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Hidden Σ_{n+1}-Actions

Olivier Mathieu

Institut de Recherches Mathèmatiques Avancées, Université Louis Pasteur et C.N.R.S., 7, rue René Descartes, F-67084 Strasbourg Cedex, France. email: mathieu@math.u-strasbg.fr

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Abstract: Let n be an integer. Denote by A_n one of the following two graded vector spaces: (a) the space of all multilinear Poisson polynomials of degree n (with a grading described below), or (b) the cohomology of the space of all n-uples of complex numbers $z_1,\dots,z_n$ with $z_i \neq z_j$ for $i \neq j$. We prove that the natural action of Σ_n on each homogeneous component of A_n can be extended to an "hidden" Σ_{n+1}-action and we compute the corresponding character (the Σ_n-character being already given by Klyaschko and Lehrer–Solomon formulas).

Introduction

Let n be an integer, let X be a symplectic manifold and let $SC_n(X)$ be the $\mathbf{Q}$-vector space generated by all multilinear maps from $(C^\infty(X))^n$ to $C^\infty(X)$ that we can obtain by composing the multiplication of functions and the Poisson bracket. It is clear that this space depends only on the dimension of X. Indeed for $\dim X \geqq (n-1)$, $SC_n(X)$ is the space of all multilinear free Poisson polynomials into n variables (see [M], Sect. 7) and it will be denoted by SC_n or by $SC_n(\infty)$. The group Σ_n acts in an obvious way on SC_n. Indeed there is a less obvious action of Σ_{n+1} on SC_n which is defined as follows. Let $p \in SC_n$ and let $w \in \Sigma_{n+1}$, where Σ_{n+1} is identified with the group of permutations of $\{0,\dots,n\}$. There exists a unique $q \in SC_n$ such that $\int_X f_{w(0)} q(f_{w(1)},\dots,f_{w(n)}) = \int_X f_0 p(f_1,\dots,f_n)$ for any compactly supported smooth functions $f_0,\dots,f_n$ on a symplectic manifold X of dimension $\geqq n-1$, where the integral over X refers to the Liouville measure (see [M], Theorem 1.5). Then the Σ_{n+1}-action is defined by the requirement $w \cdot p = q$. This "hidden" Σ_{n+1}-action extends the natural Σ_n-action. Also the space SC_n has a natural structure of graded coalgebra ([M], Sect. 3) which is preserved by the action of the symmetric group.

Denote by U_n the space of all n-uple of complex numbers $z_1,\dots,z_n$ with $z_i \neq z_j$ for $i \neq j$ and by SC_n^* the dual of SC_n. It turns out that the algebras $H^*(U_n)$ and SC_n^* have a very similar presentation (see [A] for the first one and [M] for the other one). Also it is natural to ask the following question: *can the natural Σ_n-action on $H^*(U_n)$ be extended to a Σ_{n+1}-action*? In this paper, we describe such an action on the cohomology with rational coefficients. However we prove that for $n \geqq 4$, no extension of the Σ_n-action stabilizes the integral structure of the cohomology. Thus this action does not come from an action of the group Σ_{n+1} on the topological space U_n. This is why, in order to describe the additional generator of Σ_{n+1}, we need to use a multivalued map from U_n to itself instead of an ordinary map. It is easy to prove that the inverse image of this correspondence acts as a ring automorphism of $H^*(U_n)$.

Denote by V the natural permutation Σ_{n+1}-representation on $\mathbf{Q}^{n+1}$ and define a grading $V_0 \oplus V_1$ of V by requiring that V_0 is the trivial component and V_1 is its unique equivariant complement. Another natural question is to *compute the Σ_{n+1} character of each homogenous component of $H^*(U_n)$ and SC_n.* As the Σ_n-character of these representations is already given by Lehrer–Solomon formula [LS] and the Klyaschko formula [K], the Σ_{n+1}-character can be deduced from the following:

Theorem. *As graded Σ_{n+1}-modules there are natural isomorphisms $H^*(U_{n+1}) \simeq H^*(U_n) \otimes V$ and $SC^*_{n+1} \simeq SC^*_n \otimes V$, where on the left side the actions are the natural one and on the right side they are the "hidden" actions.*

By looking at the component of higher degree, we recover the Getzler and Kapranov formula $\mathrm{Lie}(n+1) \simeq \mathrm{Lie}(n) \otimes V_1$, where $\mathrm{Lie}(n)$ denotes the space of multilinear Lie Polynomials in n-variables (see [GK], Introduction and Corollary (6.8)).

1. The Involution Associated to a Suspensive System

By definition an arrangement of hyperplanes H is a finite by collection of linear hyperplanes in a complex vector space E. We then denote U_H the complement in E of the union of all hyperplanes of H. In this section we will associate to any suspensive system v (see the definition below) an involution σ_v of $H^*(U_H)$ (unless stated otherwise, the cohomology is the $\mathbf{Q}$-valued cohomology).

(1.1). Definition of a suspensive system. Let H be an arrangement of hyperplanes in a complex vector space E. A basis $(u_1, \ldots, u_n)$ of E^* is called a suspensive system if and only if it satisfies the following three requirements:

(i) the hyperplanes $u_i = 0$ belong to H for any i,

(ii) any other hyperplane in H is defined by an equation $a \cdot u_i + b \cdot u_j = 0$ for some $i, j \in \{1, 2, \ldots, n\}$ and $a, b \in \mathbf{C}^*$,

(iii) if $\ker(a \cdot u_i + b \cdot u_j)$ belongs to H, so is $\ker(b \cdot u_i + a \cdot u_j)$ for any $a, b \in \mathbf{C}^*$, $1 \leqq i < j \leqq n$.

Only very special arrangements of hyperplanes have one or more suspensive systems. For example we can prove that the existence of a suspensive system implies that the algebra $H^*(U_H)$ is quadratic. As we will not use this fact, the proof is left to the reader.

(1.2). Multivalued functions and inverse images. Let X, Y be manifold. We will use the following formal definition of multivalued functions from X to Y. Let N be an integer. By definition a N-valued function from X to Y is a triple $F = (Z, X, Y)$ consisting of a manifold Z and two smooth maps $p: Z \to X$ and $q: Z \to Y$ such that p is an N-fold covering. The manifold Z is called the graph of F. Less formally,

we denote a N-valued map as $F: X \to Y$ and we say that F associates to any $x \in X$ the set with multiplicity $F(x) = q(p^{-1})(x)$. In order to simplify the notation we will make no differences between a N-valued function F and the NM valued function $M \cdot F$ which associates to x the same set $F(x)$ with M times the multiplicities (e.g. in Formula 2.2) because the induced maps in cohomology are the same. The composition of a N-valued map $F: X \to T$ and a N'-valued map $F': T \to Y$ is the NN'-valued map $F' \circ F: X \to Y$ whose the graph is $Z \times_T Z'$, where Z, Z' are the graphs of F and F'. Similarly one defines the product of complex valued multivalued functions. Let $F: X \to Y$ be a N-valued map. Given a form ω over Y, denote by $F(\omega)$ the form whose value at $x \in X$ is $1/N(\sum_{z \in p^{-1}(x)} q^*(\omega_z))$. Also denote by $F^*: H^*(Y) \to H^*(X)$ the map induced in cohomology. The definition of the inverse image F^* of the multivalued map F behaves like the usual inverse image of ordinary maps except that

(i) in general F^* is not a ring morphism (because of the finite integral),

(ii) in general F^* is not defined over the integral cohomology (because of the factor $1/N$).

However if $q^*(H^*(Y))$ is contained in the subspace $H^*(X)$ of $H^*(Z)$, then F^* is a ring morphism (that is why there is a factor $1/N$ in the definition of F^*).

(1.3). Let $s = (u_1, \ldots, u_m)$ be a suspensive system of an arrangement of hyperplanes H. Set $\Delta_s = \prod_{1 \leqq i \leqq m} u_i^2$. Set $F_s(u_i) = \delta_s/u_i$, where $\delta_s = \Delta_s^{1/m}$. We have $F_s(a \cdot u_i + b \cdot u_j) = \delta_s \cdot (b \cdot u_i + a \cdot u_j)/(u_i \cdot u_j)$. Hence F_s is a well-defined m-valued map from U_H to itself.

Lemma 1.3. *The inverse map F_s^* is a ring morphism. Moreover we have $(F_s^*)^2 = 1$.*

Proof. It follows from the Brieskorn Theorem that the cohomology of U_H is generated by the forms dl/l, where l runs over the space of linear forms defining the arrangement of hyperplanes (see [Br, O, OS, OT]). As $d(\delta_s)/\delta_s$ is a combination with rational coefficients of such forms, it follows that $q^*H^*(U_H) \subset H^*(U_H)$, where q is as before. Hence it follows from (1.2) that F_s^* is a ring morphism. Clearly F_s^2 is the n^2-valued map which sends $u \in U_H$ to the set with multiplicity $\{x \cdot y \cdot u | x, y \in \mu_n\}$, where μ_n is the set of n-roots of unity. As $\mathbf{C}^*$ acts trivially on $H^*(U_H)$ we have $(F_s^*)^2 = 1$. Q.E.D.

The map F_s^* will be called the involution associated with the suspensive system s.

2. Hidden Automorphisms of the Cohomology of the Arrangement Associated with a Graph

(2.1). By graph we mean non-oriented graph with simple edges and no loops. Let Γ be a graph, with a set of vertices V and set of edges E. Set $E_\Gamma = \{(z_v)_{v \in V} \in \mathbf{C}^V | \sum_{v \in V} z_v = 0\}$. For each edge (v, u) of Γ one associates the hyperplane $z_u = z_v$ of E_Γ and we denote by H_Γ the collection of all hyperplanes associated to edges of Γ. Its complement in E_Γ will be denoted by U_Γ.

(2.2). A suspension point of Γ is a vertex which is connected to all other vertices of the graph. If s is a suspension point, then the linear form $z_s - z_v$ for $v \neq s$ is a suspensive system. Denote by σ_s the associated involution of $H^*(U_\Gamma)$. We will use the following formulas. Let s, v, w be three distinct points in V, with s suspensive. We have

$$F_s(z_v - z_w) = \delta_s(z_v - z_w)/(z_s - z_v)(z_s - z_w)\,,$$

and

$$F_s(z_v - z_s) = \delta_s/(z_v - z_s)\,.$$

From this we deduce $F_s\delta_t = \delta_s \cdot \delta_t/(z_s - z_t)^2$ and $F_s\delta_s = \delta_s$, where s, t are distinct suspensive points.

(2.3). Let Γ be a graph and let S be the set of suspension points. We denote the vertices by positive integers $1, 2, \ldots, m$, where m is the number of vertices. Set $S^+ = S \cup \{0\}$. For any set Z, denote by Σ_Z the full permutation group of Z and for $z, z' \in Z$ denote by $r_{z,z'}$ the substitution exchanging z and z'. The group Σ_S acts naturally on Γ by fixing all vertices outside S. So Σ_S acts naturally on U_Γ. Let G be the group of automorphisms of $H^*(U_\Gamma)$ generated by the involutions σ_s, for $s \in S$.

Theorem 2.3. *The group G contains Σ_S and is naturally isomorphic to Σ_{S^+}. For such an isomorphism the involution σ_s is identified with $r_{0,s}$.*

Proof. Let $s, t \in S$ and let j be a vertex of Γ different from s and t. Using formulas (2.2) one gets $F_s \circ F_t \circ F_s(z_s - z_j) = (z_t - z_j)$, and $F_s \circ F_t \circ F_s(z_s - z_t) = (z_t - z_s)$, up to some multivalued constant factor. Hence we have $\sigma_s \circ \sigma_t \circ \sigma_s = r_{s,t}$. Moreover we obviously have $w\sigma_s w^{-1} = \sigma_{w(s)}$. Thus there exists a unique morphism Θ from G to Σ_{S^+} sending σ_s to $r_{0,s}$. Using the presentation of Σ_{S^+} by generators and relations, it is easy to prove that Θ is an isomorphism. Q.E.D

(2.4). Denote by K_n the complete graph with n vertices. Note that U_{K_n} is homotopic to the space U_n from the introduction. The following statement is an obvious consequence of Theorem 2.3.

Corollary 2.4. *The group of automorphisms of the algebra $H^*(U_{K_n})$ contains a subgroup Σ_{n+1} extending the natural Σ_n-action.*

(2.5). Actually no Σ_{n+1}-action extending the natural Σ_n-action comes from an action (or action up to homotopy) of Σ_{n+1} on the topological space U_{K_n} because of the following proposition.

Proposition 2.5. *Assume $n \geqq 4$. There are no actions of Σ_{n+1} on $H^1(U_{K_n})$ extending the Σ_n-action and defined over the integral cohomology.*

Proof. Denote by ρ the Σ_{n+1} action on $H^1(U_{K_n})$ defined by Theorem 2.3, and let ρ' be any other action on $H^1(U_{K_n})$ extending the natural Σ_n-action. For $1 \leqq i < j \leqq n$, set $x_{i,j} = d(z_i - z_j)/(z_i - z_j)$. Then $H^1(U_{K_n}, \mathbf{Z})$ is a free $\mathbf{Z}$-module with basis $x_{i,j}$ (Brieskorn Theorem [Br]). Let L be the hyperplane in $H^1(U_{K_n})$ containing all vectors whose sum of coordinates are 0 and set $L_\mathbf{Z} = L \cap H^1(U_{K_n}, \mathbf{Z})$. For $1 \leqq j < i \leqq n$ set $x_{i,j} = x_{j,i}$ and $x_{i,i} = 0$. Set $T_i = \sum_{1 \leqq j \leqq n} x_{i,j}$ and $T = \sum_{1 \leqq i \leqq n} T_i$.

1) As Σ_n module we have $L = L_1 \oplus L_2$, where L_1, L_2 are the simple modules with Young diagrams $(n-1,1)$ and $(n-2,2)$ and its complement in $H^1(U_{K_n})$, denoted by L_0, is the trivial module $\mathbf{Q}T$. So any Σ_{n+1}-action extending the Σ_n action will be the sum of a trivial representation and the representation with Young diagram $(n-1,2)$. Moreover for such an action L_0 will be invariant and L will be a submodule.

2) It follows from the previous point that ρ' and ρ are conjugated by some $\Phi \in GL(H^1(U_{K_n}))$. Such a Φ should act in a scalar way on L_1, L_2 and L_0. By multiplying Φ by an automorphism of ρ we can assume that Φ is the identity on L_0 and L_2, and acts as some non-zero scalar λ on L_1.

3) Set $s = \rho(r_{0,1})$. We have $s \cdot x_{1,i} = -x_{1,i} + 2/(n-1) \cdot T_1$ and $s \cdot x_{i,j} = x_{i,j} - x_{1,i} - x_{1,j} + 2/(n-1) \cdot T_1$ for $1 < i < j \leqq n$. It follows that ρ stabilizes $L_{\mathbf{Z}}$ but not $H^1(U_{K_n}, \mathbf{Z})$. As Σ_n-module, L_1 is generated by $T_1 - T_2$ and L_2 is generated by $x_{1,2} + x_{3,4} - x_{2,3} - x_{1,4}$. If π denotes the projection of $H^1(U_{K_n}, \mathbf{Z})$ over L_1, we have $\pi(x_{i,j}) = 1/(n-1)(T_i + T_j) - 2/(n(n-1))T$. Note also that we have $s(x_{1,2} + x_{3,4} - x_{2,3} - x_{1,4}) = x_{3,4} - x_{3,2}$ and $s(T_2 - T_1) = T_2 - (n-1)x_{1,2}$.

4) Set $s' = \rho'(r_{0,1})$. We have $s' \cdot x = s \cdot x + (1-\lambda)\pi \circ s \cdot x$ if $x \in L_2$ and $s' \cdot x = (1/\lambda)s \cdot x + ((1-\lambda)/\lambda)\pi \circ s \cdot x$ if $x \in L_1$. By using the previous formulas, one gets $s'(x_{1,2} + x_{3,4} - x_{2,3} - x_{1,4}) = x_{3,4} - x_{3,2} + (\lambda - 1)/(n-1)(T_4 - T_2)$, and $s'(T_2 - T_1) = (1/\lambda)(T_2 - (n-1)x_{1,2}) + ((1-\lambda)/\lambda)[(n-2)/(n-1)T_1 - 1/(n-1)(\sum_{j \geqq 3} T_j)]$.

5) Assume that ρ' stabilizes $H^1(U_{K_n}, \mathbf{Z})$. It follows from point 4 that $1/\lambda$ and $(\lambda - 1)/(n-1)$ should be integers. This implies $\lambda = 1$, i.e. $\rho = \rho'$. However ρ does not stabilize $H^1(U_{K_n}, \mathbf{Z})$. Q.E.D.

3. The Limit Ring SC_n^*

Let n be an integer and let X be a symplectic manifold. Then the product and Poisson brackets define two binary operations on $C^\infty(X)$. Consider now the space of all n-ary multilinear operators from $C^\infty \times \cdots \times C^\infty(X)$ to $C^\infty(X)$ that we can get by composing the product and the bracket. Clearly this space depends only on the dimension of X. In fact when $n \leqq \dim X + 1$, this space is independent of the dimension ([M], Theorem 7.5). It is denoted by $SC_n(\infty)$ or by SC_n. We have $\dim SC_n(X) = n!$ ([M], Lemma 3.7). Actually SC_n has a natural structure of graded cocommutative coalgebra ([M], Proposition 3.6). Let us denote by SC_n^k the component of degree k in SC_n (in [M], Sect. (3.5) this grading is called the Liouville grading). Roughly speaking SC_n^k is the space of all n-ary maps which involve exactly k brackets. The dual space SC_n^* is a commutative algebra described by the following theorem.

Theorem 3.1 ([M], Theorem 7.6). *A presentation of the limit ring SC_n^* is given by the commuting generators $x_{i,j}$ (for $1 \leqq i < j \leqq n$) and the following relations*:

(a) $x_{i,j}^2 = 0$, *for* $1 \leqq i < j \leqq n$,
(b) $x_{i,j}x_{j,k} = x_{j,k}x_{i,k} + x_{i,k}x_{i,j}$, *for any* $1 \leqq i < j < k \leqq n$.

This algebra is very similar to Arnold's algebra $H^*(U_{K_n})$ (see [A]). However SC_n^* is strictly commutative. Actually the generators are all elements of degree 1 and they can be described as follows. For $i < j$ denote by $\tau_{i,j}$ the map $(f_1, \dots, f_n) \in (C^\infty)^n \to \{f_i, f_j\} f_1 \dots f_n$ (where we omit the terms f_i and f_j in the product). Then the family $(\tau_{i,j})_{1\leqq i<j\leqq n}$ is a basis of SC_n^1 and the generators $x_{i,j}$ is the dual basis. The Σ_{n+1}-action on SC_n is described by the following proposition.

Proposition 3.2 (see [M], Theorem 1.5). *Let X be a symplectic manifold of dimension $\geqq n-1$. Let $\tau \in SC_n$ and let $\sigma \in \Sigma_{n+1}$. There exists a unique $\theta \in SC_n$ such that $\int_X f_{\sigma(0)}\tau(f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)})\cdot\omega^m = \int_X f_0\theta(f_1\otimes\cdots\otimes f_n)\cdot\omega^m$, for any compactly supported smooth functions $f_0,\ldots,f_n$ (where $2m=\dim X$).*

Then Σ_{n+1} acts on the dual SC_n^* as a group of homogeneous ring morphism. To describe the action of the symmetric group Σ_{n+1}, it will be convenient to define the elements $x_{i,j}$ for any $1\leqq i,j\leqq n$ as follows. We set $x_{i,i}=0$ and $x_{i,j}=-x_{j,i}$ for $i>j$.

Lemma 3.3. (i) *For $\omega\in\Sigma_n$, we have $w\cdot x_{i,j}=x_{w(i),w(j)}$.*
(ii) *We have $r_{0,i}\cdot x_{k,l}=x_{k,l}+x_{l,i}+x_{i,k}$ for any distinct i,j,k.*
(iii) *We have $r_{0,i}\cdot x_{i,k}=x_{k,i}$ for any distinct i,k.*

Proof. Formula (i) is obvious. Let $\tau_{i,j}$ be the dual basis of $x_{i,j}$. We have $r_{0,i}\tau_{k,l}=\tau_{k,l}$ for distinct i,k,l. Moreover we have $\int_X\{f_0,f_l\}f_1\ldots f_{l-1}\hat{f}_lf_{l+1}\ldots = \sum_{j>0}\int_X\{f_l,f_j\}f_0\ldots\hat{f}_j\ldots\hat{f}_l\ldots$. Thus we get $r_{i,0}\tau_{i,l}=\sum_{j>0}\tau_{l,j}$. So by transposition one gets the formulas (ii) and (iii).

4. Characters of the Homogeneous Components of the Σ_{n+1}-Modules SC_n and $H^*(U_n)$

(4.1). In this section we will set $S=\{1,\ldots,n\}$, $S^+=S\cup\{0\}$ and $S^{++}=S\cup\{0,-1\}$. Moreover A_n will denote one of the following two algebras (a) SC_n^* or (b) $H^*(U_{K_n})$.

(4.2). There is a natural embedding $\varepsilon: A_n\to A_{n+1}$. In case (a) it is the transposition of the natural map $\varepsilon^*: SC_{n+1}\to SC_n$ defined as follows: $\varepsilon^*P(f_1,\ldots,f_n)=P(1,f_1,\ldots,f_n)$ (denoted $R_{n+1,n}$ in [M], Sect. (3.4)). In case (b), it is the inverse map associated to the morphism $U_{K_{n+1}}\to U_{K_n}$, sending $(z_0,z_1,\ldots,z_n)$ to $(z_1,\ldots,z_n)$.

(4.3). The natural embedding ε commutes with the Σ_S action but not with the Σ_{S^+}-action. So we will twist ε to get an equivariant embedding. To do so define a morphism $\tau: A_n\to A_{n+1}$ by $\tau=r_{-1,0}\circ\varepsilon$.

Proposition 4.3. *The ring morphism τ commutes with the Σ_{V^+}-action.*

Proof. As the ring A_n is generated by its degree one component A_n^1 and as τ is a ring morphism, it suffices to check the claim on A_n^1 what is obvious (in case (a) this follows very easily from definitions as well).

(4.4). Set $V=\mathbf{Q}^{n+1}$. Consider V as a Σ_{n+1}, with action given by permuting the natural basis of V. There is a grading $V=V_0\oplus V_1$ of V in such a way that V_0 is the trivial component of V and V_1 is its unique Σ_{n+1}-complement.

Theorem 4.4. *As a graded Σ_{n+1}-module, we have $A_{n+1}=A_n\otimes V$, where the action on A_{n+1} is the natural action and the action on A_n is the hidden action described in* Sect. 2 *and* 3.

Proof. With the previous notations, consider A_n as a subalgebra of A_{n+1} by using the ring morphism τ. Define elements T_i', for $0 \leqq i \leqq n$ as follows. In case (a) set $T_i' = \sum_{0 \leqq j \leqq n} x_{i,j}$. In case (b) set $T_i' = (\sum_{0 \leqq j \leqq n} x_{i,j}) - 1/(n+1)(\sum_{i,j} x_{i,j})$. In both cases we have $\sum_{0 \leqq i \leqq n} T_i = 0$. Denote by U' the subspace of A_{n+1} generated by the T_i' and set $U = U' \oplus \mathbf{C}1$. We have $U \simeq V$. Moreover in both cases we have

(i) $A_{n+1}^1 = A_n^1 \oplus U'$.
(ii) We have $T_i \cdot T_j = b_{i,j} \sum a_{i,j,k} T_k$, for some $b_{i,j}$ and $a_{i,j,k}$ in A_n.

It follows that the natural map $\mu: A_n \otimes U \rightarrow A_{n+1}$ (given by multiplication) is onto. By comparing the dimension, μ is an isomorphism. By construction μ commutes with the Σ_{V+}-action. Q.E.D.

(4.5). The character of the graded module A_n for its natural Σ_n-action has been determined in each case. For case (a) it has been computed by Lehrer and Solomon, see [LS, CT, S]. For case (b) it is usually attributed to Klyaschko, see [Br, K, Ba, RW]. For any Σ_{n+1}-module M denote by ch(M) its character and denote by A_n^k the degree k component of A_n. Thus from Theorem 4.4, one gets a character formula for the hidden Σ_{n+1}-action on A_n as follows.

Corollary 4.5. *We have* $ch(A_n^k) = \sum_{0 \leqq l \leqq k} (-1)^l ch(A_{n+1}^{k-l}) \cdot ch(V_1)^l$, *where the character on the left side (right side) refers to the hidden (respectively natural)* Σ_{n+1}*-action.*

(4.6). The highest component of SC_n has degree $n-1$ and is isomorphic with the space of all n-ary multilinear Lie polynomials denoted Lie(n) in [GK]. Thus we get $SC_n^{n-1} \otimes V_1 \simeq SC_{n+1}^n$. This gives a quick proof of the following result of Getzler and Kapranov.

Corollary 4.6 (Getzler and Kapranov [GK]). *There is an isomorphism of* Σ_{n+1}-*modules* Lie$(n) \otimes V_1 \simeq$ Lie$(n+1)$.

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