Integrable Time-Discretisation of the Ruijsenaars–Schneider Model

F.W. Nijhoff[1], **O. Ragnisco**[2], **V.B. Kuznetsov**[3,*,**]

[1] Department of Applied Mathematical Studies, The University of Leeds, Leeds LS2 9JT, UK
[2] Dipartimento di Fisica E. Amaldi, Università di Roma III, and Istituto di Fisica Nucleare, Sezione di Roma, P. le Aldo Moro 2, I-00185 Roma, Italy
[3] Faculteit voor Wiskunde en Informatica, Universiteit van Amsterdam, Plantage Muidergracht 24, NL-1018 TV Amsterdam, The Netherlands

Abstract: An exactly integrable symplectic correspondence is derived which in a continuum limit leads to the equations of motion of the relativistic generalization of the Calogero–Moser system, that was introduced for the first time by Ruijsenaars and Schneider. For the discrete-time model the equations of motion take the form of Bethe Ansatz equations for the inhomogeneous spin-$\frac{1}{2}$ XYZ Heisenberg magnet. We present a Lax pair, the symplectic structure and prove the involutivity of the invariants. Exact solutions are investigated in the rational and hyperbolic (trigonometric) limits of the system that is given in terms of elliptic functions. These solutions are connected with discrete soliton equations. The results obtained allow us to consider the Bethe Ansatz equations as ones giving an integrable symplectic correspondence mixing the parameters of the quantum integrable system and the parameters of the corresponding Bethe wavefunction.

1. Introduction

In some previous papers, [1, 2], cf. also [3], an exact time-discretization of the famous Calogero–Moser (CM) model, [4–7], was introduced and investigated. The discrete model is an integrable symplectic correspondence, (for a definition cf. [8]), that in a well-defined continuum limit yields the classical equations of motion of the CM system.[4] A few years ago Ruijsenaars and Schneider introduced in [10], cf. also [11, 12], a relativistic variant of the CM model, which is a parameter-deformation of the original model. The equations of motion of this system in its generic (elliptic) form read

$$\ddot{q}_i = \sum_{j \neq i} \dot{q}_i \dot{q}_j v(q_i - q_j), \quad i = 1, \dots, N , \tag{1.1a}$$

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[4] It should be noted that the discrete CM model can be inferred also from the Bäcklund transformations for the continuous CM model, that were presented in [9].

where the potential $v(x)$ is given by

$$v(x) = \frac{\wp'(x)}{\wp(\lambda) - \wp(x)} \ , \tag{1.1b}$$

in which $\wp(x) = \wp(x|\omega_1, \omega_2)$ is the Weierstrass P-function, $2\omega_{1,2}$ being a pair of periods. (These are the equations of motion in the form given by Bruschi and Calogero in [13].) This multi-particle model is also integrable, and carries a representation of the Poincaré algebra in two dimensions. Moreover, a large number of the characteristics of the CM model are generalized in a natural way to the relativistic case, such as the existence of a Lax pair, a sufficient number of integrals of the motion in involution, and exact solution schemes in special cases.

In view of the results in [1–3], a natural question to ask is whether there exists a time-discrete version of the model (1.1). We have to remark that on the quantum level the transition from the usual CM system to its relativistic counterpart already amounts to a discretization (or q-deformation). In fact, the relevant operators in the quantum relativistic model are commuting difference operators, rather than differential operators, cf. [14, 15]. In view of the importance of quantum Calogero–Moser type of models in the context of representation theory, and in particular in connection with the Knizhnik–Zamolodchikov (KZ) equations as has been revealed by a large amount of recent work, cf. e.g. [16–18], these difference operators yield new interesting connections with q-special functions.

In this paper we will introduce a discrete-time version of the model (1.1), in the form of an integrable symplectic correspondence (i.e. multi-valued map) which goes in a continuum limit to the original model. This amounts to one more parameter-deformation of the CM model: apart from the "spatial" discretisation (encoded in the parameter λ), the discretisation of time constitutes another deformation where the finite step-size in time enters as the new parameter. The construction is based on an Ansatz for a Lax pair, together with an elliptic version of the Lagrange interpolation formula. We will demonstrate the integrability of this mapping along the lines of ref. [8, 19], i.e. integrability in the sense of Liouville. In contrast to the previous paper, [2], where the particle model is related to pole solutions of discrete soliton equations, in particular of the lattice Kadomtsev–Petviashvili (KP) equation, the relativistic case is related to soliton solutions of this type of discrete equations. Finally, we point out the intriguing resemblance between the equations of motion of the discrete particle model and Bethe Ansatz equations for integrable spin-$\frac{1}{2}$ quantum chains of the XYZ Heisenberg model (see [36, 39]).

2. Derivation of the Model

We will start from an *Ansatz* for a Lax pair of the form

$$L_\kappa = \sum_{i,j=1}^{N} h_i h_j \Phi_\kappa(q_i - q_j + \lambda) e_{ij} \ , \tag{2.1a}$$

$$M_\kappa = \sum_{i,j} \widetilde{h}_i h_j \Phi_\kappa(\widetilde{q}_i - q_j + \lambda) e_{ij} \ . \tag{2.1b}$$

In (2.1) the q_i denote the particle positions, and the h_i are auxiliary variables which we specify later. The tilde is a shorthand notation for the discrete-time shift, i.e.

for $q_i(n) = q_i$, we write $q_i(n + 1) = \widetilde{q}_i$, and $q_i(n - 1) = \underset{\sim}{q}_i$. The variable κ is the additional spectral parameter, whereas λ is a parameter of the system. The matrices e_{ij} are the standard elementary matrices whose entries are given by $(e_{ij})_{k\ell} = \delta_{i,k}\delta_{j,\ell}$. The function Φ_κ is defined as

$$\Phi_\kappa(x) \equiv \frac{\sigma(x + \kappa)}{\sigma(x)\sigma(\kappa)} , \qquad (2.2)$$

where $\sigma(x)$ is the Weierstrass sigma-function, given by

$$\sigma(x) = x \prod_{(k,\ell) \neq (0,0)} \left(1 - \frac{x}{\omega_{k\ell}} \right) \exp\left[\frac{x}{\omega_{k\ell}} + \frac{1}{2}\left(\frac{x}{\omega_{k\ell}} \right)^2 \right] , \qquad (2.3)$$

with $\omega_{kl} = 2k\omega_1 + 2\ell\omega_2$ and $2\omega_{1,2}$ being a fixed pair of the primitive periods, see e.g. [20]. The relations between the Weierstrass elliptic functions are given by

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \quad \wp(x) = -\zeta'(x) , \qquad (2.4)$$

where $\sigma(x)$ and $\zeta(x)$ are odd functions and $\wp(x)$ is an even function of its argument. We recall also that the $\sigma(x)$ is an entire function, and $\zeta(x)$ is a meromorphic function having simple poles at ω_{kl}, both being quasi-periodic, obeying

$$\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\eta_{1,2}, \quad \sigma(x + 2\omega_{1,2}) = -\sigma(x)e^{2\eta_{1,2}(x+\omega_{1,2})} ,$$

in which $\eta_{1,2}$ satisfy $\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}$, whereas $\wp(x)$ is periodic. From an algebraic point of view, the most important property of these elliptic functions is the existence of a number of functional relations, the most fundamental being

$$\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) - \zeta(\alpha + \beta + \gamma) = \frac{\sigma(\alpha + \beta)\sigma(\beta + \gamma)\sigma(\gamma + \alpha)}{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\alpha + \beta + \gamma)} . \qquad (2.5)$$

From this relation, one can basically derive all important identities for the Weierstrass elliptic functions. For our purpose it can be recast into the form

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x + y)[\zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa + x + y)] . \qquad (2.6)$$

The Ansatz for the Lax pair (2.1) is a natural one, in view of the fact that the matrix L_κ corresponds to the Lax matrix of the continuum Ruijsenaars–Schneider (RS) model, cf. [13], whereas the matrix M_κ is a natural choice by comparison with the earlier results obtained in [2], cf. also [3], for the discrete elliptic CM model.

Let us now consider the compatibility of the system (2.1). Assuming the Lax equation

$$\widetilde{L}_\kappa M_\kappa = M_\kappa L_\kappa , \qquad (2.7)$$

which implies the isospectrality of the discrete flow of the Lax matrix L_κ, we have from (2.6) that

$$\sum_\ell \widetilde{h}_\ell^2 [\zeta(\kappa) + \zeta(\widetilde{q}_i - \widetilde{q}_\ell + \lambda) + \zeta(\widetilde{q}_\ell - q_j + \lambda) - \zeta(\kappa + 2\lambda + \widetilde{q}_i - q_j)]$$

$$= \sum_\ell h_\ell^2 [\zeta(\kappa) + \zeta(\widetilde{q}_i - q_\ell + \lambda) + \zeta(q_\ell - q_j + \lambda) - \zeta(\kappa + 2\lambda + \widetilde{q}_i - q_j)] .$$

Noting that the conservation law $\operatorname{tr}\widetilde{L} = \operatorname{tr} L$ implies:

$$\sum_{\ell} \widetilde{h}_{\ell}^2 = \sum_{\ell} h_{\ell}^2 \,, \tag{2.8}$$

we have the identity

$$\sum_{\ell} [\widetilde{h}_{\ell}^2 \zeta(\widetilde{q}_i - \widetilde{q}_{\ell} + \lambda) - h_{\ell}^2 \zeta(\widetilde{q}_i - q_{\ell} + \lambda)]$$

$$= -\sum_{\ell} [\widetilde{h}_{\ell}^2 \zeta(\widetilde{q}_{\ell} - q_j + \lambda) - h_{\ell}^2 \zeta(q_{\ell} - q_j + \lambda)] \,, \tag{2.9}$$

for all $i, j = 1, \ldots, N$: consequently, both sides of (2.9) must be independent of the (external) particle label. Thus, we find a coupled system of equations in terms of the variables h_i, q_i of the form

$$\sum_{\ell} [\widetilde{h}_{\ell}^2 \zeta(q_i - \widetilde{q}_{\ell} - \lambda) - h_{\ell}^2 \zeta(q_i - q_{\ell} - \lambda)] = -p \,, \tag{2.10a}$$

$$\sum_{\ell} [h_{\ell}^2 \zeta(q_i - q_{\ell} + \lambda) - h_{\ell}^2 \zeta(q_i - q_{\ell} + \lambda)] = \underset{\sim}{p} \,, \tag{2.10b}$$

where p does not carry a particle label. In principle p can still depend on the discrete time-variable, but we will mostly consider it to be constant: a different choice will be taken only in Sect. 6. In that case, by eliminating the variables h_i from (2.9), we get a closed set of equations in terms of the q_l.

In order to derive this closed set of equations for the variables q_i we will make use of an elliptic version of the Lagrange interpolation formula, that was derived in [2]. In fact, we have the following statement:

Lemma. *Consider $2N$ noncoinciding complex numbers x_l, y_l ($l = 1, \ldots, N$). Then, the following formula holds true:*

$$\prod_{l=1}^{N} \frac{\sigma(\xi - x_l)}{\sigma(\xi - y_l)} = \sum_{l=1}^{N} [\zeta(\xi - y_l) - \zeta(x - y_l)] \frac{\prod_{k=1}^{N} \sigma(y_l - x_k)}{\prod_{\substack{k=1 \\ k \neq l}}^{N} \sigma(y_l - y_k)} \,,$$

$$\text{where} \quad \sum_{l=1}^{N} (y_l - x_l) = 0 \,, \tag{2.11}$$

and where x stands for any one of the zeroes x_l of the function on the left-hand side.

Equation (2.11) can be derived from an elliptic version of the Cauchy identity for the determinants of the matrix $(\Phi_\kappa(x_i - y_j))$. This Cauchy identity was proven and used in [12] to establish the commutativity of the quantum integrals for the continuum RS model. Actually, this identity goes back to Frobenius, [21], and can be proven also by purely combinatoric means starting from the fundamental identity (2.5). A statement similar to the one in the lemma was formulated in [22].

What is essential in (2.11) is the independence of the choice of x, which can be easily demonstrated by using the identity

$$\sum_{i=1}^{N} \frac{\prod_{j=1}^{N} \sigma(y_i - x_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{N} \sigma(y_i - y_j)} = 0, \qquad \sum_{i=1}^{N} (y_i - x_i) = 0 \,, \tag{2.12}$$

which can be found e.g. in [23], p. 451.

Using now Eq. (2.11) and applying the lemma to the elliptic fractional "polynomial"

$$\prod_{k=1}^{N} \frac{\sigma(\xi - q_k + \lambda)\sigma(\xi - \widetilde{q}_k - \lambda)}{\sigma(\xi - q_k)\sigma(\xi - \widetilde{q}_k)} \, ,$$

and taking $\xi = q_j - \lambda$, respectively $x = \widetilde{q}_i + \lambda$, we can by comparing with Eq. (2.9) make the identifications

$$h_\ell^2 = -p \, \frac{\prod_{k=1}^{N} \sigma(q_\ell - q_k + \lambda)\sigma(q_\ell - \widetilde{q}_k - \lambda)}{\left[\prod_{k \neq \ell} \sigma(q_\ell - q_k)\right] \prod_{k=1}^{N} \sigma(q_\ell - \widetilde{q}_k)} \, , \tag{2.13a}$$

$$\widetilde{h}_\ell^2 = p \, \frac{\prod_{k=1}^{N} \sigma(\widetilde{q}_\ell - q_k + \lambda)\sigma(\widetilde{q}_\ell - \widetilde{q}_k - \lambda)}{\left[\prod_{k \neq \ell} \sigma(\widetilde{q}_\ell - \widetilde{q}_k)\right] \prod_{k=1}^{N} \sigma(\widetilde{q}_\ell - q_k)}, \quad \ell = 1, \ldots, N \tag{2.13b}$$

from which we obtain the following system of equations:

$$\frac{p}{\underset{\sim}{p}} \prod_{\substack{k=1 \\ k \neq \ell}}^{N} \frac{\sigma(q_\ell - q_k + \lambda)}{\sigma(q_\ell - q_k - \lambda)} = \prod_{k=1}^{N} \frac{\sigma(q_\ell - \widetilde{q}_k)\, \sigma(q_\ell - q_k + \lambda)}{\sigma(q_\ell - \underset{\sim}{q}_k)\, \sigma(q_\ell - \widetilde{q}_k - \lambda)}, \quad \ell = 1, \ldots, N \, . \tag{2.14}$$

Equations (2.14) can be considered to be a product version of (2.9), and is a system of N equations for $N + 1$ unknowns, $q_1, \ldots, q_N$ and p. There is no equation for p separately, and thus it should be a priori given in order to get a closed set of equations. The most natural choice is the one for which p is constant, i.e. independent of the discrete time-variable. For convenience we will take it equal to the fixed value $p = -\sigma(\lambda)^{-1}$, but if we are interested in a continuum limit from (2.10) we should take it of the order of the reciprocal of the discrete-time step. As already mentioned, a dynamical choice of p arises at the end of Sect. 6 from the application of (2.14) to the Bethe Ansatz equations for the XYZ Heisenberg chain.

Thus, taking $p/\underset{\sim}{p}$ to be equal to unity in (2.14), we obtain the equations of motion of what we would like to call the discrete Ruijsenaars–Schneider model. It is given by a coupled set of algebraic equations, which, in fact, resemble closely the Bethe Ansatz equations (BAE's) for certain integrable quantum chains with impurities. We will make this connection more precise in Sect. 6. Here, we look at Eqs. (2.14) as defining a discrete dynamical system, which amounts to an integrable symplectic correspondence, i.e. a multi-valued map in the sense of ref. [8].

Before proving the symplecticity and integrability of the correspondence, let us show first that the interpretation of (2.14) as a discrete version of the RS model is justified, by showing that an appropriate continuum limit will give us back Eqs. (1.1) of the continuum model. In fact, taking

$$\widetilde{q}_k \mapsto q_k - \lambda + \varepsilon \dot{q}_k + \frac{1}{2} \varepsilon^2 \ddot{q}_k + \cdots \, ,$$

$$\underset{\sim}{q}_k \mapsto q_k + \lambda - \varepsilon \dot{q}_k + \frac{1}{2} \varepsilon^2 \ddot{q}_k + \cdots \, ,$$

and developing with respect to the small parameter ε, we obtain for the leading term

$$\frac{\ddot{q}_\ell}{\dot{q}_\ell} = \sum_{k \neq \ell} \dot{q}_k [2\zeta(q_\ell - q_k) - \zeta(q_\ell - q_k + \lambda) - \zeta(q_\ell - q_k - \lambda)] \, , \tag{2.15}$$

from which we recover Eqs. (1.1) using the relation

$$\zeta(x+\lambda) - \zeta(x) - \zeta(\lambda) = \frac{1}{2}\frac{\wp'(\lambda) - \wp'(x)}{\wp(\lambda) - \wp(x)} \ . \tag{2.16}$$

Let us finally remark that the non-relativistic limit is obtained by letting the parameter λ go to zero. In fact, in this limit the variables h_i^2 will behave as

$$h_l^2 \mapsto 1 + \lambda\left(\sum_{k \neq \ell}\zeta(q_\ell - q_k) - \sum_{k=1}^{N}\zeta(q_\ell - \widetilde{q}_k)\right) + \mathcal{O}(\lambda^2) \ . \tag{2.17}$$

As $\lambda \to 0$, we will obtain the matrix M_κ of the non-relativistic case immediately, whereas for L_κ we thus find

$$L_\kappa \mapsto \lambda^{-1} + \sum_{i}p_ie_{ii} + \sum_{i \neq j}\Phi_\kappa(q_i - q_j)e_{ij} + \mathcal{O}(\lambda) \ , \tag{2.18}$$

where the p_i are the expressions between brackets on the right-hand side of the arrow in (2.17). Thus, we recover the Lax representation of the non-relativistic case as given in [2].

3. Symplectic Structure and Integrability

In order to demonstrate the integrability of the discrete-time RS model, given by the equations of motion (2.14) together with $p = \widetilde{p}$, we need to establish an invariant symplectic structure. Following the philosophy of previous papers, (cf. e.g. [24] and references therein), this can be assessed on the basis of an action principle, cf. also [8, 19].

It is easy to note that an action for the discrete equations (2.14) is given by

$$S = \sum_{n}\mathscr{L}(q(n), q(n+1)) \ , \tag{3.1a}$$

in which

$$\mathscr{L}(q, \widetilde{q}) = \sum_{k, \ell=1}^{N}[f(q_\ell - \widetilde{q}_k) - f(q_\ell - \widetilde{q}_k - \lambda)] - \sum_{\substack{k, \ell=1 \\ k \neq \ell}}^{N}f(q_\ell - q_k + \lambda) \ , \tag{3.1b}$$

with the function $f(x)$ given by

$$f(x) = \int^{x}\log|\sigma(\xi)|d\xi \ . \tag{3.2}$$

Let us point out that the function f appearing here can be considered to be an elliptic version of the Euler dilogarithm function. The discrete Euler–Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial\widetilde{q}_\ell} + \frac{\widetilde{\partial\mathscr{L}}}{\partial q_\ell} = 0, \quad \ell = 1, \ldots, N \ , \tag{3.3}$$

are easily seen to lead to Eq. (2.14). The canonical momenta p_ℓ are found from

$$\widetilde{p}_\ell = \frac{\partial\mathscr{L}}{\partial\widetilde{q}_\ell} = \sum_{k=1}^{N}(-\log|\sigma(\widetilde{q}_\ell - q_k)| + \log|\sigma(\widetilde{q}_\ell - q_k + \lambda)|) \ . \tag{3.4}$$

As a consequence we have that the symplectic form $\Omega = \sum_k dp_k \wedge dq_k$ is invariant under the correspondence, which implies that any branch of the correspondence defines a canonical transformation with respect to the standard Poisson brackets given by

$$\{p_k, q_\ell\} = \delta_{k,\ell}, \qquad \{p_k, p_\ell\} = \{q_k, q_\ell\} = 0 . \tag{3.5}$$

We will now show that in terms of the canonical variables, the Lax matrix L_κ in (2.1) has the same form as the one of the continuum RS model. In fact, expressing the variables h_ℓ^2 in terms of the canonical variables, we obtain

$$h_\ell^2 = e^{p_\ell} \prod_{k \neq \ell} \frac{\sigma(q_\ell - q_k - \lambda)}{\sigma(q_\ell - q_k)} , \tag{3.6}$$

which we will refer to as the Ruijsenaars variables, [10]. In terms of these variables we have the Poisson brackets

$$\{q_k, q_\ell\} = 0, \qquad \{\log h_k^2, q_\ell\} = \delta_{k,\ell} ,$$

$$\{\log h_k^2, \log h_\ell^2\} = \zeta(q_k - q_\ell + \lambda) + \zeta(q_k - q_\ell - \lambda) - 2\zeta(q_k - q_\ell), \quad k \neq \ell . \tag{3.7}$$

Now, there remains not so much work left to establish the integrability. In fact, having expressed the Lax matrix L_κ in terms of the original variables, and having verified that it takes the same form as in the continuous case, we can rely on the proof of involutivity that was produced by Ruijsenaars (in Appendix A of [10]) to assess the involutivity of the invariants of the discrete model as well. Naturally, the invariants of the correspondence, given by

$$I_k = \operatorname{tr} L^k = \widetilde{I}_k, \quad k = 1, \dots, N , \tag{3.8}$$

as functions of the canonical variables are the same as in the continuum model.

Thus, we obtain the statement of involutivity of the invariants, i.e.,

$$\{I_k, I_\ell\} = \{\operatorname{Tr}(L^k), \operatorname{Tr}(L^\ell)\} = 0 \quad \text{for all } k, \ell = 1, 2, \dots . \tag{3.9}$$

It is a theorem by Veselov that the involutivity of the invariants lead to the linearization of the discrete flow on tori, similar to the continuous-time situation, cf. [8]. (For a simple argument see also [19].)

4. Exact Solutions

Let us now consider exact solutions of the discrete RS model in two special cases: i) the rational limit, ii) the hyperbolic (trigonometric) limit. Here, again, we restrict ourselves to the case $p = \underset{\sim}{p}$, which will then lead to an explicit integration scheme from the Lax pair, as we shall demonstrate below. If, more generally, we do not fix p, the initial value problem becomes implicit, having to take into account initial values for the p also. We shall not pursue this latter case in this paper.

The main line of reasoning in both limiting cases is very similar to the one followed in [1,2] for the non-relativistic discrete CM model. In the discrete situation the initial value problem is posed by asking to find $\{q_i(n)\}$ for given initial data $\{q_i(0)\}$ and $\{q_i(1) \equiv \widetilde{q}_i(0)\}$, where $q_i(n)$ denotes the position of the particles

q_i at the nth time-step. The solution to this problem is obtained for both the rational as well as the hyperbolic (trigonometric) limits by solving a secular problem, namely by determining the eigenvalues of an $N \times N$ matrix which can be explicitly calculated from the initial data for any discrete-time value n. However, one may ask the legitimate question what one has gained from this, as one has reduced the problem of solving one set of algebraic equations (given by the discrete equations of motion (2.14)) to another problem of solving an algebraic equation (namely the factorization of a characteristic determinant). In the first case, nonetheless, we have to solve the system step by step at each iteration of the map, whence the issue of the complexity of the solution (i.e. the growth of the number of iterates) becomes imminent. In view of the integrability of the correspondence, one can at most have only polynomial growth, as Veselov has proven in [25], and this is immediately clear from the exact solution, because the only growth comes from the permutations of eigenvalues. In fact, there are $N!$ permutations at each iteration of the map, but whatever the permutation chosen at a certain step, it will end in one of the $N!$ possible permutations at the following step: the branches will "cross" each other with no divergence.

i) *Rational Case.* This is the limit that both periods tend to infinity, i.e. $2\omega_1 \to \infty$, $2\omega_2 \to i\infty$, in which case we can make the substitutions

$$\sigma(x) \to x, \qquad \zeta(x) \to \frac{1}{x}, \qquad \wp(x) \to \frac{1}{x^2} .$$

Then, the Lax matrices take the form

$$L_\kappa = \frac{1}{\kappa} h \, {}^t h + L_0, \qquad M_\kappa = \frac{1}{\kappa} \widetilde{h} \, {}^t h + M_0 , \tag{4.1}$$

where h denotes the (column-)vector with entries h_i, the (row-)vector ${}^t h$ being its transposition, and in which

$$L_0 = \sum_{i,j=1}^{N} \frac{h_i h_j}{\lambda + q_i - q_j} e_{ij}, \qquad M_0 = \sum_{i,j=1}^{N} \frac{\widetilde{h}_i h_j}{\lambda + \widetilde{q}_i - q_j} e_{ij} . \tag{4.2}$$

From the form of the Lax matrices (4.2), we can then derive the following relations:

$$\lambda M_0 + \widetilde{Q} M_0 - M_0 Q = \widetilde{h} \, {}^t h , \tag{4.3a}$$

$$\lambda L_0 + Q L_0 - L_0 Q = h \, {}^t h , \tag{4.3b}$$

where we have set

$$Q = \sum_{i=1}^{N} q_i e_{ii} . \tag{4.4}$$

On the other hand, from the Lax equation (2.7) and inserting (4.1), we obtain the relations

$$\widetilde{L}_0 M_0 = M_0 L_0 , \tag{4.5a}$$

$$\widetilde{L}_0 \widetilde{h} - M_0 h = -p \widetilde{h} , \tag{4.5b}$$

$${}^t h L_0 - {}^t \widetilde{h} M_0 = -p \, {}^t h , \tag{4.5c}$$

together with the conservation law $\widetilde{{}^t h \cdot h} = {}^t h \cdot h$. Consequently, we can put

$$M_0 = \widetilde{U}_0 U_0^{-1}, \qquad L_0 = U_0 \Lambda U_0^{-1} , \tag{4.6}$$

where U_0 is an invertible $N \times N$ matrix, and where the matrix Λ is constant, $\widetilde{\Lambda} = \Lambda$, as a consequence of (4.5a). We assume that Λ can be chosen to be a diagonal matrix, but this is not essential for what follows. Then, introducing

$$V \equiv U_0^{-1} Q U_0, \qquad r \equiv U_0^{-1} \cdot h, \qquad {}^t s \equiv {}^t h \cdot U_0 , \tag{4.7}$$

we obtain from (4.5) and (4.6),

$$(pI + \Lambda) \cdot \widetilde{r} = r, \qquad {}^t s \cdot (pI + \Lambda) = {}^t \widetilde{s} , \tag{4.8}$$

in which I denotes the $N \times N$ unit matrix, as well as from (4.3a) and (4.3b),

$$\lambda + \widetilde{V} - V = \widetilde{r} \, {}^t s, \qquad \lambda \Lambda + [V, \Lambda] = r \, {}^t s , \tag{4.9}$$

together with the conservation law $\widetilde{r \, {}^t s} = r \, {}^t s$. Eliminating the dyadic $r \, {}^t s$ from Eq. (4.9) by making use of (4.8), we find the linear equation

$$\widetilde{V} = (pI + \Lambda)^{-1} V (pI + \Lambda) - \frac{p\lambda}{pI + \Lambda} , \tag{4.10}$$

which can be immediately solved to give

$$V(n) = (pI + \Lambda)^{-n} \left[V(0) - n \frac{p\lambda}{pI + \Lambda} \right] (\Lambda + pI)^n . \tag{4.11}$$

Thus, the solution of the discrete RS model in the rational case is given by the following statement: *the position coordinates $\{q_i(n)\}$ of the particles at the discrete time n, evolving under the discrete equations of motion (2.14) (in the rational limit $\sigma(x) \mapsto x$), are given by the eigenvalues of the matrix:*

$$Q(0) - np\lambda(pI + L_0(0))^{-1} , \tag{4.12}$$

where p can be chosen equal to unity without loss of generality.

The choice of p only affects a scaling of the variables h_i in (2.1), which enter the initial value of the matrix $L_0(0)$.

ii) *Hyperbolic (Trigonometric) Case.* In the hyperbolic limit $2\omega_1 \to \infty$, $2\omega_2 = \frac{1}{2}\pi i$, in which case we can make the substitutions

$$\sigma(x) \to \sinh(x), \qquad \zeta(x) \to \coth(x), \qquad \wp(x) \to \sinh^{-2}(x) ,$$

(after an appropriate gauge transformation of the Lax matrix with a diagonal matrix), leading to

$$\Phi_\kappa(x) \to \coth(\kappa) + \coth(x) . \tag{4.13}$$

In this case we obtain for the Lax matrices (2.1),

$$L_\kappa = L_0 + (\coth \kappa + \gamma) h \, {}^t h, \qquad M_\kappa = M_0 + (\coth \kappa + \gamma) \widetilde{h} \, {}^t h , \tag{4.14}$$

where γ can be chosen at our convenience. In this case the reduced Lax matrices L_0 and M_0 are given by

$$L_0 = \sum_{i,j=1}^{N} h_i h_j (\coth(q_i - q_j + \lambda) - \gamma) e_{ij} \, , \qquad (4.15a)$$

$$M_0 = \sum_{i,j=1}^{N} \widetilde{h}_i h_j (\coth(\widetilde{q}_i - q_j + \lambda) - \gamma) e_{ij} \, . \qquad (4.15b)$$

For them, as a consequence of the "splitting off" of the terms with the spectral parameter $\coth \kappa$, we get again the system of equations (4.5). From Eqs. (4.15) we derive that

$$e^{2\lambda} e^{2Q}(L_0 + \gamma h\,{}^t h) - (L_0 + \gamma h\,{}^t h)e^{2Q} = e^{2\lambda} e^{2Q} h\,{}^t h + h\,{}^t h e^{2Q} \, , \qquad (4.16a)$$

$$e^{2\lambda} e^{2\widetilde{Q}}(M_0 + \gamma \widetilde{h}\,{}^t h) - (M_0 + \gamma \widetilde{h}\,{}^t h)e^{2Q} = e^{2\lambda} e^{2\widetilde{Q}} \widetilde{h}\,{}^t h + \widetilde{h}\,{}^t h e^{2Q} \, , \qquad (4.16b)$$

in which Q, h and ${}^t h$ are given as before. We again make the identifications (4.6), and introduce V, r, s as in (4.7), for which we subsequently derive the relations

$$e^{2\lambda} e^{2V}(\Lambda + (\gamma - 1)r\,{}^t s) = (\Lambda + (\gamma + 1)r\,{}^t s)e^{2V} \, , \qquad (4.17a)$$

$$e^{2\lambda} e^{2\widetilde{V}}(I + (\gamma - 1)\widetilde{r}\,{}^t s) = (I + (\gamma + 1)\widetilde{r}\,{}^t s)e^{2V} \, . \qquad (4.17b)$$

Choosing now $\gamma = 1$, and using (4.8) we can again eliminate the dyadic $r\,{}^t s$ and explicitly solve for $\exp(2V)$. We obtain

$$e^{2\widetilde{V}} = (p + \Lambda)^{-1} e^{2V}(p e^{-2\lambda} + \Lambda) \, , \qquad (4.18)$$

which can be immediately integrated to yield

$$e^{2V(n)} = (\Lambda + pI)^{-n} e^{2V(0)}(\Lambda + p e^{-2\lambda} I)^n \, , \qquad (4.19)$$

the $V(0)$ and $U(0)$ as well as Λ being determined from the initial data $Q(0)$ and $Q(1)$. Thus, rewriting (4.19) we obtain the following result: *the exponentials of the position coordinates $\{e^{2q_i(n)}\}$ of the particles at the discrete time n, evolving under the discrete equations of motion (2.14) (in the hyperbolic limit $\sigma(x) \mapsto \sinh(x)$), are given by the eigenvalues of the matrix*:

$$(L_0(0) + pI)^{-n} e^{2Q(0)}(L_0(0) + p e^{-2\lambda} I)^n \, , \qquad (4.20)$$

where p can be chosen equal to unity without loss of generality.

The choice of p only affects a scaling of the variables h_i in (2.1), which can be accounted for in the calculation of the $h_i(0)$ from Eq. (2.13). We note again that at each discrete-time value the positions of the particles is uniquely determined up to a permutation.

We finally remark that the trigonometric limit of the elliptic functions $2\omega_1 = \frac{1}{2}\pi$, $2\omega_2 \to i\infty$ is integrated along similar lines after doing the replacements

$$\sigma(x) \to \sin x, \qquad \zeta(x) \to \cot x \, .$$

We shall omit the details.

5. Connection with Discrete Soliton Systems

In [1, 3], the discrete CM model was obtained from pole-solutions of a lattice version of the Kadomtsev–Petviashvili (KP) equation. The idea that integrable particle models of CM type are connected with soliton equations goes back already to the late seventies, cf. [26] and also [27, 28], but had never been applied to discrete soliton equations. In [10], but maybe a bit more transparently in [29], a connection between the relativistic particle model and soliton solutions of nonlinear integrable PDE's was established. It is a natural question to ask whether such a connection also exists on the discrete level.

   In order to establish such a connection, let us investigate more closely the trigonometric solution of the previous section. After a gauge transformation with a diagonal matrix of the form $P = [(\Lambda + pI)(\Lambda + e^{-2\lambda}pI)]^{1/2}$, we can transform $e^{2V} = P^{-n}WP^n$ into

$$W_{ij} = -2\frac{\rho_i\rho_j}{\lambda_i - \omega\lambda_j}\,, \tag{5.1}$$

in which $\rho_i \equiv (P^n \cdot r)_i = ({}^t s \cdot e^{2V}P^{-n})_i$, and $\omega \equiv e^{2\lambda}$. The dependence on the discrete (time-)variable is given by

$$\widetilde{\rho}_i = \left(\frac{\lambda_i + \omega^{-1}p}{\lambda_i + p}\right)^{1/2}\rho_i\,, \tag{5.2}$$

leading to the equations for W

$$\Lambda W - \omega W\Lambda = -2\rho\,{}^t\rho\,, \tag{5.3a}$$

$$PW - \omega\widetilde{W}P = -2\widetilde{\rho}\,{}^t\rho\,. \tag{5.3b}$$

We can then introduce the characteristic polynomial

$$\tau(\xi) = \prod_{j=1}^{N}(\xi + e^{2q_j}) = \det(\xi I + W)\,, \tag{5.4}$$

and show that $\tau(\xi)$ plays the role of the tau-function of a discrete soliton system. In fact, W is the kernel of the integral operator in the soliton sector that stands as the basis of the so-called direct linearization approach, cf. e.g. [30–32].

   To derive the relevant equations directly from the resolvent of the matrix W, we proceed as follows. First, using Eq. (5.3b), we can perform the following simple calculation:

$$\begin{aligned}
\frac{\widetilde{\tau}(\xi)}{\tau(\omega\xi)} &= \det((\omega\xi I + W)^{-1}(\xi I + \widetilde{W})) \\
&= \det[(\omega\xi I + W)^{-1}(\xi I + \omega^{-1}W + 2\omega^{-1}(\Lambda + pI)^{-1}\cdot\rho\,{}^t\rho)] \\
&= \omega^{-N}\det(I + (\omega\xi I + W)^{-1}(\Lambda + pI)^{-1}\cdot\rho\,{}^t\rho)\,,
\end{aligned}$$

leading to

$$v(\omega\xi) \equiv 1 + {}^t\rho\cdot(\omega\xi I + W)^{-1}(\Lambda + pI)^{-1}\cdot\rho = \omega^N\frac{\widetilde{\tau}(\xi)}{\tau(\omega\xi)}\,, \tag{5.5}$$

which will turn out to be one of the variables governed by the soliton system. In order to derive discrete soliton equations for $\tau(\xi)$ or v, we introduce the N-component vectors

$$u_j(\xi) \equiv (\xi I + W)^{-1} \Lambda^j \cdot \rho, \qquad {}^t u_j(\xi) \equiv {}^t\rho \cdot \Lambda^j (\xi I + W)^{-1}, \tag{5.6a}$$

as well as the scalar variables

$$U_{ij}(\xi) \equiv 2 \, {}^t\rho \cdot \Lambda^j (\xi I + W)^{-1} \Lambda^i \cdot \rho, \tag{5.6b}$$

$(i, j \in \mathbf{Z})$. Making use of the relations (5.3), we can derive the following set of recursive relations between the different vectors $u_j(\xi)$ and ${}^t u_j(\xi)$:

$$\omega P \cdot u_j(\xi) = p\widetilde{u}_j(\omega^{-1}\xi) + \widetilde{u}_{j+1}(\omega^{-1}\xi) + U_{j,0}(\xi)\widetilde{u}_0(\omega^{-1}\xi), \tag{5.7a}$$

$${}^t\widetilde{u}_j(\xi) \cdot P = \omega \, {}^t u_{j+1}(\omega\xi) + p \, {}^t u_j(\omega\xi) - {}^t u_0(\omega\xi)\widetilde{U}_{0,j}(\xi). \tag{5.7b}$$

Using the definition (5.6b) as well as (5.2), we can derive also

$$p\widetilde{U}_{ij}(\xi) + \widetilde{U}_{i+1,j}(\xi) = pU_{ij}(\omega\xi) + \omega U_{i,j+1}(\omega\xi) - U_{i,0}(\omega\xi)\widetilde{U}_{0,j}(\xi). \tag{5.7c}$$

At this point we need to reflect a moment on the role of the discrete-time shift. The shift $U_{ij}(\xi) \mapsto \widetilde{U}_{ij}(\xi)$ is "labeled" by the variable p, which can be identified as the (reciprocal) of the lattice parameter. Let us assume that the variables $U_{ij}(\xi)$ do not depend on only one discrete time-variable n, but on a number of them, say n_α, ($\alpha = 1, 2, 3, \ldots$), and that the corresponding discrete flows are compatible, each associated with its own parameter p, i.e. $p_1, p_2, p_3, \ldots$. This means that U is a *function* of these independent time-variables, i.e. $U_{ij}(\xi) = U_{ij}(\xi; n_1, n_2, \ldots)$. In that case we have for instance that $\widehat{\widetilde{U}}_{ij}(\xi; n_1, n_2, \ldots) = U_{ij}(\xi; n_1 + 1, n_2 + 1, \ldots) = \widetilde{U}_{ij}(\xi; n_1, n_2, \ldots)$, etc., the $\widehat{}$ corresponding to the translation in the second discrete variable. Of course, this puts extra compatibility conditions on the eigenvalues e^{2q_j} of the kernel W given in (5.1) of the soliton solutions. In fact, the particle-coordinates q_j can be shown to obey some two-dimensional lattice equations in terms of the shifts $\widetilde{}$ and $\widehat{}$, and the compatibility of such equations were investigated in [2] for the non-relativistic situation. Here, we restrict ourselves to obtaining from (5.7c) closed-form nonlinear partial difference equations for special elements or combinations of elements of U_{ij}, i.e. partial difference versions of the well-known soliton systems. These difference systems were investigated in detail in a number of earlier papers, (cf. [33] for a review, and references therein), on the basis of the system of equations (5.7a)–(5.7c). Thus here we will only present the results, referring to those papers for their derivation.

In fact, for the special element $u_{n_1, n_2, n_3} \equiv U_{0,0}(\omega^{-(n_1+n_2+n_3)}\xi; n_1, n_2, n_3)$, we can derive the partial difference equation

$$(p_1 - p_2 + u_{n_1+1, n_2, n_3} - u_{n_1, n_2+1, n_3})(p_3 + u_{n_1+1, n_2+1, n_3}) + \text{cycl. perm.} = 0, \tag{5.8}$$

which is a lattice version of the Kadomtsev–Petviashvili (KP) equation, cf. [30, 31]. In [2, 3] pole solutions of this equation were investigated associated with a discretization of the non-relativistic Calogero–Moser model. The relation to the

τ-function $\tau_{n_1,n_2,n_3} \equiv \tau(\omega^{-(n_1+n_2+n_3)}\xi; n_1, n_2, n_3)$ is given by

$$p_1 - p_2 + u_{n_1,n_2+1,n_3} - u_{n_1+1,n_2,n_3} = (p_1 - p_2)\frac{\tau_{n_1+1,n_2+1,n_3}\,\tau_{n_1,n_2,n_3}}{\tau_{n_1+1,n_2,n_3}\,\tau_{n_1,n_2+1,n_3}}\,, \quad (5.9)$$

from which one can derive the bilinear equation

$$(p_1 - p_2)\tau_{n_1+1,n_2+1,n_3}\,\tau_{n_1,n_2,n_3+1}(\xi) + \text{cycl. perm.} = 0\,, \quad (5.10)$$

which was first presented in [34], cf. also [35]. Equation (5.10) is related also to the following version of the lattice (modified) KP equation, [35]:

$$(p_1 - p_2)\left(\frac{v_{n_1,n_2+1,n_3+1}}{v_{n_1,n_2,n_3+1}} - \frac{v_{n_1+1,n_2+1,n_3}}{v_{n_1+1,n_2,n_3}}\right)$$

$$= (p_1 - p_3)\left(\frac{v_{n_1,n_2+1,n_3+1}}{v_{n_1,n_2+1,n_3}} - \frac{v_{n_1+1,n_2,n_3+1}}{v_{n_1+1,n_2,n_3}}\right), \quad (5.11)$$

in terms of the variable $v_{n_1,n_2,n_3} = v(\omega^{-(n_1+n_2+n_3)}\xi; n_1, n_2, n_3)$, with v being defined in (5.5), (taking $p_1 = p$).

Finally, we mention that special reductions of the above equations arising by imposing additional symmetries on the soliton solutions, will lead to the lattice soliton systems of Gel'fand–Dikii type that were introduced in [32]. In fact, taking the parameter ω equal to a root of unity, we can derive additional constraints on the system of equations given by (5.7a)–(5.7c). Thus, we find in the particular case of $\omega = -1$ for u the following lattice version of the Korteweg–de Vries equation:

$$(p_1 - p_2 + u_{n_1,n_2+1} - u_{n_1+1,n_2})(p_1 + p_2 + u_{n_1,n_2} - u_{n_1+1,n_2+1}) = p_1^2 - p_2^2\,,$$
$$(5.12)$$

cf. [32], which is related to the following bilinear equation in terms of the τ-function:

$$(p_1 + p_2)\tau_{n_1-1,n_2+1}\,\tau_{n_1+1,n_2} + (p_1 - p_2)\tau_{n_1-1,n_2}\,\tau_{n_1+1,n_2+1} = 2p_1\tau_{n_1,n_2}\,\tau_{n_1,n_2+1}\,.$$
$$(5.13)$$

It is this equation that corresponds in the continuum limit exactly to the special case considered in [29]. Similar equations can be derived for other values of ω when $|\omega| = 1$, starting from the results presented in [32].

6. Connection with Bethe Ansatz Equations

We already remarked above that the equations of motion of the discrete Ruijsenaars–Schneider model, (2.14), resemble closely the form of Bethe Ansatz equations (BAE's) for certain integrable quantum models. In this section we will make this connection more precise. Let us first focus on the hyperbolic limit of Eqs. (2.14), and connect it to the BAE's for the XXZ spin $-\frac{1}{2}$ Heisenberg magnet, cf. e.g. [36–38] and references therein. After that, we will show that the general form of (2.14) in the elliptic case is connected to the generalized Bethe Ansatz (BA) for the Heisenberg XYZ model, proposed first in [39] and developed further in [40].

We will not treat the correspondence in the rational (or XXX) case since it can be obtained by a simple limit from the hyperbolic case.

Let us, thus, consider the quadratic R-matrix algebra

$$R^{(12)}(u-v)T^{(1)}(u)T^{(2)}(v) = T^{(2)}(v)T^{(1)}(u)R^{(12)}(u-v), \qquad (6.1)$$

which is one of the central relations in the quantum inverse scattering method (QISM), (cf. [37] for an early, and [38,41] for more recent reviews), and let us consider the R-matrix for the spin $-\frac{1}{2}$ XXZ Heisenberg magnet,

$$R^{(12)}(u) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad (6.2)$$

in which

$$a = \sinh(u+\eta), \qquad b = \sinh u, \qquad c = \sinh \eta .$$

The quantum L-operator on each site k of the spin chain has the form

$$L_k(u) = \begin{pmatrix} \sinh(u+\frac{1}{2}\eta\sigma_k^3) & \sinh(\eta)\sigma_k^- \\ \sinh(\eta)\sigma_k^+ & \sinh(u-\frac{1}{2}\eta\sigma_k^3) \end{pmatrix}, \qquad (6.3)$$

where $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^1 \pm i\sigma_k^2)$, $\sigma_k^{1,2,3}$ being the Pauli matrices on site k, (i.e. $\sigma_k^{\alpha} = 1 \otimes \cdots \otimes \sigma^{\alpha} \otimes \cdots \otimes 1$ with σ^{α} on the kth entry of the tensor product). Let us now construct the following monodromy matrix:

$$T(u) = L_M(u-\delta_M)\cdots L_2(u-\delta_2)L_1(u-\delta_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \qquad (6.4)$$

where M is the length of the spin chain, and in which the δ_k, $k=1,\ldots,M$ are impurity parameters. Both the $L_k(u)$ as well as $T(u)$ obey the relation (6.1). Let us now recall the standard algebraic Bethe Ansatz construction, [37–41], i.e. there is a vacuum state Ω_M, which is an eigenvector of the diagonal entries $A(u), D(u)$ of the monodromy matrix, and which is annihilated by the operator $C(u)$

$$A(u)\Omega_M = a(u)\Omega_M, \qquad D(u)\Omega_M = d(u)\Omega_M, \qquad C(u)\Omega_M = 0 . \qquad (6.5)$$

As the monodromy matrix is a comultiplication of the L-operators along the sites of the chain, we have that

$$a(u) = \prod_{k=1}^{M} a_k(u), \qquad d(u) = \prod_{k=1}^{M} d_k(u) , \qquad (6.6)$$

where the functions $a_k(u)$ and $d_k(u)$ are the eigenvalues of the vacuum for the diagonal entries of the corresponding L-operator on site k. The algebraic Bethe Ansatz amounts to the creation of an eigenstate of the form

$$\Psi(q_1,\ldots,q_N) = \prod_{j=1}^{N} B(q_j)\Omega_M \qquad (6.7)$$

of the trace of the monodromy matrix $A(u) + D(u)$,

$$(A(u) + D(u))\Psi = t(u)\Psi \,, \tag{6.8}$$

with $t(u)$ being the corresponding eigenvalue. We have then the following proposition:

Proposition. Ψ *is an eigenfunction of the trace of the monodromy matrix iff the numbers q_l satisfy the Bethe Ansatz equations*

$$\prod_{\substack{k=1 \\ k \neq l}}^{N} \frac{\sinh(q_l - q_k + \eta)}{\sinh(q_l - q_k - \eta)} = \frac{a(q_l)}{d(q_l)}, \qquad l = 1, \dots, N \,. \tag{6.9}$$

In the case of the monodromy matrix (6.4), the functions $a(u)$, $d(u)$ take the form

$$a(u) = \prod_{k=1}^{M} \sinh\left(u - \delta_k + \frac{1}{2}\eta\right), \qquad d(u) = \prod_{k=1}^{M} \sinh\left(u - \delta_k - \frac{1}{2}\eta\right) \,, \tag{6.10}$$

which in the special case that $M = 2N$ lead to the BAE's

$$\prod_{\substack{k=1 \\ k \neq l}}^{N} \frac{\sinh(q_l - q_k + \lambda)}{\sinh(q_l - q_k - \lambda)} = \prod_{k=1}^{N} \frac{\sinh(q_l - \widetilde{q}_k)\,\sinh(q_l - q_k + \lambda)}{\sinh(q_l - q_k)\,\sinh(q_l - \widetilde{q}_k - \lambda)}, \quad l = 1, \dots, N \,, \tag{6.11}$$

i.e. Eq. (2.14) in the hyperbolic limit $(\sigma(x) \mapsto \sinh(x))$ with the identifications $\lambda = \eta$, $\widetilde{q}_k = \delta_k - \frac{1}{2}\eta$, $q_k = \delta_{k+N} + \frac{1}{2}\eta$, $(k = 1, \dots, N)$. Thus, the discrete RS system in the hyperbolic limit can be reinterpreted as the BAE's for the spin-$\frac{1}{2}$ XXZ Heisenberg chain with the number of spins (impurities) equal to twice the excitation number of the eigenstate. We should stress that within such an identification the impurities play the role of the N particle coordinates at times $n - 1$ and $n + 1$ (q resp. $\widetilde{q}$), which means that the integrable correspondence mixes the parameters of the quantum model (i.e. the impurities) and the parameters of a solution given by the Bethe wavefunction.

To make the connection with the elliptic case we have to consider the QISM for the XYZ Heisenberg model that was treated in [39] for the spin-$\frac{1}{2}$ situation. For arbitrary spin one needs to consider the algebras introduced by Sklyanin in [42]. However, we are interested here in the inhomogeneous spin-$\frac{1}{2}$ chain, i.e. including arbitrary impurities. Both extensions were treated in a recent paper by Takebe [40]. It was shown that the generalized Bethe Ansatz of Takhtajan and Faddeev is applicable to this model. The corresponding R-matrix is given by

$$R^{(12)}(u) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \tag{6.12}$$

in which

$$a = \mathrm{sn}(u + 2\eta; k), \qquad b = \mathrm{sn}(u; k), \qquad c = \mathrm{sn}(2\eta; k),$$

$$d = k\,\mathrm{sn}(2\eta; k)\mathrm{sn}(u; k)\mathrm{sn}(u + 2\eta; k).$$

The quantum L-operator on each site k of the spin chain now has the form

$$L_k(u) = \begin{pmatrix} w_0\mathbf{1} + w_3\sigma_k^3 & w_1\sigma_k^1 - iw_2\sigma_k^2 \\ w_1\sigma_k^1 + iw_2\sigma_k^2 & w_0\mathbf{1} - w_3\sigma_k^3 \end{pmatrix}, \qquad (6.13)$$

where

$$w_0 + w_3 = \mathrm{sn}(u + \eta; k), \qquad w_0 - w_3 = \mathrm{sn}(u - \eta; k),$$

$$w_1 + w_2 = \mathrm{sn}(2\eta; k), \qquad w_1 - w_2 = k\,\mathrm{sn}(2\eta; k)\mathrm{sn}(u + \eta; k)\mathrm{sn}(u - \eta; k).$$

The monodromy matrix takes again the form (6.4). The generalization of the algebraic Bethe Ansatz to the model described in (6.12),(6.13) is known only for the special eigenfunctions *for which the number of excitations is half the number of spins in the chain*. Of course, this constitutes only a special class within the total set of eigenfunctions, but no further results exist to date. Curiously, this is precisely the case in which we have a connection with the discrete RS model!

To be more precise, the generalized BA, developed in [39, 40], consists of constructing eigenfunctions of the trace of the monodromy matrix $A(u) + D(u)$ after performing a gauge transformation of the form

$$T(u) \mapsto T_{m,m'}(u) = S_m(u)^{-1}T(u)S_{m'}(u) =: \begin{pmatrix} A_{m,m'}(u) & B_{m,m'}(u) \\ C_{m,m'}(u) & D_{m,m'}(u) \end{pmatrix}, \qquad (6.14)$$

where $S_m(u)$ is an appropriate scalar matrix. (Note that we deviate from the notations in [39, 40] to be consistent with the notations used in earlier sections of the present paper.) Instead of a single vacuum there is now a set of generating (vacuum) vectors $\Omega_{M,n}$ for which one has

$$A_{M,n}(u)\Omega_{M,n} = a(u)\Omega_{M,n-1}, \qquad D_{M,n}(u)\Omega_{M,n} = d(u)\Omega_{M,n+1}$$

$$C_{M,n}(u)\Omega_{M,n} = 0. \qquad (6.15)$$

The generalized Bethe eigenfunctions are linear combinations of the vectors

$$\Psi_n(q_1, \ldots, q_N) = B_{n+1,n-1}(q_1) \cdots B_{n+N,n-N}(q_N)\Omega_{M,n-N}, \qquad (6.16)$$

of the form

$$\Psi_0(q_1, \ldots, q_N) = \sum_{n=-\infty}^{\infty} e^{2\pi in\theta}\Psi_n(q_1, \ldots, q_N), \qquad (6.17)$$

where θ is a free parameter. As it was pointed out in [39], it is possible that the series (6.17) is summable to zero for all θ except for a finite number of values θ_j. (The results of Baxter [36] show that among the θ_j there is also the value $\theta = 0$.) Then for such values of θ one can produce the corresponding eigenfunctions, provided

that the excitation numbers $q_1, \ldots, q_N$ obey the following BAE's (cf. [40]):

$$X \prod_{\substack{k=1 \\ k \neq \ell}}^{N} \frac{\sigma(q_\ell - q_k + 2\eta)}{\sigma(q_\ell - q_k - 2\eta)} = \prod_{k=1}^{2N} \frac{\sigma(q_\ell - \delta_k + 2\eta)}{\sigma(q_\ell - \delta_k)}, \qquad \ell = 1, \ldots, N, \qquad (6.18)$$

$$X = \exp\left[-4\pi i \theta + 2\eta \frac{\eta_2}{\omega_2} \left(\sum_{k=1}^{N} (2q_k + 2\eta) - \sum_{k=1}^{2N} \delta_k \right) \right] \qquad (6.19)$$

in the special case[5] that $M = 2N$, and re-expressing the equations derived in [40] in terms of σ-function. Equations (6.18) correspond to Eqs. (2.14) for the discrete RS model in the elliptic case by the following identifications: $\lambda = 2\eta$, $q_k = \delta_k$, $\widetilde{q}_k = \delta_{k+N} - 2\eta$, ($k = 1, \ldots, N$), together with

$$\frac{p}{\underset{\sim}{p}} = \exp\left[-4\pi i \theta - \lambda \frac{\eta_2}{\omega_2} (B + \widetilde{B} - 2B) \right], \qquad B = \sum_{k=1}^{N} q_k. \qquad (6.20)$$

The sum of the particle positions B plays the role of the boost generator of the underlying Poincaré algebra, cf. [10]. So, the generalized Bethe Ansatz equations correspond to the discrete-time system with the dynamics of p given in terms of the dynamics of q_k's. The results of Sect. 3 can be generalized to incorporate this case. In fact, we need to add now the term

$$4\pi i \theta B - \frac{\eta_2}{2\omega_2} \lambda (\widetilde{B} - B)^2, \qquad (6.21)$$

to the Lagrangian (3.1b) in order to get the correct factor from the Euler–Lagrange equations. If we now calculate the canonical momenta from (3.1b) we get an extra term $-\frac{\eta_2}{\omega_2} \lambda (\widetilde{B} - B)$ on the right-hand side in (3.4), but this factor cancels again when we express the h_ℓ^2 in terms of the canonical momenta, because of the p entering into Eqs. (2.13). Thus, the final result (3.6) remains unaltered.[6] Consequently, we still have the same integrals in terms of the canonical variables, and the Ruijsenaars variables do not change, hence they are in involution and we have the integrable correspondence.

It is amazing that exactly in the restricted case where the generalized BA is known to apply, i.e. the case that the number of impurities is equal to twice the number of excitations, we have a connection with an integrable correspondence of RS type. This might also suggest that there exists a deeper relation between integrable quantum models solvable by the QISM on the one hand and classical integrable multiparticle models on the other hand. The connection provided here might yield an explanation for the existence conditions of the generalized BA of Faddeev and Takhtajan and of Baxter's solution of the eight-vertex model.

7. Discussion

The connection exhibited in the previous section between the BAE's for the XXZ and XYZ model with impurities and the discrete relativistic multi-particle model is

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[5] The relation $M = 2N$ follows because of the spin-$\frac{1}{2}$ situation, while becoming $N = \ell M$ for the arbitrary spin ℓ (cf. [40]).

[6] This effect can be viewed as a consequence of the relativistic invariance of the model.

very remarkable. A similar connection seems to exist also in the non-relativistic case studied in [1, 2], which yields a link between the equations of motion of the discrete CM model and BAE's for the Gaudin model. In both cases, we thus obtain a new interpretation of the BAE's for a quantum solvable model, namely as a classical integrable dynamical model with discrete time. Such interpretations, together with the exact solutions of the discrete equations of motion that were derived in Sect. 4, might lead to possible ways of "solving" the BAE's, (which in principle constitutes a set of transcendental equations), without having to perform some thermodynamic limit, cf. [38]. This, however, requires a reformulation of the initial value problem for the discrete model rather as a boundary value problem on the discrete-time chain. Although it is not clear at this stage that such boundary value problems are explicitly solvable as was the case in Sect. 4, we nevertheless conjecture that this intriguing dynamical interpretation of the Bethe Ansatz equations will shed new light on the solvability of the corresponding quantum models. Furthermore, it would be interesting to study generalizations of our results, for instance in the directions of finding discrete particle models associated with higher-spin models and higher-rank associated Lie algebras (nested Bethe Ansatz), as well as investigating in this light the quadratic algebras associated with boundary conditions for quantum integrable systems (reflection equation algebras), cf. [43]. The latter might lead to time-discretizations of BC_N type of RS models, which were investigated in [14, 15].

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