$V(\overline{\phi}, z) = -\frac{d}{dz}V(\overline{\psi}, z) = -V(L_{-1}\overline{\psi}, z) \,, \tag{2.41}$$

so

$$\overline{\phi} = -L_{-1}\overline{\psi} \,. \tag{2.42}$$

(iii) For ψ a quasi-primary state of weight h_ψ, (2.24) gives

$$V(\overline{\psi}, z) = z^{-2h_\psi}V(\psi, 1/z^*)^\dagger \,. \tag{2.43}$$

From (i), $\overline{\psi}$ has the same conformal weight as ψ, so that a second application of (2.7) shows that the vertex operators for $\overline{\overline{\psi}}$ and ψ are equal, and so by the uniqueness theorem $\overline{\overline{\psi}} = \psi$. But from (ii), the action of L_{-1} anticommutes with the barring operation. From the decomposition of $\mathscr{F}$ into a direct sum of $su(1,1)$

highest weight representations, we thus see that for a general state $\psi \in \mathscr{F}$ (not necessarily quasi-primary) $\overline{\overline{\psi}} = \psi$.

(iv) To establish this result, we note that

$$\begin{aligned}
f_{\phi_1\phi_2\phi_3} &= \langle \overline{\phi}_1 | V(\phi_2, 1) | \phi_3 \rangle \\
&= \langle \overline{\phi}_1 | e^{L_{-1}} V(\phi_3, -1) | \phi_2 \rangle && \text{by (2.13)} \\
&= \langle \phi_2 | V(\phi_3, -1)^\dagger e^{L_1} | \overline{\phi}_1 \rangle^* \\
&= \langle \phi_2 | V(e^{-L_1} \overline{\phi}_3, -1) e^{L_1} | \overline{\phi}_1 \rangle^* && \text{by (2.24) and } \overline{\overline{\phi}}_3 = \phi_3 \\
&= \langle \phi_2 | e^{-L_{-1}} V(e^{L_1} \overline{\phi}_1, 1) e^{-L_1} | \overline{\phi}_3 \rangle^* && \text{by (2.13)} \\
&= \langle \phi_2 | e^{-L_{-1}} V(\phi_1, 1)^\dagger e^{-L_1} | \overline{\phi}_3 \rangle^* && \text{by (2.24) and } \overline{\overline{\phi}}_1 = \phi_1 \\
&= \langle \phi_1 | V(e^{-L_1} \phi_2, -1)^\dagger | \overline{\phi}_3 \rangle^* && \text{by (2.13)} \\
&= \langle \phi_1 | V(\overline{\phi}_2, -1) | \overline{\phi}_3 \rangle^* && \text{by (2.24)} \\
&= (-1)^{h_1+h_2+h_3} (f_{\overline{\phi}_1\overline{\phi}_2\overline{\phi}_3})^* \,. && (2.44)
\end{aligned}$$

(v) A quasi-primary state remains quasi-primary under the map $\psi \mapsto \overline{\psi}$, since

$$\begin{aligned}
\|L_1\overline{\psi}\|^2 &= \langle \overline{\psi} | L_{-1} L_1 | \overline{\psi} \rangle = \langle \overline{\psi} | L_1 L_{-1} | \overline{\psi} \rangle - 2 \langle \overline{\psi} | L_0 | \overline{\psi} \rangle \\
&= \langle \psi | L_1 L_{-1} | \psi \rangle - 2 \langle \psi | L_0 | \psi \rangle \,, \qquad (2.45)
\end{aligned}$$

using first $[L_1, L_{-1}] = 2L_0$ and then the facts that bar anticommutes with L_{-1} and commutes with L_0 and that barring preserves the norm (a special case of (iv) given by setting $\phi_2 = |0\rangle$ and $\phi_3 = \overline{\phi}_1$), so that

$$\|L_1\overline{\psi}\|^2 = \|L_1\psi\|^2 = 0 \,, \qquad (2.46)$$

for ψ quasi-primary. Hence we deduce $\overline{L_1\psi} = -L_1\overline{\psi}$ for an arbitrary state ψ by induction on n in considering states of the form $L_{-1}^n\phi$ with ϕ quasi-primary (as we saw in Proposition 2.7, these span $\mathscr{F}$), i.e., L_1 anticommutes with the barring operation.

(vi) From (2.7) and (2.18) we have

$$\langle 0 | V(\psi, z) V(\phi, \zeta) | 0 \rangle = (z - \zeta)^{-h_\psi - h_\phi} K_{\psi\phi} \,, \qquad (2.47)$$

where $L_0\phi = h_\phi\phi$, $L_0\psi = h_\psi\psi$ and $K_{\psi\phi} = \langle 0 | V(\psi, 1) | \phi \rangle$. But, using (2.24) and $\overline{\overline{\psi}} = \psi$, $V(\psi, 1) = V(e^{L_1}\overline{\psi}, 1)^\dagger$. So, using (2.3), $\langle 0 | V(\psi, 1) = \langle \overline{\psi} | e^{L_{-1}} e^{L_1}$, i.e.,

$$K_{\psi\phi} = \langle e^{L_1}\overline{\psi} | e^{L_1}\phi \rangle \,. \qquad (2.48)$$

Taking $\phi = \overline{\psi}$ and ψ (and hence ϕ) quasi-primary, $K_{\psi\overline{\psi}} = \|\overline{\psi}\|^2$ and $K_{\overline{\psi}\psi} = \|\psi\|^2$ (using $\overline{\overline{\psi}} = \psi$), both of which are positive (for $\psi \neq 0$). The locality relation (2.4) applied to (2.47) then implies that $K_{\psi\overline{\psi}} = (-1)^{2h_\psi} K_{\overline{\psi}\psi}$ (as $h_\psi = h_{\overline{\psi}}$), and so h_ψ must be integral. Hence the conformal weights of all quasi-primary states and hence all of their L_{-1} descendants (i.e., all states) must be integral.

(vii) For general states, locality applied to (2.47) and making use of (2.48) requires $\langle e^{L_1}\overline{\psi}|e^{L_1}(-1)^{L_0}\psi\rangle = \langle e^{L_1}\overline{\phi}|e^{L_1}(-1)^{L_0}\phi\rangle$. Replacing ϕ and ψ by $e^{L_1}\phi$ and $e^{L_1}\psi$ respectively and using the facts that L_1 anticommutes with the bar and that $(-1)^{L_0}e^{L_1} = e^{-L_1}(-1)^{L_0}$, we obtain the required result. (The factors of $(-1)^{L_0}$ cancel on either side, as the inner products clearly vanish when the conformal weights of ϕ and ψ are not equal.) □

An analogue of part (iv) of the above will be of importance later in proving one of the locality relations when we come to consider extending the CFT by a particular representation to give a new CFT.

Definition 2.10. *The* **moments** *of the vertex operators are given by*

$$V(\psi,z) = \sum_{n\in Z} V_n(\psi)z^{-n-h_\psi} \,. \tag{2.49}$$

Then, by (2.3),

$$V_{-h_\psi}(\psi)|0\rangle = \psi, \qquad V_n(\psi)|0\rangle = 0 \quad \text{for } n > -h_\psi \tag{2.50}$$

(cf. (2.1) noting that the state $\psi_L = L_{-2}|0\rangle$ has conformal weight 2). Equations (2.5) and (2.17) may then be rewritten in terms of modes as

$$\begin{aligned} [L_0, V_n(\psi)] &= -nV_n(\psi)\,, \\ [L_{-1}, V_n(\psi)] &= (1-n-h_\psi)V_{n-1}(\psi)\,, \end{aligned} \tag{2.51}$$

and the duality relation (2.11) can also be rewritten, giving

$$V(\psi,z)V(\phi,\zeta) = \sum_{n=0}^{\infty} (z-\zeta)^{n-h_\psi-h_\phi} V(\phi_n,\zeta)\,, \tag{2.52}$$

[a relation which holds, as discussed following axiom P3, at the level of analytic continuation of matrix elements] where

$$\phi_n = V_{h_\phi-n}(\psi)\phi\,, \tag{2.53}$$

and h_ϕ and h_ψ are the conformal weights of ϕ and ψ respectively, the sum being bounded below because $V_{h_\phi-n}(\psi)\phi = 0$ for $n < 0$ (otherwise we would have a non-zero state with negative conformal weight). This is a precise version of the operator product expansion (OPE), showing that this important result, often assumed in theories as an axiom, is simply a consequence of locality, emphasising further the important role played by the requirement (2.4).

Note also that for ψ quasi-primary, (2.27) gives

$$V_n(\psi)^\dagger = V_{-n}(\overline{\psi})\,. \tag{2.54}$$

A fact which will be of use later when we come to discuss sub-conformal field theories is

Proposition 2.11. *The vacuum is generated in the OPE corresponding to states ϕ and $\overline{\phi}$.*

Proof. Considering the OPE (2.52) for $\psi = \overline{\phi}$, we see that the leading term in the expansion on the right-hand side is $(z-\zeta)^{-2h_\phi} V(\phi_0,\zeta)$, where $\phi_0 = V_{h_\phi}(\overline{\phi})\phi$. ϕ_0 has conformal weight zero, from (2.51), and so must be proportional to the vacuum state $|0\rangle$ by our assumption P6 about the uniqueness of the $su(1,1)$ invariant state, i.e., $\phi_0 = k|0\rangle$, and so the leading term is $k(z-\zeta)^{-2h_\phi}$. Comparing with (2.47), we see that $k = K_{\overline{\phi}\phi} = \|e^{L_1}\phi\|^2$ from (2.48). So $k > 0$, and therefore the vertex operator for $|0\rangle$ appears in the OPE. □

Consider now the OPE (2.52) for ψ_L with ψ_L. The singular terms in the expansion are ϕ_0, ϕ_1, ϕ_2 and ϕ_3. $\phi_0 = \frac{c'}{2}|0\rangle$, where $c' = 2\|\psi_L\|^2$, $\phi_1 = L_1 L_{-2}|0\rangle = 3L_{-1}|0\rangle = 0$,
$\phi_2 = L_0\psi_L = 2\psi_L$ and $\phi_3 = L_{-1}\psi_L$. Then, setting $V(\psi_L, z) = L(z)$,

$$L(z)L(\zeta) = \frac{c'}{2}(z-\zeta)^{-4} + 2(z-\zeta)^{-2}L(\zeta) + (z-\zeta)^{-1}\frac{d}{d\zeta}L(\zeta) + O(1)\,, \qquad (2.55)$$

where the $(z-\zeta)^{-1}$ term is rewritten using (2.10) and $O(1)$ stands for terms regular at $z = \zeta$. From (2.55) we can use the usual contour manipulation arguments to derive (2.2) (with $c = c'$). That is

$$[L_n, L_m] = \frac{1}{(2\pi i)^2}\left[\oint_{|z|>|\zeta|} dz \oint d\zeta - \oint_{|\zeta|>|z|} dz \oint d\zeta\right] z^{n+1}\zeta^{m+1}L(z)L(\zeta)$$

$$= \frac{1}{(2\pi i)^2}\oint_0 d\zeta \oint_\zeta dz\, z^{n+1}\zeta^{m+1}L(z)L(\zeta)\,, \qquad (2.56)$$

where the z integral is taken on a contour positively encircling ζ excluding $z = 0$ and the ζ contour is then taken positively about $\zeta = 0$. Substituting in from (2.55) gives the required result. In other words, we can deduce the entire Virasoro algebra from the conformal field theory structure and the few simple properties used immediately above (2.55) (in particular, the relation $[L_1, L_{-2}] = 3L_{-1}$), i.e., we can weaken the requirement Pl slightly and it still holds true in full, showing once more the powerful consequences which follow from the structure of local vertex operators.

We may similarly deduce the conformal properties of vertex operators. We have

$$L(z)V(\phi,\zeta) = \cdots + (z-\zeta)^{-4}V(L_2\phi,\zeta) + (z-\zeta)^{-3}V(L_1\phi,\zeta)$$
$$+ h_\phi(z-\zeta)^{-2}V(\phi,\zeta) + (z-\zeta)^{-1}\frac{d}{d\zeta}V(\phi,\zeta) + O(1)\,, \quad (2.57)$$

for a state ϕ of weight h_ϕ. Hence, if $L_n\phi = 0$ for $n = 1, 2$ (and so for all $n \geqq 1$ by (2.2)),

$$[L_n, V(\phi,\zeta)] = \zeta^n\left\{\zeta\frac{d}{d\zeta} + (n+1)h_\phi\right\}V(\phi,\zeta)\,, \qquad (2.58)$$

for all n, by the contour manipulation argument.

Definition 2.12. *A state ϕ said to be a* **(conformal) primary state** *if it is a highest weight state for the Virasoro algebra, i.e., if $L_n\phi = 0$ for $n = 1, 2$. The corresponding vertex operator is said to be a* **(conformal) primary field**.

By using once more the fact that the conformal weights are bounded below, we see that $\mathscr{F}$ splits up into Virasoro highest weight representations, each generated by the action of the operators L_{-n} for $n>0$ on Virasoro highest weight states. If instead we have ϕ only quasi-primary, then (2.58) holds only for $n=0,\pm 1$. The relation (2.58) is a generalisation of (2.5) and (2.17), which hold for all states.

For a quasi-primary state ψ of weight h_ψ, (2.58) for $n=0,\pm 1$ is equivalent to the Möbius transformation property

$$D_\gamma V(\psi,z)D_\gamma^{-1} = \left[\frac{d\gamma(z)}{dz}\right]^{h_\psi} V(\psi,\gamma(z))\,, \tag{2.59}$$

where

$$D_\gamma = \exp\left\{\frac{b}{d}L_{-1}\right\}\left(\frac{\sqrt{ad-bc}}{d}\right)^{2L_0}\exp\left\{-\frac{c}{d}L_1\right\}\,, \tag{2.60}$$

and

$$\gamma(z) = \frac{az+b}{cz+d}\,. \tag{2.61}$$

This freedom to perform Möbius transformations on the variables for quasi-primary fields means that we can write the three-point function for quasi-primary states ϕ_1,ϕ_2 and ϕ_3 in the form

$$\begin{aligned}\langle 0|V(\phi_1,z_1)V(\phi_2,z_2)V(\phi_3,z_3)|0\rangle &= (z_1-z_2)^{h_3-h_1-h_2}(z_1-z_3)^{h_2-h_1-h_3} \\ &\quad\cdot (z_2-z_3)^{h_1-h_2-h_3} f_{\phi_1\phi_2\phi_3}\,,\end{aligned} \tag{2.62}$$

where $f_{\phi_1\phi_2\phi_3}$ is defined as above and $L_0\phi_j = h_j\phi_j$ for $1 \leqq j \leqq 3$. Locality therefore implies

$$f_{\phi_1\phi_2\phi_3} = f_{\phi_2\phi_3\phi_1} = (-1)^{h_1+h_2+h_3} f_{\phi_1\phi_3\phi_2} = (-1)^{h_1+h_2+h_3} f_{\phi_3\phi_2\phi_1}\,. \tag{2.63}$$

Definition 2.13. *We shall say that two conformal field theories $\mathscr{H}$ and $\mathscr{H}'$, with dense subspaces $\mathscr{F}$ and $\mathscr{F}'$ respectively and corresponding vertex operators $V(\psi,z)$ and $V'(\psi',z)$, are* **isomorphic** *if there exists a unitary map $u:\mathscr{H}\to\mathscr{H}'$ such that*

$$V'(u\psi,z) = uV(\psi,z)u^{-1}\,, \tag{2.64}$$

for $\psi\in\mathscr{F}$.

Proposition 2.14. *If $u:\mathscr{H}\to\mathscr{H}'$ is an isomorphism of conformal field theories $\mathscr{H},\mathscr{H}'$ then $u|0\rangle = |0'\rangle$, $u\psi_L = \psi_L'$, where $|0\rangle, |0'\rangle$ are the vacuum states and ψ_L, ψ_L' are the conformal states in $\mathscr{H}$ and $\mathscr{H}'$ respectively.*

Proof. Taking $\psi = |0\rangle$, the vacuum in $\mathscr{F}$, we have $V'(u|0\rangle,z)\equiv 1$, so by uniqueness $u|0\rangle = |0'\rangle$, the vacuum state in $\mathscr{F}'$. Also, we can show that u must map the other special state in $\mathscr{F}$, the conformal state ψ_L, into the corresponding state ψ_L' in $\mathscr{F}'$. From the action of the vertex operators on the vacuum, (2.3), together with (2.64) we see that $uL_{-1}u^{-1} = L'_{-1}$. Set $\tilde{L}_n = uL_nu^{-1}$ for all n. Then $\tilde{L}_{-1} = L'_{-1}$. By conjugation we have $\tilde{L}_1 = L'_1$, and by $[L_1,L_{-1}] = 2L_0$ we see that $\tilde{L}_0 = L'_0$.

From the OPE (2.52), we have

$$V'(\psi'_{L,z})V'(u\psi_L,\zeta) = \sum_{n=0}^{\infty}(z-\zeta)^{n-4}V'(\phi_n,\zeta)\,, \tag{2.65}$$

where

$$\phi_n = L'_{2-n} u\psi_L \,. \tag{2.66}$$

(Note that the conformal weight of $u\psi_L$ is still 2, as $\tilde{L}_0 = L'_0$ implies that u preserves coformal weights). $L'_2 u\psi_L$ has zero conformal weight, and so is $\frac{k}{2}|0'\rangle$, by the uniqueness assumption made above), for some $k \in \mathbf{C}$. $L'_1 u\psi_L = uL_1\psi_L = 0$, $L'_0 u\psi_L = 2u\psi_L$ (see comment above) and $V'(L'_{-1}u\psi_L, \zeta) = \frac{d}{d\zeta} V'(u\psi_L, \zeta)$ (cf. (2.10)), so that (2.65) becomes

$$\begin{aligned} V'(\psi'_L, z)V'(u\psi_L, \zeta) &= \frac{k}{2}(z-\zeta)^{-4} + 2(z-\zeta)^{-2}V'(u\psi_L, \zeta) \\ &\quad + (z-\zeta)^{-1}\frac{d}{d\zeta}V'(u\psi_L, \zeta) + O(1) \,. \end{aligned} \tag{2.67}$$

Hence, comparing with (2.55), we see that the contour manipulation argument gives

$$[L'_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} + \frac{k}{12}m(m^2-1)\delta_{m,-n} \,, \tag{2.68}$$

(the modes of $V'(u\psi_L, z)$ are $\tilde{L}_n$ by (2.64)).

On the other hand, we find similarly

$$\begin{aligned} V'(u\psi_L, z)V'(\psi'_L, \zeta) &= \frac{k'}{2}(z-\zeta)^{-4} + 2(z-\zeta)^{-2}V'(\psi'_L, \zeta) \\ &\quad + (z-\zeta)^{-1}\frac{d}{d\zeta}V'(\psi'_L, \zeta) + O(1) \,, \end{aligned} \tag{2.69}$$

giving

$$[\tilde{L}_m, L'_n] = (m-n)L'_{m+n} + \frac{k'}{12}m(m^2-1)\delta_{m,-n} \,. \tag{2.70}$$

Comparing (2.68) and (2.70) shows $\tilde{L}_n = L'_n$ for $n \neq 0$, and so $\psi'_L = u\psi_L$ as required (we already have the case $n = 0$ from the above discussion). □

Finally in this section, we discuss the notion of a sub-conformal field theory. This concept is not particularly exploited in the following sections, but it does provide an interesting example of the techniques and structures discussed above.

Definition 2.15. *A* **sub-conformal field theory** *of a conformal field theory* $\mathscr{H}$ *is defined to be a subspace* $\mathscr{J}$ $\mathscr{H}$ *such that*

(i) $\mathscr{J}$ *is an invariant subspace for each* $V(\phi, z)$, $\phi \in \mathscr{F}_{\mathscr{J}} \equiv \mathscr{J} \cap \mathscr{F}$
(ii) $\mathscr{J}$ *is invariant under the* $su(1,1)$ *algebra* $L_{\pm 1}, L_0$
(iii) $\overline{\mathscr{F}_{\mathscr{J}}} \equiv \{\overline{\phi} : \phi \in \mathscr{F}_{\mathscr{J}}\} = \mathscr{F}_{\mathscr{J}}$.

We noted earlier that the vertex operator for $|0\rangle$ appears in the OPE of ϕ and $\overline{\phi}$. From (i) and (iii), this immediately implies that $|0\rangle \in \mathscr{J}$. (This automatically gives invariance under L_{-1} from $V(\psi, z)|0\rangle = e^{zL_{-1}}\psi$ for $\psi \in \mathscr{J}$ together with (i).) Let us denote the orthogonal complement of $\mathscr{J}$ as $\mathscr{J}^\perp$, i.e., $\mathscr{J}^\perp = \{\psi \in \mathscr{H} : \langle\psi|\phi\rangle = 0 \ \forall\phi \in \mathscr{J}\}$. If $\mathscr{J}$ is invariant under L_n for some n, then for $\phi \in \mathscr{J}$ and $\psi \in \mathscr{J}^\perp$ $0 = \langle\phi|L_n|\phi\rangle = \langle\phi|L_{-n}|\psi\rangle^*$, i.e., $\mathscr{J}^\perp$ is invariant under L_{-n}. Hence (ii) is equivalent to saying that $\mathscr{J}^\perp$ is $su(1,1)$ invariant. Also, by (vii) of Proposition 2.9, (iii) could equally well be stated for $\mathscr{J}^\perp$.

Proposition 2.16. *A sub-conformal field theory of a (hermitian) conformal field theory is itself a (hermitian) conformal field theory.*

Proof. Suppose $\mathscr{I}$ is a sub-conformal field theory of $\mathscr{H}$. Let $P^{\mathscr{I}}$ be the orthogonal projection onto $\mathscr{I}$, and set $\psi_L^{\mathscr{I}} = P^{\mathscr{I}}\psi_L$. Write $L^{\mathscr{I}}(z) = V(\psi_L^{\mathscr{I}}, z)$ and $K(z) = V(\psi_{K,z})$, where $\psi_K = \psi_L - \psi_L^{\mathscr{I}} \in \mathscr{I}^{\perp}$. To evaluate the OPE $L(z)L^{\mathscr{I}}(\zeta)$, we require the actions of L_2, L_1 and L_0 on $\psi_L^{\mathscr{I}}$. $L_0\psi_L = 2\psi_L$ gives $L_0(\psi_L^{\mathscr{I}} + \psi_K) = 2(\psi_L^{\mathscr{I}} + \psi_K)$. But $\mathscr{I}$ and $\mathscr{I}^{\perp}$ are L_0 invariant. So, comparing both sides, we see that $L_0\psi_L^{\mathscr{I}} = 2\psi_L^{\mathscr{I}}$ (and $L_0\psi_K = 2\psi_K$), by uniqueness of the decomposition of a state into a sum of a state in $\mathscr{I}$ and a state in $\mathscr{I}^{\perp}$. $L_2\psi_L^{\mathscr{I}}$ has conformal weight zero, and so is $\frac{c^{\mathscr{I}}}{2}|0\rangle$, for some $c^{\mathscr{I}} \in \mathbf{C}$. So

$$\frac{c^{\mathscr{I}}}{2} = \langle 0|L_2|\psi_L^{\mathscr{I}}\rangle = \langle \psi_L|\psi_L^{\mathscr{I}}\rangle = \langle \psi_L|P^{\mathscr{I}}|\psi_L^{\mathscr{I}}\rangle = \|\psi_L^{\mathscr{I}}\|^2 , \tag{2.71}$$

since $P^{\mathscr{I}} = P^{\mathscr{I}\dagger}$. Finally, $L_1\psi_L = 0 = L_1\psi_L^{\mathscr{I}} + L_1\psi_K$. But $\mathscr{I}$ and $\mathscr{I}^{\perp}$ are L_1 invariant. So $L_1\psi_L^{\mathscr{I}} = -L_1\psi_K \in \mathscr{I} \cap \mathscr{I}^{\perp} = \{0\}$. From this, it follows by the usual contour manipulation argument (or comparing with (2.55)) that

$$[L_m, L_n^{\mathscr{I}}] = (m-n)L_{m+n}^{\mathscr{I}} + \frac{c^{\mathscr{I}}}{12}m(m^2-1)\delta_{m,-n} , \tag{2.72}$$

where $L_n^{\mathscr{I}}$ are the modes of $L^{\mathscr{I}}(z)$. To deduce from (2.72) that the $L_n^{\mathscr{I}}$ satisfy the (centrally extended) Virasoro algebra, we need to show that $[K_m, L_n^{\mathscr{I}}] = 0$, where K_n are the modes of $K(z)$ ($L_n = L_n^{\mathscr{I}} + K_n$). To do this, we consider the OPE $K(z)L^{\mathscr{I}}(\zeta)$. We need to prove that it contains no singular terms (at $z = \zeta$), so that the commutator vanishes on applying the contour manipulation argument, i.e., we need $K_n\psi_L^{\mathscr{I}} = 0$ for $-1 \leqq n \leqq 2$.

To show this, let us look at the action of vertex operators for states in $\mathscr{I}^{\perp}$ on states in $\mathscr{I}$ and vice versa. Let $\times$ denote the operation of taking the operator product and identifying states with the corresponding vertex operators. Then (i) becomes $\mathscr{I} \times \mathscr{I} \subset \mathscr{I}$. Therefore, for $\chi \in \mathscr{I}^{\perp}$ and $\phi, \psi \in \mathscr{I}$, $\langle \chi|V(e^{z^* L_1} z^{*-2L_0}\overline{\phi}, \frac{1}{z^*})|\psi\rangle = 0$, the state forming the argument of the vertex operator being in $\mathscr{I}$ by (ii) and (iii). So, conjugating, and using (2.24), $\langle \psi|V(\phi, z)|\chi\rangle = 0$, i.e., $\mathscr{I} \times \mathscr{I}^{\perp} \subset \mathscr{I}^{\perp}$. But, from (2.13), $V(\chi, z)\phi = e^{zL_{-1}}V(\phi, -z)\chi$. So, since $\mathscr{I}^{\perp}$ is invariant under L_{-1}, this result also implies $\mathscr{I}^{\perp} \times \mathscr{I} \subset \mathscr{I}^{\perp}$. Thus, $K_n\psi_L^{\mathscr{I}} \in \mathscr{I}^{\perp}$ and $L_n^{\mathscr{I}}\psi_L^{\mathscr{I}} \in \mathscr{I}$. But, from the above calculation of the OPE (2.72), we see that $L_n\psi_L^{\mathscr{I}} \in \mathscr{I}$ for $-1 \leqq n \leqq 2$. So $K_n\psi_L^{\mathscr{I}} = 0$ for $-1 \leqq n \leqq 2$, as required.

This gives us a type of generalised coset construction [15], with $L_n^{\mathscr{I}}$ and K_n satisfying commuting Virasoro algebras with $L_n = L_n^{\mathscr{I}} + K_n$,

$$[L_m^{\mathscr{I}}, L_n^{\mathscr{I}}] = (m-n)L_{m+n}^{\mathscr{I}} + \frac{c^{\mathscr{I}}}{12}m(m^2-1)\delta_{m,-n} , \tag{2.73}$$

$$[K_m, K_n] = (m-n)K_{m+n} + \frac{c^K}{12}m(m^2-1)\delta_{m,-n} , \tag{2.74}$$

where $c^K = c - c^{\mathscr{I}} = 2\|\psi_K\|^2$.

Since $L_n\mathscr{I} \subset \mathscr{I}$ for $n = 0, \pm 1$ by (ii) and $L_n^{\mathscr{I}}\mathscr{I} \subset \mathscr{I}$, we see that $K_n\mathscr{I} = 0$ for $n = 0, \pm 1$, i.e., we can replace $L_0, L_{\pm 1}$ by $L_0^{\mathscr{I}}, L_{\pm 1}^{\mathscr{I}}$ respectively when acting on $\mathcal{J}$. Hence, $\mathcal{J}$ becomes a conformal field theory, with vertex operators $V(\psi,z)$ for $\psi \in \mathcal{J}$ restricted to $\mathcal{J}$, vacuum state $|0\rangle$ and conformal state $\psi_L^{\mathcal{J}}$. It also retains the hermitian structure possessed by $\mathcal{H}$. (Note also that the zero conformal weight state remains unique, since $L_0 = L_0^{\mathcal{J}}$ on $\mathcal{J}$, and similarly the spectrum of $L_0^{\mathcal{J}}$ is bounded below (and hence non-negative).) □

Set $\mathcal{J}^0 = \{\psi \in \mathcal{H} : L_0^{\mathcal{J}}\psi = 0\}$. We have

Proposition 2.17.

(i) $\psi \in \mathcal{J}^0 \Leftrightarrow [L_n^{\mathcal{J}}, V(\psi,z)] = 0 \forall n \in \mathbf{Z}$. (2.75)

(ii) $\mathcal{J}^0$ *is a sub-conformal field theory of* $\mathcal{H}$.

(iii) *The vertex operators corresponding to* $\mathcal{J}$ *and* $\mathcal{J}^0$ *commute.*

(iv) $\mathcal{J} \subset (\mathcal{J}^0)^0$, *In fact,* $(\mathcal{J}^0)^0$ *is the largest sub-conformal field theory in* $\mathcal{H}$ *containing* $\mathcal{J}$ *and sharing the same conformal structure.*

Proof.

(i) If $\psi \in \mathcal{J}^0$, then $L_n^{\mathcal{J}}\psi = 0$ for $n > 0$, since the spectrum of $L_0^{\mathcal{J}}$ is non-negative. Also $L_{-1}^{\mathcal{J}}\psi = 0$, since $\|L_{-1}^{\mathcal{J}}\psi\|^2 = \|L_1^{\mathcal{J}}\psi\|^2 = 0$, using $l_1^{\mathcal{J}}, L_{-1}^{\mathcal{J}}] = 2L_0^{\mathcal{J}}$. If the commutator vanishes, applying it to the vacuum for $n \geqq -1$ (for which $L_n^{\mathcal{J}}|0\rangle =$ 0, since $L_n|0\rangle = 0$ and so $L_n^{\mathcal{J}}|0\rangle = -K_n|0\rangle \in \mathcal{J} \cup \mathcal{J}^{\perp} = \{0\}$) gives

$$L_n^{\mathcal{J}} V(\psi,z)|0\rangle = 0\,, \tag{2.76}$$

so that (2.6) gives the left-hand side of (2.75). Conversely, if $\psi \in \mathcal{J}^0$, we look at the OPE $L^{\mathcal{J}}(z)V(\psi,\zeta)$. The singular terms involve the vertex operators for the states $L_n^{\mathcal{J}}\psi$ for $n \geqq -1$, and so vanish. Therefore, the right-hand side of (2.75) follows by contour integration.

(ii) $\mathcal{J}^0$ is invariant under $L_{\pm 1}$, L_0, since if $\psi \in \mathcal{J}^0$ then $L_0^{\mathcal{J}} L_m\psi = L_m L_0^{\mathcal{J}}\psi - [L_m, L_0^{\mathcal{J}}]\psi$. The first term vanishes by definition of $\mathcal{J}^0$, and the second term, from (2.72), is $-mL_m^{\mathcal{J}}\psi$ for $-1 \leqq m \leqq 1$, which vanishes by the argument given in (i). Also, conjugation of the right-hand side of (2.75) implies that $\mathcal{J}^0$ is invariant under the bar operation. Also, for $\phi, \psi \in \mathcal{J}^0$, $L_0^{\mathcal{J}} V(\phi,z)\psi = V(\phi,z)L_0^{\mathcal{J}}\psi$, by (2.75), which vanishes by definition of $\mathcal{J}^0$, i.e., $\mathcal{J}^0$ is invariant under $V(\phi,z)$ for $\phi \in \mathcal{J}^0$.
Thus, we see that $\mathcal{J}^0$ is a sub-conformal field theory of $\mathcal{H}$.

(iii) For $\phi \in \mathcal{J}^0$, $\langle\psi_L^{\mathcal{J}}|\phi\rangle = \langle 0|L_2^{\mathcal{J}}|\phi\rangle = 0$, i.e., $\psi_L^{\mathcal{J}} \in \mathcal{J}^{0\perp}$. Also, from (2.75) and the fact that $|L_m^{\mathcal{J}}, K_n] = 0$, we see that $\psi_K \in \mathcal{J}^0$. Hence, $P^{\mathcal{J}^0}\psi_L = \psi_K$, or $L_n^{\mathcal{J}^0} = K_n$. Thus, the Virasoro algebras of $\mathcal{J}$ and $\mathcal{J}^0$ are complementary, (that is, they commute, and add to give the Virasoro algebra for $\mathcal{H}$). More generally, the vertex operators corresponding to $\mathcal{J}$ and $\mathcal{J}^0$ commute, since the singular terms in the OPE $V(\psi,z)V(\phi,\zeta)$ for $\psi \in \mathcal{J}$ and $\phi \in \mathcal{J}^0$ vanish. To see this, we note by positivity of the L_0 eigenspace that $V_n(\psi)\phi = 0$ for $n > h_\phi$. Then from $L_{-1}^{\mathcal{J}} V_n(\psi) phi = -(h_\psi + n - 1)V_{n-1}(\psi)\phi$ (since $L_{-1}^{\mathcal{J}}\phi = 0$ as $\phi \in \mathcal{J}^0$, so we can replace $L_{-1}^{\mathcal{J}} V_n(\psi)$ by its commutator, and as $\psi \in \mathcal{J}$ $L_{-1}^{\mathcal{J}}$ can be replaced by L_{-1} and we then use (2.51)) we can deduce recursively that $V_n(\psi)\phi = 0$ for $n > -h_\psi$, as required.

(iv) The above result gives, in particular, $[K_n, V(\psi,z)] = 0$ for all n and all $\psi \in \mathcal{J}$, so that by (2.75) we see that $\mathcal{J} \subset (\mathcal{J}^0)^0$. Also the conformal state for $(\mathcal{J}^0)^0$ is complementary to that for $\mathcal{J}^0$, and so coincides with that for $\mathcal{J}$. □

3. Representations of Conformal Field Theories

Definition 3.1. *A* **representation** $(\mathbf{U}, \mathscr{K})$ *of the conformal field theory* $\mathscr{H}$ *is a Hilbert space* $\mathscr{K}$ *and a set of linear operators* $U(\psi, z) : \mathscr{K} \to \mathscr{K}$ *linear in* ψ *for* $\psi \in \mathscr{F}$ *such that*

$$U(\psi,z)U(\phi,\zeta) = U(V(\psi,z-\zeta)\phi,\zeta)\,, \tag{3.1}$$

(*cf. the duality relation* (2.11)), *with* $U(|0\rangle, z) \equiv 1$ (*otherwise we could have* $U(\psi,z)$ *vanishing on some subspace of* $\mathscr{K}$).

Equivalently, using the mode expansion of V,

$$U(\psi,z)U(\phi,\zeta) = \sum_{n=0}^{\infty} (z-\zeta)^{n-h_\phi-h_\psi} U(\phi_n,\zeta)\,, \tag{3.2}$$

where h_ϕ, and h_ψ are the conformal weights of ϕ and ψ respectively and the ϕ_n are as in (2.53). The representations which we consider will be meromorphic, that is matrix elements of operators in $\mathbf{U}$ will be meromorphic functions of the complex arguments of the operators.

As a simple consequence of this definition, we have

Proposition 3.2.

(i) *The operators in* $\mathbf{U}$ *are local.*
(ii) *The modes of* $U(\psi_L, z)$ *satisfy the Virasoro algebra* (2.2).
(iii) $U(\psi,z)$ *possesses the analogous translation property to* (2.5).

Proof.

(i) First, note that by taking $\phi = |0\rangle$ in (3.1) and using (2.3) we obtain

$$U(\psi,z) = U(e^{(z-\zeta)L_{-1}}\psi,\zeta)\,. \tag{3.3}$$

Hence

$$\begin{aligned} U(\psi,z)U(\phi,\zeta) &= U(e^{(z-\zeta)L_{-1}}V(\phi,\zeta-z)\psi,\zeta) && \text{by (3.1) and (2.13)} \\ &= U(V(\phi,\zeta-z)\psi,z) && \text{by (3.3)} \\ &= U(\phi,\zeta)U(\psi,z) && \text{by(3.1)}\,, \end{aligned}$$

i.e.,

$$U(\psi,z)U(\phi,\zeta) = U(\phi,\zeta)U(\psi,z)\,, \tag{3.4}$$

again in the sense of analytic continuation of matrix elements of either side.

(ii) Set $U(\psi_L,z) = L(z) = \sum_n L_n z^{-n-2}$ (using the same notation as for $V(\psi_L,z)$, but the distinction will always be obvious from the context). Then (3.2) and the usual contour manipulation argument show that the L_n satisfy the Virasoro algebra with the same central term as for the conformal field theory $\mathscr{H}$.

(iii) (3.3) implies that

$$\frac{d}{dz}U(\psi,z) = U(L_{-1}\psi,z)\,. \tag{3.5}$$

But from (3.2) with $\psi = \psi_L$ we see that

$$[L_{-1}, U(\psi,z)] = U(L_{-1}\psi,z)\,, \tag{3.6}$$

and so the result follows. $\square$

Example 3.3. If $\mathscr{J}$ is a sub-conformal field theory of the conformal field theory $\mathscr{H}$, as defined at the end of Sect. 2, then $\mathscr{J}^\perp$ forms a representation of $\mathscr{J}$, with $U(\phi,z) = V(\phi,z)$ restricted to $\mathscr{J}^\perp$ for $\phi \in \mathscr{J}$.

Proposition 3.4. *The existence of the representation given in Definition* 3.1 *is equivalent to the existence of "intertwining" operators* $W(\chi,z) : \mathscr{H} \to \mathscr{K}$ *for* $\chi \in \mathscr{K}$ *(or rather a dense subspace of* $\mathscr{K}$*) such that*

$$U(\psi,z)W(\chi,\zeta) = W(\chi,\zeta)V(\psi,z) \tag{3.7}$$

and $W(\chi,\zeta)|0\rangle \to \chi$ *as* $\zeta \to 0$.

The locality relation (3.7), interpreted in the usual sense, is referred to as the *intertwining relation.*

Proof. Given a representation as above we define $W(\chi,z)$ by

$$W(\chi,z)\phi = e^{zL_{-1}}U(\phi,-z)\chi\,, \tag{3.8}$$

cf. the relation (2.13). We shall ultimately combine the representation and the CFT to give a new CFT, and the relation (2.13) which must hold for this CFT requires (3.8) to hold.). Given this definition, we have

$$\begin{aligned} U(\psi,z)W(\chi,\zeta)\phi &= U(\psi,z)e^{\zeta L_{-1}}U(\phi,-\zeta)\chi && \text{by (3.8)} \\ &= e^{\zeta L_{-1}}U(\psi,z-\zeta)U(\phi,-\zeta)\chi \\ &= e^{\zeta L_{-1}}U(V(\psi,z)\phi,-\zeta)\chi && \text{by (3.1)} \\ &= W(\chi,\zeta)V(\psi,z)\phi && \text{by (3.8)}\,, \end{aligned} \tag{3.9}$$

and taking $\phi = |0\rangle$ in (3.8) and letting $z \to 0$ we obtain $W(\chi,z)|0\rangle \to \chi$ as $z \to 0$, as required.

Conversely, if we are given the intertwining operators $W(\chi,z)$ satisfying the intertwining relation (3.7) for some operators $U(\psi,z)$ and also the limiting relation as $z \to 0$ on $W(\chi,z)|0\rangle$, we have

$$\begin{aligned} U(\psi,z)U(\phi,\zeta)W(\chi,w) &= W(\chi,w)V(\psi,z)V(\phi,\zeta) && \text{by (3.7)} \\ &= W(\chi,w)V(V(\psi,z-\zeta)\phi,\zeta) && \text{by (2.11)} \\ &= U(V(\psi,z-\zeta)\phi,\zeta)W(\chi,w) && \text{by (3.1)}\,. \end{aligned} \tag{3.10}$$

Apply (3.10) to $|0\rangle$ and let $w \to 0$ to give, since χ is arbitrary, the required locality relation on $\mathbf{U}$. In addition, $U(|0\rangle,z)W(\chi,\zeta) = W(\chi,\zeta)$ from the intertwining relation, and again action on $|0\rangle$ and letting $\zeta \to 0$ we obtain $U(|0\rangle,z) \equiv 1$ as required. □

Note that defining W by (3.8) from the representation $\mathbf{U}$ gives

$$W(\chi,z)|0\rangle = e^{zL_{-1}}\chi\,, \tag{3.11}$$

(cf. (2.3)). Conversely, given operators W satisfying the intertwining relation (3.7) and also (3.11) (which is consistent with the required limit as $z \to 0$ in the definition of W) we obtain, by applying (3.7) to $|0\rangle$ and using (2.3),

$$U(\psi,z)e^{\zeta L_{-1}}\chi = W(\chi,\zeta)e^{zL_{-1}}\psi\,. \tag{3.12}$$

Taking $z \to 0$, the translation property for U then gives the relation (3.8). Hence, if we impose the stronger condition (3.11) rather than just the limiting condition in the definition of the intertwining operators, we always have the relation (3.8) between W and U.

From (3.8) we have

$$\frac{d}{dz}W(\chi,z)\phi = e^{zL_{-1}}L_{-1}U(\phi,-z)\chi - e^{zL_{-1}}[L_{-1},U(\phi,-z)]\chi \tag{3.13}$$

using the translation property for U. Hence

$$\frac{d}{dz}W(\chi,z)\phi = e^{zL_{-1}}U(\phi,-z)L_{-1}\chi = W(L_{-1}\chi,z)\phi\,, \tag{3.14}$$

i.e.,

$$\frac{d}{dz}W(\chi,z) = W(L_{-1}\chi,z)\,. \tag{3.15}$$

Note also that the fact that the U's are linear operators implies, from (3.8), that $W(\chi,z)$ is linear in χ, and since $U(\phi,z)$ is linear in ϕ then the W's are linear operators.

Definition 3.5. *If $\mathscr{H}$ and $\mathscr{H}'$ are two isomorphic conformal field theories with isomorphism $u : \mathscr{H} \to \mathscr{H}'$, as above, and $(\mathbf{U},\mathscr{K})$ and $(\mathbf{U}',\mathscr{K}')$ are representations of $\mathscr{H}$ and $\mathscr{H}'$ respectively, they are said to be equivalent if there is a unitary map $\rho : \mathscr{K} \to \mathscr{K}'$ such that*

$$\rho U(\psi,z)\rho^{-1} = U'(u\psi,z)\,, \tag{3.16}$$

for all $\psi \in \mathscr{H}$.

Proposition 3.6. *If $(\mathbf{U},\mathscr{K})$ is an irreducible representation, i.e., if it has no proper subspaces invariant under all the $U(\psi,z)$ for $\psi \in \mathscr{F}$, then ρ is unique up to multiplication by a constant.*

Proof. If ρ_1 and ρ_2 are two suitable maps satisfying (3.16), then $u = \rho_2^{-1}\rho_1$ is a unitary map commuting with $U(\psi,z)$ for all $\psi \in \mathscr{F}$. By Schur's lemma (i.e., that any eigenspace of u is an invariant subspace, and so must be the whole space for an irreducible representation), we deduce that u is a multiple, κ say, $\kappa \in \mathbf{C}$, of the identity map, i.e., $\rho_1 = \kappa\rho_2$. $\square$

4. Extension of a Conformal Field Theory by a Real Hermitian Representation

Definition 4.1. *The representation* $\mathbf{U}$ *described in the preceding section is said to be* **hermitian** *if*

$$U(\overline{\phi},z) = z^{-2h_\phi}U(e^{z^*L_1}\phi, 1/z^*)^\dagger\,, \tag{4.1}$$

where ϕ is a state with conformal weight h_ϕ, (cf. (2.24)).

Further, it is said to be *real* if there is an antilinear map $\chi \mapsto \overline{\chi}$ on $\mathscr{K}$ such that $\overline{\overline{\chi}} = \chi$, $L_{-1}\overline{\chi} = -\overline{L_{-1}\chi}$ and

$$(f_{\chi_1\phi\chi_2})^* = (-1)^{h_1+h_\phi+h_2}f_{\overline{\chi_1}\overline{\phi}\overline{\chi_2}}\,, \tag{4.2}$$

(cf. (v) of Proposition (2.9)) where $L_0\chi_j = h_j\chi_j$ for $j = 1,2$, $L_0\phi = h_\phi\phi$ and $f_{\chi_1\phi\chi_2} = \langle\overline{\chi}_1|U(\phi,1)|\chi_2\rangle$. Also, we require that if $L_0\chi = h_\chi\chi$ then $L_0\overline{\chi} = h_\chi\overline{\chi}$.

Definition 4.2. *Set*

$$\overline{W}(\overline{\chi},z) = z^{-2h_\chi}W(e^{z^*L_1}\chi, 1/z^*)^\dagger\,, \tag{4.3}$$

where h_χ is the conformal weight of $\chi\in\mathscr{K}$ (i.e., eigenvalue of the zero mode L_0 of $U(\psi_L,z)$), and W is given by (3.8).

Note that this definition extends to all states by linearity of $\overline{W}(\chi,z)$ in χ, which follows from linearity of $W(\chi,z)$ in χ and the antilinearity of the map $\chi\mapsto\overline{\chi}$. Its definition is inspired by (2.24) as we wish it to be part of a vertex operator in some extended conformal field theory. The operator $\overline{W}$ is a map from $\mathscr{K}$ to $\mathscr{H}$ which intertwines the conformal field theory and the representation in the opposite sense to that in which W does in (3.7), i.e.,

$$\overline{W}(\chi,\zeta)U(\psi,z) = V(\psi,z)\overline{W}(\chi,\zeta)\,. \tag{4.4}$$

This follows simply by conjugating (3.7) and using (4.3), (4.1) and (2.24).

Proposition 4.3. *We have the locality relation*

$$\overline{W}(\chi_1,z)W(\chi_2,\zeta) = \overline{W}(\chi_2,\zeta)W(\chi_1,z)\,. \tag{4.5}$$

Proof. First, consider the action of the left-hand side on an arbitrary untwisted state ϕ, i.e.,

$$\begin{aligned}\overline{W}(\chi_1,z)W(\chi_2,\zeta)\phi &= \overline{W}(\chi_1,z)e^{\zeta L_{-1}}U(\phi,-\zeta)\chi_2 \qquad \text{by (3.8)}\\ &= e^{\zeta L_{-1}}\overline{W}(\chi_1,z-\zeta)U(\phi,-\zeta)\chi_2\,,\end{aligned} \tag{4.6}$$

where the translation property for $\overline{W}(\chi,z)$ used in the last line follows by the same arguments as in Proposition 2.8 (trivially checking that the appropriate assumptions which went into that proposition remain valid in this context). Hence, using (4.4),

$$\overline{W}(\chi_1,z)W(\chi_2,\zeta)\phi = e^{\zeta L_{-1}}V(\phi,-\zeta)\overline{W}(\chi_1,z-\zeta)\chi_2\,. \tag{4.7}$$

Therefore, the locality relation (4.5) is equivalent to

$$\begin{aligned}V(\phi,-\zeta)\overline{W}(\chi_1,z-\zeta)\chi_2 &= e^{(z-\zeta)L_{-1}}V(\phi,-z)\overline{W}(\chi_2,\zeta-z)\chi_1\\ &= V(\phi,-\zeta)e^{(z-\zeta)L_{-1}}\overline{W}(\chi_2,\zeta-z)\chi_1\,,\end{aligned} \tag{4.8}$$

and so it remains to prove

$$\overline{W}(\chi_1,z)\chi_2 = e^{zL_{-1}}\overline{W}(\chi_2,-z)\chi_1\,. \tag{4.9}$$

Note that this is analogous to the relation (2.13). Conjugating (4.9) by using (4.3), with h_1 and h_2 the conformal weights of χ_1 and χ_2 respectively, we have to show the equality of

$$z^{2h_1}\langle\chi_2|W(e^{\frac{1}{z}L_1}\overline{\chi}_1,z) \quad\text{and}\quad z^{2h_2}\langle\chi_1|W(e^{-\frac{1}{z}L_1}\overline{\chi}_2,-z)e^{\frac{1}{z}L_1}\,. \tag{4.10}$$

Acting on an arbitrary untwisted state ϕ again, and using (3.8), we have to verify

$$z^{2h_1}\langle\chi_2|e^{zL_{-1}}U(\phi,-z)e^{\frac{1}{z}L_1}|\overline{\chi}_1\rangle = z^{2h_2}\langle\chi_1|e^{-zL_{-1}}U(e^{\frac{1}{z}L_1}\phi,z)e^{-\frac{1}{z}L_1}|\overline{\chi}_2\rangle\,. \quad (4.11)$$

From the fact that the representation is hermitian, i.e., using (4.1), we see that this is equivalent to

$$z^{2h_1}\langle\chi_2|e^{zL_{-1}}U(\phi,-z)e^{\frac{1}{z}L_1}|\overline{\chi}_1\rangle = z^{2h_2-2h_\phi}\langle\chi_1|e^{-zL_{-1}}U(\overline{\phi},1/z^*)^\dagger e^{\frac{1}{z}L_1}|\overline{\chi}_2\rangle\,. \quad (4.12)$$

So, we see that for χ_1 and χ_2 quasi-primary states, we have to verify

$$\begin{aligned} z^{2h_1}\langle\chi_2|U(\phi,-z)|\overline{\chi}_1\rangle &= z^{2h_2-2h_\phi}\langle\chi_1|U(\overline{\phi},1/z^*)^\dagger|\overline{\chi}_2\rangle \\ &= z^{2h_2-2h_\phi}\langle\overline{\chi}_2|U(\overline{\phi},1/z^*)|\chi_1\rangle^*\,. \end{aligned} \quad (4.13)$$

Using (2.18), we may remove the z dependence from U, and find that the relation which we have to verify reduces to (4.2), i.e., for a real hermitian representation, the locality relation (4.5) holds for quasi-primary states.

To deduce the result in general, we make use of (3.15) and an analogous result for $\overline{W}(\chi,z)$ which we derive below. This enables us, by differentiation, to infer locality for all L_{-1} descendents of quasi-primary states, which is sufficient by Proposition 2.7 and linearity of $\overline{W}(\chi,z)$ in χ. From (2.40) together with (4.3), we see that

$$\frac{d}{dz}\overline{W}(\overline{\chi},z) = -\overline{W}(\overline{L_{-1}\chi},z)\,, \quad (4.14)$$

so that, using $\overline{\overline{\chi}} = \chi$ and $L_{-1}\overline{\chi} = -\overline{L_{-1}\chi}$,

$$\frac{d}{dz}\overline{W}(\chi,z) = -\overline{W}(L_{-1}\chi,z)\,, \quad (4.15)$$

as required. □

Our main result is

Proposition 4.4. *If we also have the locality relation*

$$W(\chi_1,z)\overline{W}(\chi_2,\zeta) = W(\chi_2,\zeta)\overline{W}(\chi_1,z)\,, \quad (4.16)$$

and the spectrum of L_0 in the representation is strictly positive, then we may extend the conformal field theory $\mathscr{H}$ to a (hermitian) conformal field theory $\widetilde{\mathscr{H}} = \mathscr{H}\oplus\mathscr{K}$, with vertex operators defined by

$$\tilde{V}(\psi,z) = \begin{pmatrix} V(\psi,z) & 0 \\ 0 & U(\psi,z)\end{pmatrix}, \qquad \tilde{V}(\chi,z) = \begin{pmatrix} 0 & \overline{W}(\chi,z) \\ W(\chi,z) & 0\end{pmatrix}, \quad (4.17)$$

where we use the notation ψ for $(\psi,0)$ with $\psi\in\mathscr{H}$ and similarly χ for $(0,\chi)$ with $\chi\in\mathscr{K}$. The vacuum and conformal states are $(|0\rangle,0)$ and $(\psi_L,0)$ respectively, which are written $|0\rangle$ and ψ_L by this convention.

Proof. The vertex operators have the required action on the vacuum, from the actions of V and W on $|0\rangle\in\mathscr{H}$. Since the modes of $U(\psi_L,z)$ satisfy the same Virasoro algebra as the modes of $V(\psi_L,z)$, then we have the required Virasoro structure. (Also note that we have uniqueness of the $su(1,1)$ invariant state and a spectrum of L_0 which is bounded below, properties which we assumed for $\mathscr{H}$ and which carry over into this new theory.) The locality relations necessary for this to be a conformal field theory reduce to the six relations

$$V(\psi,z)V(\phi,\zeta) = V(\phi,\zeta)V(\psi,z)\,, \tag{4.18}$$

$$W(\chi,z)V(\phi,\zeta) = U(\phi,\zeta)W(\chi,z)\,, \tag{4.19}$$

$$U(\psi,z)U(\phi,\zeta) = U(\phi,\zeta)U(\psi,z)\,, \tag{4.20}$$

$$V(\psi,z)\overline{W}(\chi,\zeta) = \overline{W}(\chi,\zeta)U(\psi,z)\,, \tag{4.21}$$

$$\overline{W}(\chi_1,z)W(\chi_2,\zeta) = \overline{W}(\chi_2,\zeta)W(\chi_1,z)\,, \tag{4.22}$$

$$W(\chi_1,z)\overline{W}(\chi_2,\zeta) = W(\chi_2,\zeta)\overline{W}(\chi_1,z)\,, \tag{4.23}$$

which we already have. Note also that the hermitian structure on the U's and the V's together with reality of the representation gives a hermitian structure on the new vertex operators, with $\overline{(\psi,\chi)} = (\overline{\psi},\overline{\chi})$. □

$\mathscr{H}$ is a sub-conformal field theory of $\widetilde{\mathscr{H}}$. We have $\psi_L \in \mathscr{H}$ and $\psi_K = 0$ in the previous notation, and there is a symmetric space structure

$$\mathscr{H}\times\mathscr{H}\subset\mathscr{H}, \qquad \mathscr{H}\times\mathscr{K}\subset\mathscr{K}, \qquad \mathscr{K}\times\mathscr{H}\subset\mathscr{K}, \qquad \mathscr{K}\times\mathscr{K}\subset\mathscr{H}\,. \tag{4.24}$$

$\widetilde{\mathscr{H}}$ has an automorphism ι which acts as 1 on $\mathscr{H}$ and -1 on $\mathscr{K}$.

Proposition 4.5. *If we have another definition of reality on the space $\mathscr{K}$, with the conjugation map $\chi \mapsto \widehat{\chi}$, then, if $\widetilde{\mathscr{H}}'$ is the conformal field theory obtained by using $\widehat{\chi}$ in place of $\overline{\chi}$, $\widetilde{\mathscr{H}}'$ is isomorphic to $\widetilde{\mathscr{H}}$.*

Proof. We see from

$$f_{\overline{\chi}_1\overline{\phi}\chi_2} = f_{\widehat{\chi}_1\overline{\phi}\chi_2} = \langle\chi_1|U(\overline{\phi},1)|\chi_2\rangle\,, \tag{4.25}$$

and (4.2) that

$$\langle\overline{\chi}_1|U(\phi,1)|\overline{\chi}_2\rangle = \langle\widehat{\chi}_1|U(\phi,1)|\widehat{\chi}_2\rangle\,, \tag{4.26}$$

noting that conjugation preserves the conformal weight. Let $\widehat{\chi} = u\chi$. Then (4.26) implies that

$$u^\dagger U(\phi,1)u = U(\phi,1)\,. \tag{4.27}$$

Setting $\phi = |0\rangle$, we find that the map u is unitary. So (4.27) becomes

$$U(\phi,1)u = uU(\phi,1)\,. \tag{4.28}$$

Since L_0 commutes with both conjugation operations (by definition), it commutes with u. So we may use (2.18) to deduce from (4.28) that

$$U(\phi,z)u = uU(\phi,z)\,. \tag{4.29}$$

So, if our representation is irreducible, then u must be a multiple of the identity, by Schur's lemma, i.e., $\widehat{\chi} = w^2\overline{\chi}$ for some $w \in \mathbf{C}$. Since $\overline{\overline{\chi}} = \widehat{\widehat{\chi}} = \chi$ and conjugation is antilinear, we must have $|w| = 1$, since

$$\chi = \widehat{\widehat{\chi}} = \widehat{w^2\overline{\chi}} = w^2\overline{w^2\overline{\chi}} = w^2 w^{*2}\overline{\overline{\chi}} = |w|^4\chi\,. \tag{4.30}$$

Thus

$$\tilde{V}'(v\varrho, z) = v\tilde{V}(\varrho, z)v^{-1}\,, \tag{4.31}$$

where

$$v = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \tag{4.32}$$

is the (unitary) isomorphism. □

Proposition 4.6. *Let $\mathscr{H}$ and $\mathscr{H}'$ be two isomorphic hermitian conformal field theories with isomorphism $u : \mathscr{H} \to \mathscr{H}'$. Suppose $(\mathbf{U}, \mathscr{K})$ and $(\mathbf{U}', \mathscr{K}')$ are equivalent representations of $\mathscr{H}$ and $\mathscr{H}'$ respectively, with a unitary map ρ satisfying* (3.16). *Then*

(i) *u preserves the hermitian structure,* i.e., $u\overline{\psi} = \overline{u\psi}\ \forall \psi \in \mathscr{H}$.

(ii) $\rho W(\chi, z)u^{-1} = W'(\rho\chi, z)\ \forall \chi \in \mathscr{K}$.

(iii) *If both representations are real, with conjugation denoted by barring, $\overline{\rho\chi} = \rho\overline{\chi}\ \forall \chi \in \mathscr{K}$ (rescaling ρ if necessary).*

(iv) *If, in addition, the final locality relation* (4.23) *holds in both theories (and the representations* $\mathbf{U}$ *and* $\mathbf{U}'$ *are hermitian), we may extend $\mathscr{H}$ and $\mathscr{H}'$ to conformal field theories $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{H}}'$ respectively, as in* (4.17), *(note that* (4.23) *is not affected by the redefinition $\rho \mapsto w^{-1}\rho$) and $u \oplus \rho$ defines an isomorphism $\widetilde{\mathscr{H}} \to \widetilde{\mathscr{H}}'$.*

Proof.

(i) This follows simply by conjugating (2.64) and comparing with (2.24).

(ii) For $\phi \in \mathscr{H}'$ and $\chi \in \mathscr{K}$,

$$\begin{aligned}
\rho W(\chi, z)u^{-1}\phi &= \rho e^{zL_{-1}} U(u^{-1}\phi, -z)\chi && \text{by (3.8)} \\
&= \rho e^{zL_{-1}} \rho^{-1} U'(\phi, -z)\rho\chi && \text{by (3.16)} \\
&= e^{zL'_{-1}} U'(\phi, -z)\rho\chi && \\
&= W'(\rho\chi, z)\phi && \text{by (3.8)}\,,
\end{aligned} \tag{4.33}$$

where $L'_{-1} = \rho L_{-1}\rho^{-1}$ is the appropriate moment of the vertex operator for the conformal state in $\mathscr{K}'$, from (3.16) and the fact that $u\psi_L = \psi'_L$. Thus the result follows.

(iii) We can define a second conjugation operation on $\mathscr{K}'$ by $\widehat{\chi'} = \rho\overline{\rho^{-1}\chi'}$ for $\chi' \in \mathscr{K}'$. With respect to this conjugation, $(\mathbf{U}', \mathscr{K}')$ is still real, since

$$\widehat{\widehat{\chi'}} = \rho\overline{\rho^{-1}\widehat{\chi'}} = \rho\overline{\rho^{-1}\rho\overline{\rho^{-1}\chi'}} = \rho\overline{\overline{\rho^{-1}\chi'}} = \rho\rho^{-1}\chi' = \chi'\,, \tag{4.34}$$

$$\begin{aligned}
\widehat{L'_{-1}\chi'} &= \rho\overline{\rho^{-1}L'_{-1}\chi'} = \rho\overline{L_{-1}\rho^{-1}\chi'} \\
&= -\rho L_{-1}\overline{\rho^{-1}\chi'} = -L'_{-1}\rho\overline{\rho^{-1}\chi'} \\
&= -L'_{-1}\widehat{\chi'}\,,
\end{aligned} \tag{4.35}$$

and if $L_0'\chi' = h_{\chi'}\chi'$, then

$$L_0'\widehat{\chi}' = L_0'\rho\overline{\rho^{-1}\chi'} = \rho L_0\overline{\rho^{-1}\chi'} = \rho\overline{L_0\rho^{-1}\chi'} = \rho\overline{\rho^{-1}L_0'\chi'}$$

$$= h_{\chi'}\rho\overline{\rho^{-1}\chi'} = h_{\chi'}\widehat{\chi}' \,. \tag{4.36}$$

Also, if $\widehat{f}_{\chi_1'\phi'\chi_2'} = \langle\widehat{\chi_1'}|U'(\phi',1)|\chi_2'\rangle$, for χ_1', $\chi_2' \in \mathscr{K}'$ and $\phi' \in \mathscr{H}'$, then

$$\widehat{f}_{\chi_1'\phi'\chi_2'} = \langle\overline{\rho^{-1}\chi_1'}|\rho^{-1}U'(\phi',1)|\chi_2'\rangle = \langle\overline{\rho^{-1}\chi_1'}|U(u^{-1}\phi',1)|\rho^{-1}\chi_2'\rangle$$

$$= f_{\rho^{-1}\chi_1' u^{-1}\phi'\rho^{-1}\chi_2'} \,, \tag{4.37}$$

and the reality condition (4.2) follows from that in $\mathscr{H}$, noting that ρ and u preserve the conformal weights (since $L_0' = uL_0u^{-1}$ on $\mathscr{H}'$ and $\rho L_0\rho^{-1}$ on $\mathscr{K}'$). By the previous argument leading to (4.32), we see that $\widehat{\chi}' = w^2\overline{\chi'}$ for some $w \in \mathbf{C}$ with $|w| = 1$, i.e., we just replace ρ by $w^{-1}\rho$ to obtain $\overline{\rho\chi} = \rho\overline{\chi}$ for $\chi \in \mathscr{K}$.

(iv) We have, for $\chi \in \mathscr{K}$,

$$\begin{aligned}
u\overline{W}(\chi,z)\rho^{-1} &= (\rho W(e^{z^*L_1}z^{*-2L_0}\overline{\chi}, 1/z^*)u^{-1})^\dagger && \text{by (4.3) and } u,\ \rho \text{ unitary}\\
&= W'(\rho e^{z^*L_1}z^{*-2L_0}\overline{\chi}, 1/z^*)^\dagger && \text{by (ii)}\\
&= W'(e^{z^*L_1'}z^{*-2L_0'}\rho\overline{\chi}, 1/z^*)^\dagger &&\\
&= W'(e^{z^*L_1'}z^{*-2L_0'}\overline{\rho\chi}, 1/z^*)^\dagger &&\\
&= \overline{W}'(\rho\chi,z) && \text{by (4.4)}\,.
\end{aligned} \tag{4.38}$$

Hence, together with (ii), (3.16) and (2.64), we obtain the result. Note that the requirement $\overline{\rho\chi} = \rho\overline{\chi}$ determines ρ up to a sign, which corresponds to the automorphism ι of $\widetilde{\mathscr{H}}$ which was noted earlier. □

5. Lattice Constructions

The general theory described in Sects. 2–4 is illustrated in the case of the straight and $\mathbf{Z}_2$-twisted constructions of a conformal field theory from a lattice, which we define below. For full details of these constructions and proofs of their consistencies as meromorphic conformal field theories see [11]. They can be regarded as being analogues of constructions of lattices from binary codes, and it is for this reason that we begin this section with a discussion of codes and lattices. In Sect. 6, we shall demonstrate that the connection with codes is more fundamental than at first apparent, and in fact provides a more general framework in which to consider Frenkel, Lepowsky and Meurmans' construction of the natural module for the Monster group [7].

5.1. Codes and Lattices. Let us begin with some definitions and simple facts.

A *binary linear code* is a linear subspace $\mathscr{C}$ of the vector space $\mathbf{F}_2^n$ over the two element field $\mathbf{F}_2 = \{0,1\}$. n is referred to as the *length* of the code, and $\dim\mathscr{C}$ is its *dimension*. Elements of $\mathscr{C}$ are known as *codewords*, and the *weight* of a codeword $c \in \mathscr{C}$, wt(c), is the number of non-zero coordinates of c, i.e., $c = (c_1,\ldots,c_n)$

with $c_j = 0$ or 1 and $\mathrm{wt}(c) = c^2 = \underline{1} \cdot c$, where $\underline{1} = (1, 1, \ldots, 1)$ and we use the inner product $x \cdot y = \sum_{j=1}^{n} x_j y_j$, with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ [and the arithmetic here is not performed modulo 2!] The *dual* of the code $\mathscr{C}$ is the orthogonal space $\mathscr{C}^* = \{x \in \mathbf{F}_2^n : x \cdot y \equiv 0 \bmod 2\ \forall y \in \mathscr{C}\}$, and is also clearly a binary linear code. So we have $\dim \mathscr{C}^* = n - \dim \mathscr{C}$. A code $\mathscr{C}$ is said to be *self-dual* if $\mathscr{C} = \mathscr{C}^*$ (so that $\dim \mathscr{C} = \frac{1}{2}n$, i.e., its length must be even). Clearly $\mathscr{C}$ is self-dual if and only if $\mathscr{C} \subset \mathscr{C}^*$ and $\dim \mathscr{C} = \dim \mathscr{C}^*$. $\mathscr{C}$ is said to be *even* if c^2 is even for all $c \in \mathscr{C}$. $\mathscr{C}$ is said to be *doubly-even* if c^2 is a multiple of 4 for all $c \in \mathscr{C}$. The length of any doubly-even self-dual code has to be a multiple of 8.

An *n-dimensional Euclidean lattice* Λ is a subset of n-dimensional Euclidean space which has integral coordinates in some basis e_j, $1 \leqq j \leqq n$, i.e., $\Lambda = \{\sum_{j=1}^{n} n_j e_j : n_j \in \mathbf{Z}\}$ is the integral span of a set of n linearly independent n-dimensional Euclidean vectors. (The definition can clearly be extended to the non-Euclidean case by dropping the requirement that the inner product be positive definite.) The *length* of a vector $x \in \Lambda$ is x^2. Λ is said to be *integral* if $x \cdot y \in \mathbf{Z}$ for all $x,\ y \in \Lambda$ and *unimodular* if $\|\Lambda\|^2 \equiv \det(e_i \cdot e_j) = 1$. The *dual lattice* $\Lambda^* = \{y : x \cdot y \in \mathbf{Z}\ \forall x \in \Lambda\}$ (which is obviously a lattice). Clearly Λ is integral if and only if $\Lambda \subset \Lambda^*$. Also, we see that Λ is *self-dual*, i.e., $\Lambda = \Lambda^*$, if and only if Λ is both integral and unimodular, since $\|\Lambda^*\| = \|\Lambda\|^{-1}$. The lattice Λ is said to be *even* if x^2 is even for all $x \in \Lambda$. The dimension of an even self-dual lattice has to be a multiple of 8.

We can define a construction of a lattice from a code, known as the straight construction [12]. We start from a binary linear code $\mathscr{C}$ of length d and define a lattice $\Lambda_{\mathscr{C}}$ by

$$\Lambda_{\mathscr{C}} = \tfrac{1}{\sqrt{2}}\mathscr{C} + \sqrt{2}\mathbf{Z}^d \,. \tag{5.1}$$

We see that $\Lambda_{\mathscr{C}}$ is integral if and only if $\mathscr{C} \in \mathscr{C}^*$, and that $\Lambda_{\mathscr{C}}$ is even if and only if $\mathscr{C}$ is doubly-even. Also, the length of the vector $\frac{1}{\sqrt{2}}c \in \Lambda_{\mathscr{C}}$ corresponding to the codeword $c \in \mathscr{C}$ is half the weight of c, while the dimension of $\Lambda_{\mathscr{C}}$ is clearly just the length of $\mathscr{C}$, and the Euclidean structure is preserved. The theta function

$$\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} q^{\frac{1}{2}x^2}, \qquad q = e^{2\pi i \tau}\,, \tag{5.2}$$

of the lattice is given in terms of the weight enumerator

$$W_{\mathscr{C}}(p) = \sum_{c \in \mathscr{C}} p^{\mathrm{wt}(c)}\,, \tag{5.3}$$

of the code as

$$\Theta_{\Lambda_{\mathscr{C}}}(\tau) = \Theta_3(\tau)^d\, W_{\mathscr{C}}(\Theta_2(\tau)/\Theta_3(\tau))\,, \tag{5.4}$$

where

$$\Theta_2(\tau) = \sum_{r \in \mathbf{Z} + \frac{1}{2}} q^{r^2}, \qquad \Theta_3(\tau) = \sum_{m \in \mathbf{Z}} q^{m^2}\,. \tag{5.5}$$

It is clear that $(\Lambda_{\mathscr{C}})^* = \Lambda_{\mathscr{C}^*}$, so that $\Lambda_{\mathscr{C}}$ is self-dual if and only if $\mathscr{C}$ is self-dual. Also, $\Lambda_{\mathscr{C}}$ is unimodular (which is equivalent to saying that it has one point per unit volume) if and only if $\dim \mathscr{C} = \frac{1}{2}d$ $(= \dim \mathscr{C}^*)$. Therefore, this construction implies the correspondence between the properties listed in the first two columns of Table 1. In Sect. 5.2, we shall give a corresponding construction of a conformal field theory from a lattice which justifies the correspondence between the second and third columns of the table. [We will discuss the notion of self-duality later. Bosonic

Table 1. Comparison between codes,lattices and conformal field theories

Codes	Lattices	Conformal field theories
length	dimension	c
weight	(half) length	conformal weight
$\mathscr{C} \subset \mathscr{C}^*$	integral or $\Lambda \subset \Lambda^*$	meromorphic
$\dim \mathscr{C} = \dim \mathscr{C}^*$	unimodular	?
self-dual	self-dual	self-dual
doubly-even	even	bosonic
Euclidean	Euclidean	?
$W_{\mathscr{C}}(p)$	$\Theta_\Lambda(\tau)$	$\chi_{\mathscr{H}}(\tau)$

corresponds to evenness, since we have seen that the bosonic locality relation (2.4) when combined with the hermitian condition requires all conformal weights to be integral. The function $\chi_{\mathscr{H}}(\tau)$, the character or partition function, for the conformal field theory $\mathscr{H}$ is defined to be

$$\chi_{\mathscr{H}}(\tau) = q^{-c/24}\mathrm{tr}(q^{L_0}) = q^{-c/24}\sum_h \dim \mathscr{F}_h q^h \,, \tag{5.6}$$

where $q = e^{2\pi i\tau}$ as before, using the decomposition (2.19).]

We can divide the lattice $\Lambda_{\mathscr{C}}$ into two cosets, by defining

$$\Lambda_0(\mathscr{C}) = \tfrac{1}{\sqrt{2}}\mathscr{C} + \sqrt{2}\mathbf{Z}^d_+ \,, \tag{5.7}$$

$$\Lambda_1(\mathscr{C}) = \tfrac{1}{\sqrt{2}}\mathscr{C} + \sqrt{2}\mathbf{Z}^d_- \,, \tag{5.8}$$

where

$$\mathbf{Z}^d_+ = \{x \in \mathbf{Z}^d : x^2 \in 2\mathbf{Z}\} \,, \tag{5.9}$$

$$\mathbf{Z}^d_- = \{x \in \mathbf{Z}^d : x^2 \in 2\mathbf{Z}+1\} \,, \tag{5.10}$$

so that $\Lambda_{\mathscr{C}} = \Lambda_0(\mathscr{C}) \cup \Lambda_1(\mathscr{C})$.

A second construction of an even self-dual lattice from a doubly-even self-dual code can be obtained by first defining

$$\Lambda_2(\mathscr{C}) = \tfrac{1}{\sqrt{2}}\mathscr{C} + \tfrac{1}{2\sqrt{2}}\underline{1} + \sqrt{2}\mathbf{Z}^d_{(-)^{n+1}} \,, \tag{5.11}$$

$$\Lambda_3(\mathscr{C}) = \tfrac{1}{\sqrt{2}}\mathscr{C} + \tfrac{1}{2\sqrt{2}}\underline{1} + \sqrt{2}\mathbf{Z}^d_{(-)^{n}} \,, \tag{5.12}$$

where $d = 8n$ (d must be a multiple of 8 as noted above) and setting $\widetilde{\Lambda}_{\mathscr{C}} = \Lambda_0(\mathscr{C}) \cup \Lambda_3(\mathscr{C})$. This is known as the twisted construction. It is easily seen to be even. Thus, it must be integral, and self-duality will follow if we can show that $\widetilde{\Lambda}_{\mathscr{C}}$ is unimodular, i.e., has one point per unit volume. This is clear, since $\mathscr{C}$ is self-dual.

The classification of doubly-even self-dual codes and even self-dual lattices of length (dimension) 8, 16 and 24 is known [9, 16]. If $\mathscr{C}$ is a code of length n, and π is a permutation of the n coordinates of $\mathscr{C}$, then application of π to $\mathscr{C}$ produces a code $\mathscr{C}^\pi$ (which clearly shares all the same properties as $\mathscr{C}$). $\mathscr{C}$ and $\mathscr{C}^\pi$ are said to be *equivalent* codes. If two codes are not related in this way, they are said to be *inequivalent*. There is one doubly-even self-dual linear binary code up to equivalence of length 8, 2 of length 16 and 9 of length 24 (and 85 of length 32). The one of length 8 is called the Hamming code, and is denoted by e_8 (for a reason given below). At length 16, we can take two copies of e_8 to give $e_8 \oplus e_8$. The other code is written as d_{16}. Among the 9 length 24 codes there is one which has no codewords of weight 4. This is the Golay code, g_{24}, and so can be characterised by the fact that it is the unique doubly-even self-dual binary linear code of smallest length containing no codewords of weight 4. Its symmetry group (i.e., the group of permutations π leaving it invariant) is one of the sporadic simple groups, the Mathieu group, M_{24} (see Sect. 5.1 for a brief discussion of the classification of the finite simple groups). For the even self-dual lattices, the result is that there is one in 8 dimensions, 2 in 16 dimensions and 24 in 24 dimensions. The 8-dimensional lattice is the root lattice of E_8, written as E_8, while the 16-dimensional ones are two copies of E_8, i.e., $E_8 \oplus E_8$, and D_{16}, which is the union of the root lattice of D_{16} with one of the spinor cosets of the dual of the root lattice (the weight lattice). The 24-dimensional even self-dual lattices were classified by Niemeier in 1968. Each such lattice Λ is uniquely determined by its set of minimal vectors, $\Lambda(2) = \{\lambda \in \Lambda : \lambda^2 = 2\}$, which form root systems of the following types:

$$\begin{gathered} \emptyset,\ A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_6^4,\ A_8^3,\ A_{12}^2,\ A_{24},\ D_4^6,\ D_6^4,\ D_8^3,\ D_{12}^2,\ D_{24},\ E_6^4,\ E_8^3, \\ A_5^4 D_4,\ A_7^2 D_5^2,\ A_9^2 D_6,\ A_{15} D_9,\ E_8 D_{16},\ E_7^2 D_{10},\ E_7 A_{17},\ E_6 D_7 A_{11}\ , \end{gathered} \tag{5.13}$$

where the lattice corresponding to the empty root system is the Leech lattice, Λ_{24}. It was shown by Venkov [17] that $|\Lambda(2)| = 24h$, where $|\Lambda(2)|$ is the number of elements in $\Lambda(2)$ and h is the common dual Coxeter number of the irreducible components of the corresponding root system, and that the rank of the root system was either 0 or 24. Since the algebra must be simply laced, we can then derive the above list of possibilities. We shall denote the lattices corresponding to the non-empty root systems simply by the root system itself. The Leech lattice can, similarly to the Golay code, be characterised as the unique even self-dual lattice of smallest dimension containing no points of length 2. We also similarly obtain a sporadic simple group, Conway's group Co_1, given by $\mathrm{Aut}(\Lambda_{24})/\{\pm 1\}$, where $\mathrm{Aut}(\Lambda_{24})$ is the group of automorphisms of the Leech lattice.

Clearly, we must have $\Lambda_{e_8} = \widetilde{\Lambda}_{e_8} = E_8$. In 16 dimensions, we have $\Lambda_{e_8 \oplus e_8} = E_8 \oplus E_8$ and $\Lambda_{d_{16}} = D_{16}$. The twisted construction interchanges the two lattices, giving $\widetilde{\Lambda}_{e_8 \oplus e_8} = D_{16}$ and $\widetilde{\Lambda}_{d_{16}} = E_8 \oplus E_8$.

For the length 24 codes, we look at the points of length 2 in the lattice, and use the results of Venkov. We find $|\Lambda_{\mathscr{C}}(2)| = 48 + 16|\mathscr{C}_4|$, where $|\mathscr{C}_4|$ is the number of codewords of weight 4, and $|\widetilde{\Lambda}_{\mathscr{C}}(2)| = 8|\mathscr{C}_4|$. This, together with a computation of the number of orthogonal components into which $\Lambda(2)$ decomposes, is sufficient to identify the lattice.

The results of the two constructions in 24 dimensions are summarised in Figs. 1, 2 and 3, where we have the codes on the left, with the values of $|\mathscr{C}_4|$ noted, and the lattices on the right, with straight arrows denoting the straight construction and wavy arrows the twisted construction.

Since there are 24 lattices and only 9 codes, the two constructions can produce at most 18 of the lattices, and in fact are found to produce only 12. This is due to some overlap between the two constructions, which enables us to exhibit the

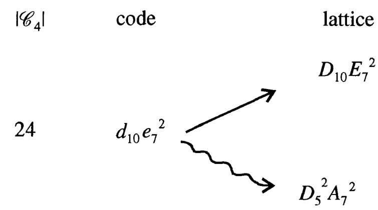

Fig. 1.

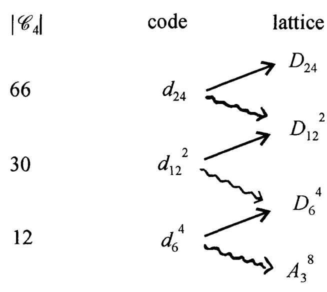

Fig. 2.

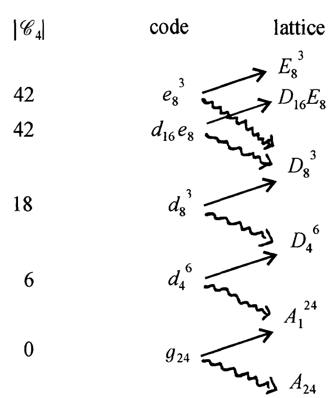

Fig. 3.

results in the form of Figs. 1–3. Note that the two exceptional structures which we discussed earlier, i.e., the Golay code and the Leech lattice, are connected by the twisted construction.

We shall construct in Sects. 5.2 and 5.3 corresponding straight and twisted constructions value of c and no states of conformal weight one, and we would expect the automophism group of this theory to be in some way connected to one of the sporadic finite simple groups. These conjectures, essentially due to FLM, are examined in Sect. 6, where we look at the results of the two constructions and also at the Monster group.

5.2. Untwisted Construction of a Conformal Field Theory. In this section, we discuss the straight or untwisted construction of a conformal field theory from a lattice, the construction being analogous to the straight construction (5.1) of a lattice from a binary linear code described in the previous section. We shall define the space of states and the corresponding vertex operators, as well as the vacuum and conformal states, and discuss the hermitian structure of the theory. Finally we shall consider the concept of self-dual theories. The proofs that this is consistent as a conformal field theory are given in [11].

We start with a Euclidean lattice Λ of dimension d, and introduce orthonormal states $|\lambda\rangle \equiv \Psi_\lambda$, $\lambda \in \Lambda$, $\langle\lambda|\lambda'\rangle = \delta_{\lambda\lambda'}$, in Dirac's notation, and oscillators a_n^j, $n \in \mathbf{Z}$, $1 \leqq j \leqq d$, satisfying the commutation relations

$$[a_m^i, a_n^j] = m\delta^{ij}\delta_{m,-n}\,, \tag{5.14}$$

and $a_n^{j\dagger} = a_{-n}^j$, $a_n^j|\lambda\rangle = 0$, $n > 0$, $p^j|\lambda\rangle = \lambda^j|\lambda\rangle$, where $p^j \equiv a_0^j$. The space of states $\mathscr{H}(\Lambda)$ is then defined to be generated by the action of the oscillators a_{-n}^j, $n > 0$, on the momentum states $|\lambda\rangle$, $\lambda \in \Lambda$. $\mathscr{H}(\Lambda)$ has a basis consisting of states of the form

$$\psi = \left(\prod_{a=1}^{M} a_{-m_a}^{j_a}\right)|\lambda\rangle\,, \tag{5.15}$$

where $\lambda \in \Lambda$ and the m_a and j_a are positive integers, $1 \leqq j_a \leqq d$.

We define the position operator q, with $q^\dagger = q$, which is a d-dimensional vector and only appears in the form $e^{i\lambda\cdot q}$, by

$$e^{i\lambda\cdot q}|\mu\rangle = |\lambda + \mu\rangle\,, \tag{5.16}$$

and define the field

$$X^j(z) = q^j - ip^j \ln z + i\sum_{n\neq 0}\frac{a_n^j}{n}z^{-n}\,, \tag{5.17}$$

which similarly only appears in an exponential or as a derivative (so that the arbitrariness in the definition of $\ln z$ is irrelevant). The vertex operator corresponding to the state (5.5) is then defined to be

$$V(\psi, z) =: \left(\prod_{a=1}^{M}\frac{i}{(m_a-1)!}\frac{d^{m_a}X^{j_a}}{dz^{m_a}}(z)\right)\exp\{i\lambda\cdot X(z)\} : \sigma_\lambda\,, \tag{5.18}$$

where the normal ordering denoted by the colons indicates that q^j is written to the left of p^j as well as the creation operators being written to the left of the annihilation operators, i.e., if

$$X_{\lessgtr}^j(z) = i\sum_{n\lessgtr 0}\frac{a_n^j}{n}z^{-n}\,, \tag{5.19}$$

then

$$:\exp\{i\lambda\cdot X(z)\}:= \exp\{i\lambda\cdot X_<(z)\}e^{i\lambda\cdot q}z^{\lambda\cdot p}\exp\{i\lambda\cdot X_>(z)\}\,. \tag{5.20}$$

The operator σ_λ is a cocycle operator, such that

$$\hat\sigma_\lambda\hat\sigma_\mu = (-1)^{\lambda\cdot\mu}\hat\sigma_\mu\hat\sigma_\lambda\,, \tag{5.21}$$

where $\hat{\sigma}_\lambda = e^{i\lambda \cdot q}\sigma_\lambda$, a property which is necessary for the bosonic locality relation (2.4), demonstrated below. The construction and properties of cocycle operators for both this construction and the twisted construction are discussed in [11].

The vacuum is the state $|0\rangle$ and the conformal state is taken to be the state

$$\psi_L = \frac{1}{2} a_{-1} \cdot a_{-1}|0\rangle . \tag{5.22}$$

We have [11]

Theorem 5.1. *If Λ is even, $\mathscr{H}(\Lambda)$, with the structure defined above, forms a conformal field theory with central charge $c = d$ and hermitian structure given by*

$$\overline{\phi} = (-1)^{L_0}\theta\phi , \tag{5.23}$$

where ϕ is a real linear combination of states of the form (5.15) (*the result can clearly be extended to include complex combinations, since we know that the map $\phi \mapsto \overline{\phi}$ is antilinear*), *and θ is given by*

$$\theta a_n^j \theta^{-1} = -a_n^j \qquad \theta\Psi_\lambda = \Psi_{-\lambda} . \tag{5.24}$$

Thus, we have a construction of a chiral bosonic conformal field theory with a hermitian structure from an even lattice. This is analogous to the construction (5.1) of a lattice from a doubly-even binary linear code. Here, the lattice plays the role of the code, and the d-dimensional Heisenberg algebra plays an analogous role to the cubic lattice $\mathbf{Z}^d$. The construction provides justification for the correspondences postulated in Table 1 between properties for lattices and conformal field theories. In particular, the dimension of the lattice becomes the value of the central charge c in the conformal field theory, while the length of a point in the lattice is related to the conformal weight of a state, i.e., the state (5.15) has conformal weight given by

$$h_\psi = \frac{1}{2}\lambda^2 + \sum_{a=1}^{M} m_a . \tag{5.25}$$

We have also seen that the bosonic locality relation holds due to the fact that the lattice is even, and we may write the partition function for the conformal field theory $\mathscr{H}(\Lambda)$ in terms of the theta function of the corresponding lattice Λ as

$$\chi_{\mathscr{H}(\Lambda)}(\tau) = \Theta_\Lambda(\tau)/\eta(\tau)^d , \tag{5.26}$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) . \tag{5.27}$$

We consider the two transformations S and T of the parameter τ, defined by

$$S(\tau) = -1/\tau, \qquad T(\tau) = \tau + 1 . \tag{5.28}$$

These generate the modular group $\Gamma = PSL(2, \mathbf{Z})$. The lattice is clearly even if and only if $\Theta_\Lambda(\tau + 1) = \Theta_\Lambda(\tau)$. Then

$$\chi_{\mathscr{H}(\Lambda)}(\tau + 1) = \chi_{\mathscr{H}(\Lambda)}(\tau)e^{i\pi d/12} . \tag{5.29}$$

Also

$$\Theta_\Lambda(-1/\tau) = (-i\tau)^{\frac{1}{2}d}\|\Lambda^*\|\Theta_{\Lambda^*}(\tau) \tag{5.30}$$

together with $\eta(-1/\tau) = (i\tau)^{\frac{1}{2}}\eta(\tau)$ gives

$$\chi_{\mathscr{H}(\Lambda)}(-1/\tau) = \|\Lambda^*\|\chi_{\mathscr{H}(\Lambda^*)}(\tau)\,. \tag{5.31}$$

Hence, we see that the partition function for the conformal field theory is invariant under the modular group Γ, i.e., under the transformations of (5.28), if the lattice is not only even but, in addition, self-dual with dimension a multiple of 24. It is invariant under the transformation S and changes by a phase under T if we only require Λ to be even and self-dual (i.e., allow d to be an arbitrary multiple of 8). [Note that the full partition function, including the anti-holomorphic factor, remains modular invariant since the phases cancel.]

Definition 5.2. *A conformal field theory is said to be* ***self-dual*** *if its character is invariant under the transformation S.*

[Note that this definition is consistent with the identification made in Table 1.]

5.3. Z_2-Twisted Construction of a Conformal Field Theory. In this section, we shall define a representation of a sub-conformal field theory of the lattice conformal field theory $\mathscr{H}(\Lambda)$. This satisfies the necessary properties, as discussed in Sect. 4, to extend the sub-conformal field theory to a new theory, which we shall call $\widetilde{\mathscr{H}}(\Lambda)$, the twisted conformal field theory, provided that Λ is even (necessary for $\mathscr{H}(\Lambda)$ to be a conformal field theory), that the dimension of Λ is a multiple of 8 and that $\sqrt{2}\Lambda^*$ is even. The requirement that $\sqrt{2}\Lambda^*$ be even comes from the verification of the final locality relation (4.23), which involves the lattice in a non-trivial way as described in [11]. Note that this is almost a modular invariance condition, i.e., almost requires self-duality of Λ, but arises from a consideration of locality on the Riemann sphere alone. This construction is the analogue of the twisted construction of a lattice from a binary code described in Sect. 5.1. Again, the relevant proofs of our results may be found in [11].

Let Λ be an even Euclidean lattice of dimension d. As noted in [11], the map θ defined by (5.24) is an involution (automorphism of order 2) of the conformal field theory $\mathscr{H}(\Lambda)$. We write

$$\mathscr{H}^{\pm}(\Lambda) = \{\psi \in \mathscr{H}(\Lambda) : \theta\psi = \pm\psi\}\,, \tag{5.32}$$

so that $\mathscr{H}(\Lambda) = \mathscr{H}^{+}(\Lambda) \oplus \mathscr{H}^{-}(\Lambda)$.

Lemma 5.3. *$\mathscr{H}^{+}(\Lambda)$ is a sub-conformal field theory of $\mathscr{H}(\Lambda)$.*

Hence $\mathscr{H}^{+}(\Lambda)$ is a conformal field theory, from Proposition 2.16. It has vacuum $|0\rangle$, conformal state ψ_L and vertex operators those of $\mathscr{H}(\Lambda)$ restricted to $\mathscr{H}^{+}(\Lambda)$ (which we shall still write as $V(\psi,z)$ for $\psi \in \mathscr{H}^{+}(\Lambda)$). $\mathscr{H}^{+}(\Lambda)$ consists of states of the form $|\lambda\rangle + |-\lambda\rangle$ acted on by an even number of creation operators and $|\lambda\rangle - |-\lambda\rangle$ acted on by an odd number of creation operators. We have observed that the d-dimensional Heisenberg algebra plays an analogous role to the lattice $\mathbf{Z}^d$ in the construction of the lattice $\Lambda_{\mathscr{C}}$ from a binary code $\mathscr{C}$, and we see that picking out $\mathscr{H}^{+}(\Lambda)$ from $\mathscr{H}(\Lambda)$ corresponds to selecting out the coset $\Lambda_0(\mathscr{C})$ (in which we restrict $\mathbf{Z}^d$ to $\mathbf{Z}^d_+$) from $\Lambda_{\mathscr{C}}$.

To obtain the lattice $\widetilde{\Lambda}_{\mathscr{C}}$, we then added in the lattice $\Lambda_3(\mathscr{C})$ which is obtained by shifting $\Lambda_0(\mathscr{C})$ by $\frac{1}{2\sqrt{2}}\underline{1}$ if d is an even multiple of 8 and $\frac{1}{2\sqrt{2}}\underline{1} + \sqrt{2}e_j$ for any j with $1 \leq j \leq d$ if d is an odd multiple of 8. So, firstly, we suspect that the corresponding construction we produce here will only make sense if d is a multiple of 8. Analogous to the shifting by $\frac{1}{2\sqrt{2}}\underline{1}$, we define a Hilbert space $\mathscr{H}_T(\Lambda)$ created by the action of half-integrally model oscillators c_r^j, $r \in \mathbf{Z}+\frac{1}{2}$, $1 \leqq j \leqq d$, satisfying

$$[c_r^i, c_s^j] = r\delta_{r,-s}\delta^{ij} \tag{5.33}$$

and $c_r^{j\dagger} = c_{-r}^j$, on an irreducible representation space $\chi(\Lambda)$ of the gamma matrix algebra $\Gamma(\Lambda) = \{\pm\gamma_\lambda : \lambda \in \Lambda\}$ with

$$\gamma_\lambda\gamma_\mu = (-1)^{\lambda\cdot\mu}\gamma_\mu\gamma_\lambda = \varepsilon(\lambda,\mu)\gamma_{\lambda+\mu}, \qquad \gamma_\lambda^2 = (-1)^{\frac{1}{2}\lambda^2}, \tag{5.34}$$

where $\lambda, \mu \in \Lambda$ and $\varepsilon(\lambda,\mu) = \pm 1$ (see [11]), with $c_r^j\chi_0 = 0$ for $r > 0$ and $\chi_0 \in \chi(\Lambda)$. The introduction of the space $\chi(\Lambda)$ is necessary, since we have no zero-moded oscillators, and so no momentum space on which to represent the cocycles, and we introduce the algebra by analogy with the cocyle operators in the straight construction.

The operators

$$L_n = \frac{1}{2}\sum_{\substack{r=-\infty \\ r\in\mathbf{Z}+\frac{1}{2}}}^{\infty} : c_r \cdot c_{n-r} : + \frac{d}{16}\delta_{n0} \tag{5.35}$$

($r, s, \ldots$ will usually denote elements of $\mathbf{Z}+\frac{1}{2}$ in this chapter) satisfy the Virasoro algebra (2.2) with $c = d$, and these turn out to be the moments of the operator corresponding to the conformal state ψ_L when we define the representation. Thus, the ground state sector $\chi(\Lambda)$ has conformal weight $d/16$, and so we see that if we wish to extend $\mathscr{H}^+(\Lambda)$ by some subspace of $\mathscr{H}_T(\Lambda)$ to give a new conformal field theory as described in Sect. 4, then we must have d a multiple of 8 as postulated above, since the conformal weights must be integral, and the conformal weight of the state

$$\chi = \left(\sum_{a=1}^{M} c_{-r_a}^{j_a}\right)\chi_0, \tag{5.36}$$

where $\chi_0 \in \chi(\Lambda)$, $1 \leqq j_a \leqq d$ and $r_a = m_a + \frac{1}{2}$ with the m_a positive integers, is

$$h_\chi = \frac{d}{16} + \sum_{a=1}^{M} r_a. \tag{5.37}$$

This also tells us that we must consider only those states with $\theta = 1$, where

$$\theta c_r^j \theta^{-1} = -c_r^j, \qquad \theta\chi_0 = (-1)^{d/8}\chi_0, \tag{5.38}$$

gives us an extension of θ defined in (5.24) from $\mathscr{H}(\Lambda)$ to $\mathscr{H}(\Lambda) \oplus \mathscr{H}_T(\Lambda)$. Define

$$\mathscr{H}_T^{\pm}(\Lambda) = \{\chi \in \mathscr{H}_T(\Lambda) : \theta_\chi = \pm\chi\}, \tag{5.39}$$

(cf. (5.32)). $\mathscr{H}_T^+(\Lambda)$ is the subspace of $\mathscr{H}_T(\Lambda)$ consisting of states with integral conformal weight. It is seen to be analogous to $\Lambda_3(\mathscr{C})$, in a similar way to the correspondence which we have noted between $\mathscr{H}^+(\Lambda)$ and $\Lambda_0(\mathscr{C})$. So, we would expect to obtain an analogue of the twisted construction of a lattice from a binary code by adjoining the twisted sector $\mathscr{H}_T^+(\Lambda)$ to $\mathscr{H}^+(\Lambda)$. This is the extension which was discussed in Sect. 4. We saw that to do so we must have $\mathscr{H}_T^+(\Lambda)$ a real, hermitian representation of $\mathscr{H}^+(\Lambda)$, satisfying an additional locality relation.

Define the field

$$R^j(z) = i \sum_{r=-\infty}^{\infty} \frac{c_r^j}{r} z^{-r}, \tag{5.40}$$

by analogy with (5.17) (note here we sum over $r \in \mathbf{Z} + \frac{1}{2}$). Then, corresponding to the state ψ given by (5.15), we define, by analogy with the definition (5.18) of the vertex operators in the straight theory,

$$V_T^0(\psi, z) =: \left(\prod_{a=1}^{M} \frac{i}{(m_a - 1)!} \frac{d^{m_a} R^{j_a}}{dz^{m_a}}(z) \right) \exp\{i\lambda \cdot R(z)\} : \gamma_\lambda, \tag{5.41}$$

where we use the obvious normal ordering, i.e.,

$$: \exp\{i\lambda \cdot R(z)\} := \exp\{i\lambda \cdot R_<(z)\} \exp\{i\lambda \cdot R_>(z)\}, \tag{5.42}$$

where

$$R^j_{\lessgtr}(z) = i \sum_{r \lessgtr 0} \frac{c_r^j}{r} z^{-r}. \tag{5.43}$$

Then set

$$V_T(\psi, z) = V_T^0(e^{A(-z)}\psi, z), \tag{5.44}$$

where

$$A(z) = \frac{1}{2} \sum_{\substack{n,m \geqq 0 \\ m+n>0}} \binom{-\frac{1}{2}}{m} \binom{\frac{1}{2}}{n} \frac{(-z)^{-m-n}}{m+n} a_m \cdot a_n - \frac{1}{2} a_0 \cdot a_0 \ln(-4z). \tag{5.45}$$

[Note that the operators L_n which we wrote down in (5.35) are actually the modes of the operator $V_T(\psi_L, z)$, so that these *will* be the Virasoro generators in the twisted sector, and the conformal weights are as stated. The normal ordered sum in (5.35) arises from $V_T^0(\psi_L, z)$ as in the untwisted case. However, $e^{A(-z)}\psi_L = \psi_L + \frac{d}{16} z^{-2}|0\rangle$, so we have the extra term $V_T^0(\frac{d}{16}z^{-2}|0\rangle, z) = \frac{d}{16}z^{-2}$, which accounts for the shift in L_0.]

Let M be a symmetric, unitary matrix satisfying

$$M\gamma_\lambda{}^* = \gamma_\lambda M. \tag{5.46}$$

Then the main result of [11] is

Theorem 5.4. *The operators $V_T(\psi, z)$ define a real, hermitian representation of the conformal field theroy $\mathscr{H}^+(\Lambda)$, with the conjugation map on the twisted sector $\mathscr{H}_T^+(\Lambda)$ given by*

$$\overline{\chi}_u = (-1)^{L_0} \theta \chi_{Mu^*}, \tag{5.47}$$

for

$$\chi_u = \left(\prod_{a=1}^{M} c_{-r_a}^{j_a} \right) u, \tag{5.48}$$

the extension to all twisted states following by antilinearity. Further, if $\sqrt{2}\Lambda^*$ is even, $\widetilde{\mathscr{H}}(\Lambda) = \mathscr{H}^+(\Lambda) \oplus \mathscr{H}_T^+(\Lambda)$ is a hermitian conformal field theory, which is self-dual if Λ is self-dual.

[Note that it has been shown in [23] that the condition that $\sqrt{2}\Lambda^*$ be even is also *necessary* for consistency of the conformal field theory.]

6. Results of the Constructions and Connections with the Monster

6.1. The Monster Module. The result of the recently completed classification of finite simple groups [10] is that there are 16 infinite families of groups of Lie type, the alternating groups on n elements for $n \geqq 5$ and 26 so-called sporadic simple groups, which do not fit into any systematic classification. One of the sporadic groups is the Mathieu group, M_{24}, the symmetry group of the Golay code, as discussed in Sect. 5.1. Conway's group Co_0 is the automorphism group of the Leech lattice. It involves, as quotients of subgroups, 12 sporadic simple groups including the Mathieu group. In 1973, Fischer and Griess predicted independently the existence of what would turn out to be the largest of the sporadic groups, the Monster, F_1, (which turns out to involve 19 of the other sporadic groups) which would have order $2^{46}3^{20}5^9 7^7 11^2 13^3 17.19.23.29.31.41.47.59.71 \approx 8.10^{53}$. It was observed by Griess, Conway and Norton that the smallest non-trivial irreducible representation of the Monster would have dimension $d_1 \geqq 196883$. Norton showed that this representation would have the structure of a real commutative non-associative algebra, and Griess [8] explicitly constructed such an algebra of dimension 196883 and verified enough of its symmetries to prove the existence of the Monster (and also that $d_1 = 196883$). (Tits subsequently showed that the Monster is the full automorphism group of the Griess algebra.) However, the construction of Griess is inelegant. From the point of view which we have been pursuing in this work, i.e., the analogies between codes, lattices and conformal field theories, we would expect, as was stated towards the end of Sect. 5.1, the automorphism group of the conformal field theory obtained by the twisted construction from the Leech lattice to be somehow related to a sporadic simple group, since the Golay code was related to the Mathieu group and yielded the Leech lattice under the twisted construction, which was related to Conway's group Co_1.

Further evidence for this point of view is provided by the theory of modular functions. The modular group

$$\Gamma = PSL(2,\mathbf{Z}) = SL(2,\mathbf{Z})/\langle \pm 1 \rangle \tag{6.1}$$

has an action on the upper half complex plane H given by

$$g \cdot \tau = \frac{a\tau + b}{c\tau + d} \quad \text{for } g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \tau \in H\,. \tag{6.2}$$

Dedekind and, independently, Klein produced a function $j(\tau)$ on H invariant under Γ. Set $q = e^{2\pi i \tau}$. Then

$$j(\tau) = \frac{\Theta_{E_8^3}(\tau)}{\eta(\tau)^{24}}\,. \tag{6.3}$$

Lemma 6.1. *The modular functions* (*the meromorphic modular-invariant functions on* $H \cup \{i\infty\}$) *are given by the field of rational functions of* $j(\tau)$. *Up to an additive constant,* $j(\tau)$ *is the unique such function having a simple pole at* $i\infty$ *with residue* 1 *in* q.

We see from (5.26) that $j(\tau)$ is the partition function for the conformal field theory $\mathscr{H}(E_8^3)$. We showed there that $\chi_{\mathscr{H}(\Lambda)}(\tau)$ was modular invariant for Λ an even self-dual lattice of dimension a multiple of 24. So, by the above lemma, taking another even self-dual 24-dimensional lattice in place of E_8^3 in (6.3) will give $j(\tau)$ up to an additive constant. The constant term in the expansion of the character as a power series in q is the number of states of conformal weight 1 in the theory, which is at least 24 since we always have the states $a_{-1}^j|0\rangle$ for $1 \leqq j \leqq 24$. For the Leech lattice, these are the only such states, but for other lattices we have the states $|\lambda\rangle$ for λ a lattice point of length 2. We set $J(\tau)$ to be $j(\tau)$ with zero constant term, giving

$$J(\tau) = q^{-1} + 0 + 196884q + 21493760q^2 + \cdots \equiv \sum_n a_n q^n , \tag{6.4}$$

where $a_n \geqq 0$ for all $n \in \mathbf{Z}$. It was noticed by McKay that

$$a_1 = d_0 + d_1 , \tag{6.5}$$

where $d_0 = 1$ can be interpreted as the dimension of the trivial representation of the Monster. McKay and Thompson [18] soon noticed

$$a_2 = d_0 + d_1 + d_2 , \tag{6.6}$$

where d_2 is the dimension of the next largest irreducible Monster module, and similarly for other terms. It was conjectured from this that there exists a natural *infinite-dimensional* representation of the *finite-dimensional* Monster

$$V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots , \tag{6.7}$$

such that $\dim V_n = a_n$, $n = -1, 1, 2, \ldots$. From (6.3), it would be imagined that the natural choice for $V^\natural$ would be the conformal field theory $\mathscr{H}(\Lambda)$ associated by the straight construction with a 24-dimensional even self-dual lattice Λ. However, as mentioned above, none of these provide zero constant term in their characters. Inspired by the analogies to codes and lattices as we have mentioned, the obvious thing to consider is the twisted construction $\widetilde{\mathscr{H}}(\Lambda_{24})$ from the Leech lattice. The weight one states in the twisted theory $\widetilde{\mathscr{H}}(\Lambda)$ are given in 24 dimensions by $|\lambda\rangle + |-\lambda\rangle$ for $\lambda \in \Lambda$ a vector of length 2, since the states $a_{-1}^j|0\rangle$ are projected out by taking the $\theta = 1$ subspace. (There is no contribution from the twisted sector, since in 24 dimensions the twisted ground state has conformal weight $\frac{3}{2}$, and so the smallest weight of a twisted state is 2 for the states $c_{-\frac{1}{2}}^j \chi_0$, $\chi_0 \in \chi(\Lambda)$, $1 \leqq j \leqq 24$.) So $\widetilde{\mathscr{H}}(\Lambda_{24})$ has no weight one states and the character is $J(\tau)$ as required, i.e., it is conjectured that $\widetilde{\mathscr{H}}(\Lambda_{24})$ provides the natural module $V^\natural$ for the Monster, with V_n the space of states in the conformal field theory of weight $n+1$.

This conjecture was proved by Frenkel, Lepowsky and Meurman [6, 7]. The basic idea of their work is to construct an involution σ known as a triality operator of the conformal field theory which extends the natural action of $\mathrm{Aut}(\Lambda_{24})$ on the theory to the Monster. What we shall show in Sect. 6 and 7 is that this triality can be understood in a more general context than the specific case considered for the Monster, for which properties special to this case were used. In the remainder of this section we shall discuss in more detail how the Monster arises as the automorphism group of $\widetilde{\mathscr{H}}(\Lambda_{24})$. In Sect. 6.2, we discuss the results of the straight and twisted constructions of conformal field theories, and in particular the coincidences between the straight and twisted theories.

$\mathrm{Aut}(\Lambda)$ is the group of automorphisms of the d-dimensional even self-dual lattice Λ, i.e.,

$$\mathrm{Aut}(\Lambda) = \{R \in SO(d) : R\lambda \in \Lambda \ \forall \lambda \in \Lambda\} . \tag{6.8}$$

Its centre is $\mathbf{Z}_P \equiv \{\pm 1\} \cong \mathbf{Z}_2$. As already remarked, in the case of the Leech lattice the automorphism group is Co_0, and the group $\mathrm{Aut}(\Lambda_{24})/\mathbf{Z}_P$ is Conway's group Co_1. Co_1 is a simple group of order $2^{21}3^9 5^4 7^2 11.13.23 \approx 8.10^{18}$. We have a representation of $\mathrm{Aut}(\Lambda)$ on the Hilbert space $\mathscr{H}(\Lambda)$ given by $R \mapsto u_R$ with

$$u_R a_n^j u_R^{-1} = R_{ij} a_n^i, \qquad u_R|\lambda\rangle = |R\lambda\rangle , \tag{6.9}$$

for $R \in \mathrm{Aut}(\Lambda)$. However, this is not a group of automorphisms of the conformal field theory because we must consider the cocycle operators.

We have from [19] that if γ_λ and γ'_λ are irreducible representations of gamma matrices with the same symmetry factor ($(-1)^{\lambda \cdot \mu}$ in this case) then there exists a unitary transformation S such that

$$S\gamma_\lambda S^{-1} = v(\lambda)\gamma'_\lambda , \tag{6.10}$$

where $v(\lambda) = \pm 1$ and

$$\varepsilon'(\lambda,\mu) = \frac{v(\lambda+\mu)}{v(\lambda)v(\mu)}\, \varepsilon(\lambda,\mu) . \tag{6.11}$$

Take $\gamma'_\lambda = \gamma_{R\lambda}$ for some $R \in \mathrm{Aut}(\Lambda)$. Let

$$\hat{C}(\Lambda) = \{(R,S) : R \in \mathrm{Aut}(\Lambda),\ S\gamma_\lambda S^{-1} = v_{R,S}(\lambda)\gamma_{R\lambda}\} , \tag{6.12}$$

for some $v_{R,S}(\lambda) = \pm 1$. Note that in the trivial case $\gamma'_\lambda = \gamma_\lambda$, $v(\lambda+\mu) = v(\lambda)v(\mu)$ and $S \in \Gamma(\Lambda)$. Thus, the kernel of the homomorphism $(R,S) \mapsto R$ is $\Gamma(\Lambda)$, and so we have the exact sequence

$$1 \to \Gamma(\Lambda) \to \hat{C}(\Lambda) \to \mathrm{Aut}(\Lambda) \to 1 . \tag{6.13}$$

This provides a group of automorphisms of $\mathscr{H}(\Lambda) \oplus \mathscr{H}_T(\Lambda)$ given by

$$u_{R,S} a_n^j u_{R,S}^{-1} = R_{ij} a_n^i, \qquad u_{R,S} c_r^j u_{R,S}^{-1} = R_{ij} c_r^i ,$$

$$u_{R,S}|\lambda\rangle = v_{R,S}(\lambda)|R\lambda\rangle, \qquad u_{R,S}\chi = S\chi . \tag{6.14}$$

Since we have $\gamma_\lambda = \gamma_{-\lambda}$, then $\iota_P = (-1,1) \in \hat{C}(\Lambda)$ acts trivially on $\widetilde{\mathscr{H}}(\Lambda)$. Thus, we have a group of automorphisms $C(\Lambda) \cong \hat{C}(\Lambda)/\hat{\mathbf{Z}}_P$, where $\hat{\mathbf{Z}}_P \equiv \{(\pm 1, 1)\} \cong \mathbf{Z}_2$, and a homomorphism $(\pm R, S) \mapsto R$ of $C(\Lambda) \to \mathrm{Aut}(\Lambda)/\mathbf{Z}_P$ with kernel $\Gamma(\Lambda)$, giving us the exact sequence

$$1 \to \Gamma(\Lambda) \to C(\Lambda) \to \mathrm{Aut}(\Lambda)/\mathbf{Z}_P \to 1 . \tag{6.15}$$

In the case of the Leech lattice, we obtain

$$1 \to \Gamma(\Lambda_{24}) \to C(\Lambda_{24}) \to Co_1 \to 1 , \tag{6.16}$$

and $C(\Lambda_{24})$ together with the triality operator (involution) σ generates the Monster as a group of automorphisms of $\widetilde{\mathscr{H}}(\Lambda_{24})$. $C(\Lambda_{24})$ is the centraliser of the involution

$\iota = (1,-1) \in \Gamma$ inside the Monster, i.e., it preserves the "fermion number," or in other words it maps states in the straight sector into the straight sector and similarly for the twisted sector, while the triality operator mixes the straight and twisted sectors.

The group $\Gamma(\Lambda)$ is an extra-special 2-group, i.e., $|\Gamma(\Lambda)| = 2^{d+1}$ with centre $\mathbf{Z}_2$ and $\Gamma(\Lambda)/\mathbf{Z}_2 = \Lambda/2\Lambda \cong \mathbf{Z}_2^d$.

On the other hand, it is ι which has trivial action when $\hat{C}(\Lambda)$ acts on $\mathscr{H}(\Lambda)$. So we have a representation of $Co(\Lambda) = \hat{C}(\Lambda)/\mathbf{Z}_\iota$, where $\mathbf{Z}_\iota = \{1, \iota\} \cong \mathbf{Z}_2$. The homomorphism $(R, \pm S) \mapsto R$ leads to the exact sequence

$$1 \to \Lambda/2\Lambda \to Co(\Lambda) \to \mathrm{Aut}(\Lambda) \to 1 \,. \tag{6.17}$$

6.2. Results of the Straight and Twisted Lattice Constructions. In this section, we shall describe the results of the straight and twisted constructions of a conformal field theory from an even self-dual lattice in 8, 16 and 24 dimensions.

Let us begin with a few standard results which are relevant.

Proposition 6.2. *Let ψ^a, $1 \leqq a \leqq N$, be an orthogonal real basis for the weight one states of a conformal field theory $\mathscr{H}$,* i.e., *$\overline{\psi^a} = \psi^a$ and $\langle \psi^a | \psi^b \rangle = k\delta^{ab}$. Set*

$$V(\psi^a, z) = T^a(z) \equiv \sum_n T_n^a z^{-n-1} \,. \tag{6.18}$$

Then these modes obey the affine Kac–Moody algebra

$$[T_m^a, T_n^b] = i f^{abc} T_{m+n}^c + km\delta^{ab}\delta_{m,-n} \,. \tag{6.19}$$

Proof. The operator product expansion (2.52) gives

$$T^a(z)T^b(\zeta) = k\delta^{ab}(z-\zeta)^{-2} + i f^{abc} T^c(\zeta)(z-\zeta)^{-1} + O(1) \,, \tag{6.20}$$

where $T_0^a \psi^b = i f^{abc} \psi^c$, since this must be a state of weight one, and also $T_1^a \psi^b = \lambda|0\rangle$ similarly, where $\lambda = \langle 0 | T_1^a | \psi^b \rangle$, i.e., $\lambda^* = \langle \psi^b | V_{-1}(\psi^a) | 0 \rangle$ by (2.54) if the states ψ^a are quasi-primary. So $\lambda^* = \langle \psi^b | \psi^a \rangle$ by (2.50), and so $\lambda = k\delta^{ab}$. We have that the weight one states ψ^a are quasi-primary, since $L_1 \psi^a = \lambda^a |0\rangle$ say, so $\lambda^a = \langle 0 | L_1 | \psi^a \rangle$, i.e., $\lambda^{a*} = \langle \psi^a | L_{-1} | 0 \rangle = 0$, since $|0\rangle$ is $su(1,1)$ invariant. The usual contour manipulation argument then shows that (6.20) is equivalent to (6.19). □

The zero modes define a compact Lie algebra with structure constants f^{abc}, i.e., we have a continuous group of automorphisms of the conformal field theory. We shall denote this Lie algebra by $g_{\mathscr{H}}$.

Proposition 6.3. *For Λ an even lattice, the affine algebra of $\mathscr{H}(\Lambda)$ in Proposition 6.2 is the affinization $\hat{g}_\Lambda$ of the Lie algebra g_Λ with root system $\Lambda(2)$. In particular, $g_{\mathscr{H}(\Lambda)} = g_\Lambda$.*

Proof. The weight one states are given by $a_{-1}^j|0\rangle$ and $|\lambda\rangle$ for $\lambda \in \Lambda$ a vector of length 2. The appropriate operator products may easily be evaluated, for example

$$\begin{aligned} V(\lambda, z)V(\mu, \zeta) = {} & e^{i\lambda \cdot X_<(z)} e^{i\lambda \cdot q} z^{\lambda \cdot p} e^{i\lambda \cdot X_>(z)} \sigma_\lambda \\ & \cdot e^{i\mu \cdot X_<(\zeta)} e^{i\mu \cdot q} \zeta^{\mu \cdot p} e^{i\mu \cdot X_>(\zeta)} \sigma_\mu \,. \end{aligned} \tag{6.21}$$

The σ_λ commutes through to the right and we use $\sigma_\lambda \sigma_\mu = \varepsilon(\lambda, \mu)\sigma_{\lambda+\mu}$. Moving $z^{\lambda \cdot p}$ past $e^{i\mu \cdot q}$ produces a factor $z^{\lambda \cdot \mu}$, while

$$e^{i\lambda \cdot X_>(z)} e^{i\mu \cdot X_<(\zeta)} = e^{i\mu \cdot X_<(\zeta)} e^{i\lambda \cdot X_>(z)} e^{[i\lambda \cdot X_>(z), i\mu \cdot X_<(\zeta)]}, \tag{6.22}$$

where the commutator is given by

$$[i\lambda \cdot X_>(z), i\mu \cdot X_<(\zeta)] = -\sum_{n>0} \lambda \cdot \mu \frac{1}{n}\left(\frac{\zeta}{z}\right)^n = \lambda \cdot \mu \ln\left(1 - \frac{\zeta}{z}\right), \tag{6.23}$$

from (5.19) and (5.14). Hence we obtain

$$\begin{aligned} V(\lambda, z)V(\mu, \zeta) = (z-\zeta)^{\lambda \cdot \mu} \varepsilon(\lambda, \mu) e^{i\lambda \cdot X_<(z) + i\mu \cdot X_<(\zeta)} e^{i(\lambda+\mu) \cdot q} \\ \cdot z^{\lambda \cdot p} \zeta^{\mu \cdot p} e^{i\lambda \cdot X_>(z) + i\mu \cdot X_>(\zeta)} \sigma_{\lambda+\mu} \, . \end{aligned} \tag{6.24}$$

Since λ, μ have length 2, $\lambda \cdot \mu = \pm 2,\ \pm 1, 0$. So we only obtain singular terms in the operator product for $\lambda \cdot \mu = -1$ and $\lambda \cdot \mu = -2$ (i.e., $\mu = -\lambda$). In the first case,

$$V(\lambda, z)V(\mu, \zeta) = \frac{\varepsilon(\lambda, \mu)}{z - \zeta} V(\lambda + \mu, \zeta) + O(1)\,, \tag{6.25}$$

while in the second case

$$V(\lambda, z)V(-\lambda, \zeta) = \frac{\varepsilon(\lambda, -\lambda)}{(z-\zeta)^2}\left[1 + (z-\zeta)i\lambda \cdot \frac{d}{d\zeta}X(\zeta)\right] + O(1)\,, \tag{6.26}$$

Taylor expanding the right-hand side of (6.24) about $z = \zeta$. The vertex operator for the state $\varepsilon \cdot a_{-1}|0\rangle$ is $i\varepsilon \cdot \frac{d}{d\zeta}X(\zeta)$, which has modes a_n. Similarly to the above, we obtain

$$i\varepsilon \cdot \frac{d}{dz}X(z)V(\lambda, \zeta) = \frac{\varepsilon \cdot \lambda}{z - \zeta} V(\lambda, \zeta) + O(1)\,. \tag{6.27}$$

In commutator form, the algebra of the weight one states is, in the gauge in which $\varepsilon(\lambda, -\lambda) = 1$,

$$\begin{aligned} [V_n(\lambda), V_m(\mu)] &= \begin{cases} \varepsilon(\lambda, \mu)V_{m+n}(\lambda + \mu) & \lambda \cdot \mu = -1 \\ \lambda \cdot a_{n+m} + n\delta_{n,-m} & \lambda \cdot \mu = -2\,, \\ 0 & \lambda \cdot \mu \geqq 0 \end{cases} \\ [\varepsilon \cdot a_n, V_m(\lambda)] &= \varepsilon \cdot \lambda V_{n+m}(\lambda)\,, \\ [\varepsilon \cdot a_n, \eta \cdot a_m] &= \varepsilon \cdot \eta \delta_{n,-m}\,. \end{aligned} \tag{6.28}$$

This is recognisable as $\hat{g}_\Lambda$ as required. □

If we denote [20] the space of conformal fields corresponding to states of conformal weight at most n by W_n, then we have a map

$$W_n \times W_m \longrightarrow W_{n+m-1}\,, \tag{6.29}$$

given by the singular part of the operator product expansion. Thus for $n = m = 1$, the operator algebra, for the modes of the vertex operators given by taking moments with the usual contour manipulation argument, as we have seen above, closes. In the case $n = m = 2$, we see that it does not close in general, but we can follow FLM and define a new product such that this *cross-bracket* algebra does close on W_2. We remove the term corresponding to a state of conformal weight 3 in the operator product expansion for two states of conformal weight 2 by multiplying by $(z-\zeta)$ to obtain

$$(z-\zeta)V(\psi,z)V(\phi,\zeta) = \sum_{n=0}^{2}(z-\zeta)^{n-3}V(V(\psi)_{2-n}\phi,\zeta) + O(1)\,. \tag{6.30a}$$

For the modes of the corresponding vertex operators, this is

$$\psi_m \times \phi_n = [\psi_{m+1}, \phi_n] - [\psi_m, \phi_{n+1}]\,, \tag{6.30b}$$

i.e., it is composed of two brackets which "cross."

In the case of the theory $\widetilde{\mathscr{H}}(\Lambda_{24})$, there are no weight one states, and so we do not have the continuous group of automorphisms corresponding to these fields, nor indeed any continuous automorphisms [7] (see also [27]), but only discrete automorphisms which, as we have stated in Sect. 6.1, close to form the Monster group. We have an algebra on the space of states V_1 of conformal weight 2 (cf. the decomposition (6.7)) given by $\psi \times \phi = V_0(\psi)\phi$, for $\psi, \phi \in V_1$, of which (6.30) is the commutative affinization [7] $\frac{1}{2}\psi_L$ acts as the identity element. The algebra is commutative and non-associative. (Commutativity follows from the mode expansion of (2.13), remembering that there are no states of conformal weight one.) We may call this the Griess algebra. In fact, it is a slight modification of the algebra originally defined by Griess [8], incorporating a natural identity element [7]. Note that Tits' proof [21] that the Monster is the full automorphism group of the Griess algebra, together with the observation that the modes of the vertex operators in W_2 generate $V^\natural$, implies that the Monster is in fact the full automorphism group of the conformal field theory.

This completes the necessary review of the standard concepts which are relevant to this section.

If the rank of $g_{\mathscr{H}}$, the Lie algebra corresponding to the affine algebra generated by the weight one fields in a conformal field theory $\mathscr{H}$, is equal to the value of the central charge c of the theory, then we have c weight one fields $P^j(z)$ corresponding to states ψ^j (i.e., $P^j(z) = V(\psi^j, z)$) corresponding to a Cartan subalgebra of $g_{\mathscr{H}}$. We choose the states ψ^j to be real and orthonormal, and the moments of $P^j(z)$, i.e.,

$$P^j(z) = \sum_n a_n^j z^{-n-1}\,, \tag{6.31}$$

satisfy

$$[a_m^i, a_n^j] = m\delta_{m,-n}\delta^{ij}\,, \tag{6.32}$$

(which we see immediately from (6.19)).

Proposition 6.4. *The simultaneous eigenvalues of the* $p^j \equiv a_0^j$ *form an even lattice* Λ, *and if* $\dim \Lambda = c$, *then* $\mathscr{H} \cong \mathscr{H}(\Lambda)$.

Proof. We have that the modes L_n' of

$$L'(z) = \frac{1}{2} : P(z) \cdot P(z) := \sum_n L_n' z^{-n-2}\,, \tag{6.33}$$

where the normal ordering is defined as in Sect. 5.2, satisfy the Virasoro algebra (2.2) with central charge c, i.e.,

$$[L'_m, L'_n] = (m-n)L'_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \,. \tag{6.34}$$

Also, $P^j(z)|0\rangle = e^{zL_{-1}}a^j_{-1}|0\rangle$ from (2.3) implies that $a^j_n|0\rangle = 0$ for $n \geqq 0$. This gives

$$L'(z)|0\rangle \to L'_{-2}|0\rangle = \frac{1}{2}a_{-1}\cdot a_{-1}|0\rangle \equiv \psi'_L \,, \tag{6.35}$$

as $z \to 0$. Also note $L'_n|0\rangle = 0$ for $n \geqq -1$ and $L'^{\dagger}_n = L'_{-n}$ (since $a^{j\dagger}_n = a^j_{-n}$, as the states corresponding to the $P^j(z)$ were chosen to be real and we use (2.54)).

The weight one states are Virasoro primary states, i.e., $L_n\phi = 0$ for $n > 0$ if ϕ is a state of conformal weight one (this is obvious for $n \geqq 2$ as the conformal weights are non-negative. For $n = 1$, the result follows by the argument used in Proposition 6.1 to show that the states ψ_a are annihilated by L_1). Thus (2.58) holds for all $n \in \mathbf{Z}$, and we may deduce for $V(\phi, z) = P^j(z)$ by taking modes that

$$[L_m, \alpha^j_n] = -na^j_{m+n} \,. \tag{6.36}$$

We may thus deduce that

$$[L_{-1}, L'(z)] = \frac{d}{dz}L'(z) \,. \tag{6.37}$$

Equations (6.37) and (6.34), together with the fact that $L'(z)$ is clearly local with respect to the vertex operators (since the $P^j(z)$ are), shows that, by the uniqueness theorem, $L'(z) = V(\psi'_L, z)$.

Equation (6.36) also implies that

$$[L_m, L'_n] = (m-n)L'_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \,, \tag{6.38}$$

so that, setting $\tilde{L}_n = L_n - L'_n$, (6.34) and (6.38) imply

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} \,. \tag{6.39}$$

Set $\tilde{L}(z) = \sum_n \tilde{L}_n z^{-n-2} = L(z) - L'(z) = V(\psi_L - \psi'_L, z)$. But $\|\psi_L - \psi'_L\|^2 = \langle 0|\tilde{L}_2\tilde{L}_{-2}|0\rangle = \langle 0|4\tilde{L}_0|0\rangle + \langle 0|\tilde{L}_{-2}\tilde{L}_2|0\rangle$ by (6.39), i.e., $\|\psi_L - \psi'_L\|^2 = 0$ by $L_n|0\rangle = L'_n|0\rangle = 0$ for $n \geqq -1$. Thus $\psi_L = \psi'_L$ and so $L'_n = L_n$.

Since the action of a^j_n on a state decreases the L_0 eigenvalue by n, we can deduce from the non-negative spectrum of L_0 that the space may be built up from states Ψ^k_K satisfying

$$p^j\Psi^k_K = K^j\Psi^k_K, \qquad \langle\Psi^k_K|\Psi^{k'}_K\rangle = \delta^{kk'}, \qquad a^j_n\Psi^k_K = 0 \quad \text{for } n > 0 \,, \tag{6.40}$$

by the action of a^j_n for $n < 0$. (The states have been decomposed into simultaneous eigenstates of the commuting hermitian operators $p^j \equiv a^j_0$.) The k on Ψ^k_K is a degeneracy label, which will be shown below to be unnecessary. The space $\mathscr{H}$ decomposes into a direct sum of spaces $\mathscr{H}^k_K$ for $K \neq 0$ (generated from Ψ^k_K by the action of the creation operators a^j_n) and $\mathscr{H}_0$ generated by the creation operators from the vacuum (there is no degeneracy label due to our uniqueness assumption about the state of conformal weight zero).

From the OPE (2.52) (noting that Ψ_K^k has conformal weight $\frac{1}{2}K^2$),

$$\begin{aligned} P^j(z)V(\Psi_K^k,\zeta) &= \sum_{n=0}^{\infty} V(a^j_{\frac{1}{2}K^2-n}\Psi_K^k,\zeta)(z-\zeta)^{n-\frac{1}{2}K^2-1} \\ &= \sum_{n=0}^{\infty} V(a^j_{-n}\Psi_K^k,\zeta)(z-\zeta)^{n-1} \\ &= (z-\zeta)^{-1}K^jV(\Psi_K^k,\zeta)+O(1)\,, \end{aligned} \tag{6.41}$$

so that

$$[p^j,V(\Psi_K^k,\zeta)] = K^jV(\Psi_K^k,\zeta)\,, \tag{6.42}$$

i.e., acting with $V(\Psi_K^k,\zeta)$ on a state with p^j eigenvalue λ^j maps it to a state with p^j eigenvalue λ^j+K^j (if the state is non-zero).

Now, since Ψ_K^k is quasi-primary and has weight $\frac{1}{2}K^2$, $V(\overline{\Psi_K^k},z)=z^{-K^2}V(\Psi_K^k,1/z^*)^\dagger$. Equation (6.42) then implies that $V(\overline{\Psi_K^k},z)$ lowers the eigenvalues of p^j by K^j. So $\overline{\Psi_K^k}$, given by the action of $V(\overline{\Psi_K^k},z)$ on $|0\rangle$, must be a state with p^j eigenvalue $-K^j$. Also, since from Proposition 2.9 it has the same conformal weight as Ψ_K^k, it must be a state of the form $\sum_{k'}a_{k'}\Psi_{-K}^{k'}$, for some $a_{k'}\in\mathbf{C}$. We choose the degeneracy labels such that $\overline{\Psi_K^k}=\Psi_{-K}^k$. (Note that the map $\psi\mapsto\overline{\psi}$ is invertible (as it preserves norms, from Proposition 2.9) and so there are the same number of degeneracy labels in the $-K$ eigenspace as in the K eigenspace.)

Now, from the OPE,

$$V(\overline{\Psi_K^k},z)V(\Psi_K^{k'},\zeta)\sim\lambda(z-\zeta)^{-K^2}\,, \tag{6.43}$$

where $\lambda=\delta^{kk'}$ from (2.48).

Consider

$$V(\Psi_K^j,z)V(\Psi_{-K}^k,\zeta)|\Psi_K^l\rangle\,. \tag{6.44}$$

We see from (6.42) that $V(\Psi_{-K}^k,\zeta)|\Psi_K^l\rangle\in\mathscr{H}_0$, and so $V(\Psi_K^j,z)V(\Psi_{-K}^k,\zeta)|\Psi_K^l\rangle\in\mathscr{H}_K^j$. However, (6.43) implies that, as $z\to\zeta$, (6.44) goes like $(z-\zeta)^{-K^2}\delta^{jk}\Psi_K^l$. Thus, we see that there can be only one degeneracy label, and we drop them from now on.

The momenta K form a lattice also, since the 4-point function

$$\langle 0|V(\Psi_{-K},z_1)V(\Psi_{-L},z_2)V(\Psi_L,z_3)V(\Psi_K,z_4)|0\rangle$$

must be non-zero (see below for the details), implying that $V(\Psi_L,z)V(\Psi_K,\zeta)|0\rangle$ is non-zero, a momentum eigenstate with momentum $K+L$. The requirement of integral conformal weights fixes this lattice Λ to be even, and so we have the desired Fock space structure. However, we must still verify that we have the isomorphism $\mathscr{H}\cong\mathscr{H}(\Lambda)$ of conformal field theories. In particular, we need to consider the cocycle structure.

From the additivity of momenta, we see that we must have the OPE

$$V(\Psi_\lambda,z)V(\Psi_\mu,\zeta)=\varepsilon(\lambda,\mu)(z-\zeta)^{\lambda\cdot\mu}V(\Psi_{\lambda+\mu},\zeta)+\ldots\,, \tag{6.45}$$

where $\varepsilon(\lambda,\mu)$ is not necessarily non-zero. (We shall show below that it is in fact of unit modulus.) Consider the 4-point function

$$\langle 0|V(\Psi_{-\mu},z_1)V(\Psi_{-\lambda},z_2)V(\Psi_\lambda,z_3)V(\Psi_\mu,z_4)|0\rangle = (z_1-z_4)^{-\mu^2}(z_2-z_3)^{-\lambda^2}$$
$$\cdot\,(z_1-z_2)^{\lambda\cdot\mu}(z_1-z_3)^{-\lambda\cdot\mu}(z_2-z_4)^{-\lambda\cdot\mu}(z_3-z_4)^{\lambda\cdot\mu}G(x)\,, \tag{6.46}$$

using $su(1,1)$ invariance, with

$$x \equiv \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \tag{6.47}$$

and $G(x)$ having potential poles at $x=0,1$ and ∞. We consider the limits $z_1\to z_4$, $z_2\to z_3$; $z_1\to z_2$, $z_3\to z_4$; $z_1\to z_3$, $z_2\to z_4$. The first of these gives, on using (6.43),

$$\langle 0|V(\Psi_{-\mu},z_1)V(\Psi_{-\lambda},z_2)V(\Psi_\lambda,z_3)V(\Psi_\mu,z_4)|0\rangle \sim (z_1-z_4)^{-\lambda^2}(z_2-z_3)^{-\mu^2}\,, \tag{6.48}$$

and we deduce that $G(x)$ is regular at infinity.

Considering the second limit and using (6.45) gives

$$\langle 0|V(\Psi_{-\mu},z_1)V(\Psi_{-\lambda},z_2)V(\Psi_\lambda,z_3)V(\Psi_\mu,z_4)|0\rangle$$
$$\sim \varepsilon(-\mu,-\lambda)\varepsilon(\lambda,\mu)\cdot(z_1-z_2)^{\lambda\cdot\mu}(z_3-z_4)^{\lambda\cdot\mu}\langle 0|V(\Psi_{-\lambda-\mu},z_2)V(\Psi_{\lambda+\mu},z_4)|0\rangle\,. \tag{6.49}$$

But

$$\langle 0|V(\Psi_{-\lambda-\mu},z_2)V(\Psi_{\lambda+\mu},z_4)|0\rangle = z_2^{-(\lambda+\mu)^2}\langle\Psi_{\lambda+\mu}|e^{L_1/z_2}e^{z_4L_{-1}}|\Psi_{\lambda+\mu}\rangle\,, \tag{6.50}$$

by (2.43) and the creation property (2.3) (noting that the states Ψ_λ are (quasi-) primary). Applying (2.29) then gives us

$$\langle 0|V(\Psi_{-\mu},z_1)V(\Psi_{-\lambda},z_2)V(\Psi_\lambda,z_3)V(\Psi_\mu,z_4)|0\rangle$$
$$\sim \varepsilon(-\mu,-\lambda)\varepsilon(\lambda,\mu)\cdot(z_1-z_2)^{\lambda\cdot\mu}(z_3-z_4)^{\lambda\cdot\mu}(z_2-z_4)^{-(\lambda+\mu)^2}\,. \tag{6.51}$$

Hence $G(x)$ must be regular at $x=0$.

Similarly, the third limit gives us that $G(x)$ is regular at $x=1$. Hence, by Liouville's theorem, $G(x)$ must be a constant, in fact $G(x)\equiv 1$ from (6.48). So we deduce that

$$\langle 0|V(\Psi_{-\mu},z_1)V(\Psi_{-\lambda},z_2)V(\Psi_\lambda,z_3)V(\Psi_\mu,z_4)|0\rangle = (z_1-z_4)^{-\mu^2}(z_2-z_3)^{-\lambda^2}$$
$$\cdot\,(z_1-z_2)^{\lambda\cdot\mu}(z_1-z_3)^{-\lambda\cdot\mu}(z_2-z_4)^{-\lambda\cdot\mu}(z_3-z_4)^{\lambda\cdot\mu}\,, \tag{6.52}$$

and

$$\varepsilon(-\mu,-\lambda)\varepsilon(\lambda,\mu)=1\,. \tag{6.53}$$

However, the OPE (6.45) together with the hermitian property (2.43) tell us that $\varepsilon(\lambda,\mu)^*=\varepsilon(-\mu,-\lambda)$, and so by (6.53) we see that $\varepsilon(\lambda,\mu)$ is of unit modulus. The locality property applied to (6.45) gives immediately that

$$\varepsilon(\lambda,\mu)=(-1)^{\lambda\cdot\mu}\varepsilon(\mu,\lambda)\,, \tag{6.54}$$

and associativity implies

$$\varepsilon(\lambda,\mu)\varepsilon(\lambda+\mu,\nu)=\varepsilon(\lambda,\mu+\nu)\varepsilon(\mu,\nu)\,. \tag{6.55}$$

We thus see, from [11], that we may make a "gauge transformation" of the cocycles for $\mathscr{H}(\Lambda)$ if necessary so that the symmetry factors are given by the above set of $\varepsilon(\lambda,\mu)$. Hence, our proposed isomorphism $u:\mathscr{H}\to\mathscr{H}(\Lambda)$ is given by

$$ua_n^j u^{-1}=\hat{a}_n^j;\qquad u|\Psi_\lambda=|\hat{\lambda}\rangle\,, \tag{6.56}$$

using the notation of Sect. 5.2 for $\mathscr{H}(\Lambda)$ (where we use a hatted notation to distinguish the theory $\mathscr{H}(\Lambda)$). All that we are now required to show to complete the proof is that

$$uV(\psi,z)u^{-1}=\hat{V}(u\psi,z)\,, \tag{6.57}$$

for all $\psi\in\mathscr{H}$. From the definition of the $P^j(z)$, we see that this is true for the states $a_{-1}^j|0\rangle$. Further, we may act with the modes of the $P^j(z)$ (i.e., the a_n^j) on the states Ψ_λ to generate all states in $\mathscr{H}$ by use of the duality relation (cf. the argument given in [11] to simplify the final locality relation), which holds also in $\mathscr{H}(\Lambda)$. Hence, we need only verify (6.57) for $\psi=\Psi_\lambda,\lambda\in\Lambda$. Further, we see that it is only necessary to verify (6.57) for such states acting on states $|\mu\rangle,\mu\in\Lambda$, since we may act on the left with suitable combinations of the operators $\hat{V}(\hat{a}_{-1}^j|0\rangle,\zeta)$ and move them to the right by locality to act on $|\mu\rangle$ and raise it to an arbitrary state in $\mathscr{H}(\Lambda)$. So, we need only check

$$uV(\Psi_\lambda,z)|\Psi_\mu\rangle=\hat{V}(|\hat{\lambda}\rangle,z)|\hat{\mu}\rangle\,, \tag{6.58}$$

for all $\lambda,\mu\in\Lambda$.

Let us verify (6.58) by induction on the modes. From (6.45) we have

$$V(\Psi_\lambda,z)|\Psi_\mu\rangle=z^{\lambda\cdot\mu}\varepsilon(\lambda,\mu)|\Psi_{\lambda+\mu}\rangle+\cdot\,, \tag{6.59}$$

i.e., the first non-zero term in the expansion is

$$V_{-\frac{1}{2}(\lambda+\mu)^2+\frac{1}{2}\mu^2}(\Psi_\lambda)|\Psi_\mu\rangle=\varepsilon(\lambda,\mu)|\Psi_{\lambda+\mu}\rangle\,, \tag{6.60}$$

similarly for the right-hand side of (6.58). We have chosen the $\varepsilon(\lambda,\mu)$ of $\mathscr{H}(\Lambda)$ such that (6.58) holds for this mode. Now, from (2.51),

$$[L_{-1},V_n(\Psi_\lambda)]=(1-n-\tfrac{1}{2}\lambda^2)V_{n-1}(\Psi_\lambda)\,. \tag{6.61}$$

Hence

$$\begin{aligned}L_{-1}V_n(\Psi_\lambda)|\Psi_\mu\rangle&=(1-n-\tfrac{1}{2}\lambda^2)V_{n-1}(\Psi_\lambda)|\Psi_\mu\rangle+V_n(\Psi_\lambda)\mu\cdot a_{-1}|\Psi_\mu\rangle\\&=(1-n-\tfrac{1}{2}\lambda^2-\lambda\cdot\mu)V_{n-1}(\Psi_\lambda)|\Psi_\mu\rangle+\mu\cdot a_{-1}V_n(\Psi_\lambda)|\Psi_\mu\rangle\,,\end{aligned} \tag{6.62}$$

using $[a_m^j,V_n(\Psi_\lambda)]=\lambda^jV_{m+n}(\Psi_\lambda)$ (which follows from the OPE (6.41)), and similarly for $V_n(\lambda)|\mu\rangle$. So, if we have $(1-n-\frac{1}{2}\lambda^2-\lambda\cdot\mu)\neq0$ for all relevant n then the required result follows by induction. Thus, we require the coefficient to be positive for the first non-zero term, i.e., for $n=-\frac{1}{2}(\lambda+\mu)^2+\frac{1}{2}\mu^2$, and so we have the desired conclusion. □

In [20] the twist invariant subalgebras for the theories $\mathscr{H}(\Lambda)$ were evaluated, i.e., the algebras generated by the weight one states in $\mathscr{H}^+(\Lambda)$, which survive into the twisted theory $\widetilde{\mathscr{H}}(\Lambda)$, i.e., the states $|\lambda\rangle + |-\lambda\rangle$ for $\lambda \in \Lambda(2)$. For dimensions of 24 or greater, the minimal conformal weight in the twisted sector is at least 2, and so the algebra $g_{\widetilde{\mathscr{H}}(\Lambda)}$ is just the twist invariant subalgebra of $g_{\mathscr{H}(\Lambda)}$. But in 8 and 16 dimensions, the twist invariant subalgebras become "enhanced." There are twisted states with conformal weight one in these dimensions, and including the corresponding vertex operators in the operator algebra extends the twist invariant subalgebra. $\mathscr{H}(E_8)$ as we have seen has $g_{\mathscr{H}(E_8)} = E_8$, and the twist invariant subalgebra D_8 is enhanced to E_8. By Proposition 6.4, we see that $\widetilde{\mathscr{H}}(E_8) \cong \mathscr{H}(E_8)$, in complete analogy with the result for the constructions of lattices from codes. Similarly, and again mirroring the results for codes, it is found that $\widetilde{\mathscr{H}}(E_8 \oplus E_8) \cong \mathscr{H}(D_{16})$ and $\widetilde{\mathscr{H}}(D_{16}) \cong \mathscr{H}(E_8 \oplus E_8)$. The results for the 24-dimensional even self-dual lattices are shown in Table 2. The lattices Λ (except for the Leech lattice Λ_{24}) are denoted by the algebra whose root system $\Lambda(2)$ forms, while the straight and twisted theories $\mathscr{H}(\Lambda)$ and $\widetilde{\mathscr{H}}(\Lambda)$ are labelled by the algebra corresponding to the weight one fields (except for $\widetilde{\mathscr{H}}(\Lambda_{24}) \equiv V^\natural$, the natural Monster module).

Table 2. Straight and twisted constructions from even self-dual lattices in 24 dimensions

Λ	$\mathscr{H}(\Lambda)$	$\widetilde{\mathscr{H}}(\Lambda)$
E_8^3	E_8^3	D_8^3
$D_{16}E_8$	$D_{16}E_8$	D_8^3
D_8^3	D_8^3	D_4^6
D_4^6	D_4^6	A_1^{24}
A_1^{24}	A_1^{24}	Λ_{24}
Λ_{24}	$U(1)^{24}$	$V^\natural$
D_{24}	D_{24}	D_{12}^2
D_{12}^2	D_{12}^2	D_6^4
D_6^4	D_6^4	A_3^8
A_3^8	A_3^8	A_1^{16}
$D_{10}E_7^2$	$D_{10}E_7^2$	$D_5^2A_7^2$
$D_5^2A_7^2$	$D_5^2A_7^2$	$D_4^2B_2^4$
A_{24}	A_{24}	B_{12}
$A_{17}E_7$	$A_{17}E_7$	D_9A_7
$A_{15}D_9$	$A_{15}D_9$	$D_8B_4^2$
A_{12}^2	A_{12}^2	B_6^2
$A_{11}D_7E_6$	$A_{11}D_7E_6$	$D_6B_3^2C_4$
$A_9^2D_6$	$A_9^2D_6$	$D_5^2A_3^2$
A_8^3	A_8^3	B_4^3
A_6^4	A_6^4	B_3^4
$A_5^4D_4$	$A_5^4D_4$	$A_3^4A_1^4$
A_4^6	A_4^6	B_2^6
A_2^{12}	A_2^{12}	A_1^{12}
E_6^4	E_6^4	C_4^4

We see that there are 9 coincidences between a twisted theory and a straight theory (Proposition 6.4 shows that the theories are indeed isomorphic) and the other 15 twisted theories aredistinct, as their algebras are distinct. This gives us a total of 39 self-dual bosonic theories with $c = 24$. We also remember that there are 9 doubly-even self-dual binary codes of length 24. By comparing with the results from Sect. 5.1 given in Figs. 1, 2 and 3 we see that these coincidences between the straight and twisted theories are such that the conformal field theory given by the straight construction from the lattice obtained by the twisted construction applied to a doubly-even self-dual binary code is isomorphic to the theory given by the twisted construction from the lattice obtained by the straight construction applied to the same code. We may thus extend Figs. 1–3 to give Figs. 4–6, where there is included on the right hand side the dimension of the appropriate Lie algebra. Again, wavy arrows denote the twisted construction and straight arrows the straight construction. ($\dim g_{\mathcal{H}(\Lambda)} = |\Lambda(2)| + 24$ and $\dim g_{\widetilde{\mathcal{H}}}(\Lambda) = \frac{1}{2}|\Lambda(2)|$, in at least 24 dimensions, so that, from our discussion in Sect. 5.1, $\dim g_{\mathcal{H}(\Lambda_{\mathcal{C}})} = 16|\mathcal{C}_4| + 72$, $\dim g_{\widetilde{\mathcal{H}}(\Lambda_c)} = \dim g_{\mathcal{H}(\widetilde{\Lambda}_c)} = 8|\mathcal{C}_4| + 24$ and $\dim g_{\widetilde{\mathcal{H}}(\widetilde{\Lambda}_c)} = 4|\mathcal{C}_4|$ in 24 dimensions). Also, Fig. 7 shows the results in 16 dimensions. (Note that Fig. 7 may be extended indefinitely by adjoining copies of itself, a property which is not shared by the graphs in 24 dimensions, since in that case $\dim g$ decreases as one descends the graph).

We thus see that the connection with codes is more than just an analogy. Codes can be used to understand the structure and symmetries of conformal field theories. In Sect. 7, we prove that $\mathcal{H}(\widetilde{\Lambda}_{\mathcal{C}}) \cong \widetilde{\mathcal{H}}(\Lambda_{\mathcal{C}})$ for any doubly-even self-dual binary code $\mathcal{C}$ (of any length – a multiple of 8). Here, we show

Proposition 6.5. *Any coincidence between a twisted theory and a straight theory must be due to the existence of a doubly-even self-dual binary code.*

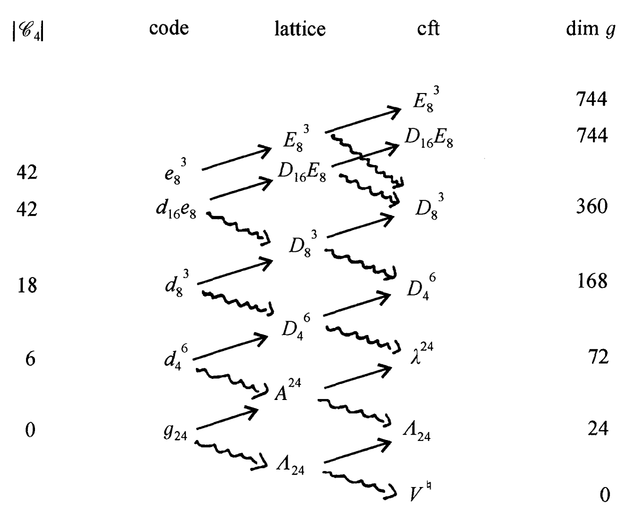

Fig. 4.

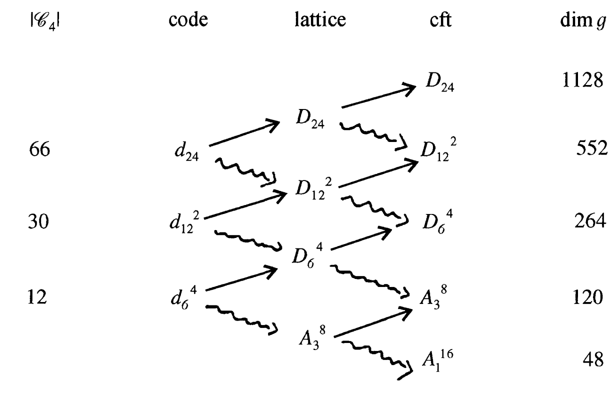

Fig. 5.

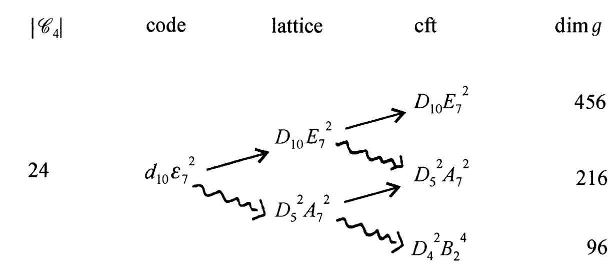

Fig. 6.

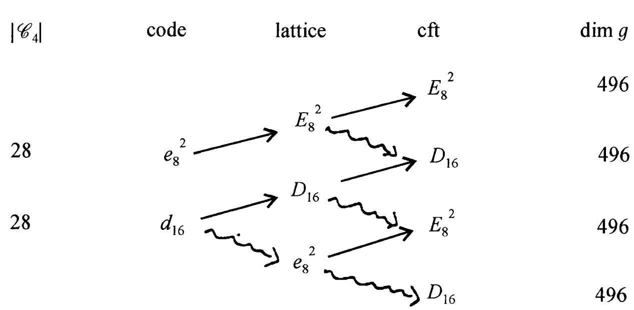

Fig. 7.

First we need

Lemma 6.6. *Let $\mathscr{H}$ be a self-dual bosonic conformal field theory with central charge $c = d$. Then $\mathscr{H}$ is isomorphic to $\mathscr{H}(\Lambda_{\mathscr{C}})$ for $\mathscr{C}$ some doubly-even self-dual binary code of length d if and only if $g_{\mathscr{H}} \supset su(2)^d$.*

Proof. The proof in one direction is immediate (see (7.1–3)). Conversely, if $g_{\mathscr{H}} \supset su(2)^d$ then rank $g_{\mathscr{H}} = d$. So, from Proposition 6.4, we see that $\mathscr{H} \cong \mathscr{H}(\Lambda)$ for some even lattice Λ. Λ is a weight lattice for $g_{\mathscr{H}} \supset su(2)^d$, and so is contained between the root lattice of $su(2)^d$ and its dual, i.e.,

$$\sqrt{2}\mathbf{Z}^d \subset \Lambda \subset \frac{1}{\sqrt{2}}\mathbf{Z}^d . \tag{6.63}$$

So we can write

$$\Lambda = \frac{\mathscr{C}}{\sqrt{2}} + \sqrt{2}\mathbf{Z}^d \tag{6.64}$$

for some doubly-even linear binary code $\mathscr{C}$ (these properties of $\mathscr{C}$ follow since Λ is an even lattice). Since $\mathscr{H}$ is self-dual, $\chi_{\mathscr{H}}(-1/\tau) = \chi_{\mathscr{H}}(\tau)$, and so we see from (5.31) that Λ, and hence $\mathscr{C}$, must be self-dual. Note that this also implies that d must be a multiple of 8 from the lattice theory discussed in Sect. 5.1. □

Proof of Proposition 6.5. If a twisted theory $\widetilde{\mathscr{H}}(\Lambda)$ in d-dimensions coincides with a straight theory $\mathscr{H}(\Lambda')$ then the corresponding algebra must have rank d (since $\mathscr{H}(\Lambda')$ contains the states $a_{-1}^j|0\rangle$, $1 \leqq j \leqq d$, for which the corresponding fields commute). The weight one states in $\widetilde{\mathscr{H}}(\Lambda)$ are of the form $|\lambda_+\rangle \equiv |\lambda\rangle + |-\lambda\rangle$ for $\lambda \in \Lambda(2)$, and these satisfy the commutation relations

$$[V_n(\lambda_+), V_m(\mu_+)] = \begin{cases} 0 & \lambda \cdot \mu = 0 \\ \varepsilon(\lambda, \mp\mu)V_{n+m}((\lambda \mp \mu)_+) & \lambda \cdot \mu = \pm 1, \end{cases} \tag{6.65}$$

from (6.28). Hence, since the algebra has rank d, we must have d orthogonal vectors in $\Lambda(2)$ (since for $\lambda \cdot \mu = \pm 2$, $\lambda = \pm\mu$ and the states λ_+ and μ_+ are not independent). So, denoting these by $\sqrt{2}e_j$ for $1 \leqq j \leqq d$, we see from (7.1–3) that $g_{\mathscr{H}(\Lambda)} \supset su(2)^d$, and, since $\mathscr{H}(\Lambda)$ is self-dual due to the fact that $\widetilde{\mathscr{H}}(\Lambda)$ is, the above result tells us that $\Lambda = \Lambda_{\mathscr{C}}$ for some doubly-even self-dual binary code $\mathscr{C}$. But, as is shown in Sect. 7, $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) \cong \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$. So we deduce also that Λ' is equivalent to $\widetilde{\Lambda}_{\mathscr{C}}$. Therefore, all coincidences are labelled by codes in the manner noted from the results above. □

We also note those twisted theories which are distinct from untwisted theories must have rank strictly less than d, from Proposition 6.4, and this is consistent with Table 2, since all of the 15 new twisted theories have algebras of rank less than 24.

For $\mathscr{C}$ a doubly-even self-dual binary code, we have the decomposition $\Lambda_{\mathscr{C}} = \Lambda_0(\mathscr{C}) \cup \Lambda_1(\mathscr{C}), \widetilde{\Lambda}_{\mathscr{C}} = \Lambda_0(\mathscr{C}) \cup \Lambda_3(\mathscr{C})$. So we can divide $\mathscr{H}^{\pm}(\Lambda_{\mathscr{C}})$ and $\mathscr{H}^{\pm}(\widetilde{\Lambda}_{\mathscr{C}})$ into two subspaces each according to whether the momentum is in Λ_0 or not, i.e., we define $\mathscr{V}_a^{\pm} = \mathscr{H}^{\pm}(\Lambda_a)$ to be the subspace generated by the Heisenberg algebra from the states $|\lambda\rangle$ with $\lambda \in \Lambda_a$ and with $\theta = \pm 1$, $0 \leqq a \leqq 3$ (note that we also include the lattice Λ_2). Similarly we can define twisted spaces $\mathscr{T}_a^{\pm}$ (for more details see the next section). Then we have the decompositions

$$\begin{aligned} \mathscr{H}(\Lambda_{\mathscr{C}}) &= \mathscr{V}_0^+ \oplus \mathscr{V}_0^- \oplus \mathscr{V}_1^+ \oplus \mathscr{V}_1^- , \\ \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}}) &= \mathscr{V}_0^+ \oplus \mathscr{V}_0^- \oplus \mathscr{V}_3^+ \oplus \mathscr{V}_3^- , \\ \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) &= \mathscr{V}_0^+ \oplus \mathscr{V}_1^+ \oplus \mathscr{T}_0^+ \oplus \mathscr{T}_1^+ , \\ \widetilde{\mathscr{H}}(\widetilde{\Lambda}_{\mathscr{C}}) &= \mathscr{V}_0^+ \oplus \mathscr{V}_3^+ \oplus \mathscr{T}_0^+ \oplus \mathscr{T}_3^+ . \end{aligned} \tag{6.66}$$

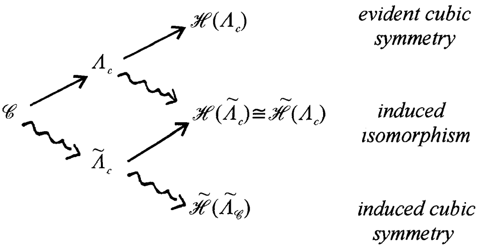

Fig. 8.

The triality structure of FLM is an involution acting on $\tilde{\mathscr{H}}(\tilde{\Lambda}_{\mathscr{C}})$ which mixes the straight and twisted spaces, e.g. maps $\mathscr{V}_3^+ \leftrightarrow \mathscr{T}_0^+$, $\mathscr{V}_0^+ \to \mathscr{V}_0^+$, $\mathscr{T}_3^+ \to \mathscr{T}_3^+$. So, we might postulate that an extension of this operator to the whole structure (6.66) would provide the isomorphism between $\mathscr{H}(\tilde{\Lambda}_{\mathscr{C}})$ and $\tilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ and also an automorphism of $\mathscr{H}(\Lambda_{\mathscr{C}})$. In the next section, we show that this is true, but give rather a converse argument to this, i.e., we observe that there is an obvious triality structure of $\mathscr{H}(\Lambda_{\mathscr{C}})$ which, using the isomorphism $\mathscr{H}(\tilde{\Lambda}_{\mathscr{C}}) \cong \tilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ which we prove directly, can be extended to the first three rows of (6.66) and then finally to $\tilde{\mathscr{H}}(\tilde{\Lambda}_{\mathscr{C}})$, providing a simple construction of the triality structure of FLM, which serves to generate the Monster, and also generalising this structure beyond the particular case associated with the Golay code. In Sect. 8, we define further involutions to give a cubic group of automorphisms. This strategy is summarised in Fig. 8.

[Note that if we let g_0 be the Lie algebra corresponding to the sub-conformal field theory $\mathscr{V}_0^+$, and g_a for $a = 1,2,3$ the algebras corresponding to $\mathscr{V}_0^-$, $\mathscr{V}_1^+$ and $\mathscr{V}_1^-$ respectively, then we have a sort of "elaborated symmetric space structure"

$$\begin{aligned} &[g_0, g_0] \subset g_0 \quad [g_0, g_a] \subset g_a\,, \\ &[g_a, g_a] \subset g_0 \quad [g_a, g_b] \subset g_c\,, \end{aligned} \tag{6.67}$$

where (a,b,c) is a permutation of $(1,2,3)$. $g = \bigoplus_{a=0}^{3} g_a$ can be divided into a symmetric space in three isomorphic ways, i.e., $g/g_0 \oplus g_a$.]

7. Construction of the Triality Operator

For $\mathscr{C}$ a doubly-even self-dual binary code of length d, we defined the spaces $\mathscr{V}_a^\pm$ for $a = 0,1,2,3$ in the previous section. (Note that we suppress the $\mathscr{C}$ dependence for ease of notation.) We may also define 8 corresponding twisted spaces starting from an irreducible representation $\mathscr{X} \equiv \mathscr{X}(\Lambda_0^*)$ of the gamma matrix algebra $\Gamma \equiv \Gamma(\Lambda_0^*) = \{\gamma_\lambda : \lambda \in \Lambda_0^*\}$ (noting that $\Lambda_0^* = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$), which is described in the appendix. $\mathscr{X}$ is of dimension $2^{1+d/2}$ and splits into four irreducible representations $\mathscr{X}_a$, $0 \leqq a \leqq 3$, of $\Gamma_0 = \Gamma(\Lambda_0)$, with $\mathscr{X}_0 \oplus \mathscr{X}_a$ an irreducible representation of dimension $2^{d/2}$ of $\Gamma(\Lambda_0 \cup \Lambda_a)$ for $a = 1,2,3$. (Note that the lattice is only even for $a = 1,3$). Define $\theta = (-1)^{d/8}$ on $\mathscr{X}_a$ for $a = 0,1,3$ and $\theta = -(-1)^{d/8}$ on $\mathscr{X}_2$. Set $\mathscr{T}_a^\pm = \mathscr{H}_T^\pm(\Lambda_a)$ for $0 \leqq a \leqq 3$, the subspace with $\theta = \pm 1$ generated by the c-oscillators from $\mathscr{X}_a$.

Define the weight one states

$$\zeta_1^j = \frac{1}{\sqrt{2}} a_{-1}^j |0\rangle \in \mathscr{V}_0^- , \tag{7.1}$$

$$\zeta_2^j = -\frac{1}{2}(|\sqrt{2}e_j\rangle + |-\sqrt{2}e_j\rangle) \in \mathscr{V}_1^+ , \tag{7.2}$$

$$\zeta_3^j = \frac{i}{2}(|\sqrt{2}e_j\rangle - |-\sqrt{2}e_j\rangle) \in \mathscr{V}_1^- , \tag{7.3}$$

$1 \leqq j \leqq d$, where the e_j are the unit vectors in the direction of the axes (which we defined in Sect. 5.1). Set $J^{ja}(z) = V(\zeta_a^j, z)$, $1 \leqq a \leqq 3$, $1 \leqq j \leqq d$. Then these currents define an affine $su(2)^d$ algebra. This follows from the relations (6.28), which give

$$[J_m^{ja}, J_n^{kb}] = i\varepsilon^{abc} J_{m+n}^{kc} \delta^{jk} + \frac{1}{2} m \delta_{m,-n} \delta^{ab} \delta^{jk} , \tag{7.4}$$

where there is an implicit sum over c, $1 \leqq c \leqq 3$. For each $su(2)$, i.e., for each j, we can define a rotation in that $su(2)$,

$$\sigma^j = \exp\left\{ \frac{i\pi}{\sqrt{2}} (J_0^{j1} + J_0^{j2}) \right\} , \tag{7.5}$$

which rotates by π about the axis equally inclined to the first and second axes. Set

$$\sigma = \prod_{j=1}^{d} \sigma^j . \tag{7.6}$$

Then we have, since the distinct $su(2)$'s commute,

$$\sigma J_m^{j1} \sigma^{-1} = J_m^{j2}, \qquad \sigma J_m^{j2} \sigma^{-1} = J_m^{j1}, \qquad \sigma J_m^{j3} \sigma^{-1} = -J_m^{j3} , \tag{7.7}$$

and σ defines an automorphism of $\mathscr{H}(\Lambda_{\mathscr{C}})$, provided the cocycles are chosen appropriately, which has order 2 ($\sigma^2 = 1$), i.e., σ is an involution of $\mathscr{H}(\Lambda_{\mathscr{C}})$.

When d is an odd multiple of 8, we shall modify the definition of σ given above slightly, by redefining for some l, $1 \leqq l \leqq d$,

$$\sigma^l = \exp\left\{ \frac{3\pi i}{\sqrt{2}} (J_0^{l1} + J_0^{l2}) \right\} , \tag{7.8}$$

σ still being given by (7.6). This still gives $\sigma^2 = 1$. [Although each individual $su(2)$, with generators J_0^{ja} for some j, has half-integral spins on $\mathscr{H}(\Lambda_{\mathscr{C}})$, the diagonal group, with generators $\sum_j J_0^{ja}$, has only integral spins, due to the way in which the occurrence of the half-integral spins is correlated by the codewords. The redefinition (7.8) for d an odd multiple of 8 changes σ by a factor of -1 on states with half-integral spin with respect to the $su(2)$ labelled by l (but leaves it unchanged on states with integral spin with respect to this $su(2)$).]

Proposition 7.1. *The spaces $\mathscr{V}_a^{\pm}$ and $\mathscr{T}_a^{\pm}$ for $0 \leqq a \leqq 3$ are irreducible as representation spaces for $\mathscr{V}_0^+$* [7].

Proof. Consider initially the space $\mathscr{V}_a^P$ for some a, $0 \leqq a \leqq 3$, and some parity $P = \pm$. Suppose that U is an irreducible representation space for $\mathscr{V}_0^+$ contained in

$\mathcal{V}_a^P$. Set

$$\psi^J = \frac{1}{2} a_{-1}^j a_{-1}^j |0\rangle, \qquad L^j(z) \equiv V(\psi^j, z) = \frac{1}{2} : P^j(z) P^j(z) : , \tag{7.9}$$

where there is no sum over j (we drop the summation convention unless otherwise stated) and

$$P^j(z) = i \frac{d}{dz} X^j(z) , \tag{7.10}$$

from (5.18). Also, for $j \neq k$, set

$$\psi^{jk} = a_{-1}^j a_{-1}^k |0\rangle, \qquad L^{jk}(z) \equiv V(\psi^{jk}, z) = P^j(z) P^k(z) , \tag{7.11}$$

(for $j \neq k$, P^j and P^k commute, so no normal ordering is necessary). Then, since U is irreducible, $L^j(z)$ and $L^{jk}(z)$ map $U \to U$, since ψ^j, $\psi^{jk} \in \mathcal{V}_0^+$. Denoting the modes of the vertex operators for ψ^j and ψ^{jk} similarly, we have $[L_0^j, L_0^k] = 0$ and so we may write U as the direct sum of simultaneous eigenspaces of the L_0^j. If $\phi \in U$ is such a state, say $L_0^j \phi = v^j \phi$, $1 \leqq j \leqq d$. Then

$$L_M^{jk} \phi = \sum_n a_n^j a_{M-n}^k \phi \tag{7.12}$$

$(j \neq k)$ must be a state in U, since $L^{jk}(z) : U \to U$. But

$$L_0^j a_n^j a_{M-n}^k \phi = (v^j - n) a_n^j a_{M-n}^k \phi \, . \tag{7.13}$$

The projection of the state (7.12) onto the simultaneous eigenspaces must be in U, so we see from (7.13) that $a_n^j a_{M-n}^k \phi \in U$, i.e., $a_m^j a_n^k$ maps $U \to U$ for $j \neq k$. Thus, so does

$$[a_m^j a_{-M}^k, a_n^j a_M^k] = -M a_n^j a_m^j + m \delta_{m,-n} a_{-M}^k a_M^k \, , \tag{7.14}$$

for $j \neq k$. Hence $a_n^j a_m^j$ maps $U \to U$ for $m \neq -n$. Putting $m = -n$ in (7.14) we have that

$$m a_{-M}^k a_M^k - M a_{-m}^j a_m^j \tag{7.15}$$

maps $U \to U$ for $j \neq k$. Since $\psi_L \in \mathcal{V}_0^+$, then we can decompose U into eigenspaces of $L_0 : U \to U$. Applying (7.15) to a state $\phi \in U$ of conformal weight h_ϕ, we see that a_M^k must annihilate ϕ for sufficiently large M, i.e., for $M > h_\phi$. Thus $a_{-m}^j a_m^j$ maps $U \to U$, and we have that $a_m^j a_n^k$ maps $U \to U$ for all $1 \leqq j$, $k \leqq d$ and $m, n \in \mathbf{Z}$.

Therefore, since $S^{jk} \equiv a_0^j a_0^k$ maps $U \to U$ and the operators S^{jk} commute, we may decompose U into the direct sum of the simultaneous eigenspaces of these operators. By application of $a_m^j a_n^k$ with m, $n \geqq 0$ to a state in such an eigenspace, we see that each such eigenspace contains a state of the form $|\lambda_a\rangle + P| - \lambda_a\rangle$, where $\lambda_a \in \Lambda_a(\mathscr{C})$. The vectors $\sqrt{2} e_j \pm \sqrt{2} e_k$ and $\frac{1}{\sqrt{2}} c$ for $1 \leqq j < k \leqq d$, $c \in \mathscr{C}$, generate the lattice $\Lambda_0(\mathscr{C})$, and generate from λ_a the lattice $\Lambda_a(\mathscr{C})$. Set

$$\zeta_{jk}^\pm = |\sqrt{2} e_j \pm \sqrt{2} e_k\rangle + | - (\sqrt{2} e_j \pm \sqrt{2} e_k)\rangle, \qquad \zeta_c = |\tfrac{1}{\sqrt{2}} c\rangle + | - \tfrac{1}{\sqrt{2}} c\rangle \, . \tag{7.16}$$

Then $V(\zeta_{jk}^\pm, z)$ and $V(\zeta_c, z)$ map $U \to U$ (since the states are in $\mathcal{V}_0^+$) and, projecting onto the simultaneous eigenspaces of the operators S^{jk}, we see that application of these vertex operators takes us from the eigenspace containing $|\lambda_a\rangle + P|-\lambda_a\rangle$ to all eigenspaces $|\lambda\rangle + P|-\lambda\rangle$ with $\lambda \in \Lambda_a$. Applying $a_m^j a_n^k$ with $m, n \leqq 0$ generates $\mathscr{V}_a^P$ from these states (e.g. taking $m = 0$, $n < 0$ maps us to a state with an odd number of creation operators and zero mode piece $|\lambda\rangle - P|-\lambda\rangle$), i.e., $U = \mathscr{V}_a^P$, and so the spaces $\mathscr{V}_a^\pm$ are irreducible as representation spaces for $\mathscr{V}_0^+$.

The argument works in a similar way for the spaces $\mathscr{T}_a^\pm$. For U an irreducible representation space for $\mathscr{V}_0^+$ contained in T_a^P, $c_r^j c_s^k$ maps $U \to U$, as above. Then by application of $c_r^j c_s^k$ with $r,\ s > 0$ we see that U contains a state χ for $P = P_0$ or a state $c^j_{-\frac{1}{2}}\chi$ for $P = -P_0$, where $\chi \in \mathscr{X}_a$ and $P_0 = (-1)^{d/8}$. Acting with the vertex operators $V(\zeta_\lambda, z)$ for $\zeta_\lambda = |\lambda\rangle + |-\lambda\rangle \in \mathscr{V}_0^+$ in the first case shows that the set of all such χ appearing in U must form an irreducible representation space for the gamma matrix algebra Γ_0, i.e., U must contain $\mathscr{X}_a$, and then acting with $c_r^j c_s^k$ for $r, s < 0$ we deduce that $U = \mathscr{T}_a^P$. In the second case, acting with $c^k_{-\frac{1}{2}} c^j_{\frac{1}{2}}$ shows that U contains all the states $c^k_{-\frac{1}{2}}\chi$, $1 \leqq k \leqq d$. We may, as above, deduce that χ ranges over all of $\mathscr{X}_a$ (act with $V(\zeta_\lambda, z)$ for $\lambda \cdot e_k = 0$) and hence that $U = \mathscr{T}_a^P$. $\square$

Proposition 7.2. *σ maps $\mathscr{V}_0^+$ to itself. Further, $\sigma : \mathscr{V}_0^- \to \mathscr{V}_1^+$, $\mathscr{V}_1^+ \to \mathscr{V}_0^-$ and $\mathscr{V}_1^- \to \mathscr{V}_1^-$.*

Proof. From the above argument, $a_m^j a_n^k$, $V(\zeta_{jk}^\pm, z)$ and $V(\zeta_c, z)$, $1 \leqq j < k \leqq d, m, n \in \mathbf{Z}$, $c \in \mathscr{C}$, generate $\mathscr{V}_0^+$ from $|0\rangle$. So, since it is clear that $\sigma|0\rangle = |0\rangle$, it is only necessary to show that σ transforms these operators into operators which map $\mathscr{V}_0^+$ into itself in order to establish the result. Equation (7.7) gives

$$\begin{aligned} &\sigma V(\zeta_1^j, z)\sigma^{-1} = V(\zeta_2^j, z)\,, \qquad \sigma V(\zeta_2^j, z)\sigma^{-1} = V(\zeta_1^j, z)\,, \\ &\sigma V(\zeta_3^j, z)\sigma^{-1} = -V(\zeta_3^j, z)\,. \end{aligned} \tag{7.17}$$

Now $V(\zeta_a^j, z)V(\zeta_a^k, \zeta)$ for $a = 1,2,3$, $1 \leqq j,\ k \leqq d$, maps $\mathscr{V}_0^+$ to itself, since $\mathscr{V}_0^+$ is defined as a subspace of $\mathscr{H}(\Lambda_\mathscr{C})$ by the conditions $\theta = 1$ and $\theta_1 = 1$, where

$$\theta_1 = \exp\left\{ i\frac{\pi}{\sqrt{2}} \underline{1} \cdot p \right\}, \tag{7.18}$$

and each of these products commutes with both θ and θ_1. ($\theta_1 = 1$ fixes the momentum to lie in $\Lambda_0(\mathscr{C})$.) Taking moments of the product with $a = 1$ shows that $a_m^j a_n^k$ maps $\mathscr{V}_0^+$ to itself, and from

$$V(\zeta_{jk}^\pm, z) = 2V(\zeta_2^j, z)V(\zeta_2^k, z) \mp 2V(\zeta_3^j, z)V(\zeta_3^k, z) \tag{7.19}$$

we see that $V(\zeta_{jk}^\pm, z)$ does also. Finally, we must consider $V(\zeta_c, z)$. Set $\Psi_c = |\frac{1}{\sqrt{2}} c\rangle$. Then for j such that $e_j \cdot c = 1$,

$$[J_0^{j1}, V(\Psi_c, z)] = \tfrac{1}{2} V(\Psi_c, z), \qquad [J_0^{j1}, V(\Psi_{c'}, z)] = -\tfrac{1}{2} V(\Psi_{c'}, z)\,, \tag{7.20a}$$

$$[J_0^{j2}, V(\Psi_c, z)] = \tfrac{i}{2} \varepsilon V(\Psi_{c'}, z), \qquad [J_0^{j2}, V(\Psi_{c'}, z)] = -\tfrac{i}{2} \varepsilon V(\Psi_c, z)\,, \tag{7.20b}$$

where $c' = c - 2e_j$ and $\varepsilon = \varepsilon(-\sqrt{2}e_j, \frac{1}{\sqrt{2}}c) = -\varepsilon(\sqrt{2}e_j, \frac{1}{\sqrt{2}}c')$. Therefore

$$\sigma^j \begin{pmatrix} V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix} \sigma_j^{-1} = \exp\left\{\frac{i\pi}{2\sqrt{2}}\begin{pmatrix} 1 & i\varepsilon \\ -i\varepsilon & -1 \end{pmatrix}\right\}\begin{pmatrix} V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix}$$

$$= \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & i\varepsilon \\ -i\varepsilon & -1 \end{pmatrix}\begin{pmatrix} V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix}. \tag{7.21}$$

Set $\Delta(c) = \{c' : c'_k = \pm c_k,\ 1 \leqq k \leqq d\}$, i.e., the set of all d-tuples which can be obtained from c by application of (7.20) for various j. We see that

$$\sigma V(\Psi_c,z)\sigma^{-1} = 2^{-\frac{1}{2}|c|}\sum_{c'\in\Delta(c)} i^{n(c,c')}\eta(c,c')V(\Psi_{c'},z), \tag{7.22}$$

where $n(c,c')$ is the number of j, $1 \leqq j \leqq d$, such that $c'_j \neq c_j$ and $\eta(c,c')$ is given by

$$\left(\prod_{c'_j=-1} \gamma_{\sqrt{2}e_j}\right)\gamma_{\frac{c}{\sqrt{2}}} = \eta(c,c')\gamma_{\frac{c'}{\sqrt{2}}}. \tag{7.23}$$

Similarly, we find

$$\sigma V(\Psi_{-c},z)\sigma^{-1} = 2^{-\frac{1}{2}|c|}\sum_{c'\in\Delta(c)} (-i)^{n(c,c')}\eta(c,c')V(\Psi_{-c'},z). \tag{7.24}$$

But $\mathscr{C}$ is doubly-even, so that $n(c,c') + n(c,-c') = |c| \in 4\mathbf{Z}$, i.e., $(-i)^{n(c,-c')} = i^{n(c,c')}$. From the appendix,

$$\eta(c,c') = (-1)^{n(c,c')}\eta(c,-c'), \tag{7.25}$$

and so (7.22) and (7.24) may be added to give

$$\sigma V(\zeta_c,z)\sigma^{-1} = 2^{-\frac{1}{2}|c|}\sum_{c'\in\Delta_+(c)} i^{n(c,c')}\eta(c,c')V(\zeta_{c'},z), \tag{7.26}$$

where $\Delta_+(c) = \{c' \in \Delta(c) : n(c,c') \in 2\mathbf{Z}\}$. When $n(c,c')$ is even, $\zeta_{c'} \in \mathscr{V}_0^+$, since $\frac{c'}{\sqrt{2}} = \frac{c}{\sqrt{2}} + \lambda$, where $\lambda \in \sqrt{2}\mathbf{Z}_+^d$. Therefore, $V(\zeta_c,z)$ maps to operators which map V_0^+ to itself. Thus, $\sigma : \mathscr{V}_0^+ \to \mathscr{V}_0^+$ as required.

Since $\mathscr{V}_0^-$, $\mathscr{V}_1^+$ and $\mathscr{V}_1^-$ are irreducible as representation spaces for $\mathscr{V}_0^+$ and σ maps $\mathscr{V}_0^+$ to itself, it is only necessary to check the transformation of one state in each space to show that $\sigma : \mathscr{V}_0^- \to \mathscr{V}_1^+$, $\mathscr{V}_1^+ \to \mathscr{V}_0^-$ and $\mathscr{V}_1^- \to \mathscr{V}_1^-$. Since $\sigma\zeta_1^j = \sigma\zeta_2^j$, $\sigma\zeta_2^j = \sigma\zeta_1^j$ and $\sigma\zeta_3^j = -\zeta_3^j$, then this result follows. □

Hence, σ gives an isomorphism between $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$ and $\mathscr{V}_0^+ \oplus \mathscr{V}_1^+$ (note that these are conformal field theories ($\psi_L \in \mathscr{V}_0^+$)).

As stated in the previous section, we shall now show

Proposition 7.3. *$\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}}) \cong \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$, for $\mathscr{C}$ a doubly-even self-dual binary code.*

Proof. $\sqrt{2}J_0^{j1} = p_0^j$, $1 \leqq j \leqq d$, provides a Cartan subalgebra for $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$, and we see from (6.65) that $\sqrt{2}J_0^{j2}$, $1 \leqq j \leqq d$, provides a Cartan subalgebra for $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$

(i.e., both $g_{\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})}$ and $g_{\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})}$ have rank d). The corresponding lattice of eigenvalues in the first case on $\mathscr{V}_0^+$ is $\Lambda_0(\mathscr{C})$, i.e., is d-dimensional. So, by (7.7), the lattice of eigenvalues of $\sqrt{2}J_0^{j2}$ in $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ is also d-dimensional and contains $\Lambda_0(\mathscr{C})$. By Proposition 6.4, $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ is isomorphic to a theory $\mathscr{H}(\Lambda)$ for some d-dimensional even lattice Λ. Λ contains $\Lambda_0(\mathscr{C})$, and since it is even it must be integral, and hence $\Lambda \subset \Lambda_0(\mathscr{C})^* = \Lambda_0(\mathscr{C}) \cup \Lambda_1(\mathscr{C}) \cup \Lambda_2(\mathscr{C}) \cup \Lambda_3(\mathscr{C})$. Note that $\Lambda_1(\mathscr{C})$, $\Lambda_2(\mathscr{C})$ and $\Lambda_3(\mathscr{C})$ are all shifts of $\Lambda_0(\mathscr{C})$ so that, since $\Lambda \supset \Lambda_0(\mathscr{C})$, if Λ contains any element of $\Lambda_i(\mathscr{C})$ then $\Lambda \supset \Lambda_i(\mathscr{C})$ for $i = 1,2,3$. Thus, there are three possibilities for Λ even, i.e., $\Lambda = \Lambda_0(\mathscr{C})$, $\Lambda = \Lambda_0(\mathscr{C}) \cup \Lambda_1(\mathscr{C}) = \Lambda_{\mathscr{C}}$ and $\Lambda = \Lambda_0(\mathscr{C}) \cup \Lambda_3(\mathscr{C}) = \widetilde{\Lambda}_{\mathscr{C}}$. For $d \geqq 24$, these three possibilities can be easily distinguished by considering their partition functions. $\dim g_{\mathscr{H}(\Lambda_{\mathscr{C}})} = 16|\mathscr{C}_4| + 3d$, while $\dim g_{\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})} = \dim g_{\mathscr{H}(\Lambda_0(\mathscr{C}))} = 8|\mathscr{C}_4| + d$ from earlier discussions. Also, since $\Lambda_0(\mathscr{C})$ is strictly contained in $\widetilde{\Lambda}_{\mathscr{C}}$, the partition functions for $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ and $\mathscr{H}(\Lambda_0(\mathscr{C}))$ are distinct. Hence, all three cases have distinct partition functions, and so to complete the proof in $d \geqq 24$ it is only necessary to verify that $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ and $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ have equal partition functions. (Note that since $\dim g_{\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})} = 8|\mathscr{C}_4| + d$ we can immediately exclude the possibility $\Lambda = \Lambda_{\mathscr{C}}$.) Since $\mathscr{V}_0^+ \oplus \mathscr{V}_0^- \cong \mathscr{V}_0^+ \oplus \mathscr{V}_1^+$ from the above, then the partition functions for these parts coincide, and it remains to consider $\mathscr{V}_3^+ \oplus \mathscr{V}_3^-$ and $\mathscr{T}_0^+ \oplus \mathscr{T}_1^+$. The corresponding partition functions are

$$\tfrac{1}{2}2^{\frac{d}{2}}q^{\frac{d}{16}-1}\prod_{n=1}^{\infty}(1-q^n)d\left(\sum_{m=-\infty}^{\infty}q^{\frac{1}{2}m+m^2}+(-1)^{\frac{d}{8}}\sum_{m=-\infty}^{\infty}(-q)^{\frac{1}{2}m+m^2}\right) \tag{7.27}$$

and

$$\tfrac{1}{2}2^{\frac{d}{2}}q^{\frac{d}{16}-1}\left(\prod_{r=\frac{1}{2}}^{\infty}(1-q^r)^{-d}+(-1)^{\frac{d}{8}}\prod_{r=\frac{1}{2}}^{\infty}(1+q^r)^{-d}\right), \tag{7.28}$$

and these are equal by virtue of the identity

$$\sum_{m=-\infty}^{\infty}q^{\frac{1}{2}m+m^2}=\prod_{n-1}^{\infty}(1-q^{2n})\prod_{r=\frac{1}{2}}^{\infty}(1+q^r)\,. \tag{7.29}$$

Alternatively, we may see the equality immediately since, as we have argued previously, $\chi_{\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})}$ and $\chi_{\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})}$ are modular invariant, and so equal up to a constant, from the discussion of Sect. 6.1. The constant terms coincide, both being equal to $8|\mathscr{C}_4| + d$, the dimension of the corresponding Lie algebra. The cases $d = 8$ and $d = 16$ can be considered separately. There is only one code to consider in the first case and two in the second, and the results quoted in Sect. 6.2 show that $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) \cong \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ here also. So we have $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) \cong \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ as required. □

Proposition 7.4. *σ can be extended to an isomorphism $\widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) \cong \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$.*

Proof. From Proposition 7.3, we have an isomorphism $\sigma_0 : \mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}}) \to \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ with $\sigma_0 J^{j1}(z)\sigma_0^{-1} = J^{j2}(z)$. Restricted to $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$, $\sigma = \sigma_0 u$, where u is an automorphism of $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$ commuting with $J^{j1}(z)$, $1 \leqq j \leqq d$, i.e., u preserves the eigenspaces of the p^j. Thus $u|\lambda\rangle = v(\lambda)|\lambda\rangle$ for $\lambda \in \Lambda_0(\mathscr{C})$, where $|v(\lambda)| = 1$. From

$uV(|\mu\rangle,z)u^{-1} = V(u|\mu\rangle,z)$ applied to $|\lambda\rangle$, we see that

$$v(\lambda+\mu) = v(\lambda)v(\mu) \tag{7.30}$$

is required. u can be extended to an automorphism of $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ by choosing $\mu \in \Lambda_3(\mathscr{C})$ and $v(\mu)$ such that $v(\mu)^2 = v(2\mu)$ (since $2\mu \in \Lambda_0(\mathscr{C})$ this is known). This ensures that (7.30) holds. Then replacing σ_0 by $\sigma_0 u$ gives an isomorphism $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}}) \to \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}})$ which coincides with σ on $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$. Thus, σ can be extended to the whole of $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ as required. □

Therefore $\mathscr{V}_3^+ \oplus \mathscr{V}_3^-$ and $\mathscr{T}_0^+ \oplus \mathscr{T}_1^-$ are equivalent as representations of the isomorphic conformal field theories $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$ and $\mathscr{V}_0^+ \oplus \mathscr{V}_1^+$. It is shown in Sect. 8 that σ maps $\mathscr{V}_3^+ \to \mathscr{T}_0^+$ and $\mathscr{V}_3^- \to \mathscr{T}_1^+$ (in fact, a more generalised form of this result is demonstrated). (σ is unique up to the automorphism ι of $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$ which is 1 on $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$ and -1 on $\mathscr{V}_3^+ \oplus \mathscr{V}_3^-$.) σ thus gives an isomorphism of $\mathscr{V}_0^+ \oplus \mathscr{V}_3^+$ and $\mathscr{V}_0^+ \oplus \mathscr{T}_0^+$. For $\psi \in \mathscr{V}_0^+$, $\phi \in \mathscr{V}_3^+$,

$$\sigma^{-1}V(\psi,z)\sigma\phi = V(\sigma\psi,z)\phi\,. \tag{7.31}$$

In [24] it is shown that any irreducible hermitian real meromorphic representation $\mathscr{K}$ of $\mathscr{H}(\Lambda)_+$ is equivalent either to $\mathscr{H}(\Lambda)_\pm$ or $\mathscr{H}_T(\Lambda)_+$, a twisted analogue of Proposition 6.4. The argument used there proceeds by showing that any representation U of a conformal field theory $\mathscr{H}$ can be characterised by a state in $\mathscr{H}$; to do this the expectation value in a state χ of the representation of a product of the operators $U(\psi_j,z_j)$, representing states $\psi_j \in \mathscr{H}$, is rewritten using (3.8) and locality as the scalar product of a suitable state of $\mathscr{H}$ on the product of vertex operators of $\mathscr{H}$ acting on one of the ψ_j's. The essential representation property (3.1) can be translated into properties of this state. This argument can be reversed to define a representation by a suitable state of $\mathscr{H}$ having these properties. This procedure can be applied to associate a state to any meromorphic representation of $\mathscr{H}(\Lambda)_+$. The properties required for this state to define a representation are such that the state defines also a representation of $\mathscr{H}(\Lambda)$. This representation is an extension of the initial representation of $\mathscr{H}(\Lambda)_+$, and must restrict to this, in particular restricting to a meromorphic representation of $\mathscr{H}(\Lambda)_+$. The (non-meromorphic) representations of $\mathscr{H}(\Lambda)$ are easily classified (see e.g. [25]), and those that restrict to a meromorphic representation of $\mathscr{H}(\Lambda)_+$ are simply $\mathscr{H}(\Lambda)$ itself and $\mathscr{H}_T(\Lambda)$ (modulo inequivalent ground state representations of the twisted cocycles). The possible cases quoted then follow. These cases are clearly distinguished by a simple count of the number of states of conformal weight one. (Note that the uniqueness of the twisted representation has been demonstrated previously in the case of the Leech lattice in [26] using specific features of this model, so allowing a demonstration of triality for the Monster theory. The arguments of [24] allow the result to be extended to encompass all even lattices. This is in the spirit of this paper of extending results for the Monster to a broader class of theories by abstracting the general properties required.) Thus, we deduce that $\mathscr{T}_0^+ \oplus \mathscr{T}_3^+$ and $\mathscr{V}_3^+ \oplus \mathscr{T}_3^+$ are equivalent as representations of $\mathscr{V}_0^+ \oplus \mathscr{V}_3^+$ and $\mathscr{V}_0^+ \oplus \mathscr{T}_0^+$ respectively, identified by σ. Since $\sigma(\mathscr{T}_0^+) = \mathscr{V}_3^+$, then $\mathscr{T}_3^+$ corresponds to $\mathscr{T}_3^+$. So σ can be extended by a map ρ from $\mathscr{T}_0^+ \to \mathscr{V}_3^+$ and $\mathscr{T}_3^+ \to \mathscr{T}_3^+$ into an automorphism of $\widetilde{\mathscr{H}}(\widetilde{\Lambda}_{\mathscr{C}})$, by the arguments at the end of Sect. 4. For $\psi \in \mathscr{V}_0^+$, $\phi \in \mathscr{V}_3^+$,

$$\rho V(\psi,z)\rho^{-1}\phi = V(\sigma\psi,z)\phi\,. \tag{7.32}$$

Considering (7.31) and (7.32) together, we deduce that acting on $\mathscr{T}_0^+$,

$$\beta V(\psi,z)\beta^{-1} = V(\psi,z)\,, \tag{7.33}$$

where $\beta = \sigma\rho$. But we know from the above arguments that the action of $\mathscr{V}_0^+$ on $\mathscr{T}_0^+$ is irreducible, and so it must follow from a Schur's lemma type argument that $\beta = \kappa 1$ for some $\kappa \in \mathbf{C}$. From the arguments at the end of Sect. 4, we see that we must have $\rho = \pm\sigma^{-1}$ since both ρ and σ are compatible with the conjugation and this fixes the map up to a sign. Therefore, we may choose $\rho = \sigma^{-1}$ on $\mathscr{T}_0^+$ and rename $\rho : \mathscr{T}_0^+ \to \mathscr{V}_3^+$ as σ. σ squares to 1 on $\mathscr{V}_0^+$, $\mathscr{V}_3^+$ and $\mathscr{T}_0^+$, and so to demonstrate that it is an involution, we must check that $\sigma^2 = 1$ on $\mathscr{T}_3^+$. For $\psi \in \mathscr{V}_3^+$, $\phi \in \mathscr{T}_0^+$,

$$\sigma^2 V(\psi,z)\phi = \sigma V(\sigma\psi,z)\sigma\phi = V(\sigma^2\psi,z)\sigma^2\phi = V(\psi,z)\phi\,, \tag{7.34}$$

and so, since $\mathscr{V}_0^+$ acts irreducibly on $\mathscr{T}_3^+$, (7.34) is equivalent to saying that $\sigma^2 = 1$ on $\mathscr{T}_3^+$. This establishes that σ is the triality operator postulated, i.e., an automorphism of order 2 of $\widetilde{\mathscr{H}}(\tilde{\Lambda}_{\mathscr{C}})$ such that it maps $\mathscr{V}_0^+$ and $\mathscr{T}_3^+$ into themselves, and interchanges $\mathscr{V}_3^+$ and $\mathscr{T}_0^+$. It is defined up to an automorphism α (induced by ι on $\mathscr{H}(\tilde{\Lambda}_{\mathscr{C}})$) equal to 1 on $\mathscr{V}_0^+ \oplus \mathscr{T}_3^+$ and -1 on $\mathscr{V}_3^+ \oplus \mathscr{T}_0^+$, with which it commutes.

8. Extension to a Cubic Group

Renaming the triality operator σ constructed in the previous section of σ_3, we may also construct, by permuting ζ_1^j, ζ_2^j and ζ_3^j, automorphisms σ_1 and σ_2 of $\mathscr{H}(\Lambda_{\mathscr{C}})$. Then 1, σ_a, $\sigma_1\sigma_2$, $\sigma_1\sigma_3\iota_b$, $1 \leqq a,\ b \leqq 3$, (where $\iota_1 = 1$ on $\mathscr{V}_0^+ \oplus \mathscr{V}_0^-$ and -1 on $\mathscr{V}_1^+ \oplus \mathscr{V}_1^-$ and ι_2 and ι_3 are defined by cyclically permuting $\mathscr{V}_0^-, \mathscr{V}_1^+$ and $\mathscr{V}_1^-$) generate a group isomorphic to the symmetry group of the cube, S_4. For (a,b,c) a permutation of $(1,2,3)$,

$$\sigma_a\sigma_b\sigma_a = \iota_c\sigma_c, \qquad \iota_a\iota_b = \iota_c, \qquad \sigma_a\iota_a = \iota_a\sigma_a, \qquad \sigma_a\iota_b = \iota_c\sigma_a\,. \tag{8.1}$$

There exist subgroups isomorphic to S_3, e.g. $\{1,\sigma_1,\sigma_2\iota_2,\sigma_3,\sigma_1\sigma_2\iota_2,\sigma_1\sigma_3\}$.

We therefore extend the spaces which were considered in the previous section to give the following "magic square" diagram:

$$\begin{array}{cccccccccc}
 & & & & (1) & & (2) & & (3) \\
 & & & & \mathscr{H}(\Lambda_{\mathscr{C}}) & & \widetilde{\mathscr{H}}(\tilde{\Lambda}_{\mathscr{C}}) & & \widetilde{\mathscr{H}}'(\tilde{\Lambda}_{\mathscr{C}}) \\
 & & & & \| & & \| & & \| \\
 & & & & \mathscr{V}_0^+ & & \mathscr{V}_0^+ & & \mathscr{V}_0^+ \\
 & & & & \oplus & & \oplus & & \oplus \\
(1) & \mathscr{H}(\tilde{\Lambda}_{\mathscr{C}}) & = & \mathscr{V}_0^+ \;\oplus & \mathscr{V}_0^- & \oplus & \mathscr{V}_3^+ & \oplus & \mathscr{V}_3^- \\
 & & & & \oplus & & \oplus & & \oplus \\
(2) & \widetilde{\mathscr{H}}(\Lambda_{\mathscr{C}}) & = & \mathscr{V}_0^+ \;\oplus & \mathscr{V}_1^+ & \oplus & \mathscr{T}_0^+ & \oplus & \mathscr{T}_1^+ \\
 & & & & \oplus & & \oplus & & \oplus \\
(3) & \widetilde{\mathscr{H}}'(\tilde{\Lambda}_{\mathscr{C}}) & = & \mathscr{V}_0^+ \;\oplus & \mathscr{V}_1^- & \oplus & \mathscr{T}_3^+ & \oplus & \mathscr{T}_2^-
\end{array} \tag{8.2}$$

Let $\mathscr{V}_{ab}$ be the space in row a and column b and set $\mathscr{V}_{ab} = \mathscr{V}_0^+$ if $a \cdot b = 0$, for $0 \leqq a, b \leqq 3$. For (a,b,c) a permutation of $(1,2,3)$, $\sigma_a : \mathscr{V}_{01} \to \mathscr{V}_{01}, \mathscr{V}_{a1} \to \mathscr{V}_{a1}$, $\mathscr{V}_{b_1} \leftrightarrow \mathscr{V}_{c1}$, i.e., it defines an automorphism of $\mathscr{V}_{a0} \oplus \mathscr{V}_{a1}$ and an isomorphism $\mathscr{V}_{b0} \oplus \mathscr{V}_{b1} \to \mathscr{V}_{c0} \oplus \mathscr{V}_{c1}$. Define $\mathscr{H}_a$ to be the space given by row a and $\mathscr{H}^b$ to be the space given by column b. Then σ_a induces an automorphism of $\mathscr{H}_a$ and an isomorphism $\mathscr{H}_b \to \mathscr{H}_c$. The generalisation of the result that $\sigma : \mathscr{V}_3^+ \to \mathscr{T}_0^+$ which was required in the last section is that σ_a preserves the columns, in the sense that $\sigma_a : \mathscr{V}_{a\alpha} \to \mathscr{V}_{a\alpha}$ and $\sigma_a : \mathscr{V}_{b\alpha} \leftrightarrow \mathscr{V}_{c\alpha}$ for $\alpha = 1,2$. This induces an automorphism of the space $\mathscr{H}^2 = \widetilde{\mathscr{H}}(\widetilde{\Lambda}_{\mathscr{C}})$. In a similar way, an automorphism of the third column is induced. Note that σ_a, as an automorphism of the second column, is defined up to the involution ι_a, which is 1 on $\mathscr{V}_{02} \oplus \mathscr{V}_{a2}$ and -1 on $\mathscr{V}_{b2} \oplus \mathscr{V}_{c2}$. Thus ι_a commutes with σ_a. Either $\sigma_1\sigma_3\sigma_1$ is equal to σ_2 or it is equal to $\iota_2\sigma_2$, depending on the sign choices made in the definitions. Whichever choice is made though, $\sigma_1^2 = \sigma_3^2 = (\sigma_1\sigma_3\sigma_1)^2 = 1$, so that $\{1, \sigma_1, \sigma_3, \sigma_1\sigma_3\sigma_1, \sigma_1\sigma_3, \sigma_3\sigma_1\}$ forms a group isomorphic to S_3. Thus, we obtain a triality group of automorphisms of $\mathscr{H}(\widetilde{\Lambda}_{\mathscr{C}})$, explaining the origin of the term triality operator which has been used so far. In the case $\mathscr{C} = g_{24}$, the Golay code, the triality group, or just σ_3, then, as explained in Sect. 6.1, together with the extension of Conway's group defined there generates the Monster.

Note that if we reverse the situation of the previous section, and use the modified definition (7.8) if and only if d is an *even* multiple of 8, then the induced maps, say $\tilde{\sigma}_a$, interchange columns 2 and 3 rather than preserve them, i.e., the $\tilde{\sigma}_a$ induce isomorphisms $\mathscr{H}^2 \to \mathscr{H}^3$.

Finally in this section, we verify that σ_a does in fact preserve the columns as stated. In other words, we wish to show

Proposition 8.1. *For* $\alpha = 1,2$, *and* (a,b,c) *a permutation of* $(1,2,3)$,

$$\sigma_a : \mathscr{V}_{b\alpha} \to \mathscr{V}_{c\alpha}, \qquad \sigma_a : \mathscr{V}_{a\alpha} \to \mathscr{V}_{a\alpha}\,, \tag{8.3}$$

Proof. From the arguments of the previous section, we see that for σ_3 (and analogously for all the σ_a) that it either preserves or interchanges the columns, i.e., it satisfies either (8.3) or

$$\sigma_a : \mathscr{V}_{b\alpha} \to \mathscr{V}_{c\beta}, \qquad \sigma_a : \mathscr{V}_{a\alpha} \to \mathscr{V}_{a\beta}\,, \tag{8.4}$$

where $\alpha = 2$ and $\beta = 3$ or vice versa. Since, on $\mathscr{H}(\Lambda_{\mathscr{C}})$ we have $\sigma_a\sigma_b\sigma_a = \iota_a\sigma_c$ then we see that on $\mathscr{V}_{d2} \oplus \mathscr{V}_{d3}$, for $1 \leqq d \leqq 3$, we must have $\sigma_a\sigma_b\sigma_a = \pm\sigma_c$, thus showing that if either one of σ_b or σ_c preserves the columns, then both do, since with two applications of σ_a whether it preserves or interchanges the columns is of no consequence. Thus, we can deduce that σ_1, σ_2 and σ_3 preserve the columns if we can show that any one of them does. σ_1 is the simplest of the three to consider.

By cyclic permutation of the corresponding property for $\sigma \equiv \sigma_3$ from Sect. 7, we have the relations

$$\sigma_1\zeta_1^j = -\zeta_1^j, \qquad \sigma_1\zeta_2^j = \zeta_3^j, \qquad \sigma_1\zeta_3^j = \zeta_2^j\,, \tag{8.5}$$

or

$$\sigma_1 a_n^j \sigma_1^{-1} = -a_n^j, \qquad \sigma_1|\sqrt{2}e_j\rangle = i|-\sqrt{2}e_j\rangle, \qquad \sigma_1|-\sqrt{2}e_j\rangle = -i|\sqrt{2}e_j\rangle\,. \tag{8.6}$$

Its action on $V(\Psi_c, z)$ is given by an argument similar to that given in Sect. 7 in showing that σ mapped $\mathscr{V}_0^+$ to itself, i.e., for $e_j \cdot c = 1$, we use

$$[J_0^{j3}, V(\Psi_c, z)] = -\tfrac{1}{2}\varepsilon V(\Psi_{c'}, z), \qquad [J_0^{j3}, V(\Psi_{c'}, z)] = -\frac{1}{2}\varepsilon V(\Psi_c, z)\,, \tag{8.7}$$

together with (7.20) to give

$$\sigma_1^j \begin{pmatrix} V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix} = \exp\left\{ \frac{i\pi}{2\sqrt{2}} \begin{pmatrix} 0 & (i-1)\varepsilon \\ -(i+1)\varepsilon & 0 \end{pmatrix} \right\} \begin{pmatrix} V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix}$$
$$= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -(1+i)\varepsilon \\ (1-i)\varepsilon & 0 \end{pmatrix} \begin{pmatrix} (V(\Psi_c,z) \\ V(\Psi_{c'},z) \end{pmatrix}, \tag{8.8}$$

where

$$\sigma_1^j = \exp\left\{ \frac{i\pi}{\sqrt{2}} (J_0^{j2} + J_0^{j3}) \right\}, \tag{8.9}$$

(cf. (7.5)). So it follows by successive application of (8.8) that

$$\sigma_1 V(\Psi_c,z)\sigma_1^{-1} = (-1)^{\frac{1}{4}|c|} V(\Psi_{-c},z). \tag{8.10}$$

Together with (8.6), this gives

$$\sigma_1 = \theta \exp\{i\pi w \cdot p\}, \tag{8.11}$$

where $w = \frac{1}{2\sqrt{2}}\underline{1}$, which is in $\Lambda_3(\mathscr{C})$ for d a multiple of 16. This defines σ_1 on $\mathscr{H}(\Lambda_\mathscr{C})$, although it can also be used to define σ_1 on $\mathscr{V}_3^{\pm}$, giving an automorphism of $\mathscr{H}_1 = \mathscr{H}(\widetilde{\Lambda}_\mathscr{C})$ which preserves $\mathscr{V}_0^{\pm}$, $\mathscr{V}_3^{\pm}$. On the twisted spaces $\mathscr{T}_0^+, \mathscr{T}_1^+, \mathscr{T}_2^-$ and $\mathscr{T}_3^+ \sigma_1$ may be defined by the analogous relation

$$\sigma_1 = \theta \gamma_w . \tag{8.12}$$

This provides an isomorphism $\mathscr{H}_2 \to \mathscr{H}_3$ which preserves the columns, and the required result follows. When d is an odd multiple of 8, if we use the same definition of σ_1, then we have that $w \in \Lambda_2(\mathscr{C})$, and σ_1 (and hence σ_2 and σ_3) interchanges the columns, as noted above. Otherwise, we redefine σ_1^l for some l by analogy with (7.8), which serves to modify w to $\frac{1}{2\sqrt{2}}\underline{1} + \sqrt{2}e_l$, which is once more an element of $\Lambda_3(\mathscr{C})$, and hence σ_1 preserves the columns. (Making this modification when d is a multiple of 16 gives a σ_1 which interchanges the columns.) □

9. Conclusions

The main result of this paper is the demonstration that the remarkable results of Frenkel, Lepowsky and Meurman on the construction of the natural representation of the Monster group as a conformal field theory generalise to a wider class of theories. This generalization exhibits the features which lead to the existence of the "triality" structure more clearly, and specific features of particular models are not required.

Following in this spirit, the discussion of the structure and representations of chiral bosonic meromorphic conformal field theories and the construction of orbifolds illustrates a general program and approach which it is hoped to take further.

The nature of what were previously thought of as merely useful analogies of conformal field theory with the theories of lattices and codes has been extended to deeper links between their structures, and it will prove interesting in the future to extend the depth and, more importantly, the understanding of such connections.

Acknowledgement. Paul Montague is grateful to the S.E.R.C. for a Research Studentship, and to Gonville and Caius College for a Research Fellowship. The authors are also grateful to Klaus Lucke for useful comments.

Appendix

In this appendix we wish to define the gamma matrix algebra $\Gamma \equiv \Gamma(\Lambda_0^*) = \{\pm\gamma_\lambda : \lambda \in \Lambda_0^*\}$ required for the extension of the triality structure in Sect. 7. In order to do this, we must specify the symmetry factor $S(\lambda, \mu)$ [22] in

$$\gamma_\lambda \gamma_\mu = S(\lambda, \mu) \gamma_\mu \gamma_\lambda \,, \tag{A.1}$$

such that

$$S(\lambda, \nu) S(\mu, \nu) = S(\lambda + \mu, \nu), \qquad S(\lambda, \mu) = 1/S(\mu, \lambda), \qquad S(\lambda, \lambda) = 1 \,, \tag{A.2}$$

for $\lambda, \mu, \nu \in \Lambda_0^*$. The definition of [11] requires modification, since the lattice is not even in this case. Following [11], $S(\lambda, \mu) = (-1)^{\lambda \cdot \mu}$ for $\lambda, \mu \in \Lambda_{\mathscr{C}} = \Lambda_0 \cup \Lambda_1$ or $\widetilde{\Lambda}_{\mathscr{C}} = \Lambda_0 \cup \Lambda_3$. Making the choices $S(\sqrt{2}e_j, \frac{1}{2\sqrt{2}}\underline{1}) = i = -S(\frac{1}{2\sqrt{2}}\underline{1}, \sqrt{2}e_j)$ for $1 \leqq j \leqq d$ gives, from (A.2), $S(\lambda, \mu)$ as follows:

$$\begin{array}{ccccc} & \mu \in \Lambda_0 & \mu \in \Lambda_1 & \mu \in \Lambda_2 & \mu \in \Lambda_3 \\ \lambda \in \Lambda_0 & (-1)^{\lambda \cdot \mu} & (-1)^{\lambda \cdot \mu} & (-1)^{\lambda \cdot \mu} & (-1)^{\lambda \cdot \mu} \\ \lambda \in \Lambda_1 & (-1)^{\lambda \cdot \mu} & (-1)^{\lambda \cdot \mu} & -e^{i\lambda \cdot \mu\pi} & -e^{i\lambda \cdot \mu\pi} \\ \lambda \in \Lambda_2 & (-1)^{\lambda \cdot \mu} & e^{i\lambda \cdot \mu\pi} & (-1)^{\lambda \cdot \mu + 1} & -e^{i\lambda \cdot \mu\pi} \\ \lambda \in \Lambda_3 & (-1)^{\lambda \cdot \mu} & e^{i\lambda \cdot \mu\pi} & e^{i\lambda \cdot \mu\pi} & (-1)^{\lambda \cdot \mu} . \end{array} \tag{A.3}$$

If $\mathscr{X}$ is an irreducible representation of Γ, then we consider the division of $\mathscr{X}$ into irreducible representations of $\Gamma(\Lambda_{\mathscr{C}})$. Since $\Lambda_{\mathscr{C}}$ is an even self-dual lattice, we see from Appendix C of [11] that such a representation $\mathscr{X}(\Lambda_{\mathscr{C}}) \subset \Gamma$ is of dimension $2^{\frac{1}{2}d}$. Then $\mathscr{X}'(\Lambda_{\mathscr{C}}) = \gamma_w \mathscr{X}(\Lambda_{\mathscr{C}})$ for $w = \frac{1}{2\sqrt{2}}\underline{1}$ is also an irreducible representation space for $\mathscr{X}(\Lambda_{\mathscr{C}})$. Noting that $\gamma_w \mathscr{X}'(\Lambda_{\mathscr{C}}) = \mathscr{X}(\Lambda_{\mathscr{C}})$ (since $2w \in \Lambda_{\mathscr{C}}$ as $\underline{1} \in \mathscr{C}$ for $\mathscr{C}$ doubly-even) and that γ_w and $\Gamma(\Lambda_{\mathscr{C}})$ generate Γ, gives $\mathscr{X} = \mathscr{X}(\Lambda_{\mathscr{C}}) \oplus \mathscr{X}'(\Lambda_{\mathscr{C}})$. In other words, $\mathscr{X}$ is of dimension $2^{\frac{1}{2}d+1}$. Also, from Appendix C of [11], an irreducible representation of $\Gamma(\Lambda_0)$ has dimension $2^{\frac{1}{2}d-1}$, so similarly $\mathscr{X}(\Lambda_{\mathscr{C}}) = \mathscr{X}_0(\Lambda_0) \oplus \mathscr{X}_1(\Lambda_0)$, where $\mathscr{X}_0(\Lambda_0)$ and $\mathscr{X}_1(\Lambda_0)$ are irreducible representations of $\Gamma(\Lambda_0)$. Similarly, $\mathscr{X}'(\Lambda_{\mathscr{C}}) = \mathscr{X}_2(\Lambda_0) \oplus \mathscr{X}_3(\Lambda_0)$. Also, we have a decomposition of $\mathscr{X}$ into irreducible representations of $\mathscr{X}(\widetilde{\Lambda}_{\mathscr{C}})$, which must be different sums of the $\mathscr{X}_j(\Lambda_0)$ (because $\Gamma(\Lambda_{\mathscr{C}})$ and $\Gamma(\widetilde{\Lambda}_{\mathscr{C}})$ together generate Γ). Thus, we take $\mathscr{X}_0 \oplus \mathscr{X}_3$ and $\mathscr{X}_1 \oplus \mathscr{X}_2$ (dropping the explicit Λ_0 dependence) to be such sums.

The gauge choice $\gamma_\lambda = \gamma_{-\lambda}$ made in [11] cannot be extended to the whole of Γ. We have $\gamma_\lambda \gamma_{-\lambda} = \varepsilon(\lambda, -\lambda)\gamma_0 = \varepsilon(\lambda, -\lambda)1$. So $\gamma_\lambda = \gamma_{-\lambda}$ implies that $\gamma_\lambda^2 = \pm 1$, and so is a central element of Γ. From the table (A.3) we can see that this can only be true for $\lambda \in \Lambda_0$. With the definition of θ given in Sect. 7 however, i.e., inserting an additional -1 on $\mathscr{X}_2$, then $\theta\gamma_\lambda\theta\gamma_\lambda$ is central, and so, by irreducibility and Schur's lemma, must be a multiple of the identity. Since $\gamma_{-\lambda} = \pm\gamma_\lambda^{-1}$, from the above, then

$\gamma_\lambda = \pm\theta\gamma_{-\lambda}\theta^{-1}(\theta = \theta^{-1}$ as θ is an involution). By a choice of gauge, we take

$$\theta\gamma_{-\lambda}\theta^{-1} = \gamma_\lambda \,. \tag{A.4}$$

This gives the involution

$$\theta V(\psi, z)\theta^{-1} = V(\theta\psi, z)\,, \tag{A.5}$$

for $\psi \in \mathscr{H}(\Lambda_a)$, $a = 0, 1, 3$, acting on the full twisted space $\mathscr{T}_0^+ \oplus \mathscr{T}_1^+ \oplus \mathscr{T}_2^+ \oplus \mathscr{T}_3^+$, explaining the reasoning behind the apparently unnatural definition of θ. This still gives $\gamma_\lambda = \gamma_{-\lambda}$ restricted to $\mathscr{X}_0 \oplus \mathscr{X}_j$ for $j = 1, 3$ and $\lambda \in \Lambda_0 \cup \Lambda_j$, as we required for such self-dual lattices in Appendix B of [11].

To make the above analysis perhaps a bit clearer, an explicit construction is now given of $\Gamma \equiv \Gamma(\Lambda_0^*)$ in terms of $\Gamma(\Lambda_0)$. Suppose $\Gamma(\Lambda_0) = \{\pm s_\lambda : \lambda \in \Lambda_0\}$, $s_{2\nu}$ is central in $\Gamma(\Lambda_0)$, for $\nu \in \Lambda_{\mathscr{C}}$, as $2\nu \cdot \lambda$ is even for all $\lambda \in \Lambda_0$. So s_λ is proportional to $s_{\lambda+2\nu}$, and we arrange the gauge so that they coincide, i.e.,

$$s_\lambda s_\mu = (-1)^{\lambda\cdot\mu} s_\mu s_\lambda, \qquad s_\lambda^2 = (-1)^{\frac{1}{2}\lambda^2}, \qquad s_\lambda = s_{\lambda+2\nu} = s_{-\lambda}\,, \tag{A.6}$$

for $\lambda, \mu \in \Lambda_0$. Choose $\kappa \in \Lambda_1$ and $\rho \in \Lambda_3$ with $\kappa \cdot \rho = \frac{1}{2}$. Then $S(\kappa, \rho) = -S(\rho, \kappa) = i$, from the table. We can take $s_{2\rho} = 1$ as it is central in $\Gamma(\Lambda_0)$ (since $2\rho \cdot \lambda$ is even for all $\lambda \in \Lambda_0$). Then we have a representation of $\Gamma(\Lambda_{\mathscr{C}}) = \{\pm\beta_\nu : \nu \in \Lambda_{\mathscr{C}}\}$:

$$\beta_\lambda = \begin{pmatrix} s_\lambda & 0 \\ 0 & \varepsilon_\lambda^\kappa s_\lambda \end{pmatrix}, \qquad \beta_{\lambda+\kappa} = \begin{pmatrix} 0 & \varepsilon_\lambda^\kappa s_\lambda \\ s_\lambda & 0 \end{pmatrix}, \tag{A.7}$$

for $\lambda \in \Lambda_0$, $\varepsilon_\lambda^\kappa = (-1)^{\kappa\cdot\lambda}$. This satisfies $\beta_\lambda = \beta_{-\lambda}$ and $\beta_\lambda^2 = (-1)^{\frac{1}{2}\lambda^2}$, as required. A representation of $\Gamma(\Lambda_0^*)$ is then given by defining

$$\gamma_\lambda = \begin{pmatrix} s_\lambda & 0 & 0 & 0 \\ 0 & \varepsilon_\lambda^\kappa s_\lambda & 0 & 0 \\ 0 & 0 & \varepsilon_\lambda^{\kappa+\omega} s_\lambda & 0 \\ 0 & 0 & 0 & \varepsilon_\lambda^\omega s_\lambda \end{pmatrix},$$

$$\gamma_{\lambda+\kappa} = \begin{pmatrix} 0 & \varepsilon_\lambda^\kappa s_\lambda & 0 & \\ s_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varepsilon_\lambda^\omega s_\lambda \\ 0 & 0 & i\varepsilon_\lambda^{\kappa+\omega} s_\lambda & 0 \end{pmatrix},$$

$$\gamma_{\lambda+\omega} = \begin{pmatrix} 0 & 0 & 0 & \varepsilon_\lambda^\omega s_\lambda \\ 0 & 0 & -\varepsilon_\lambda^{\kappa+\omega} s_\lambda & 0 \\ 0 & \varepsilon_\lambda^\kappa s_\lambda & 0 & 0 \\ s_\lambda & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_{\lambda+\kappa+\omega} = \begin{pmatrix} 0 & 0 & i\varepsilon_\lambda^{\kappa+\omega} s_\lambda & 0 \\ 0 & 0 & 0 & -i\varepsilon_\lambda^\omega s_\lambda \\ s_\lambda & 0 & 0 & 0 \\ 0 & -\varepsilon_\lambda^\kappa s_\lambda & 0 & 0 \end{pmatrix}, \tag{A.8}$$

with $\varepsilon_\lambda^\omega = (-1)^{\omega\cdot\lambda}$ and $\varepsilon_\lambda^{\kappa+\omega} = \varepsilon_\lambda^\kappa \varepsilon_\lambda^\omega$. This gives $\theta\gamma_{-\lambda}\theta^{-1} = -\gamma_\lambda$ for $\lambda \in \Lambda_2$, although (A.4) holds for the remaining sectors. This could be corrected by a change of gauge, although $\mathscr{H}(\Lambda_2)$ need not be considered anyway, since it corresponds to a fermionic conformal field theory.

Finally, we consider the cocycles for $\Lambda_{\mathscr{C}}$ necessary for defining the triality operator σ. Set

$$\eta_c = \prod_{j:c_j=1} \gamma_{\sqrt{2}e_j}\,, \tag{A.9}$$

for $c \in \mathscr{C}$. Then $\eta_c = \pm 1$, since it is proportional to $\gamma_{\sqrt{2}c}$, which is central. We wish to choose the gauge such that $\eta_c = 1$ for all $c \in \mathscr{C}$, and also preserve the gauge choice $\gamma_\lambda = \gamma_{-\lambda}$. This means that we may change $\gamma_{\sqrt{2}e_j}$ by a factor ε_j only if we change $\gamma_{-\sqrt{2}e_j}$ by the same factor. Let $\mathscr{C}_1 = \{c \in \mathscr{C} : \eta_c = 1\}$. This is a sub-code of $\mathscr{C}$, since $\eta_{c+c'} = \eta_c \eta_{c'}$, where $c + c'$ is performed modulo 2 (i.e., inside $\mathscr{C}$) (since the $\gamma_{\sqrt{2}e_j}$ commute). Then $\mathscr{C} = \mathscr{C}_1$ or $\mathscr{C} = \mathscr{C}_1 \cup (\mathscr{C}_1 + c_o)$, where $\eta_{c_0} = -1$. In the case $\mathscr{C}_1 \neq \mathscr{C}$, $\mathscr{C}_i^* \supset \mathscr{C}$. Choose $c_2 \in \mathscr{C}_1^*$ such that $c_2 \notin \mathscr{C}$. Then $c_2 \cdot c \in 2\mathbf{Z}$ for all $c \in \mathscr{C}_1$ and $c_2 \cdot c_0 \in 2\mathbf{Z} + 1$ (otherwise $c_2 \in \mathscr{C}^* = \mathscr{C}$). Set $\varepsilon_j = -1$ for $(c_2)_j = 1$ and $\varepsilon_j = 1$ otherwise. Then $\eta_c = 1$ for all $c \in \mathscr{C}$ as required. If $\eta(c, c')$ is defined as in (5.23), then

$$\left(\prod_{c'_j=1} \gamma_{\sqrt{2}e_j}\right)\gamma_{\frac{c}{\sqrt{2}}} = \eta(c, -c')\gamma_{\frac{c'}{\sqrt{2}}}\,. \tag{A.10}$$

Noting $\gamma^2_{\frac{c}{\sqrt{2}}} = \gamma^2_{\frac{c'}{\sqrt{2}}} = (-1)^{\frac{1}{4}|c|}$, multiply (5.23) and (A.10) to give

$$\eta(c, c')\eta(c, -c') = \left(\prod_{c_j \neq c'_j} \gamma_{\sqrt{2}e_j}\right)\gamma_{\frac{c}{\sqrt{2}}}\left(\prod_{c_j \neq c'_j} \gamma_{\sqrt{2}e_j}\right)\gamma_{\frac{c}{\sqrt{2}}}(-1)^{\frac{1}{4}|c|} = \eta_c(-1)^{n(c,c')}\,, \tag{A.11}$$

and (5.25) follows.

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Communicated by R.H. Dijkgraaf