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Percolation Techniques in Disordered Spin Flip Dynamics: Relaxation to the Unique Invariant Measure

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Abstract: We consider lattice spin systems with short range but random and unbounded interactions. We give criteria for ergodicity of spin flip dynamics and estimate the speed of convergence to the unique invariant measure. We find for this convergence a stretched exponential in time for a class of "directed" dynamics (such as in the disordered Toom or Stavskaya model). For the general case, we show that the relaxation is faster than any power in time. No assumptions of reversibility are made. The methods are based on relating the problem to an oriented percolation problem (contact process) and (for the general case) using a slightly modified version of the multiscale analysis of e.g. Klein (1993).

1. Introduction

Adding disorder to a system of many interacting particles may in general be a highly non-trivial perturbation. The study of spin flip systems with quenched disorder is certainly not so well developed as their corresponding versions without disorder. In this paper we show how percolation techniques can be useful for investigating that part of the phase diagram in which the disordered dynamics typically forgets about initial data. In this uniqueness regime (high temperature, high noise, low density, strong magnetic field, strong bias,...) the main problem consists in circumventing the dynamical consequences of the so-called Griffiths' singularities, see Griffiths (1969). A consequence of the disorder is that there will typically be large regions on which the spins are strongly interacting. This is completely analogous to the situation in equilibrium. What is far worse here however, is that these "bad" regions should be thought of as infinitely extending in the time direction, i.e., spins there will relax only very slowly in the course of time. Depending on the size and the "badness" of these regions, the relaxation time may become arbitrarily large. Therefore we cannot expect to see *typically* an exponential decay to the invariant measure.

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As far as we know, there exist very few rigorous results for dynamics governed by random parameters. Mostly they concern Glauber (or heat-bath) dynamics for spin glasses with nearest neighbor interactions. For one-dimensional spin glasses with unbounded pair interactions, Zegarlinski (1994) recently found a convergence to equilibrium faster than $c \exp -t^\delta$ with $\delta \in (0, 1)$ and c dependent on the realization. Moreover he argues that for any dimension this is the best possible result: the dynamics has no spectral gap. This assertion is confirmed by the extensive computer simulations of spin glasses by Ogielski (1985). He observed the time dependent correlation function $q(t) = \mathbf{E}(\langle \sigma_x(0), \sigma_x(t) \rangle)$ for large systems during long observation times and also found a stretched exponential decay. There, $q(t)$ is calculated in thermal equilibrium and is an average over all realizations. Randeria et al. (1985) argued for a lower bound for this same correlation function, by considering the contribution to the relaxation process of independent tightly coupled and unfrustrated clusters. The energy needed to flip a cluster with diameter L typically goes as $\sim \sigma L^{d-1}$, with σ the surface tension. The relaxation time thus goes as $\exp(-c\sigma L^{d-1})$. Using this and the fact that the probability to find such a cluster is $\sim \exp(-\lambda L^d)$ they get $q(t) > \exp -c(\log t)^{\frac{d}{d-1}}$. This does not agree with the simulations of Ogielski (1985). Palmer et al. (1984) introduced different degrees of freedom to elucidate the relaxation in strongly interacting glassy materials. They propose a serial relaxation process, slower modes are constrained by the faster ones: the former can't decay before the latter are finished. The formula they obtain for $q(t)$ contains two parameters that can be chosen so that a stretched exponential decay appears.

Here we do not restrict ourselves to *reversible* continuous time spin glasses. Also other continuous and discrete time spin flip dynamics are regarded. The common characteristic is a short range interaction governed by random and possibly unbounded parameters.

We describe the influence of the initial data on the asymptotic state via oriented percolation (contact process) in a random environment. For the subclass of dynamics in which the spatial dependence of the transition rates is "directed", we get a stretched exponential for the decay in time. For the more general case, we only get a decay faster than any power of time. An important ingredient already appeared in the work of Campanino–Klein (1991), Klein (1993) and Klein (1994).

The outline of this paper is as follows: Section 2 contains general definitions. Section 3 is devoted to the main results for both discrete time and continuous time spin flip dynamics. In the Appendix we collect the more technical arguments and modifications with respect to the multiscale analysis of Klein (1993).

2. Definitions

We consider spin systems defined on the regular d-dimensional lattice $\mathbb{Z}^d$. Nearest neighbor sites $x, y \in \mathbb{Z}^d$ are connected by bonds $\langle x, y \rangle$; we write $x \sim y$ if two sites x and y are nearest neighbors (or adjacent).

A configuration σ puts a spin value $\sigma(x) = 1$ or $\sigma(x) = -1$ on every site $x \in \mathbb{Z}^d$. $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ is the set of all configurations. Our results can easily be extended to other finite single site state spaces.

We study quenched disorder. This means that the disorder is frozen in the interactions or transition rates. The relevant objects for our analysis are the (random) transition kernels (see below) and we do not need to refer to specific forms of the interaction. We therefore write π to denote such a general (random) realization (of the disorder). Π is the set of all these realizations. $\mathbf{Q}$ is the probability law on the realizations. $\mathbf{E}$ the expectation value with respect to the distribution $\mathbf{Q}$.

Independent point percolation plays a crucial role in our construction. The points are the vertices of the space-time lattice associated to the dynamics. One independently assigns to every point a value 0 or 1 with a certain density. A point with value one is called open. An open path is a sequence of neighboring open points. We say that percolation occurs if there is a positive probability for the event that there exists an infinite open path. The percolation problem will be time-oriented: only paths with decreasing time coordinate on the space-time lattice are considered. For more details and definitions we refer to e.g. Grimmett (1989).

The densities at each point may (and frequently will) differ, depending on the (random) realization.

Some more notations:

$$\mathrm{dist}(f,g) = \min_{\substack{x\in \mathrm{supp} f\\ x\in \mathrm{supp} g}} |x-y| \ , \tag{1}$$

with supp$f$ the support of f and $|x-y| = \sum_{\alpha=1}^{d} |x_\alpha - y_\alpha|$. $\|f\|$ is the usual supremum norm of f and $\delta_x = \sup_\eta |f(\eta^x) - f(\eta)|$ the oscillation of f at $x \in \mathbb{Z}^d$. The total oscillation is then $|||f||| = \sum_{x\in\mathbb{Z}^d} \delta_x f$.

3. Dynamics with Random Transition Rates

3.1. Disordered Probabilistic Cellular Automata. A Probabilistic Cellular Automaton (PCA) is a parallel updating discrete time evolution $\sigma_n, n = 0, 1, \ldots$ on Ω. σ_n is a Markov process defined by transition probabilities $p_x(\pm 1, \eta)$ and for every $\Lambda \in \mathbb{Z}^d$,

$$\mathrm{Prob}[\sigma_n(x) = \xi(x), \forall x \in \Lambda | \sigma_{n-1} = \eta] = \prod_{x\in\Lambda} p_x(\xi(x)|\eta) \tag{2}$$

for $\xi, \eta \in \Omega$.

For simplicity we assume that the $p_x(\,\cdot\,|\eta)$ depend only on the nearest neighbor spins $\eta(y), y \sim x$ and on $\eta(x)$. While the dynamics is time homogeneous we add quenched disorder by letting the transition probabilities $\{p_x(\,\cdot\,|\eta)\}$ depend on the realization π of the disorder. It may for example enter in the coupling between $x \sim y$ or in the bias (see the examples below). There typically are extra parameters (such as the noise level or temperature) available to modify the behavior of our PCA for the same distribution of realizations.

We use P_π to denote the transfer operator of the PCA and $P_\pi^N = P_\pi^{N-1} P_\pi$. A probability measure ν_π on Ω is invariant if $\nu_\pi P_\pi = \nu_\pi$. The PCA is ergodic if there exists a unique invariant measure ν_π, such that for all probability measures μ on Ω,

$$\mu P_\pi^N \longrightarrow \nu_\pi \ . \tag{3}$$

See Lebowitz et al. (1990) for more details.

The PCA evolution is of course not deterministic because P_π applied to a configuration $\eta \in \Omega$ (the delta measure concentrated at η) gives the product measure with densities $\{p_x(\cdot\,|\eta)\}$. Our object of study is therefore a stochastic dynamics in which the degree and/or nature of the stochasticity (noise,...) itself is randomly determined (via the quenched disorder realized by π).

Define

$$k_x \equiv \max_{\eta,\eta'} \mathrm{var}(p_x(\,\cdot\,|\eta), p_x(\,\cdot\,|\eta')) \ , \tag{4}$$

where var(· , ·) is the variational distance. k_x is a function of the realization π but we assume that π only enters locally: $\{k_x\}$ is a one-dependent random field. In particular, the k_x, x in the even (odd) sublattice of $\mathbb{Z}^d$, are jointly independent. At the same time there typically will be large regions on which the k_x are large (close to one). This is similar to what happens in the equilibrium case (see Gielis–Maes (1995)) but, as remarked before, these regions are copied at every time step and therefore give rise to infinite "cylinders" on the space-time lattice. This is the reason why, contrary to the equilibrium case, we cannot allow a k_x to be one with positive probability.

Example 1 (Discrete time spin glass). Let $J_{xy} = J_{yx}$ for $x \sim y$ be an independent family of real-valued random couplings, the transition probability is

$$p_x(\sigma(x)|\eta) = \frac{1}{2}[1 + \sigma(x)\tanh(\beta \sum_{y\sim x} J_{xy}\eta(y))] \ . \tag{5}$$

Here,

$$k_x = \tanh \beta \sum_{y\sim x} |J_{xy}| \ . \tag{6}$$

Example 2 (A random version of Stavskaya's PCA). The $\{\gamma_x\}$ are independent and identically distributed non-negative random variables. The transition probability is

$$p_x(+1|\eta) = \begin{cases} 1 & \text{if } \eta(x) = \eta(x+1) = +1 \\ \mathrm{e}^{-\lambda\gamma_x} & \text{otherwise} \end{cases} \ . \tag{7}$$

In the case $\gamma_x = \gamma$ with γ large, the PCA has more than one invariant measure. Here,

$$k_x = 1 - \mathrm{e}^{-\lambda\gamma_x} \ . \tag{8}$$

We define the space-time graph of a PCA as follows. The set of vertices are the space-time points (x,n), $x \in \mathbb{Z}^d$, $n = 0,1,\ldots$ in the time-ordered stacking of the spatial lattice $\mathbb{Z}^d$; n is the time-coordinate. The graph is completed by drawing arrows (oriented edges) from each space-time point (x,n) to another point $(y,n-1)$ ($y = x$ or $y \sim x$ in so far that there is a non-trivial dependence of $p_x(+1|\eta)$ on $\eta(y)$ for some realization π).

A path ω on the space-time graph (starting at (x_0,N)) is an (oriented) sequence of space-time points $\omega : (x_0,N),(x_1,N-1),\ldots,(x_k,N-k)$, $x_l \in \mathbb{Z}^d, l = 0,\ldots,k < N$ in which at each step (x,n) to $(y,n-1)$ there is an arrow in the graph from the point (x,n) to the point $(y,n-1)$.

Consider now the independent oriented site percolation problem on this space-time graph with (random but highly correlated) densities $\{k_{(x,N)}\} = \{k_x\}$; each point (x,N) is open (closed) with probability k_x $(1-k_x)$. Let $G_\pi^N(x,y)$ denote the probability to have an open path from (x,N) to $(y,0)$.

The basic coupling of a PCA with itself is a new Markov process (a new PCA) (σ_N, σ'_N) on the product space $\Omega \times \Omega$, whose transition probabilities satisfy

$$\mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'_N(x)|\sigma_{N-1} = \eta, \sigma'_{N-1} = \eta'] = \mathrm{var}(p_x(\,\cdot\,|\eta), p_x(\,\cdot\,|\eta')) \leqq k_x \ . \tag{9}$$

Lemma 1. *In the basic coupling*

$$Prob_\pi[\sigma_N(x) \neq \sigma'_N(x) | \sigma_0, \sigma'_0] \leqq \sum_{y \in \mathbb{Z}^d} G_\pi^N(x,y) \ . \tag{10}$$

Proof. From (9) it follows that

$$\mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'_N(x) | \sigma_{N-1}, \sigma'_{N-1}] \neq 0 \tag{11}$$

if and only if $\sigma_{N-1}(x) \neq \sigma'_{N-1}(x)$ for some neighboring space-time point $(x, N-1)$. Hence,

$$\begin{aligned}\mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'_N(x) | \sigma_0, \sigma'_0] &\leqq \mathrm{Prob}_\pi[\text{There is a path of disagreement} \\ &\qquad \text{from } (x,N) \text{ to } (\,\cdot\,,0)] \\ &\leqq \sum_{y \in \mathbb{Z}^d} G_\pi^N(x,y) \ . \qquad \square\end{aligned} \tag{12}$$

Note that the basic coupling also has (as every coupling) the property that

$$|P_\pi^N f(\sigma) - P_\pi^N(\sigma')| \leqq \sum_x \delta_x f \mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'_N(x) | \sigma, \sigma'] \ . \tag{13}$$

Combining Lemma 1 with (13) and using that there is at least one invariant measure for the dynamics, we get Proposition 1 and see that all depends on how well we are able to control the connectivity function $G_\pi(\cdot\ ,\ \cdot)$. This is the same idea as in Gielis–Maes (1995) and is the dynamical version of the concept of disagreement percolation in van den Berg–Maes (1994).

Proposition 1. *For every local function f,*

$$\|P_\pi^N f - \nu_\pi(f)\| \leqq |||f||| \sup_{x \in \mathrm{supp} f} \sum_{y \in \mathbb{Z}^d} G_\pi^N(x,y) \ , \tag{14}$$

with ν_π an invariant measure for the dynamics.

3.1.1. Ergodicity for a "directed" PCA. We consider here PCA with transition probabilities of the form

$$p_x(\,\cdot\,|\eta) = p_x(\,\cdot\,|\eta_x, \eta(x+e_\alpha), \alpha = 1, \ldots, d) \ , \tag{15}$$

where the $\{e_\alpha\}$ are the unit vectors on $\mathbb{Z}^d$.

The nice thing about "directed" PCA is that its space-time graph is not only timelike oriented but it is also spatially directed : there is no path connecting (x,N) with (x,M) passing by another space-time point (y,n) with $y \neq x$. In particular, when we project this graph on the spatial lattice (by identifying all points (x,n), $n = 0,1,\ldots$ with $(x,0)$) we get an oriented spatial lattice. Every path $\omega : (x,N), (x_1, N-1), \ldots, (x_l, N-l), \ldots, (y,0)$ from the point (x,N) to the point $(y,0)$ on the space-time graph gives rise to a directed path on this spatial lattice. The opposite relation can also be uniquely defined if in addition we specify the times $n_l = 1,2,\ldots$ the path stays on the site x_l visited by the spatial path.

Here we restrict ourselves to PCA's for which $\{k_x\}$ is a set of independent identically distributed random variables. Stavskaya's PCA (Example 2) is a well-known example. Another one (in two dimensions) is Toom's model.

Example 3 (Toom's model). The transition probability is derived from a majority rule. With $\{\gamma_x\}$ as in Example 2:

$$p(\sigma(x)|\eta) = \frac{1}{2}[1 + \sigma(x)(1 - \mathrm{e}^{-\lambda\gamma_x})\mathrm{sgn}(\eta(x) + \eta(x + e_1) + \eta(x + e_2))] \ . \tag{16}$$

Then,

$$k_x = 1 - \mathrm{e}^{-\lambda\gamma_x} \ . \tag{17}$$

In Propositions 2 and 3 we estimate the connectivity functions $G_\pi^N(x, y)$ of a directed PCA. First we derive an upperbound uniform in N.

Proposition 2. *Suppose*

$$\mathbf{E}\left(\frac{k_x}{1 - k_x}\right) < \frac{1}{d} \ , \tag{18}$$

then

$$\mathbf{E}[G_\pi^N(x, y)] \leqq C_1 e^{-\lambda'|x-y|} \tag{19}$$

for constants $C_1 < \infty$, $\lambda' > 0$.

Proof. Let $|x - y| = m$. Every path $\omega : (x, N) \rightsquigarrow (y, 0)$ on the directed space-time graph gives rise to a spatial path $\omega' : x \rightsquigarrow y$ on $\mathbb{Z}^d$ of length $|\omega'| = m$. Following the constuction explained above we may thus write that

$$G_\pi^N(x, y) \leqq \sum_{\substack{\omega' : x \rightsquigarrow y \\ |\omega'| = m}} \ \sum_{\substack{l_1, \ldots, l_m \geqq 1 \\ l_1 + \ldots + l_m = N+1}} \ \prod_{i=1}^{m} k_{\omega'_i}^{l_i} \leqq \sum_{\substack{\omega' : x \rightsquigarrow y \\ |\omega'| = m}} \ \prod_{i=1}^{m} \frac{k_{\omega'_i}}{1 - k_{\omega'_i}} \ , \tag{20}$$

from which the conclusion readily follows. □

Remarks

1. From the Borel–Cantelli lemma we have from Proposition 2 that **Q**-a.s. there is a $N_{x,1} = N_{x,1}(\pi, \mathbf{E}(\frac{k_x}{1-k_x}), d) < \infty$ such that if $|x - y| > N_{x,1}$, then

$$G_\pi^N(x, y) \leqq \mathrm{e}^{-\lambda|x-y|} \ , \tag{21}$$

with $\lambda < \lambda'$.

2. The same conclusions remain valid in $d = 1$ under the assumption that

$$\mathbf{E}(\log \frac{k_x}{1 - k_x}) < 0 \ . \tag{22}$$

This is clear from (20) (but now there is only one path $\omega' : x \rightsquigarrow y$) by applying the strong law of large numbers.

Define l_x^* such that for all $l > l_x^*$, $k_x^l < \exp(-\lambda' l^\delta)$, with $0 \leqq \delta < 1$, i.e.

$$l_x^* = (\frac{-\lambda'}{\log k_x})^{(\frac{1}{1-\delta})} \ . \tag{23}$$

Lemma 2. *Take* $0 \leqq \beta \leqq 1$ *and* $v > 0$ *such that* $(1-\beta)(1-\delta)v > \beta d + 1$. *If*

$$\mathbf{Q}(k_x \geqq 1-\tau) < \tau^v , \tag{24}$$

then – with **Q**-*probability one – there exists a* $L_0 = L_0(\pi)$ *such that for* $L > L_0$,

$$l_x^* \leqq L^{1-\beta} \quad \textit{for all } x \in [-L^\beta, L^\beta]^d \cap \mathbb{Z}^d . \tag{25}$$

Proof. A straightforward calculation shows that

$$\mathbf{Q}\{l_x^* \leqq L^{1-\beta} : x \in [L^\beta, L^\beta]^d\} \geqq \left(1 - \left(1 - \exp(-\lambda' L^{(\delta-1)(1-\beta)v})\right)\right)^{(2L^\beta+1)^d} . \tag{26}$$

Remember that $e^{-x} \geqq 1-x$ for $0 \leqq x \leqq 1$ and that $\lambda' L^{(1-\beta)(\delta-1)} < 1$ for L large enough. So,

$$\mathbf{Q}\{\exists x \in [L^\beta, L^\beta]^d \cap \mathbb{Z}^d : l_x^* > L^{1-\beta}\} \leqq 1 - (1 - \lambda' L^{(1-\beta)(\delta-1)v})^{(2L^\beta+1)^d} . \tag{27}$$

This is summable if $(1-\beta)(1-\delta)v > \beta d + 1$. Hence we can use the Borel–Cantelli lemma to conclude. □

The second step in the study of $G_\pi^N(x,y)$ concerns deriving an upperbound uniformly in $|x-y|$.

Proposition 3. *If*

$$\mathbf{E}(\frac{k_x}{1-k_x}) < \frac{1}{d} , \tag{28}$$

then for every $0 < \varepsilon < 1/2$ *we have that if*

$$\mathbf{Q}\{k_x > 1-\tau\} < \tau^v , \tag{29}$$

with $v = v(\varepsilon)$ *high enough, there exist – with* **Q**-*probability one – a time* $N_{x,2} = N_{x,2}(\pi, d, \mathbf{E}(\frac{k_x}{1-k_x})) < \infty$ *and a constant* λ, *independent of* x *and* π *such that for* $N > N_{x,2}$,

$$G_\pi^N(x,y) \leqq \exp(-\lambda N^\varepsilon) . \tag{30}$$

Proof. For $0 < \beta < 1$ (will be specified later on) take $N'_{x,2}$ such that $(N'_{x,2})^\beta > N_{x,1}$ (see Proposition 2) and $\left((N'_{x,2})^{1-\beta} - 1\right)^{\frac{1}{1-\beta}} > L_0$ (see Lemma 2). Let $N > N'_{x,2}$. First we assume that $m = |x-y| < N^\beta$ and start again from

$$G_\pi^N(x,y) \leqq \sum_{\omega' : x \rightsquigarrow y} \sum_{\substack{l_1,\dots,l_m \geqq 1 \\ l_1+\dots+l_m = N+1}} \prod_{i=1}^m k_{\omega'_i}^{l_i} . \tag{31}$$

In every path ω there is at least one site ω_i for which l_i exceeds $(N+1)/N^\beta > N^{1-\beta}$. So,

$$(31) \leqq \sum_{\omega' : x \rightsquigarrow y} \sum_{i=1}^m k_{\omega'_i}^{N^{1-\beta}-1} \prod_{i=1}^m \left(\frac{k_{\omega'_i}}{1-k_{\omega'_i}}\right) . \tag{32}$$

Using Lemma 2 with $\lambda < \lambda'$ and δ as specified later on, and summing over y gives

$$(32) \leqq \exp -(\lambda (N^{1-\beta} - 1)^\delta) \sum_{y \in \mathbb{Z}^d} \sum_{\omega' : x \rightsquigarrow y} \sum_{i=1}^m \prod_{i=1}^m \frac{k_{\omega_i}}{1-k_{\omega_i}} . \tag{33}$$

Condition (28) assures that the sum is finite with **Q**-probability one. Hence for $\beta' > \beta$ and N large enough,

$$G_\pi^N(x,y) \leqq \exp -(\lambda N^{(1-\beta')\delta}) \ . \tag{34}$$

In the case $|y-x| \geqq N^\beta$ we can use Proposition 2 to see that

$$G_\pi((x,0),(y,N)) \leqq \exp -(\lambda N^\beta) \ . \tag{35}$$

Now for $0 < \varepsilon < 1/2$ we can choose δ and $\beta' > \beta$ such that $\beta > \varepsilon$ and $(1-\beta')\delta > \varepsilon$. This proves the proposition for $N_{2,x} > N'_{2,x}$ large enough. □

Theorem 1. *If*

$$\mathbf{E}\left(\frac{k_x}{1-k_x}\right) < \frac{1}{d} \ , \tag{36}$$

then the "directed" PCA has a unique invariant measure ν_π. *Moreover for every* $0 < \delta < 1/2$ *we have that if*

$$\mathbf{Q}\{k_x > 1-\tau\} < \tau^v \ , \tag{37}$$

with $v = v(\delta)$ *high enough, then for every local function* f *there exist – with* **Q**-*probability one – a time* $N_0 = N_0(\pi, \mathrm{supp} f, d, \mathbf{E}(\frac{k_x}{1-k_x})) < \infty$ *and a constant* $m > 0$ *such that for* $N > N_0$,

$$\|P_\pi^N f - \nu_\pi(f)\| \leqq \exp(-mN^\delta) \ . \tag{38}$$

Proof. Combining Proposition 1 with Proposition 3 we see that with **Q**-probability one there is a unique invariant measure. Furthermore,

$$\begin{aligned}\|P_\pi^N f - \nu_\pi(f)\| \leqq |||f||| \Bigg\{ &\sup_{z\in\mathrm{supp} f} \sum_{|y-z|\leqq N^\varepsilon} G_\pi^N(z,y) \\ &+ \sup_{z\in\mathrm{supp} f} \sum_{|y-z|> N^\varepsilon} G_\pi^N(z,y)\Bigg\} \ .\end{aligned} \tag{39}$$

Take $N_0 > \max\{\sup_{x\in\mathrm{supp} f} N_{x,1}, \sup_{x\in\mathrm{supp} f} N_{x,2}\}$, then for $N^\varepsilon > N_0$ and $\delta < \varepsilon$,

$$(39) \leqq \exp -(mN^\delta) \tag{40}$$

with $0 < m < \lambda$ and N_0 large enough. □

3.1.2. Ergodicity for a general PCA. The paths considered in the associated percolation problem are not spatially directed and k_x and k_y can be mutually dependent for $x \sim y$.

Proposition 4. *Take*

$$K > 2d^2\left(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}\right).$$

If

$$\mathbf{E}[\{\log(1 - \log(1 - k_x))\}^K] < \infty , \tag{41}$$

then – for $\mathbf{Q}$*-almost every realization – there exists a* $v(K,d) > 1$ *such that for every* $1 < v < v(K,d)$ *and* $m > 0$ *we can find constants* $0 < C_1, C_2 < 1$ *such that when*

$$\mathbf{Q}\{k_x > C_1\} < C_2 , \tag{42}$$

there exists a $N_{0,x} = N_{0,x}(\pi)$ *for which*

$$G_\pi^N(x,y) \leqq \exp(-m \max\{|x-y|, (\log(1+N))^v\}) \tag{43}$$

whenever $N > N_{0,x}$.

Campanino–Klein (1991) proved an analogous result for independent bond percolation and Klein (1993) for (continuous time) percolation on $\mathbb{Z}^d \times \mathbb{R}$. In their problems the set corresponding to $\{k_x\}$ contains *independent* random variables. Although we deal with directed site percolation and k_x and k_y are *dependent* if $x \sim y$, we can almost copy their proof. The modifications are given in Appendix A.

Theorem 2. *Take*

$$K > 2d^2\left(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}\right).$$

If

$$\mathbf{E}[\{\log(1 - \log(1 - k_x))\}^K] < \infty , \tag{44}$$

then – for $\mathbf{Q}$*-almost every realization – there exists a* $v(K,d) > 1$ *such that for every* $1 < v < v(K,d)$ *and* $c > 0$ *we can find constants* $0 < C_1, C_2 < 1$ *such that when*

$$\mathbf{Q}\{k_x > C_1\} < C_2 \tag{45}$$

the PCA has a unique invariant measure ν_π. *Moreover, for every local function* f *we can find a time* $N_0 = N_0(\pi, f)$ *and a constant* $B(d,f) < \infty$ *so that for all* $N > N_c$,

$$\|P_\pi^N f - \nu_\pi(f)\| \leqq B(d,f)\exp(-c(\log(1+N))^v). \tag{46}$$

Proof. Write (14) as

$$\|P_\pi^N f - \nu_\pi(f)\| \leqq |||f||| \left\{ \sup_{z\in \mathrm{supp} f} \sum_{|y-z| \leqq (\log(1+N))^v} G_\pi^N(z,y) \right.$$
$$\left. + \sup_{z\in \mathrm{supp} f} \sum_{|y-z| > (\log(1+N))^v} G_\pi^N(z,y)) \right\} . \tag{47}$$

Using (43) and taking $N_0 > \sup_{x \in \mathrm{supp} f} N_{0,x}$, then for $N > N_0$,

$$\|P_\pi^N f - v_\pi(f)\| \leqq B(d,f)\exp -c(\log(1+N))^v \ , \tag{48}$$

with $0 < c < m$ and N_0 large enough. □

3.2. Disordered Interacting Particle Systems. The continuous time version of the previous dynamics is defined in terms of spin flip rates $c(x,\eta)$. They give the probability per unit time of flipping the value of the spin at site x if the configuration is η,

$$\mathrm{Prob}_\pi[\sigma_t(x) = \eta(x)|\sigma_0 = \eta] = c(x,\eta)t + o(t) \ . \tag{49}$$

See Liggett (1985) for more details on these spin flip dynamics as a class of interacting particle systems (IPS). Also here we have that $c(x,\eta)$ only depends on η via $\eta(y)$, $y = x$ or $y \sim x$, via a random interaction. Define

$$\begin{cases} \lambda_x \equiv \sup\limits_{\eta,\eta'}\{|c(x,\eta) - c(x,\eta')|, \eta(x) = \eta'(x)\} \\ \delta_x \equiv \inf\limits_{\eta,\eta'}\{c(x,\eta) + c(x,\eta'), \eta(x) \neq \eta'(x)\} . \end{cases} \tag{50}$$

Example 4 (Continuous time spin glass). The spin flip rate is of Glauber type:

$$c(x,\eta) = \exp -(\frac{\beta}{2}[H_\pi(\eta^x) - H_\pi(\eta)]) \tag{51}$$

with η^x, the configuration that differs from η in x and

$$H_\pi(\eta) = -\sum_{y\sim x} J_{xy}\eta(x)\eta(y) \ . \tag{52}$$

The $\{J_{xy}\}$ is a family of real valued independent and identically distributed random variables. Here,

$$\begin{aligned} \delta_x &= 2\exp(-\beta \sum_{y\sim x} |J_{xy}|), \\ \lambda_x &= \exp(\beta \sum_{y\sim x} |J_{xy}|) - \exp(-\beta \sum_{y\sim x} |J_{xy}|) \ . \end{aligned} \tag{53}$$

Example 5 (Random version of a majority vote IPS). Let $\{\gamma_x\}_{x\in\mathbb{Z}^d}$ be independent identically distributed positive random variables. The spin flip rate is

$$c(x,\eta) = \begin{cases} \frac{2-\mathrm{e}^{-\lambda\gamma_x}}{2} & \text{if } \eta_x \neq \mathrm{sign}\ (\eta(x) + \sum_{y\sim x}\eta(y)) \\ \frac{\mathrm{e}^{-\lambda\gamma_x}}{2} & \text{if } \eta_x = \mathrm{sign}\ (\eta(x) + \sum_{y\sim x}\eta(y)) \end{cases} . \tag{54}$$

Here,

$$\begin{aligned} \delta_x &= \mathrm{e}^{-\lambda\gamma_x} \ , \\ \lambda_x &= 1 - \mathrm{e}^{-\lambda\gamma_x} \ . \end{aligned} \tag{55}$$

To show the ergodicity for the continuous time dynamics, we connect to the IPS a Contact Process on the configuration space $\{0,1\}^{\mathbb{Z}^d}$. (See Liggett (1985).) The extinction of the population of 1's in this process, or absence of percolation in its graphical representation, implies ergodicity of the original IPS. The Contact Process is defined by the following rates:

$$\bar{c}(x,\xi) = \begin{cases} \delta_x & \text{if } \xi(x) = 1 \\ \lambda_x \sum\limits_{y \in A_x} \xi(y) & \text{if } \xi(x) = 0 \end{cases} \tag{56}$$

with λ_x and δ_x as defined in (50) and A_x the set of all neighbors y of x for which $c(x,\eta)$ depends on $\eta(x)$.

The graphical representation of the process on the continuous time lattice $\mathbb{Z}^d \times \mathbb{R}$ is obtained by putting cuts on every line $\{x\} \times \mathbb{R}$ according to a Poisson process with rate δ_x and placing arrows from the line $y \times \mathbb{R}$, $y \in A_x$ to the line $x \times \mathbb{R}$ with intensity λ_x.(See e.g. Liggett (1985), Klein (1993)). $G_\pi^t(x,y)$ is the probability that (x,t) and $(y,0)$ are connected in this construction. Absence of percolation (no infinite clusters) implies extinction of the Contact Process with the same parameters.

Proposition 5. *For every local function f,*

$$\|P_\pi^t f - \nu_\pi(f)\| \leqq |||f||| \sup_{z \in \operatorname{supp} f} \sum_{y \in \mathbb{Z}^d} G_\pi^t(z,y) \tag{57}$$

with ν_π an invariant measure for the dynamics.

Proof. We couple three IPS: $\eta_t, {\eta'}_t$ and ξ_t with rates $c(x,\eta), c(x,\eta')$ and $\bar{c}(x,\xi)$ in such a way that if $|\eta_0(x) - {\eta'}_0(x)| \leqq 2\xi_t(x)$ for all $x \in \mathbb{Z}^d$, then $|\eta_t(x) - {\eta'}_t(x)| \leqq 2\xi_t(x)$ with probability one for all $t \geqq 0$. This can be done by using the coupling described in Liggett (1985) (p. 130).

Then for every local function,

$$\begin{aligned}
|P_\pi^t f(\sigma) - P_\pi^t f(\sigma')| &\leqq \sum_x \delta_x f \operatorname{Prob}_\pi[\sigma_t(x) \neq {\sigma'}_t(x) | \sigma_0 = \sigma, {\sigma_0}' = \sigma'] \\
&\leqq \sum_x \delta_x f \operatorname{Prob}_\pi[\xi_t(x) | \xi(0)] \leqq |||f||| \sup_{x \in \operatorname{supp} f} \operatorname{Prob}_\pi[\xi_t(x) = 1] \\
&\leqq |||f||| \sup_{x \in \operatorname{supp} f} \sum_{y \in \mathbb{Z}^d} G_\pi^t(x,y) ,
\end{aligned} \tag{58}$$

which is stronger than (57). $\square$

3.2.1. Ergodicity in a "directed" IPS. Just as for the PCA, it is worthwhile considering the class of IPS with transition rates for which

$$A_x = \{x + e_\alpha, \alpha = 1, \ldots, d\} . \tag{59}$$

Then, in the graphical representation, there are only arrows pointing in the opposite direction of one of the unit vectors and there are no paths that reach one spatial point twice. Further, we suppose that $\{(\delta_x, \lambda_x)\}$ is a set of independent couples of random variables. (Of course, δ_x and λ_x can be mutually dependent.)

We discretize the corresponding Contact Process to apply the arguments used for a directed PCA. The space time lattice becomes $\mathbb{Z}^d \times a\mathbb{Z}$. To each space-time point $(x,t) \in \mathbb{Z}^d \times a\mathbb{Z}$ we assign the occupation value $1(0)$ with probability $e^{-a\delta_x}(1 - e^{-a\delta_x})$. Between each two time layers $\mathbb{Z}^d \times \{na\}$ and $\mathbb{Z}^d \times \{(n+1)a\}$

we draw arrows from $x+e_\alpha$ to x with probability $1-\mathrm{e}^{-a\lambda_x}$. Two sites (x,t_1) and (y,t_2) are connected if there exists a time- and space-directed path composed of arrows and open sites. The probability that these two points are linked is denoted by $G_{a,\pi}^{t_2-t_1}((x,y))$. Finally, for $\Lambda=\mathbb{Z}^d$, one can show that $G_{a,\pi}^t(x,y)\Rightarrow G_\pi^t(x,y)$ when $a\downarrow 0$. (For more details, see Bezuidenhout–Grimmett (1991).)

Proposition 6. *Suppose*

$$\mathbf{E}\left(\frac{\lambda_x}{\delta_x}\right)<\frac{1}{d}\,, \tag{60}$$

then

$$\mathbf{E}(G_\pi^t(x,y))\leqq C_1\exp(-\lambda'|x-y|) \tag{61}$$

for constants $C_2<\infty$ *and* $\lambda'>0$.

Proof. Let $m=|x-y|$ and ω' the projection of the space-time path ω on $\mathbb{Z}^d$,

$$\begin{aligned} G_{a,\pi}^t(x,y) &\leqq \sum_{\omega':x\rightsquigarrow y}\ \sum_{\substack{l_1,\ldots,l_m\geqq 1\\ l_1+\ldots+l_m=\frac{t}{a}+1}}\ \prod_{i=1}^m 1-\mathrm{e}^{-a\lambda_{\omega'_i}}\prod_{j=1}^m \mathrm{e}^{-a\delta_{\omega'_j}l_{\omega'_j}} \\ &\leqq \sum_{\omega':x\rightsquigarrow y}\ \prod_{i=1}^m\frac{1-\mathrm{e}^{-a\lambda_{\omega'_i}}}{1-\mathrm{e}^{-a\delta_{\omega'_i}}}\prod_{j=1}^m \mathrm{e}^{-a\delta_{\omega'_j}}\ . \end{aligned} \tag{62}$$

Letting $a\downarrow 0$ the proposition easily follows. □

Remark. From Proposition 6 we can deduce that there exists a $T_{x,1}=T_{x,1}(\pi,\mathbf{E}(\frac{\lambda_x}{\delta_x}),d)$ such that for $|x-y|>T_{x,y}$

$$G_\pi^t(x,y)\leqq\exp\lambda|x-y| \tag{63}$$

with $\lambda<\lambda'$.

Proposition 7. *If*

$$\mathbf{E}\left(\frac{\lambda_x}{\delta_x}\right)<\frac{1}{d}\,, \tag{64}$$

then for every $0<\varepsilon<1/2$ *we have that if*

$$\mathbf{Q}\{e^{-\delta_x}>1-\tau\}<\tau^v\,, \tag{65}$$

with $v=v(\varepsilon)$ *high enough, there exist – for* **Q***-almost every realization – a time* $T_{x,2}=T_{x,2}(\pi,d,\mathbf{E}(\frac{\lambda_x}{\delta_x}))<\infty$ *and a constant* λ, *independent of* x *and* π *such that for* $t>T_{x,2}$

$$G_\pi^t(x,y)\leqq\exp(-\lambda t^\varepsilon\ )\ . \tag{66}$$

Proof. Take m and ω' as before. In the same way as we did in the proof of Proposition 3 for a PCA, we can show that for T high enough and $|x-y|<T^\beta$,

$$G_\pi^t(x,y)\leqq\sum_{\omega':x\rightsquigarrow y}\ \sum_{i=1}^m\left(\mathrm{e}^{-a\delta_{\omega'_i}}\right)^{\frac{T^{1-\beta}}{a}-1}\prod_{i=1}^m\frac{1-\mathrm{e}^{-a\lambda_{\omega'_i}}}{1-\mathrm{e}^{-a\delta_{\omega'_i}}}\ . \tag{67}$$

Let $a \downarrow 0$, sum over $y \in \mathbb{Z}^d$ and use Lemma 2, with $\lambda < \lambda'$, to see that

$$G^t_\pi(x,y) \leqq \exp\left(-\lambda t^{(1-\beta)\delta}\right) \sum_{y \in \mathbb{Z}^d} \sum_{\omega' : x \rightsquigarrow y} \sum_{i=1}^{m} \prod_{i=1}^{m} \left(\frac{\lambda_{\omega'_i}}{\delta_{\omega'_i}}\right) . \tag{68}$$

The sum is finite with **Q**-probability one. Hence for $\beta' > \beta$,

$$G^t_\pi(x,y) \leqq \exp(-\lambda t^{(1-\beta')\delta}) . \tag{69}$$

If $|x-y| \geqq T^\beta$, Proposition 6 says that for T large enough

$$G^T_\pi(x,y) \leqq \exp(-\lambda T^\beta) . \tag{70}$$

Hence, for a given $0 < \varepsilon < 1/2$ choose δ and $\beta' > \beta$ such that $\beta > \varepsilon$ and $(1 - \beta')\delta > \varepsilon$. □

Propositions 6 and 7 allow us to prove the main result for a "directed" IPS.

Theorem 3. *If*

$$\mathbf{E}\left(\frac{\lambda_x}{\delta_x}\right) < \frac{1}{d} , \tag{71}$$

then the IPS has a unique invariant measure ν_π. Moreover for every $0 < \delta < 1/2$ we have that if

$$\mathbf{Q}\{e^{-\delta_x} > 1 - \tau\} < \tau^v , \tag{72}$$

*with $v = v(\delta)$ high enough, then for every local function f there exist – for **Q**-almost every realization – a time $T_0 = T_0(\pi, \operatorname{supp} f, d, \mathbf{E}(\frac{\lambda_x}{\delta_x})) < \infty$ and a constant $m > 0$ such that for $N > N_0$,*

$$\|P^N_\pi f - \nu_\pi(f)\| \leqq \exp(-mN^\delta) . \tag{73}$$

The proof of Theorem 7 is similar to the one of Theorem 1.

3.2.2. Ergodicity in a general IPS. Define

$$\lambda_{\langle x,y\rangle} = \lambda_x + \lambda_y . \tag{74}$$

Note that $\lambda_{\langle xy\rangle}$ and δ_x can be *correlated* if x and y are (next-) nearest neighbors.

Proposition 8. *Let*

$$K > 2d^2\left(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}}\right).$$

Suppose

$$\Gamma \equiv \max\{\mathbf{E}\ [\{\log(1+\lambda_{\langle x,y\rangle})\}^K]\, \mathbf{E}[\{\log(1+1/\delta_x)\}^K]\} < \infty . \tag{75}$$

*Then – for **Q**-almost every realization – there exists a $v(K,d) > 1$ so that for every $1 < v < v(K,d)$ and $m > 0$ we can find a constant $C_1 > 0$ such that when*

$$\mathbf{E}\{(\log(1 + \frac{\lambda_{\langle x,y\rangle}}{\delta_x}))^K\} < C_1 , \tag{76}$$

there exists a $T_{c,x} = T_{c,x}(\pi)$ for which

$$G^t_\pi(x,y) \leqq \exp(-m \max[|x-y|, (\log(1+t|))^v]) \tag{77}$$

whenever $|t-s| > T_{c,x}$.

Proposition 8 is formally the same as Theorem 3.2 in Klein (1993). However, we must take care of the consequences of the non-trivial correlations between δ_x and $\lambda_{\langle x,y\rangle}$. In Appendix B we briefly show the necessary modifications to the argument of Klein (1993).

Theorem 4. *Let*

$$K > 2d^2\left(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}}\right).$$

Suppose

$$\Gamma \equiv \max\{\ \mathbf{E}\ [\{\log(1 + \lambda_{\langle x,y\rangle})\}^K], \mathbf{E}\ [\{\log(1 + 1/\delta_x)\}^K]\} < \infty\ . \tag{78}$$

Then – for $\mathbf{Q}$*-almost every realization – there exists a* $v(K,d) > 1$ *so that for every* $1 < v < v(K,d)$ *and* $c > 0$ *we can find constants* $C_1 > 0$ *such that if*

$$\mathbf{E}((\log(1 + \frac{\lambda_{\langle x,y\rangle}}{\delta_x}))^K) < C_1, \tag{79}$$

for every local function f *there is a time* $T_0 = T_0(\pi, f)$ *and a constant* $B(d,f) < \infty$ *so that*

$$\|P^t_\pi f - \nu_\pi(f)\| \leqq B(d,f)\exp(-c(\log(1+t))^v) \tag{80}$$

when $t > T_0$.

Proof. Combining (57) and (77) in the same way as we did for Theorem 2 gives a proof of Theorem 4. □

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Appendix A

Remark. In the Appendices A and B we give the necessary modifications to the argument of Klein (1993) and Campanino–Klein (1991).

As we already mentioned, the proof of Proposition 4 is similar to the one in Klein (1993) and Campanino–Klein (1991). In this paragraph the argument of (mainly) Klein (1993) is summarized. The most important lemmas are presented, with emphasis on the necessary modifications due to the mutual dependence of the k_x's. The same notations and definitions are used as in Klein (1993).

Klein (1993) uses a multiscale analysis. The proof is by induction. In Lemma 3 the induction hypothesis is proven: for every (m_0, L_0) we can find constants $0 < C_1$, $C_2 < 1$ such that if $\mathbf{Q}\{k_x > C_1\} < C_2$, then

$$\mathbf{Q}\{\text{The origin is } (m_0, L_0)\text{-regular}\} \geqq 1 - \frac{1}{L_0^p}\ . \tag{81}$$

In the induction step (Lemma 4, Sublemmas 3.2.2 and 3.2.2) it is proven that if

$$\mathbf{Q}\{\text{The origin is } (m_k, L_k)\text{-regular}\} \geqq 1 - \frac{1}{L_k^p}\ , \tag{82}$$

then

$$\mathbf{Q}\{\text{The origin is } (m_{k+1}, L_{k+1})\text{-regular}\} \geqq 1 - \frac{1}{L_{k+1}^p} , \tag{83}$$

with $L_{k+1} = L_k^\alpha$, $\alpha > 1$ and $m_{k+1} < m_k$. One can show that for every choice of $0 < m_\infty < m_0$ we can choose L_0 such that $m_\infty < \lim_{k\uparrow\infty} m_k < m_0$. Then

$$\mathbf{Q}\{\text{The origin is } (m_\infty, L_k)\text{-singular}\} \leqq \frac{1}{L_k^p} , \tag{84}$$

and using the Borel–Cantelli Lemma we know that – with **Q**-probability one – there exists a K such that if $k > K$ the origin is (m_∞, L_k)-regular. Klein (1993) estimates $G_\pi^N(x, y)$ in such a (m_∞, L_k)-regular region.

Lemma 3. *For every m, $L > 0$, there exist constants $0 < C_1, C_2 < 1$ such that if*

$$\mathbf{Q}\{k_x > C_1\} < C_2 , \tag{85}$$

then

$$\mathbf{Q}\{\textit{The origin is } (m, L)\textit{-regular}\} \geqq 1 - 1/L^p . \tag{86}$$

Proof. For any $C > 0$ there exist constants $0 < C_1, C_2 < 1$ such that if $\mathbf{Q}\{k_x > C_1\} < C_2$, then

$$\mathbf{Q}\left\{\sup_{x\in\Lambda_L(0)} k_x < C\right\} \leqq \frac{1}{L^p} . \tag{87}$$

Because $G_{\pi,B_L(0)}$ becomes arbitrary small for decreasing C, the lemma is proven by taking C small enough. □

Lemma 4. *Suppose that condition* (41) *holds. Let*

$$\alpha = d + \sqrt{d^2 + d} , \tag{88}$$

and choose v and p such that

$$\frac{\alpha d(d + \beta + 1)}{\beta(\alpha - d + \alpha d)} < 1/v < 1 , \tag{89}$$

$$\alpha d < p < \frac{\beta(1/v(\alpha - d + \alpha d) - \alpha) - \alpha d}{\alpha} . \tag{90}$$

For every m_0 and m_∞, with $m_\infty < m_0$, there exists $\bar{L} = \bar{L}(d, \beta, \Gamma, 1/v, p, m_0, m_\infty) < \infty$, such that if for some $L_0 > \bar{L}$ we have

$$\mathbf{Q}\{0 \textit{ is } (m_0, L_0)\textit{-regular }\} \geqq 1 - \frac{1}{L_0^p} . \tag{91}$$

Taking $L_{k+1} = L_k^\alpha$, $k = 1, 2, \ldots$, we also have

$$\mathbf{Q}\{0 \textit{ is } (m_\infty, L_k)\textit{-regular }\} \geqq 1 - \frac{1}{L_k^p} , \tag{92}$$

for all $k = 0, 1, 2, \ldots$.

Proof of Lemma 4. To prove Lemma 4 we need Sublemmas 3.2.2 and 3.2.2.

Sublemma 1. *Pick positive integers R, κ and b such that*

$$\alpha < \frac{(R+1)p}{p+(R+1)d} , \tag{93}$$

$$1 < \alpha(1-1/v) + 1/v < \kappa < \frac{b}{d} < \frac{\alpha}{vd} . \tag{94}$$

Let $m_0 \geqq m_k \geqq 1/L_k^\theta$, with $0 < \theta < \theta_0 = \min\{\alpha(1-1/v), 1\}$.

Suppose there exist $x_1, \dots, x_R \in \Lambda_{L_{k+1}}(0)$ such that $\Lambda_L \setminus \bigcup_{j=0}^R \Lambda_{2L_k+1}(x_j)$

$$\text{is a } (m_k, L_k)\text{-regular region} . \tag{95}$$

Define

$$\tilde{\Lambda} = \bigcup_{j=1}^{R} \Lambda_{l^\kappa(x_j)} \cap \Lambda_L(0) . \tag{96}$$

Suppose

$$\sum_{x \in \tilde{\Lambda}} \log(1-k_x)^{-1} \leqq L_k^b . \tag{97}$$

Then, if L_k is large enough $\Lambda_{L_{k+1}}(0)$ is a (m_{k+1}, L_{k+1})-regular region with $m_{k+1} \geqq 1/L_{k+1}^\theta$.

Proof. The proof is analogous to the proofs of Sublemmas 3.6 and 3.7 in Campanino–Klein (1991) and Sublemmas 4.2 and 4.3 in Klein (1993). However, there is one necessary modification that should be mentioned. Consider the sets $V_1, \dots, V_k \subset V_0 \subset \mathbb{Z}^d$ with $V_i \cap V_j = \emptyset$. For $i : 1, \dots, k$ we define

$$B_i = V_i \times \{(-T, T) \cap \mathbb{Z}\} , \tag{98}$$

$$\partial B = \partial(B_i, B) \quad \text{and} \quad B' = B \setminus \bigcup_{i=1}^{k} B_i . \tag{99}$$

Let $\{(x, N_1) \xrightarrow{B} (y, N_2)\}$ the event that there is an oriented open path in B from (x, N_1) to (y, N_2). For $(y, t) \in \partial B$,

$$\begin{aligned}
&\{(0,0) \xrightarrow{B} (y,N)\} \\
&\subseteq \bigcup_{r=0}^{k} \bigcup_{\{i_1,\dots,i_k\} \subset \{1,\dots,k\}} \{\text{There exist oriented open paths } (0,0) \xrightarrow{B'} \partial B_{i_1} , \\
&\partial B_{i_1} \xrightarrow{B'} \partial B_{i_2}, \dots, \partial B_{i_r} \xrightarrow{B'} (y,N)\} \\
&\subseteq \bigcup_{r=0}^{k} \bigcup_{\{i_1,\dots,i_k\} \subset \{1,\dots,k\}} \{(0,0) \xrightarrow{B'} \partial B_{i_1}\} \circ \{\partial B_{i_1} \xrightarrow{B'} \partial B_{i_2}\} \circ \dots \\
&\dots \circ \{\partial B_{i_r} \xrightarrow{B'} (y,N)\} .
\end{aligned} \tag{100}$$

Note that we used the time-oriented character of the paths. The van den Berg–Kesten inequality can be used to estimate the probability of the event $\{(0,0) \xrightarrow{B} (y,N)\}$ and we can complete the proof as in Klein (1993) and Campanino–Klein (1991). □

Sublemma 2. *If*

$$\mathbf{Q}\{0 \text{ is } (m_k, L_k) - \textit{regular}\} \geqq 1 - \frac{1}{L_k^p}, \tag{101}$$

then $\mathbf{Q}\{(95)$ *and* $(97)\} \geqq 1 - \frac{1}{L_k^p}$.

Proof. We call two points x_1 and x_2 non-l-touching if it is not possible to walk from the box $\Lambda_l(x_1)$ to $\Lambda_l(x_2)$ without passing a point that does not belong to any of the sets. If $R+1$ points $x_1, \ldots, x_{R+1}$ are non-l-touching, the events $\{x_i$ is (m,l)-regular$\}$ i:1,2,... are independent,

$$\mathbf{Q}\{\exists x_1, x_2, \ldots, x_{R+1} \in \Lambda_{L_{k+1}}(0) \text{ that are non } L_k\text{-touching and } (m_k, L_k)\text{-singular}\}$$

$$\leqq \frac{(2L_{k+1}+1)^{d(R+1)}}{L_k^{p(R+1)}} < \frac{1}{L_k^p} \tag{102}$$

for L_k sufficiently large by the choice of R. Hence

$$\mathbf{Q}\ \{\exists x_1, x_2, \ldots, x_R \in \Lambda_{L_k}(0) \text{ such that } \Lambda_{L_{k+1}}(0) \setminus \bigcup_{j=0}^{R} \Lambda_{2L_k+1}(x_j)$$

$$\text{is a } (m_k, L_k)\text{-regular region}\} \geqq 1 - \frac{1}{2L_{k+1}^p}. \tag{103}$$

For the $\mathbf{Q}$-probability of (97), we have that

$$\mathbf{Q}\left\{\sum_{x \in \tilde{\Lambda}} \log(1-k_x)^{-1} > L_k^b\right\} \leqq \mathbf{Q}\left\{\exists x \in \tilde{\Lambda} : \log(1-k_x)^{-1} > \frac{L_k^b}{(2L_k^\kappa+1)^d}\right\}$$

$$\leqq (2L_k^\kappa+1)^d \mathbf{Q}\{\log(1+\log(1-k_x)^{-1}) > \log(1+\frac{L_k^b}{(2L_k^\kappa+1)^d})\}$$

$$\leqq (2L_k^\kappa+1)^d \mathbf{Q}\{\log(1+\log(1-k_x)^{-1}) > \frac{L_k^b}{(2L_k^\kappa+1)^d}\}$$

$$\leqq \frac{(2L_k^\kappa+1)^d (2L_k^\kappa+1)^{Kd}}{L_k^{Kb}} \mathbf{E}([\log(1+\log(1-k_x)^{-1})]^K)$$

$$\leqq \frac{\mathbf{E}([\log(1+\log(1-k_x)^{-1})]^K)}{L_k^{K(b-\kappa d)-\kappa d}} \leqq \frac{1}{2L_{k+1}^p}. \tag{104}$$

To prove the sublemma, note that both (95) and (97) are decreasing events. □

Appendix B

The only thing to be done is to observe that the events (1.6), (4.1) and (4.2) in Klein (1993) have the same $\mathbf{Q}$ probability.

Choose κ, b, γ, τ such that

$$\alpha(1 - 1/v) + 1/v < \kappa < \frac{b}{d} < \frac{\alpha}{vd} ,$$

$$0 < \gamma < b - \kappa d, 1/v < \tau < \alpha(1 - 1/v) . \tag{105}$$

Consider the following events:

$$8d\rho_{L_0} < e^{2m_0} , \tag{106}$$

$$\delta_{L_0} > e^{-1/2L_0^{1/v}} , \tag{107}$$

$$\lambda_{L_0} < e^{L_0^{1/v}} , \tag{108}$$

with

$$\rho_L = \sup_{x\in\Lambda_L(0)} \left\{ \frac{1}{\delta_x} \max_{y\sim x} \lambda_{\langle xy\rangle} \right\} , \tag{109}$$

$$\delta_L = \inf_{x\in\Lambda_L(0)} \delta_x , \tag{110}$$

$$\lambda_L = \max_{(x,y)\in\partial(\Lambda_L(0),\mathbb{Z}^d)} \lambda_{\langle xy\rangle} , \tag{111}$$

$$\partial(\Lambda_L(0), \mathbb{Z}^d) = \{(x, y) \in (\mathbb{Z}^d)^2 : x \in \Lambda_L(0), y \in \Lambda_L^c(0) : y \sim x\}, \tag{112}$$

and the events

$$\lambda_{L_{k+1}} < e^{L_{k+1}^{1/v}} , \tag{113}$$

$$\{\text{There exist } x_1, x_2, \ldots, x_R \in \Lambda_{L_{k+1}}(0) \text{ such that } \Lambda_{L_{k+1}}(0) \setminus \bigcup_{j=0}^{R} \Lambda_{2L_k+1}$$

$$\text{is a } (m_k, L_k)\text{-regular region}\} , \tag{114}$$

$$e^{L_k^\gamma} \sum_{\langle x,y\rangle\subset\tilde{\Lambda}} \lambda_{\langle x,y\rangle} - \sum_{x\in\tilde{\Lambda}} \log(1 - e^{-\delta_x \exp L_k^\gamma}) \leqq L_k^b , \tag{115}$$

with

$$\tilde{\Lambda} = \bigcup_{j=1}^{R} \Lambda_{L_k}^{\kappa}(x_j) \cap \Lambda_{L_{k+1}}(0) . \tag{116}$$

We only have to prove that there exists an L_0, large enough such that

$$\mathbf{Q}\{(106),(107),(108)\} \geqq 1 - \frac{1}{L_0^p} , \tag{117}$$

and that if

$$\mathbf{Q}\{0 \text{ is } (m, L_k)\text{-regular}\} \geqq 1 - \frac{1}{L_k^p} , \tag{118}$$

then

$$\mathbf{Q}\{(113),(114),(115)\} \geqq 1 - \frac{1}{L_{k+1}^p} . \tag{119}$$

The lower bounds for $\mathbf{Q}\{(106)\}$, $\mathbf{Q}\{(107)\}$, $\mathbf{Q}\{(108)\}$, $\mathbf{Q}\{(113)\}$ and $\mathbf{Q}\{(115)\}$ are calculated in Klein (1993).

To estimate $\mathbf{Q}\{(114)\}$ we modify the definition of non-l-touching points: Two points x_i, x_j are non-l-touching if it is not possible to walk from the box $\Lambda_l(x_i)$ to the box $\Lambda_l(x_j)$ without passing two points that don't belong to any of the boxes. Note that (106) to (115) are all decreasing events. Hence, we can use the FKG-inequality to prove (117) and (119).

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