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Fredholm Determinants and the mKdV/ Sinh–Gordon Hierarchies

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Abstract: For a particular class of integral operators K we show that the quantity

$$\phi := \log \det(I+K) - \log \det(I-K)$$

satisfies both the integrated mKdV hierarchy and the Sinh–Gordon hierarchy. This proves a conjecture of Zamolodchikov.

I. Introduction

In recent years it has become apparent that there is a fundamental connection between certain Fredholm determinants and total systems of differential equations. This connection first appeared in work on the scaling limit of the 2-point correlation function in the 2D Ising model [7, 15] and the subsequent generalization to n-point correlations and holonomic quantum fields [12]. In applications the Fredholm determinants are either correlation functions or closely related to correlation functions in various statistical mechanical or quantum field-theoretic models. In the simplest of cases the differential equations are one of the Painlevé equations. Some, but by no means a complete set of, references to these further developments are [2–5, 13, 14, 16] The review paper [6] can be consulted for more examples of this connection.

In recent work by the present authors on random matrices, techniques were developed that gave simple proofs of the connection between a large class of Fredholm determinants and differential equations [13, 14]. In this paper we show how the philosophy of [3, 5, 13, 14] can be applied to study Fredholm determinants which are associated with operators K having kernel of the form

$$K(x,y) = \frac{E(x)\,E(y)}{x+y}\ ,$$

where

$$E(x) = e(x)\,\exp\left(\sum \tfrac{1}{2}\, t_k\, x^k\right)\ .$$

The (finite) sum is taken over odd positive and negative integers k. The domain of integration for the operator is $(0, \infty)$, and the function $e(x)$ can be very general. All that is required is that the operator be trace class for a range of values of the t_k so the Fredholm determinants are defined. The quantity of interest is

$$\phi := \log\det(I+K) - \log\det(I-K)\ . \tag{1}$$

We shall show that ϕ satisfies the equations of the integrated mKdV hierarchy if t_1 is the space variable and $t_3, t_5, \ldots$ the time variables, and that it satisfies the Sinh–Gordon hierarchy when $t_{-1}, t_{-3}, \ldots$ are the time variables.

To state the results precisely, the first assertion is that for $n \geqq 1$,

$$\frac{\partial\phi}{\partial t_{2n+1}} = \left(D^2 - 4\frac{\partial\phi}{\partial t_1}D^{-1}\frac{\partial\phi}{\partial t_1}D\right)^n \frac{\partial\phi}{\partial t_1}\ , \tag{2}$$

where D denotes $\partial/\partial t_1$ and D^{-1} denotes the antiderivative which vanishes at $t_1 = -\infty$. (Observe that ϕ and all its derivatives vanish at $t_1 = -\infty$.) This is the integrated mKdV hierarchy of equations,

$$\frac{\partial\phi}{\partial t_3} = \frac{\partial^3\phi}{\partial t_1^3} - 2\left(\frac{\partial\phi}{\partial t_1}\right)^3,$$

$$\frac{\partial\phi}{\partial t_5} = \frac{\partial^5\phi}{\partial t_1^5} - 10\left(\frac{\partial^2\phi}{\partial t_1^2}\right)^2\frac{\partial\phi}{\partial t_1} - 10\left(\frac{\partial\phi}{\partial t_1}\right)^2\frac{\partial^3\phi}{\partial t_1^3} + 6\left(\frac{\partial\phi}{\partial t_1}\right)^5,$$

etc. (In general there are constant factors on the left sides which can be removed by changes of scale in the time variables; e.g. [1].)

To go in the other direction we introduce the inverse of the operator appearing in (2), which is given by

$$\left(D^2 - 4\frac{\partial\phi}{\partial t_1}D^{-1}\frac{\partial\phi}{\partial t_1}D\right)^{-1} = \frac{1}{2}(D^{-1}e^{2\phi}D^{-1}e^{-2\phi} + D^{-1}e^{-2\phi}D^{-1}e^{2\phi})\ . \tag{3}$$

(Precisely, this is the inverse in a suitable space of functions. See Lemma 4 below.) We shall show that for $n \geqq 1$ we have the Sinh–Gordon hierarchy of equations

$$\frac{\partial\phi}{\partial t_{-2n+1}} = 2^{-n}(D^{-1}e^{2\phi}D^{-1}e^{-2\phi} + D^{-1}e^{-2\phi}D^{-1}e^{2\phi})^n\frac{\partial\phi}{\partial t_1}\ . \tag{4}$$

The case $n = 1$ of this is equivalent to the Sinh–Gordon equation

$$\frac{\partial^2\phi}{\partial t_{-1}\partial t_1} = \frac{1}{2}\sinh 2\phi\ . \tag{5}$$

Observe that (2) and (4) can be combined into the single statement that either of them holds for all values of the integer n. Further observe that these results hold independently of the function $e(x)$ appearing in the kernel $K(x, y)$. The function $e(x)$ affects the boundary conditions for (2) and (4) at $t_k = -\infty$.

That ϕ satisfies the integrated mKdV hierarchy was conjectured in [16], and that it satisfies the Sinh–Gordon equation (5) was conjectured in [16] and proved in [2].

A related identity,

$$-\frac{\partial^2}{\partial t_{-1}\partial t_1}\log\det(I-K)=\frac{e^{2\phi}-1}{4}\,, \tag{6}$$

was also conjectured in [16] and proved in [2], and will be rederived here.

We prove our results by expressing all relevant quantities in terms of inner products

$$u_{i,j}:=((I-K^2)^{-1}E_i,\,E_j)\,,\qquad v_{i,j}:=((I-K^2)^{-1}K\,E_i,\,E_j)\,, \tag{7}$$

where $E_i(x):=x^i\,E(x)$, and showing that these quantities satisfy nice differentiation and recursion formulas. Observe that both $u_{i,j}$ and $v_{i,j}$ are symmetric in the indices, since the operator K is symmetric. That these inner products are basic is expected from earlier investigations; e.g. [3, 5, 13, 14].

II. Recursion and Differentiation Formulas

If we denote by M multiplication by the independent variable, then the form of the kernel of K shows that

$$MK+KM=E\otimes E\,, \tag{8}$$

where in general we denote by $X\otimes Y$ the operator with kernel $X(x)Y(y)$. Applying this twice we see that, with brackets denoting the commutator as usual,

$$[M,\,K^2]=E\otimes KE-KE\otimes E\,.$$

It follows immediately that if $Q_i:=(I-K^2)^{-1}E_i$ and $P_i:=(I-K^2)^{-1}KE_i$, then

$$[M,\,(I-K^2)^{-1}]=Q_0\otimes P_0-P_0\otimes Q_0\,.$$

Applying these operators to the function E_j gives the recursion formula

$$x\,Q_j(x)-Q_{j+1}(x)=Q_0(x)\,v_j-P_0(x)\,u_j\,, \tag{9}$$

where we write u_j for $u_{j,0}$ and v_j for $v_{j,0}$. Taking inner products with E_i gives

$$u_{i+1,\,j}-u_{i,\,j+1}=u_i\,v_j-v_i\,u_j\,. \tag{10}$$

To obtain the analogous relations for the $v_{i,j}$ we temporarily define

$$w_i:=((I-K^2)^{-1}KE,\,KE_i)\,,$$

and take inner products with KE_i in (9), obtaining

$$(MKE_i,\,Q_j)-v_{i,\,j+1}=v_i\,v_j-w_i\,u_j\,.$$

The identity $(I-K^2)^{-1}K^2=(I-K^2)^{-1}-I$ gives

$$w_i=u_i-(E,\,E_i),$$

and by (8)

$$(MKE_i,\,Q_j)=-(KE_{i+1},\,Q_j)+(E,\,E_i)(E,\,Q_j)=-v_{i+1,\,j}+(E,\,E_i)\,u_j.$$

Thus we obtain

$$v_{i+1,\,j} + v_{i,\,j+1} = u_i\, u_j - v_i\, v_j\ . \tag{11}$$

For the differentiation formulas we use the fact

$$\frac{\partial}{\partial t_k} E(x)\,E(y) = \frac{1}{2}\,(x^k + y^k)\,E(x)\,E(y)$$

and elementary algebra to deduce that for $k > 0$,

$$\frac{\partial K}{\partial t_k} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i\, E_i \otimes E_j, \qquad \frac{\partial K}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-k-1} (-1)^{i+1}\, E_i \otimes E_j. \tag{12}$$

In the first sum we take $i, j \geqq 0$ and in the second $i, j \leqq -1$. This will be our convention throughout. (Here we use the fact that k is odd; the reader will find other such places later.) Since, with $t = t_k$ or t_{-k},

$$\frac{\partial \phi}{\partial t} = \text{tr}\,(I+K)^{-1}\frac{\partial K}{\partial t} + \text{tr}\,(I-K)^{-1}\frac{\partial K}{\partial t} = 2\ \text{tr}(I-K^2)^{-1}\frac{\partial K}{\partial t}\ ,$$

we find that

$$\frac{\partial \phi}{\partial t_k} = \sum_{i+j=k-1} (-1)^i\, u_{i,\,j}\ , \qquad \frac{\partial \phi}{\partial t_{-k}} = \sum_{i+j=-k-1} (-1)^{i+1}\, u_{i,\,j}\ . \tag{13}$$

Notice especially the important fact

$$\frac{\partial \phi}{\partial t_1} = u_0\ . \tag{14}$$

To obtain differentiation formulas for the $u_{i,\,j}$ and $v_{i,\,j}$ themselves we use

$$\frac{\partial}{\partial t_k}(I-K^2)^{-1} = (I-K^2)^{-1}\frac{\partial K^2}{\partial t_k}(I-K^2)^{-1}$$

and, by (12),

$$\frac{\partial K^2}{\partial t_k} = K\,\frac{\partial K}{\partial t_k} + \frac{\partial K}{\partial t_k}\,K = \frac{1}{2}\sum_{i+j=k-1} (-1)^i\,(K\,E_i \otimes E_j + E_i \otimes K\,E_j)$$

to deduce

$$\frac{\partial}{\partial t_k}(I-K^2)^{-1} = \frac{1}{2}\sum_{i+j=k-1} (-1)^i\,(P_i \otimes Q_j + Q_i \otimes P_j)\ .$$

From this and the fact $\partial E_i/\partial t_k = \frac{1}{2}E_{i+k}$ we deduce from the definition (7) that

$$\frac{\partial u_{p,\,q}}{\partial t_k} = \frac{1}{2}\sum_{i+j=k-1} (-1)^i\,(u_{p,\,j}\,v_{q,\,i} + v_{p,\,j}\,u_{q,\,i}) + \frac{1}{2}\,(u_{p+k,\,q} + u_{p,\,q+k})\ . \tag{15}$$

If we introduce $R_i := (I-K^2)^{-1}K^2E_i = Q_i - E_i$, then we find similarly first

$$\begin{aligned}\frac{\partial}{\partial t_k}(I-K^2)^{-1}K &= \frac{1}{2}\sum_{i+j=k-1} (-1)^i\,(Q_i \otimes R_j + P_i \otimes P_j) + \frac{1}{2}\sum_{i+j=k-1} (-1)^i\, Q_i \otimes E_j\\ &= \frac{1}{2}\sum_{i+j=k-1} (-1)^i\,(Q_i \otimes Q_j + P_i \otimes P_j)\ ,\end{aligned}$$

and then

$$\frac{\partial v_{p,q}}{\partial t_k} = \frac{1}{2} \sum_{i+j=k-1} (-1)^i (u_{p,j}\, u_{q,i} + v_{p,j}\, v_{q,i}) + \frac{1}{2} (v_{p+k,q} + v_{p,q+k}) \,. \tag{16}$$

In a completely analogous fashion, using the second part of (12), we obtain formulas for differentiation with respect to the t_{-k}:

$$\frac{\partial u_{p,q}}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-k-1} (-1)^{i+1} (u_{p,j}\, v_{q,i} + v_{p,j}\, u_{q,i}) + \frac{1}{2} (u_{p-k,q} + u_{p,q-k}) \,, \tag{17}$$

$$\frac{\partial v_{p,q}}{\partial t_{-k}} = \frac{1}{2} \sum_{i+j=-k-1} (-1)^{i+1} (u_{p,j}\, u_{q,i} + v_{p,j}\, v_{q,i}) + \frac{1}{2} (v_{p-k,q} + v_{p,q-k}) \,. \tag{18}$$

III. The mKdV Hierarchy

We begin by showing how to derive the first of the integrated mKdV equations,

$$\frac{\partial \phi}{\partial t_3} = \frac{\partial^3 \phi}{\partial t_1^3} - 2 \left(\frac{\partial \phi}{\partial t_1} \right)^3 .$$

This will illustrate the procedure. By (14) $\partial\phi/\partial t_1 = u_0$, and we differentiate twice more with respect to t_1, using (15) and (16). We find that the quantities u_0, u_1, $u_{1,1}$, v_0 and v_1 arise. But the recursion formulas (10) and (11) allow us to express two of these in terms of the others:

$$v_1 = (u_0^2 - v_0^2)/2, \qquad u_{1,1} = u_2 + u_0\, v_1 - u_1\, v_0 = u_2 + \frac{1}{2}\, u_0\, (u_0^2 - v_0^2) - u_1\, v_0 \,.$$

In the end the formula becomes

$$\frac{\partial^3 \phi}{\partial t_1^3} = \frac{3}{2}\, u_0^3 + \frac{1}{2}\, u_0\, v_0^2 + u_1\, v_0 + u_2 \,.$$

Now from (13), $\partial\phi/\partial t_3 = 2\,u_2 - u_{1,1}$ and by the above representation of $u_{1,1}$ this may be written

$$\frac{\partial \phi}{\partial t_3} = -\frac{1}{2} u_0^3 + \frac{1}{2} u_0\, v_0^2 + u_1\, v_0 + u_2 \,.$$

This gives

$$\frac{\partial^3 \phi}{\partial t_1^3} - \frac{\partial \phi}{\partial t_3} = 2\, u_0^3 = 2 \left(\frac{\partial \phi}{\partial t_1} \right)^3 ,$$

which is the desired equation.

The proof of the general formula (2) follows from a series of three lemmas.

Lemma 1. *We have*

$$2\, u_0\, \frac{\partial u_0}{\partial t_k} = \frac{\partial}{\partial t_1} \sum_{i+j=k-1} (-1)^i\, u_i\, u_j \,. \tag{19}$$

Proof. We begin by noting that from (15)

$$\frac{\partial u_0}{\partial t_k} = \sum_{i+j=k-1} (-1)^i \, u_i \, v_j + u_k$$

and, from (15), (16), (10) and (11),

$$\frac{\partial u_p}{\partial t_1} = u_0 \, v_p + u_{p+1}, \qquad \frac{\partial v_p}{\partial t_1} = u_0 \, u_p \, . \tag{20}$$

We find that the right side of (19) equals

$$\sum_{i+j=k-1} (-1)^i \, [u_i \, (u_0 \, v_j + u_{j+1}) + u_j \, (u_0 \, v_i + u_{i+1})]$$

$$= 2 \, u_0 \sum_{i+j=k-1} (-1)^i \, u_i \, v_j + 2 \sum_{i+j=k-1} (-1)^i \, u_i \, u_{j+1} \, .$$

The last sum equals

$$u_0 \, u_k - u_1 \, u_{k-1} + u_2 \, u_{k-2} - \cdots - u_{k-2} \, u_2 + u_{k-1} \, u_1 = u_0 \, u_k \, .$$

It follows that the right side of (19) equals the left side of (19).

Lemma 2. *We have*

$$2 \, v_k = \sum_{i+j=k-1} (-1)^i \, (u_i \, u_j - v_i \, v_j) \, . \tag{21}$$

Proof. By the recursion formulas (11),

$$v_{k,0} + v_{k-1,1} = u_0 \, u_{k-1} - v_0 \, v_{k-1}$$

$$-(v_{k-1,1} + v_{k-2,2}) = -(u_1 \, u_{k-2} - v_1 \, v_{k-2})$$

$$\vdots$$

$$-(v_{2,k-2} + v_{1,k-1}) = -(u_{k-2} \, u_1 - v_{k-2} \, v_1)$$

$$v_{1,k-1} + v_{0,k} = u_{k-1} \, u_0 - v_{k-1} \, v_0 \, .$$

Adding gives (21).

Lemma 3. *We have for* $k \geqq 1$,

$$\frac{\partial \phi}{\partial t_{k+2}} = D \frac{\partial u_0}{\partial t_k} - 4 \, u_0 \, D^{-1} \left(u_0 \, \frac{\partial u_0}{\partial t_k} \right) . \tag{22}$$

Proof. By Lemma 1 and the differentiation formula (15) the right side of (22) equals

$$\frac{\partial}{\partial t_1} \left(\sum_{i+j=k-1} (-1)^i \, u_i \, v_j + u_k \right) - 2 \, u_0 \sum_{i+j=k-1} (-1)^i \, u_i \, u_j \, ,$$

and by (20) this equals

$$\sum_{i+j=k-1} (-1)^i (u_i\, u_0\, u_j + u_0\, v_i\, v_j + u_{i+1}\, v_j) + u_0\, v_k + u_{k+1} - 2\, u_0 \sum_{i+j=k-1} (-1)^i\, u_i\, u_j$$

$$= u_0 \sum_{i+j=k-1} (-1)^i (v_i\, v_j - u_i\, u_j) + \sum_{i+j=k-1} (-1)^i\, u_{i+1}\, v_j + u_0\, v_k + u_{k+1}\, .$$

This is the right side of (22). By (13) the left side equals

$$u_{k+1} - (u_{1,\,k} - u_{2,\,k-1}) - (u_{3,\,k-2} - u_{4,\,k-3}) - \cdots - (u_{k,\,1} - u_{k+1,\,0})\, ,$$

and by (10) this equals

$$u_{k+1} + (u_1\, v_{k-1} - u_{k-1}\, v_1) + (u_3\, v_{k-3} - u_{k-3}\, v_3) + \cdots + (u_k\, v_0 - u_0\, v_k)$$

$$= u_{k+1} - \sum_{i+j=k} (-1)^i\, u_i\, v_j = u_{k+1} + \sum_{i+j=k-1} (-1)^i\, u_{i+1}\, v_j - u_0\, v_k\, .$$

Thus the difference between the right and left sides of (22) equals

$$u_0 \sum_{i+j=k-1} (-1)^i (v_i\, v_j - u_i\, u_j) + 2\, u_0\, v_k\, ,$$

and by Lemma 2 this equals 0.

The proof of (2) is now immediate. In fact (22) may be rewritten

$$\frac{\partial \phi}{\partial t_{k+2}} = (D^2 - 4\, u_0\, D^{-1}\, u_0\, D)\, \frac{\partial \phi}{\partial t_k}\, , \tag{23}$$

and this together with (14) gives (2).

IV. The Sinh–Gordon Hierarchy

We begin by deriving (3).

Lemma 4. *The operator* $D^2 - 4\, u_0\, D^{-1}\, u_0\, D$ *is invertible in the space of smooth functions all of whose derivatives are rapidly decreasing as* $t_1 \to -\infty$, *and its inverse is given by* (3).

Remark. The function ϕ and all the $u_{i,\,j}$ and $v_{i,\,j}$ belong to the space of functions in the statement of the lemma.

Proof. We have

$$D^2 - 4\, u_0\, D^{-1}\, u_0\, D = (I - 4\, u_0\, D^{-1}\, u_0\, D^{-1})\, D^2\, .$$

Both factors on the right are invertible (the Neumann series inverts the first factor) so the operator on the left is also, and its inverse is equal to

$$\begin{aligned} D^{-2}\, (I - 4\, u_0\, D^{-1} u_0\, D^{-1})^{-1} &= \frac{1}{2} D^{-2} [(I - 2\, u_0\, D^{-1})^{-1} + (I + 2\, u_0\, D^{-1})^{-1}] \\ &= \frac{1}{2} D^{-1} [(D - 2\, u_0)^{-1} + (D + 2\, u_0)^{-1}]\, . \end{aligned}$$

Since $(D + p)^{-1} = e^{-D^{-1} p}\, D^{-1}\, e^{D^{-1} p}$ and $D^{-1}\, u_0 = \phi$, the above is equal to

$$\frac{1}{2} (D^{-1}\, e^{2\phi}\, D^{-1}\, e^{-2\phi} + D^{-1}\, e^{-2\phi}\, D^{-1}\, e^{2\phi})\, .$$

Lemma 5. *Relation* (23) *holds for* $k \leqq -1$.

The proof of this is almost exactly the same as for $k \geqq 1$ and so is omitted. Lemma 5 is equivalent to the statement that for $k = 1, 3, 5, \ldots$,

$$\frac{\partial \phi}{\partial t_{-k+2}} = (D^2 - 4\,u_0\,D^{-1}\,u_0\,D)\,\frac{\partial \phi}{\partial t_{-k}}\,,$$

or by (3),

$$\frac{\partial \phi}{\partial t_{-k}} = \frac{1}{2}(D^{-1}\,e^{2\phi}\,D^{-1}\,e^{-2\phi} + D^{-1}\,e^{-2\phi}\,D^{-1}\,e^{2\phi})\,\frac{\partial \phi}{\partial t_{-k+2}}\,.$$

This establishes (4) by induction.

The case $n = 1$ of (4) is

$$\frac{\partial \phi}{\partial t_{-1}} = \frac{1}{2}(D^{-1}\,e^{2\phi}\,D^{-1}\,e^{-2\phi} + D^{-1}\,e^{-2\phi}\,D^{-1}\,e^{2\phi})\,\frac{\partial \phi}{\partial t_1}\,,$$

which gives (keep in mind that D^{-1} is the antiderivative which vanishes at $-\infty$)

$$\begin{aligned} 4\,\frac{\partial^2 \phi}{\partial t_{-1}\partial t_1} &= 4\,D\,\frac{\partial \phi}{\partial t_{-1}} = 2\,(e^{2\phi}\,D^{-1}\,e^{-2\phi} + e^{-2\phi}\,D^{-1}\,e^{2\phi})\,\frac{\partial \phi}{\partial t_1} \\ &= e^{2\phi}\,(1 - e^{-2\phi}) + e^{-2\phi}\,(e^{2\phi} - 1) = 2\,\sinh\,2\phi\,. \end{aligned}$$

This is (5).

Finally we derive (6). By (17) we have

$$\frac{\partial^2 \phi}{\partial t_{-1}\partial t_1} = \frac{\partial u_0}{\partial t_{-1}} = u_{-1}\,(1 + v_{-1})\,,$$

and so we know that

$$u_{-1}\,(1 + v_{-1}) = \frac{1}{2}\,\sinh\,2\phi\,.$$

Now we use a special case of (11), $2\,v_{-1} = u_{-1}^2 - v_{-1}^2$, which has the more useful form

$$(1 + v_{-1})^2 = 1 + u_{-1}^2\,.$$

These equations can be solved for u_{-1} and v_{-1}, giving

$$u_{-1} = \sinh\phi, \qquad v_{-1} = \cosh\phi - 1\,. \tag{24}$$

Now we use the fact $(I - K)^{-1} = (I - K^2)^{-1} + (I - K^2)^{-1}K$ and (12) to obtain

$$-2\frac{\partial}{\partial t_1}\,\log\det\,(I - K) = ((I - K)^{-1}E,\,E) = u_0 + v_0\,.$$

Therefore by (17) and (18),

$$-2\frac{\partial^2}{\partial t_{-1}\partial t_1}\,\log\det\,(I - K) = u_{-1}(v_{-1} + 1 + u_{-1})\,.$$

Using (24) we find that the right side equals $(e^{2\phi} - 1)/2$, which gives (6).

Note added in proof. After this work was completed, the authors became aware of the work [8–11]) which also considers integral equations, similar to the ones considered here, which yield solutions of a broad class of nonlinear evolution equations. In these papers one finds methods for deriving differentiation formulas for quantities similar to our $u_{i,j}$ and $v_{i,j}$.

Using the Miura transformation,

$$u_0 \to u_0^2 + \frac{\partial u_0}{\partial t_1}$$

we can show that

$$2\frac{\partial^2}{\partial t_1^2} \log\det(I-K) = \frac{\partial}{\partial t_1}(u_0+v_0)$$

satisfies the KdV hierarchy.

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