Commun. Math. Phys. 175, 433 – 455 (1996)

Communications in
Mathematical
Physics

Blow-up Results of Viriel Type for Zakharov Equations

Frank Merle

Université de Cergy-Pontoise, Centre de Mathématiques, Avenue du Parc, 8, Le Campus, F-95033 Cergy-Pontoise Cedex, France

Received: 30 October 1994

Abstract: We consider the Zakharov equations in $\mathbb{R}^N$ (for $N = 2, N = 3$). We first establish a viriel identity for such equations and then prove a blow-up result for solutions with a negative energy.

In this paper, we consider Zakharov equations in $\mathbb{R}^N$ (for $N = 2, 3$):

$$iu_t = -\Delta u + nu ,$$

$$(I'_{c_0}) \qquad \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 ,$$

$$u(0) = \phi_0,\ n(0) = n_0,\ n_t(0) = n_1 ,$$

where $c_0 > 0$, Δ is the Laplacian operator on $\mathbb{R}^N, u : [0, T) \times \mathbb{R}^N \to \mathbb{C}$, $n : [0, T) \times \mathbb{R}^N \to \mathbb{R}$ and ϕ_0, n_0, n_1 are initial data.

In fact, we consider equation (I'_{c_0}) in the Hamiltonian case. That is, we assume that there is a $w_0 : \mathbb{R}^N \to \mathbb{R}$ such that

$$n_t(0) = n_1 = -\Delta w_0 . \tag{1.1}$$

Then $\forall t$, there is a $w(t)$ such that

$$n_t(t) = -\Delta w(t) = -\nabla \cdot v(t) ,$$

where $v(t) = \nabla w(t)$. In this case, (I'_{c_0}) can be written in the form

$$iu_t = -\Delta u + nu ,$$

$$n_t = -\nabla \cdot v ,$$

$$(I_{c_0}) \qquad \frac{1}{c_0^2} v_t = -\nabla n - \nabla |u|^2 ,$$

$$u(0) = \phi_0, \qquad n(0) = n_0, \qquad v(0) = v_0 .$$

Local existence in time of solutions of (I_{c_0}) $(u(t), n(t), v(t))$ has been studied in various papers (see [1, 2, 4, 6, 12, 13, 19]) in spaces $H_2 = H^2 \times H^1 \times H^1$ for $N = 2$ and $N = 3$, where $H^i = H^i(\mathbb{R}^N)$. Moreover, let $m = 1$ for $N = 2$ and $m = 2$ for $N = 3$. We have for all $(\phi_0, n_0, v_0) \in H_2$, there is a unique solution $(u, n, v)(t)$ in H_2 on $[0, T)$ and

$$\cdot \; T = +\infty$$

or

$$\cdot \; |(u, n, v)|_{H_m} \underset{t \to T}{\longrightarrow} +\infty \, .$$

In the paper, $(u, n, v)(t)$ will be a regular solution of (I_{c_0}) on $[0, t_0]$ if $(u, n, v)(t) \in C([0, t_0], H_2)$. Moreover, conservations of mass and energy yield $\forall t \in [0, T)$,

$$\int |u(t,x)|^2 dx = \int |\phi_0(x)|^2 dx$$

and

$$H(t) = H(u(t), n(t), v(t)) = H(\phi_0, n_0, v_0) = H_0 \, ,$$

where

$$H(u, n, v) = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + n(x)|u(x)|^2 + \frac{1}{2} n^2(x) + \frac{1}{2c_0^2} |v(x)|^2 dx \, .$$

We are interested in this paper on existence of singular solutions of equation (I_{c_0}). Few results are known in this direction.

In the surcritical case ($N = 3$) there are no results on existence of blow solutions. We can mention numerical simulations which suggest a finite time blow-up for some initial data (see Landman, Papanicolaou, C. and P.L. Sulem, Wang [9, 16]).

In the critical case ($N = 2$), using a perturbed conformal structure of (I_{c_0}), Glangetas and Merle in [5] have exhibited a family of blow-up solutions of the form

$$u(t,x) = \frac{\omega}{T-t} e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)}\right)} P\left(\frac{x\omega}{T-t}\right) , \tag{1.2}$$

$$n(t,x) = \left(\frac{\omega}{T-t}\right)^2 N\left(\frac{x\omega}{T-t}\right) , \tag{1.3}$$

where $\omega > \omega_0$, $\theta \in \mathbb{R}$, $T > 0$ and

$$P(x) = P(|x|), \qquad N(x) = N(|x|), \qquad \Delta P - P = NP,$$

$$\frac{1}{(c_0\omega)^2}(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta P^2$$

with $r = |x|$, $w_r = \frac{\partial w}{\partial r}$, $\Delta w = w_{rr} + \frac{1}{r}\omega_r$. Nevertheless, there is no result of existence of a large class of initial data such that blow-up in finite time occurs (even if from numerical simulations [16] suggest the asymptotic forms of blow-up solutions are of the form (1.2)–(1.3)).

Let us recall the situation in the case $c_0 = +\infty$, that is when Zakharov equations reduce to the cubic nonlinear Schrödinger equation:

$$(I_\infty) \qquad iu_t = -\Delta u - |u|^2 u$$

$$u(0) = \phi_0 \, ,$$

where $u : [0, T) \times \mathbb{R}^N \to \mathbb{C}$ (with $N = 2$ or $N = 3$).

Equation (I_∞) has a unique solution in H^1 and there is a $T > 0$ such that $u(t)$ is defined on $[0, T)$ and

$$T = +\infty$$

or

$$\lim_{t \to T} |u(t)|_{H^1} = +\infty$$

(see Ginibre and Velo [4], Kato [8]). Moreover, $\forall t$,

$$\cdot \int |u(t,x)|^2 dx = \int |\phi_0(x)|^2 dx ,$$

$$\cdot \ E(u(t)) = E_0 ,$$

where

$$E(u) = \frac{1}{2}\int |\nabla u(x)|^2 dx - \frac{1}{4}\int |u(x)|^4 dx .$$

In addition, if $\phi_0 \in \Sigma = \{u | x | u \in L^2\} \cap H^1$, then $\forall t,\ u(t) \in \Sigma$ and we have the viriel identity

$$\frac{d^2}{dt^2}\int |x|^2 |u(t,x)|^2 dx = 16E(\phi_0) - C(N)\int |u(t,x)|^4 dx , \tag{1.4}$$

where $C(2) = 0$ and $C(3) > 0$.

From this identity, Zakharov, Sobolev, Synakh [17] and later Glassey [7] derived the existence of singular solutions of (I_∞). Indeed, if

$$E_0 < 0 \tag{1.5}$$

then

$$T < +\infty .$$

(If $T = +\infty$ integration of (1.4) yields that $\int |x|^2 |u(t,x)|^2 dx < 0$ for t large which is a contradiction.)

In the case of Equation (I_{c_0}), we want to prove the existence of a "large" class of initial data such that the solution $(u(t), n(t), v(t))$ blows up. From the results of [17, 7], we ask and answer in a certain sense the following:

Question : *If* $H_0 < 0$ *then the solution* $(u(t), n(t), v(t))$ *blows up* .

Unfortunately such identity like (1.4) for the nonlinear Schrödinger equation which allows us to conclude does not exist for the Zakharov equation (I_{c_0}). Indeed terms which cannot be controlled interfere in the time variations of $\int |x|^2 |u(t,x)|^2 dx$ (see Sect. II). Nevertheless, we have the following perturbed viriel identity for solutions of (I_{c_0}).

Let $\Sigma' = \{(u, n, v) / \int |x|^2 |u|^2 + |x|(n^2 + |v|^2) < +\infty\}$.

Proposition (Viriel Identity). *Let* $(u, n, v)(t)$ *be a regular solution of* (I_{c_0}) *on* $[0, t_0]$. *Assume in addition that* $(\phi_0, n_0, v_0) \in \Sigma'$ *and* $\||x|^{1/2} u(t,x)|_{L^\infty}$ *is uniformly bounded in time on* $[0, t_0]$. *We then have* $\forall t \in [0, t_0]$,

i) $$(u(t), n(t), v(t)) \in \Sigma' ,$$

ii) $$\frac{d}{dt}\left(\frac{1}{4}\int |x|^2 |u|^2 + \int_0^t \frac{1}{c_0^2}\int (x \cdot v) n\right) = \operatorname{Im}\int (x \cdot \nabla u)\bar{u} + \frac{1}{c_0^2}\int (x \cdot v) n ,$$

$$\frac{d^2}{dt^2}\left(\frac{1}{4}\int |x|^2 |u|^2 + \int_0^t \int \frac{1}{c_0^2}(x \cdot v) n\right) = NH_0 - (N-2)\int |\nabla u|^2 - \frac{1}{c_0^2}(N-1)\int |v|^2 ,$$

where $u = u(t,x),\ n = n(t,x),\ v = v(t,x)$.

Remark. There is also a version of this viriel identity with the term

$$\frac{1}{4}\int|x|^2\left(|u|^2+\frac{1}{c_0^2}n^2+\frac{1}{c_0^4}|v|^2\right)+\frac{1}{2c_0^2}\int_0^t\int\nabla|u|^2\cdot v|x|^2$$

(see Sect. III).

Remark. When $c_0=+\infty$, we find the usual viriel identity for Equation (I_∞).

This identity from the fact that $\int(x\cdot v)n$ has no sign (and cannot be controlled) does not yield a blow-up theorem. Nevertheless in the radial situation, we claim the following blow-up theorem (from a local version of the viriel identity).

Theorem (Blow-up Theorem for radial solution of the Zakharov equation (I_{c_0})). *Assume that for all time, $(u,n,v)(t)$ are radially symmetric functions. Moreover, assume that*

$$H(0)=H_0<0\,. \tag{1.6}$$

Then $(u,n,v)(t)$ blows up. More precisely, we have the following alternative:

i) $(u,n,v)(t)$ blows up **in finite time**.

ii) $(u,n,v)(t)$ blows up **in infinite time in** H_1 :

$(u,n,v)(t)$ is defined for all t and $\lim_{t\to+\infty}|(u,n,v)(t)|_{H_1}=+\infty$.

Remark. If (ϕ_0,n_0,v_0) are radially symmetric and in a Cauchy space for the equation (I_{c_0}) then for all time $(u,n,v)(t)$ has the same property.

Remark. We expect that the blows-up is **always** in finite time but we are not able to prove it (see Sect. V for partial results in this direction).

In dimension two, we have from the theorem and [6] that (in both cases $T=+\infty$ and $T<+\infty$)

$$\lim_{t\to T}|(u,n,v)(t)|_{H_1}=+\infty\,.$$

From variational arguments [6], we have a concentration phenomenon in L^2 of $u(t)$ (in particular in the case $T=+\infty$). That is there is a $x(t)\in\mathbb{R}^2$ such that

$$\forall R>0,\quad \lim_{t\to T}\int_{|x-x(t)|<R}|u(t,x)|^2dx\geqq|Q|_{L^2}^2\,,$$

where Q is the radially symmetric positive solution of

$$\Delta z+|z|^2z=z \text{ in } \mathbb{R}^2\,. \tag{1.7}$$

Remark. (Instability of periodic solutions of equation (I_{c_0})). In dimension two, the theorem implies that the periodic solutions of (I_{c_0}) of the form

$$(u,n,v)(t)=\left(e^{i\omega^2t}\omega z(\omega x),-\omega^2|z(x\omega)|^2,0\right)\,,$$

where z is a radially symmetric solution of Eq. (1.7) and $\omega>0$, are orbitally instable. We can remark that instability has been proved in [6] under a nondegeneracy condition on z.

Indeed, from Pohozaev ((1.7)) we have

$$H(u(t), n(t), v(t)) = 2E(\omega z(\omega x)) = 2\omega^2 E(z) = 0\,,$$

and there is a sequence of initial data

$$(u_\varepsilon, n_\varepsilon, v_\varepsilon)(0) = ((1+\varepsilon)u(0), (1+\varepsilon)^2 n(0), 0)$$

such that

(i) $$\forall \varepsilon > 0, \quad H(u_\varepsilon(0), n_\varepsilon(0), v_\varepsilon(0)) < 0\,,$$

(ii) $$(u_\varepsilon(0), n_\varepsilon(0), v_\varepsilon(0)) \underset{\varepsilon \to 0}{\longrightarrow} (u(0), n(0), v(0)) \text{ in } H_2\,.$$

The result follows from the theorem.

In dimension three, an adaptation of arguments of Berestycki and Cazenave [3] and the proof of the theorem yield the instability of only the ground state periodic solution (see [3] for more details).

The plan of the paper is the following:

- In Sect. II, we check that the variation in time of $\int |x|^2 |u(t,x)|^2 dx$ does not yield a blow-up result.
- In Sect. III, we establish the viriel identity.
- In Sect. IV, we prove the blow-up Theorem.
- Section V is devoted to some open problems and comments.

II. Breakdown of the Standard Proof

In this section, we briefly check that $\frac{d^2}{dt^2}\int |x|^2 |u(t,x)|^2 dx$ has no sign under the constraint $H(u,n,v) < 0$. Therefore, we cannot apply the same arguments for the nonlinear Schrödinger equation.

We will prove in Sect. III that under regularity conditions, we have:

$$\frac{d^2}{dt^2}\int |x|^2 |u(t,x)|^2 dx = 4Q(u,n,v)$$
$$= 4\left\{N\int |\nabla u|^2 + N\int n|u|^2 - (N-2)\int |\nabla u|^2 + \int (x \cdot \nabla |u|^2) n\right\}\,. \quad (2.1)$$

We now claim the following lemma.

Lemma 2.1. *There is a* $(u,n,v) \in H^1 \times L^2 \times L^2$ *such that*

$$H(u,n,v) < 0\,,$$

and

$$Q(u,n,v) > 0\,.$$

Proof. Let us consider $n = -\alpha |u|^2$, $v = 0$ with

$$0 < \alpha < 2 - \frac{N}{2}\,,$$

$$H(u,n,v) = \int |\nabla u|^2 - \alpha \int |u|^4 + \frac{\alpha^2}{2}\int |u|^4 = \int |\nabla u|^2 - \alpha\left(1 - \frac{\alpha}{2}\right)\int |u|^4\,, \quad (2.2)$$

and

$$Q(u,n,v) = N\int |\nabla u|^2 - \alpha N\int |u|^4 - (N-2)\int |\nabla u|^2 - \alpha \int x \cdot \nabla |u|^2 |u|^2$$
$$= 2\int |\nabla u|^2 - \frac{\alpha}{2} N \int |u|^4 .$$

We remark that it is sufficient to find u, α such that $H(u,n,v) = 0$ and $Q(u,n,v) > 0$ with the condition $H(u,n,v) = \int |\nabla u|^2 - \alpha(1 - \frac{\alpha}{2}) \int |u|^4 = 0$, scaling arguments to prove the lemma and using

$$Q(u,n,v) = \left\{2\alpha\left(1 - \frac{\alpha}{2}\right) - \frac{\alpha}{2}N\right\} \int |u|^4 = \left\{\left(2 - \frac{N}{2}\right)\alpha - \alpha^2\right\} \int |u|^4$$
$$= \alpha\left\{\left(2 - \frac{N}{2}\right) - \alpha\right\} \int |u|^4 > 0 .$$

Therefore the lemma follows from the existence of a u such that

$$\int |\nabla u|^2 - \alpha\left(1 - \frac{\alpha}{2}\right) \int |u|^4 = 0 ,$$

(since $\alpha(1 - \frac{\alpha}{2}) > \alpha(1 - \frac{1}{2}(2 - \frac{N}{2})) > 0$ and scaling arguments).

III. Viriel Type Identity for Zakharov Equations

For the nonlinear Schrödinger equation (Eq. (I_∞)), we have the well-known viriel identity. That is for a solution u of (I_∞),

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 16E(\phi_0) - C(N) \int |u|^4 , \tag{3.1}$$

where $C(2) = 0$ and $C(3) > 0$.

This result was discovered by Sobolev, Synach, Zakharov [17] and later by Glassey [7]. This identity yields a blow-up result for the solution of (I_∞) in the case

$$E(\phi_0) < 0$$

by integration in time.

Unfortunately, such an identity does not exist for the Zakharov equation (I_{c_0}): clearly $\int |x|^2 |u(t,x)|^2 dx$ by itself does not satisfy some simple equation. Nevertheless, it satisfies a perturbed identity (with terms involved in the wave equation).

We first establish such an identity. We then write a local version in space of this identity. Through this section, we assume that $(u,n,v)(t)$ is a regular solution of Eq. (I_{c_0}) on $[0,t_0]$ in the sense

$$\text{for } N = 2,\ (u,n,v)(t) \in \mathscr{C}([0,t_0], H_2) , \tag{3.2}$$

$$\text{for } N = 3,\ (u,n,v)(t) \in \mathscr{C}([0,t_0], H_2) . \tag{3.3}$$

In particular $\forall t$,

$$\int |u(t,x)|^2 dx = \int |\phi(x)|^2 dx ,$$

$$H(t) = H_0 , \tag{3.4}$$

where

$$H(t) = \int |\nabla u(t,x)|^2 + n(t,x)|u(t,x)|^2 + \frac{1}{2}n^2(t,x) + \frac{1}{2c_0^2}|v(t,x)|^2 dx ,$$

$$H(0) = H_0 . \tag{3.5}$$

III.A. Global Viriel Identity

We claim the following.

Theorem (Viriel identity). *Let $(u,n,v)(t)$ be a regular solution of (I_{c_0}) on $[0,t_0]$. Assume in addition that*

$$\int |x|^2 |\phi_0(x)|^2 dx + \int |x| (|n_0(x)|^2 + |v_0(x)|^2) \, dx < +\infty \tag{3.6}$$

and

$$\exists c / \forall t \in [0,t_0], \ \left| (1+|x|^{1/2}) |u(t,x)| \right|_{L^\infty} \leqq c .$$

We then have $\forall t \in [0,t_0]$,

$$\int |x||u(t,x)|^2 dx + \int |x| (|n(t,x)|^2 + |v(t,x)|^2) \, dx < +\infty ,$$

$$\frac{d}{dt}\left(\frac{1}{4}\int |x|^2|u|^2 + \int_0^t \frac{1}{c_0^2}\int (x \cdot v) n\right) = \operatorname{Im}\int (x \cdot \nabla u)\bar{u} + \frac{1}{c_0^2}\int (x,v)n , \tag{3.7}$$

$$\frac{d^2}{dt^2}\left(\frac{1}{4}\int |x|^2|u|^2 + \int_0^t \frac{1}{c_0^2}\int (x \cdot v) n\right) = NH_0 - (N-2)\int |\nabla u|^2 - \frac{1}{c_0^2}(N-1)\int |v|^2 , \tag{3.8}$$

where $u = u(t,x), n = n(t,x), v = v(t,x)$.

Remark. Of course for $c_0 = +\infty$, $v = 0$, (3.8) reduces to (3.1).

Remark. The condition on the L^∞ norm of $u(t,x)$ is implied for example by the fact that $u(t)$ is radially symmetric and in H^2.

If we assume more regularity on the solution, we have

Proposition 3.1. *Let (u,n,v) such that*

$$\forall t \in [0,t_0], \quad |xu(t,x)| \leqq c_1 , \tag{3.9}$$

$$\int |x|^2 \left(|\phi(x)|^2 + |n_0(x)|^2 + |v_0(x)|^2 \right) dx < +\infty . \tag{3.10}$$

We then have $\forall t \in [0, t_0]$,

i)
$$\int |x|^2(|u(t,x)|^2 + |n(t,x)|^2 + |v(t,x)|^2)dx < +\infty ,$$

ii)
$$\frac{d}{dt}\left(\frac{1}{4}\int |x|^2\left(|u|^2 + \frac{1}{c_0^2}n^2 + \frac{1}{c_0^4}|v|^2\right) + \frac{1}{2c_0^2}\int_0^t\int \nabla|u|^2 \cdot v|x|^2\right)$$

$$= \mathrm{Im}\int (x \cdot \nabla u)\bar{u} + \frac{1}{c_0^2}\int (x \cdot v) , \tag{3.11}$$

$$\frac{d^2}{dt^2}\left(\frac{1}{4}\int |x|^2\left(|u|^2 + \frac{1}{c_0^2}n^2 + \frac{1}{c_0^4}|v|^2\right) + \frac{1}{2c_0^2}\int_0^t\int \nabla|u|^2 \cdot v|x|^2\right)$$

$$= NH_0 - (N-2)\int |\nabla u|^2 - \frac{(N-1)}{c_0^2}\int |v|^2 , \tag{3.12}$$

where $u = u(t,x)$, $n = n(t,x)$, $v = v(t,x)$.

Remark. Equation (3.9) is implied by

- $(u,n,v)(t) \in \mathscr{C}([0,t_0], H_2)$
- $(u,n,v)(t)$ are radially symmetric in space for all $t \in [0,t_0]$.

Proof of the Theorem. We prove the identities involved in the theorem by a regularising procedure. We approximate functions $|x|^2, x$ by regular and bounded truncatures of these, make the calculation for such truncatures (see Sect. II.B), and then go the limit to obtain the result. These techniques are classical and we omit them.

Let us show first

Lemma 3.2. (Uniform bound in time for $\int |x|^2|u|^2 + \int |x|(n^2 + |v|^2)$). *There is a* $c > 0$ *such that* $\forall t \in [0, t_0]$,

$$\int |x|^2|u(t,x)|^2dx + \int |x|(n^2(t,x) + |v(t,x)|^2)\,dx \leqq c . \tag{3.13}$$

Proof. We have on one hand,

$$\frac{d}{dt}\int |x|^2|u(t,x)|^2dx = 4\ \mathrm{Im}\int (x \cdot \nabla u)\bar{u} \leqq \int |\nabla u|^2 + \int |x|^2|u(t,x)|^2dx$$

$$\leqq c + \int |x|^2|u(t,x)|^2dx . \tag{3.14}$$

On the other hand,

$$\frac{d}{dt}\left(\int (1+|x|^2)^{1/2}\left(n^2(t,x) + \frac{|v|^2}{c_0^2}(t,x)\right)dx\right)$$

$$= 2\int (1+|x|^2)^{1/2}\left(nn_t + \frac{1}{c_0^2}v \cdot v_t\right)$$

$$= 2\int (1+|x|^2)^{1/2}(-n\nabla \cdot v - \nabla nv - \nabla|u|^2 \cdot v)$$

$$= -2\int (1+|x|^2)^{1/2}\nabla|u|^2 \cdot v + \int 2\frac{x \cdot v}{(1+|x|^2)^{1/2}}n . \tag{3.15}$$

Therefore

$$\frac{d}{dt}\int(1+|x|^2)^{1/2}\left(n^2+\frac{|v|^2}{c_0^2}\right)$$

$$\leqq 2\left(\int v^2+\int n^2+\left|u(1+|x|^2)^{1/4}\right|_{L^\infty}\left(\int|\nabla u|^2+\int(1+|x|^2)^{1/2}|v|^2\right)\right) . \qquad (3.16)$$

Since

$$\left||u(x)|(1+|x|^2)^{1/4}\right|_{L^\infty}\leqq c , \qquad (3.17)$$

and (3.16)–(3.17)

$$\frac{d}{dt}\int(1+|x|^2)^{1/2}\left(n^2+\frac{|v|^2}{c_0^2}\right)\leqq c+\int(1+|x|^2)^{1/2}|v|^2 . \qquad (3.18)$$

The conclusion follows from (3.14), (3.18) and a Gronwall lemma.

Let us prove the identities of the theorem. It follows from

Lemma 3.3.

i) $$\frac{d}{dt}\int|x|^2|u|^2=4\ \mathrm{Im}\int(x\cdot\nabla u)\bar{u} ,$$

ii) $$\frac{d^2}{dt^2}\int|x|^2|u|^2$$

$$=4\{N\int|\nabla u|^2+N\int n|u|^2-(N-2)\int|\nabla u|^2+\int(x\cdot\nabla|u|^2)n\}$$

$$=4\left\{NH_0-(N-2)\int|\nabla u|^2-\frac{N}{2c_0^2}\int|v|^2-\frac{1}{c_0^2}\int(x\cdot v_t)n\right\} .$$

Lemma 3.4.

$$\frac{d}{dt}\int(x\cdot v)n=\int(x\cdot v_t)n-\frac{(N-2)}{2}\int|v|^2 .$$

Before proving Lemmas 3.3–3.4, let us show a useful identity.

Lemma 3.5.

$$-\mathrm{Re}\int(x\cdot\overline{\nabla\theta})\Delta\theta=-\frac{(N-2)}{2}\int|\nabla\theta|^2 , \qquad (3.19)$$

for regular functions θ with a good decay at infinity.

Proof of Lemma 3.5. By integration by part, we have

$$I_1 = \int -\operatorname{Re}\,(x\cdot\overline{\nabla\theta})\Delta\theta = \operatorname{Re}\int\nabla(x\cdot\overline{\nabla\theta})\nabla\theta$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{Re}\int\frac{\partial}{\partial x_j}\left(x_i\frac{\overline{\partial\theta}}{\partial x_i}\right)\frac{\partial\theta}{\partial x_j}$$

$$= \int|\nabla\theta|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{Re}\int x_i\frac{\partial^2\bar\theta}{\partial x_i\partial x_j}\frac{\partial\theta}{\partial x_j}$$

$$= \int|\nabla\theta|^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int x_i\frac{\partial}{\partial x_i}\left|\frac{\partial\theta}{\partial x_j}\right|^2$$

$$= \int|\nabla\theta|^2 - \frac{1}{2}\int\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\frac{\partial\theta}{\partial x_j}\right|^2 = \int|\nabla\theta|^2 - \frac{N}{2}\int|\nabla\theta|^2$$

$$= -\frac{(N-2)}{2}\int|\nabla\theta|^2\,.$$

Proof of Lemma 3.3.

i) $$\frac{d}{dt}\int|x|^2|u|^2 = 2\operatorname{Re}\int|x|^2\left(\frac{\partial u}{\partial t}\bar u\right) = 2\operatorname{Re}\int|x|^2(i\Delta u - inu)\bar u$$

$$= 2\operatorname{Re}\,i\int|x|^2\Delta u\bar u = 4\operatorname{Re} - i\int x\nabla u\bar u = 4\operatorname{Im}\int(x\cdot\nabla u)\bar u\,.$$

ii) We have

$$\frac{d^2}{dt^2}\int|x|^2|u|^2 = 4\frac{d}{dt}\operatorname{Im}\int(x\cdot\overline{\nabla u})u$$

$$= 4\operatorname{Im}\left\{\int\left(x\cdot\nabla\frac{\partial u}{\partial t}\right)\bar u + \int(x\cdot\nabla u)\overline{\frac{\partial u}{\partial t}}\right\}$$

$$= 4\left\{2\operatorname{Im}\int(x\cdot\nabla u)\overline{\frac{\partial u}{\partial t}} - N\operatorname{Im}\int\frac{\partial u}{\partial t}\bar u\right\}\,. \tag{3.20}$$

On the one hand,

$$-N\int\operatorname{Im}\frac{\partial u}{\partial t}\bar u = N\operatorname{Re}\int i\frac{\partial u}{\partial t}\bar u = N\operatorname{Re}\int(-\Delta u + nu)\bar u$$

$$= N\left\{\int|\nabla u|^2 + \int n|u|^2\right\}\,. \tag{3.21}$$

On the other hand, by a direct calculation and (3.19),

$$2\ \mathrm{Im} \int (x \cdot \nabla u)\overline{\frac{\partial u}{\partial t}} = -2\ \mathrm{Im} \int (x \cdot \overline{\nabla u})\frac{\partial u}{\partial t}$$

$$= \mathrm{Re}\ 2\int (x \cdot \overline{\nabla u})(-\Delta u + nu)$$

$$= 2\{-\mathrm{Re} \int (x \cdot \overline{\nabla u})\Delta u + \mathrm{Re} \int (x \cdot \overline{\nabla u})un\}$$

$$= 2\left\{-\frac{(N-2)}{2}\int |\nabla u|^2 + \frac{1}{2}\int x \cdot \nabla |u|^2 n\right\}$$

$$= -(N-2)\int |\nabla u|^2 + \int x \cdot \nabla |u|^2 n\,. \tag{3.22}$$

In conclusion,

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 4\{N\int |\nabla u|^2 + N\int n|u|^2 - (N-2)\int |\nabla u|^2 + \int x \cdot \nabla |u|^2 n\}\,. \tag{3.23}$$

Moreover we have

$$\int (x \cdot \nabla |u|^2)n = -\int (x \cdot \nabla n)n - \frac{1}{c_0^2}\int (x \cdot v_t)n$$

$$= -\frac{1}{2}\int x \cdot \nabla n^2 - \frac{1}{c_0^2}\int (x \cdot v_t)n$$

$$= \frac{N}{2}\int n^2 - \frac{1}{c_0^2}\int (x \cdot v_t)n \tag{3.24}$$

and

$$\frac{d^2}{dt^2}\int |x|^2|u|^2$$

$$= 4\left\{N\int |\nabla u|^2 + N\int n|u|^2 + \frac{N}{2}\int n^2 - (N-2)\int |\nabla u|^2 - \frac{1}{c_0^2}\int (x \cdot v_t)n\right\}$$

$$= 4\left\{NH_0 - (N-2)\int |\nabla u|^2 - \frac{N}{2c_0^2}\int |v|^2 - \frac{1}{c_0^2}\int (x \cdot v_t)n\right\}\,,$$

which concludes the proof of Lemma 3.3.

Proof of Lemma 3.4. We recall that there is a $w(t)$ such that

$$v(t) = \nabla w(t)\,. \tag{3.25}$$

Indeed, let $w(t)$ solution

$$-\frac{1}{c_0^2}w_t = n(t) + |u(t)|^2 \quad \text{and} \quad -\Delta w(0) = n_1\,,$$

then

$$-\Delta w(t) = \frac{\partial n}{\partial t} \quad \text{and} \quad v(t) = \nabla w(t)\,.$$

We have from (3.19),

$$\frac{d}{dt}(x \cdot v)n = \int (x \cdot v_t)n + \int (x \cdot v)n_t = \int (x \cdot v_t)n - \int (x \cdot v)\nabla \cdot v$$

$$= \int (x \cdot v_t)n - \int (x \cdot \nabla w)\nabla \cdot \nabla w = \int (x \cdot v_t)n - \int (x \cdot \nabla w)\Delta w$$

$$= \int (x \cdot v_t)n - \frac{(N-2)}{2}\int |\nabla w|^2 = \int (x \cdot v_t)n - \frac{(N-2)}{2}\int |v|^2 ,$$

which concludes the proof of Lemma 3.4 and the theorem. Proposition 3.1 follows from the theorem and the identity

$$\frac{d}{dt}\int \frac{1}{4}|x|^2\left(n^2 + \frac{|v|^2}{c_0^2}\right) = \frac{1}{2}\int |x|^2(-n \cdot \nabla \cdot v - \nabla n \cdot v - \nabla |u|^2 \cdot v)$$

$$= -\frac{1}{2}\int |x|^2 \nabla |u|^2 \cdot v + \int (x \cdot v)n ,$$

and the fact that

$$\frac{d}{dt}\frac{1}{4}\int |x|^2\left(n^2 + \frac{|v|^2}{c_0^2}\right) \leqq \int |x|^2|v|^2 + \int |x|^2 n^2$$

$$+ \frac{1}{2}|xu|_{L^\infty}\left(\int |\nabla u|^2 + \int |x|^2|v|^2\right) .$$

This concludes Sect. III.A.

III.B. Local Viriel Identity

As in the case of nonlinear Schrödinger equation (I_∞), we use local viriel identities in various problems related to blow-up. That is quantities of the form

$$\int \psi(x)|u|^2 dx + \int_0^t \frac{1}{c_0^2}\int (\nabla\psi \cdot v)n , \tag{3.26}$$

where ψ behaves like $|x|^2$ near zero, and like $|x|$ at infinity (see [10, 11, 14, 15]).

We claim

Proposition 3.6 (Local viriel identity).
a) General case. *Under the assumptions of Proposition* 3.1 *for* $(u, n, v)(t)$ *and for* ψ *such that* $\forall x$,

$$|\psi(x)| \leqq c(1 + |x|^2), \qquad |\nabla\psi(x)| \leqq c(1 + |x|), \qquad |\Delta\psi(x)| + |\Delta^2\psi(x)| \leqq c ,$$

we have $\forall t > 0$,
i)

$$\frac{d}{dt}\left\{\frac{1}{2}\int \psi(x)|u|^2 + \frac{1}{c_0^2}\int_0^t \int (\nabla\psi \cdot v)n\right\} = \operatorname{Im}\int (\nabla\psi \cdot \nabla u)\bar{u} + \frac{1}{c_0^2}\int (\nabla\psi \cdot v)n$$

ii)

$$\frac{d^2}{dt^2}\left\{\frac{1}{2}\int\psi(x)|u|^2+\frac{1}{c_0^2}\int_0^t\int(\nabla\psi\cdot v)n\right\}$$

$$=\left\{\int\Delta\psi\left(n|u|^2+\frac{1}{2}n^2\right)+2\,\mathrm{Re}\int\sum_{i=j}^{N}\sum_{j=1}^{N}\partial_i\partial_j\psi\overline{\partial_i u}\partial_j u\right.$$

$$\left.+\frac{1}{c_0^2}\int\sum_j\sum_i\partial_i\partial_j\psi\, v_iv_j-\frac{1}{2c_0^2}\int\Delta\psi|v|^2-\frac{1}{2}\int|u|^2\Delta^2\psi\right\}. \tag{3.27}$$

b) The radial case. Assume in addition that all functions u,n,v,ψ are radially symmetric in x, we have

$$\frac{d^2}{dt^2}\left\{\frac{1}{2}\int\psi(x)|u|^2+\frac{1}{c_0^2}\int_0^t(\nabla\psi\cdot v)n\right\}=\left\{\int\Delta_r\psi\left(n|u|^2+\frac{n^2}{2}\right)+2\int\partial_r^2\psi|\partial_r u|^2\right.$$

$$\left.+\frac{1}{2c_0^2}\int\left(\partial_{rr}^2\psi-\frac{(N-1)}{r}\partial_r\psi\right)|v|^2-\frac{1}{2}\int\Delta_r^2\psi|u|^2\right\}, \tag{3.28}$$

where $\Delta_r=\partial_r^2+\frac{(N-1)}{r}\partial_r$.

Proof. It follows from a similar calculation as in Sect. III.A.

Lemma 3.7.

$$-\mathrm{Re}\int(\nabla\psi\cdot\overline{\nabla\theta})\Delta\theta=\left(\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i\theta}\partial_j\theta\right)-\frac{1}{2}\int\Delta\psi|\nabla\theta|^2 \tag{3.29}$$

for regular functions θ with a good decay at infinity. Indeed,

$$\begin{aligned}-\mathrm{Re}\int(\nabla\psi\cdot\overline{\nabla\theta})\Delta\theta &= -\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int(\partial_i\psi\,\partial_i\bar\theta)\partial_j^2\theta\\
&=\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta+\int\partial_i\psi\overline{\partial_i(\partial_j\theta)}\partial_j\theta\right)\\
&=\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta+\frac{1}{2}\int\partial_i\psi\,\partial_i|\partial_j\theta|^2\\
&=\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta-\frac{1}{2}\int\partial_{ii}^2\psi|\partial_j\theta|^2\\
&=\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta-\frac{1}{2}\int\Delta\psi|\nabla\theta|^2.\end{aligned}$$

Lemma 3.8.

i)
$$\frac{d}{dt}\int\psi(x)|u|^2=2\mathrm{Im}\int(\nabla\psi\cdot\nabla u)\bar u, \tag{3.30}$$

ii)
$$\frac{d^2}{dt^2}\int\psi|u|^2 = 2\left\{\int\Delta\psi\left(n|u|^2+\frac{n^2}{2}\right)+2\mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i u}\partial_j u\right.$$
$$\left.-\frac{1}{2}\int|u|^2\Delta^2\psi-\frac{1}{c_0^2}\int(\nabla\psi\cdot v_t)n\right\}. \tag{3.31}$$

i) follows from direct calculation.
ii) We have as before

$$\frac{d^2}{dt^2}\int\psi|u|^2 = 2\left\{\mathrm{Im}\int\nabla\psi\bar{u}\nabla\frac{\partial u}{\partial t}+\mathrm{Im}\int\nabla\psi\frac{\partial\bar{u}}{\partial t}\nabla u\right\}$$
$$= 2\left\{2\ \mathrm{Im}\int\nabla\psi\nabla u\frac{\partial\bar{u}}{\partial t}-\mathrm{Im}\int\Delta\psi\bar{u}\frac{\partial u}{\partial t}\right\}. \tag{3.32}$$

On the one hand,

$$\begin{aligned}-\mathrm{Im}\int\Delta\psi\bar{u}\frac{\partial u}{\partial t} &= \mathrm{Re}\int\Delta\psi(-\Delta u+nu)\bar{u}\\ &= \int\Delta\psi|\nabla u|^2+\int\Delta\psi\, n|u|^2+\int\nabla(\Delta\psi)\nabla u\bar{u}\\ &= \int\Delta\psi|\nabla u|^2+\int\Delta\psi\, n|u|^2-\frac{1}{2}\int|u|^2\Delta^2\psi\,.\end{aligned} \tag{3.33}$$

On the other hand,

$$\begin{aligned}2\ \mathrm{Im}\int\nabla\psi\nabla u\frac{\partial\bar{u}}{\partial t} &= 2\ \mathrm{Re}\int\Delta\psi\overline{\nabla u}(-\Delta u+nu)\\ &= -2\ \mathrm{Re}\int\nabla\psi\overline{\nabla u}\Delta u+\int(\nabla\psi\nabla|u|^2)n\\ &= 2\ \mathrm{Re}\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\overline{\partial_i u}\partial_j u-\int\Delta\psi|\nabla u|^2\\ &\quad+\int(\nabla\psi\nabla|u|^2)n\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\int(\nabla\psi\nabla|u|^2)n &= -\int(\nabla\psi\cdot\nabla n)n-\frac{1}{c_0^2}\int(\nabla\psi\cdot v_t)n\\ &= -\frac{1}{2}\int\nabla\psi\nabla n^2-\frac{1}{c_0^2}\int(\nabla\psi\cdot v_t)n\\ &= \frac{1}{2}\int\Delta\psi\, n^2-\frac{1}{c_0^2}\int(\nabla\psi\cdot v_t)n\,.\end{aligned} \tag{3.35}$$

Therefore (3.32) follows from (3.33)–(3.35).

Lemma 3.9.

$$\begin{aligned}\frac{d}{dt}\frac{1}{c_0^2}\int(\nabla\psi\cdot v)n &= \frac{1}{c_0^2}\int(\nabla\psi\cdot v_t)n\\ &= \frac{1}{c_0^2}\int\left(\sum_i\sum_j\partial_i\partial_j\psi\, v_i v_j\right)-\frac{1}{2c_0^2}\int\Delta\psi|v|^2\,.\end{aligned} \tag{3.36}$$

Indeed,

$$\frac{d}{dt}\int(\nabla\psi\cdot v)n=\int(\nabla\psi\cdot v_t)n+\int(\nabla\psi\cdot v)n_t$$
$$=\int(\nabla\psi\cdot v_t)n-\int(\nabla\psi\cdot v)\nabla\cdot v\,.$$

Since $v=\nabla w$ and (3.29),

$$-\int(\nabla\psi\cdot v)\nabla\cdot v=-\int(\nabla\psi\cdot\nabla w)\Delta w$$
$$=\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\,\partial_i w\partial_j w\right)-\frac{1}{2}\int\Delta\psi|\nabla w|^2$$
$$=\sum_{i=1}^{N}\sum_{j=1}^{N}\int\partial_i\partial_j\psi\,v_iv_j-\frac{1}{2}\int\Delta\psi|v|^2\,,$$

and the conclusion follows.

Parts i) and ii) of Proposition 3.6 are implied by Lemmas 3.8 and 3.9.

b) We can easily check that in the radially symmetric case, we can choose w radially symmetric such that

$$v(t)=\nabla w(t)\,.$$

Therefore, the conclusion is implied by the following lemma.

Lemma 3.10. *Let* $\theta=\theta(r)$ *and* $\psi=\psi(r)$,

$$\operatorname{Re}\int\sum_{i=1}^{N}\sum_{j=1}^{N}\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta=\int\partial_{rr}\psi|\partial_r\theta|^2\,.$$

Indeed, we have

$$\partial_i\theta=\partial_r\theta\frac{x_i}{r}$$

and

$$\partial_i\partial_j\psi=\partial_i\left(\frac{\partial_r\psi}{r}\right)x_j+\frac{\partial_r\psi}{r}\partial_i x_j=\partial_r\left(\frac{\partial_r\psi}{r}\right)\frac{x_ix_j}{r}+\delta_{i=j}\frac{\partial_r\psi}{r}\,.$$

Thus,

$$\operatorname{Re}\int\sum_{i=1}^{N}\sum_{j=1}^{N}\partial_i\partial_j\psi\overline{\partial_i\theta}\,\partial_j\theta$$
$$=\operatorname{Re}\int\sum_{i=1}^{N}\sum_{j=1}^{N}r\partial_r\left(\frac{\partial_r\psi}{r}\right)|\partial_r\theta|^2\frac{x_i^2x_j^2}{r^4}+\operatorname{Re}\int\sum_{i=1}^{N}\sum_{j=1}^{N}\delta_{i=j}\frac{\partial_r\psi}{r}|\partial_r\theta|^2\frac{x_ix_j}{r^2}$$
$$=\int r\partial_r\left(\frac{\partial_r\psi}{r}\right)|\partial_r\theta|^2\frac{\left(\sum_{i=1}^{N}x_i^2\right)\left(\sum_{j=1}^{N}x_j^2\right)}{r^4}+\int\frac{\partial_r\psi}{r}|\partial_r\theta|^2\frac{\left(\sum_{i=1}^{N}x_i^2\right)}{r^2}$$
$$=\int|\partial_r\theta|^2\left\{r\partial_r\left(\frac{\partial_r\psi}{r}\right)+\frac{\partial_r\psi}{r}\right\}=\int|\partial_r\theta|^2\partial_r\left(r\frac{\partial_r\psi}{r}\right)$$
$$=\int\partial_r^2\psi|\partial_r\theta|^2\,.$$

Therefore the proof of Proposition 3.6 is concluded.

IV. Blow-up Results for Zakharov Equations

As we have seen in Sect. II, the variation in time of

$$\int |x|^2 |u(t,x)|^2 dx \tag{4.1}$$

does not allow us to conclude on existence of blow-up solutions for equation (I_{c_0}). We in fact use in this section identities local in space of the type

$$\frac{d^2}{dt^2}\left(\frac{1}{4}\int |x|^2|u|^2 + \int_0^t \frac{1}{c_0^2}\int (x\cdot v)n\right) = NH_0 - (N-2)\int |\nabla u|^2 - \frac{1}{c_0^2}(N-1)\int |v|^2 . \tag{4.2}$$

We assume in this section that

$$H_0 < 0 , \tag{4.3}$$

and we want to prove that the solution ever blows up in finite time or blows up as t goes to $+\infty$ in H_1. That is

– there is a $T > 0$ such that either $T < +\infty$ and

$$\int n^2 + |\nabla u|^2 + |v|^2(t) \underset{t\to T}{\longrightarrow} +\infty \quad \text{for } N = 2 ,$$

$$|(u,n,v)|_{H_2} \underset{t\to T}{\longrightarrow} +\infty \quad \text{for } N = 3 ,$$

or $T = +\infty$ and

$$\int n^2 + |\nabla u|^2 + |v|^2(t) \underset{t\to T}{\longrightarrow} +\infty .$$

Therefore, assuming that the solution is defined for all time, we want to prove

$$\int n^2 + |\nabla u|^2 + |v|^2(t) \underset{t\to+\infty}{\longrightarrow} +\infty . \tag{4.4}$$

Indeed, it is shown in [6] (using techniques of [1,9]) that for $N = 2$, if the solution $(u,n,v)(t)$ blows-up in finite time in H_2 as $t \to T$, then $|(u,n,v)|_{H_1} \underset{t\to T}{\longrightarrow} +\infty$.

We remark that (4.2) yields by integration

$$\int_0^t \frac{1}{c_0^2}\int (x\cdot v)n \underset{t\to+\infty}{\longrightarrow} -\infty , \tag{4.5}$$

which is weaker version of (4.4).

To prove (4.4), we in fact use a local version in space of (4.2). In order to control the perturbation term in this identity, we assume in this section that all functions are radial (in particular $\vec{v} = v(r)\frac{\vec{x}}{r}$).

Let us consider as in [10,11] a function $h : \mathbb{R}^N \to \mathbb{R}$ such that

- $h(x) = h(r)$ where $r = |x|$,
- $h \in C^3(\mathbb{R}^N, \mathbb{R})$,

- $h(r) = r, \quad 0 \leqq r < 1$,
- $h(r) = r - (r-1)^4$ for $r \geqq 1$ and near 1 ,
- $h(r) = \frac{3}{2}$ for $r \geqq 3$,
- $0 \leqq h'(r) \leqq 1$,
- $h'(r) < 1$ for $r > 1$. (4.6)

We can see easily that such h exist (see also [14, 15]). Let

$$\psi(x) = \psi(r) \text{ with } \psi(0) = 0 \text{ and } \psi_r = h . \tag{4.7}$$

We can easily check from (4.6) that

$$-\Delta\psi + N \geqq 0 , \tag{4.8}$$

$$(-\Delta\psi + N)^2 \leqq c_1(1 - \partial_{rr}\psi) . \tag{4.9}$$

Finally consider,

$$\psi_m = m^2\psi\left(\frac{x}{m}\right) \quad \text{and} \quad h_m = mh\left(\frac{x}{m}\right) . \tag{4.10}$$

We then have

$$\psi_{m_r} = h_m \quad \text{and} \quad \frac{\partial^k\psi_m}{\partial r^k} = m^{2-k}\frac{\partial^k\psi}{\partial r^k}\left(\frac{x}{m}\right) . \tag{4.11}$$

We claim the following

Proposition 4.1 (Existence of Lyapunov function in time). *Let (u, n, v) a solution of equation (I_{c_0}) defined for all $t > 0$. We then have for $c_1 = c_1(\int |u|^2) = c_1(\int |\varphi_0|^2) > 0$ and $c_2 > 0, \forall m, \forall t$,*

i) *Critical case* $(N = 2)$

$$\frac{d}{dt}\left(-\mathrm{Im}\int(\nabla\psi_m\nabla u)\bar{u} - \frac{1}{c_0^2}\int(\nabla\psi_m \cdot v)n\right) \geqq -NH_0 + \frac{c_2}{c_0^2}\int|v|^2 - c_1\left(\frac{1}{m} + \frac{1}{m^2}\right) + 2\left(1 - \frac{c_1}{m}\right)\int(1 - \partial_{rr}\psi_m)|\nabla u|^2 . \tag{4.12}$$

ii) *Surcritical case* $(N = 3)$

$$\frac{d}{dt}\left(-\mathrm{Im}\int(\nabla\psi_m\nabla u)\bar{u} - \frac{1}{c_0^2}\int(\nabla\psi_m \cdot v)n\right) \geqq -NH_0 + \frac{1}{4}\int|\nabla u|^2 + \frac{c_2}{c_0^2}\int|v|^2 - \frac{c_1}{m^2} - \frac{c_1}{m^4} . \tag{4.13}$$

Remark. c_1 is in particular independent of c_0.

As a corollary of Proposition 4.1, we have

Corollary 4.2 (Blow-up theorem for Zakharov equation). *Assume that $E_0 < 0$ and u, n, v are radially symmetric functions defined for all time. We have*

$$\int n^2 + |v|^2 + |\nabla u|^2(t) \xrightarrow[t \to +\infty]{} +\infty .$$

We first show how Proposition 4.1 implies Corollary 4.2 and then prove Proposition 4.1. This will conclude the proof of the blow-up theorem given in the introduction.

Proof of Corollary 4.2. We first remark that $\forall t$,

$$\begin{aligned} y_m(t) &= -\mathrm{Im} \int (\nabla \psi_m \cdot \nabla u)\bar{u} - \frac{1}{c_0^2} \int (\nabla \psi_m \cdot v) n \\ &\leqq |\nabla \psi_m|_{L^\infty} \left\{ \left(\int u^2\right)^{1/2} \left(\int |\nabla u|^2\right)^{1/2} + \frac{1}{c_0^2} \left(\int v^2\right)^{1/2} \left(\int n^2\right)^{1/2} \right\} \\ &\leqq c|h_m|_{L^\infty} \left\{ \left(\int |\nabla u|^2\right)^{1/2} + \left(\int v^2\right)^{1/2} \left(\int n^2\right)^{1/2} \right\} \\ &\leqq cm \left\{ \left(\int |\nabla u|^2\right)^{1/2} + \left(\int v^2\right)^{1/2} \left(\int n^2\right)^{1/2} \right\} . \end{aligned} \tag{4.14}$$

Therefore

$$y_m(t) \leqq cm \left(1 + \int (|\nabla u|^2 + n^2 + v^2)(t)\right) \tag{4.15}$$

and to have the conclusion, it is sufficient to prove that for a given $m > 0$,

$$y_m(t) \xrightarrow[t \to +\infty]{} +\infty . \tag{4.16}$$

Equation (4.16) follows in fact from integration of (4.12)–(4.13) with a suitable $m > 0$. Let us consider two cases

Case 1: Surcritical case ($N = 3$). Since $H_0 < 0$ we have $-NH_0 > 0$ and for m large enough,

$$-NH_0 - c_1 \left(\frac{1}{m^2} + \frac{1}{m^4} \right) \geqq -\frac{N}{2} H_0 . \tag{4.17}$$

Thus by integration of (4.13), $\forall t$

$$\begin{aligned} y_m(t) - y_m(0) &\geqq \int_0^t \left(\frac{1}{4} \int |\nabla u|^2(s) + \frac{c_2}{c_0^2} \int |v(s)|^2 - NH_0 - \frac{c_1}{m^4} - \frac{c_1}{m^2} \right) ds \\ &\geqq \int_0^t -\frac{N}{2} H_0 ds = -\frac{NH_0}{2} t \xrightarrow[t \to +\infty]{} +\infty , \end{aligned}$$

which concludes the proof in the case of the space dimension three.

Case 2: Critical case ($N = 2$). For this case for m large enough, we have

- $1 - \dfrac{c_1}{m} > 0$,
- $-NH_0 - \dfrac{c_1}{m} - \dfrac{c_1}{m^2} > -\dfrac{NH_0}{2}$,

and we conclude the proof as before.

This concludes the proof of the corollary.

Proof of Proposition 4.1. It follows from the local variance identity obtained in Sect. III and in a crucial way control of $|u|^4$ by $|\nabla u|^2$ away from the origin in the radial case.

From (3.28), we have

$$\frac{d}{dt}\left(-\operatorname{Im}\int(\nabla\psi_m\nabla u)\bar{u}-\frac{1}{c_0^2}\int(\nabla\psi_m v)n\right)$$

$$=-\int\Delta_r\psi_m\left(n|u|^2+\frac{n^2}{2}\right)-2\int\partial_r^2\psi_m|\partial_r u|^2$$

$$-\frac{1}{2c_0^2}\int\left(\partial_{rr}^2\psi_m-\frac{(N-1)}{r}\partial_r\psi_m\right)|v|^2+\frac{1}{2}\int\Delta_r^2\psi_m|u|^2$$

$$=-\int\Delta_r\psi_m\left(n|u|^2+\frac{n^2}{2}+\frac{1}{2c_0^2}|v|^2\right)-2\int\partial_r^2\psi_m|\partial_r u|^2$$

$$+\frac{1}{c_0^2}\int\frac{(N-1)}{r}\partial_r\psi_m|v|^2+\frac{1}{2}\int\Delta_r^2\psi_m|u|^2+NH_0-NH_0$$

$$=-NH_0+\int(N-\Delta_r\psi_m)\left(n|u|^2+\frac{n^2}{2}+\frac{1}{2c_0^2}|v|^2\right)+2\int(1-\partial_r^2\psi_m)|\partial_r u|^2$$

$$+(N-2)\int|\partial_r u|^2+\frac{1}{c_0^2}\int\frac{(N-1)}{r}\partial_r\psi_m|v|^2+\frac{1}{2}\int\Delta_r^2\psi_m|u|^2\,. \tag{4.19}$$

From (4.6), we have

– The existence of $c_2>0$ such that $\forall m$,

$$\frac{1}{2}(N-\Delta_r\psi_m)+\frac{(N-1)}{r}\partial_r\psi_m \geqq c_2>0\,. \tag{4.20}$$

– The existence of c such that $\forall m$, $\forall t$,

$$\left|\frac{1}{2}\int\Delta_r^2\psi_m|u|^2\right| \leqq \frac{1}{2}\left|\Delta_r^2\psi_m\right|_{L^\infty}\int|u|^2 \leqq \frac{c}{m^2}\,. \tag{4.21}$$

Since $N-\Delta_r\psi_m \geqq 0$ and $(1-\partial_r^2\psi_m)\geqq 0$, the only term to control is

$$\int(N-\Delta_r\psi_m)n|u|^2\,. \tag{4.22}$$

We have

$$\int(N-\Delta_r\psi_m)n|u|^2 \geqq \int(N-\Delta_r\psi_m)\left(-\frac{1}{2}n^2-\frac{1}{2}|u|^4\right)$$

and

$$\int(N-\Delta_r\psi_m)\left(n|u|^2+\frac{1}{2}n^2\right) \geqq -\frac{1}{2}\int(N-\Delta_r\psi_m)|u|^4\,. \tag{4.23}$$

Thus

$$y_m'(t) \geqq -NH_0 - \frac{c}{m^2} + \frac{c_2}{c_0^2}\int |v|^2 + 2\int (1-\partial_r^2\psi_m)|\partial_r u|^2$$

$$-\frac{1}{2}\int (N-\Delta_r\psi_m)|u|^4 + (N-2)\int |\partial_r u|^2\,, \tag{4.24}$$

where

$$y_m(t) = -\mathrm{Im}\int (\nabla\psi_m \cdot \nabla u)\bar{u} - \frac{1}{c_0^2}\int (\nabla\psi_m \cdot v)n\,.$$

Let us consider two cases: $N = 3$ and $N = 2$.

Case ii): Surcritical case ($N = 3$). In this case, we have

$$y_m'(t) \geqq -NH_0 - \frac{c}{m^2} + \frac{c^2}{c_0^2}\int |v|^2 + \int |\partial_r u|^2 - \frac{1}{2}\int (N-\Delta_r\psi_m)|u|^4\,. \tag{4.25}$$

Since

$$N - \Delta_r\psi_m(x) = 0, \quad \text{for } |x| \leqq m\,,$$

$$C \geqq N - \Delta_r\psi_m \geqq 0, \quad \forall x \in \mathbb{R}^N\,,$$

and

Lemma 4.3 (Strauss [18]). *If u is radially symmetric function in H^1,*

$$r^2|u(r)|^2 \leqq \left(\int u^2\right)^{1/2}\left(\int |\partial_r u|^2\right)^{1/2}\,, \tag{4.26}$$

we have

$$y_m'(t) \geqq -NH_0 - \frac{c}{m^2} + \frac{c_2}{c_0^2}\int |v|^2 + \int |\partial_r u|^2 - \frac{1}{2}\int_{|x|\geqq m} (N-\Delta_r\psi_m)|u|^4$$

$$\geqq -NH_0 - \frac{c}{m^2} + \frac{c_2}{c_0^2}\int |v|^2 + \int |\partial_r u|^2 - c\left(\int |u|^2\right)|u|^2_{L^\infty(|x|\geqq m)}$$

$$\geqq -NH_0 - \frac{c}{m^2} + \frac{c_2}{c_0^2}\int |v|^2 + \int |\partial_r u|^2 - \frac{c}{m^2}\left(\int |\partial_r u|^2\right)^{1/2}$$

$$\geqq -NH_0 - c\left(\frac{1}{m^2} + \frac{1}{m^4}\right) + \frac{c_2}{c_0^2}\int |v|^2 + \frac{1}{4}\int |\partial_r u|^2\,,$$

which concludes the proof in the case ii).

Case i): Critical ($N = 2$). In this case, we have

$$y_m'(t) \geqq -NH_0 - \frac{c}{m^2} + \frac{c_2}{c_0^2}\int |v|^2$$

$$+ 2\int (1-\partial_r^2\psi_m)|\partial_r u|^2 - \frac{1}{2}\int (N-\Delta_r\psi_m)|u|^4\,. \tag{4.27}$$

We remark that

$$\left|-\frac{1}{2}\int(N-\Delta_r\psi_m)|u|^4\right| \leqq c\int(1-\partial_{rr}\psi_m)^{1/2}|u|^4 \leqq c\int_{|x|\geqq m}(1-\partial_{rr}\psi_m)^{1/2}|u|^4$$
$$\leqq c\left|(1-\partial_{rr}\psi_m)^{1/4}|u|\right|^2_{L^\infty(|x|\geqq m)}. \tag{4.28}$$

We claim the following lemma (see also [14]).

Lemma 4.4. *Let u be an H^1 radially symmetric function,*

$$\left|(1-\partial_{rr}\psi_m)^{1/4}|u|\right|^2_{L^\infty(|x|\geqq m)} \leqq \frac{c}{m}\left[\frac{1}{m}\int|u|^2+\int|u|^2+\int(1-\partial_{rr}\psi_m)|\nabla u|^2\right]. \tag{4.29}$$

Proof. For $|x|\geqq m$,

$$\left|(1-\partial_{rr}\psi_m(r))^{1/2}u^2(r)\right| = \int_r^{+\infty}\partial_r\left((1-\partial_{rr}\psi_m)^{1/2}u^2\right)dt$$
$$\leqq \frac{1}{m}\left\{\int\left|\nabla\left((1-\partial_{rr}\psi_m)^{1/2}\right)u^2\right| + 2\int(1-\partial_{rr}\psi_m)^{1/2}|u|\,|\nabla u|\right\}$$
$$\leqq \frac{1}{m}\left\{\frac{1}{m}\int|u|^2+\int(1-\partial_{rr}\psi_m)|\nabla u|^2+\int|u|^2\right\}.$$

Therefore

$$y_m'(t) \geqq -NH_0 - c\left(\frac{1}{m}+\frac{1}{m^2}\right)+\frac{c_2}{c_0^2}\int|v|^2+2\left(1-\frac{c}{m}\right)\int(1-\partial_{rr}\psi_m)|\nabla u|^2,$$

which concludes the proof of Proposition 4.1 and the blow-up theorem.

V. Comments and Open Problems

In this section, we briefly give some other extensions of the previous results and some open problems.

1. We have works in the Hamiltonian situation of Equation (I'_{c_0}). We expect that the results are still true in the nonhamiltonian case, that is (see also [6]) when, $\forall t$

$$\forall t, n_t(t) = -\nabla\cdot v(t)+v_1, \tag{5.1}$$

where v_1 does not depend on t.

2. All the results are the radial case. We conjecture that in the nonradial situation, the blow-up theorem still holds.

3. We suspect strongly that in the alternative of the theorem, we always have blow-up in finite time. Unfortunately, the estimates we obtained do not yield this result.

In this direction, we just have partial results in the supercritical case. Indeed, in this case we have ((4.13), (4.14), (4.17) with a suitable m) for a $c > 0, \forall t > 0$,

$$\left(\int |\nabla u|^2(t)\right)^{1/2} + \left(\int n^2(t)\right)^{\frac{1}{2}} \left(\int v^2(t)\right)^{1/2} \geqq \int_0^t c\left[\int |\nabla u|^2(s) + \int |v|^2(s) - H_0\right] ds\,. \tag{5.2}$$

Proposition 5.1 (Partial results of blow-up in finite time). *Under the assumptions of the theorem, for $N = 3$, if for $\varepsilon > 0, c > 0, \forall t > 0$,*

$$\int_0^t \int |\nabla u|^2(s) \geqq c\int_0^t \left(\int n^2(s)\right)^{1+\varepsilon}, \tag{5.3}$$

then blow-up in finite time occurs.

Remark. Variational estimates from the conservation of the Hamiltonian give that

$$\forall s, \left(\int |\nabla u|^2(s)\right)^{3/2} \geqq c\int n^2(s)$$

which is far from (5.3). Nevertheless in [16], the blow-up solutions numerically observed (for $N = 3$) are such that

$$\forall s, \int |\nabla u|^2(s) \geqq c\left(\int n^2(s)\right)^2,$$

which implies (5.3).

In this sense, Proposition 5.1 is a partial result of blow-up in finite time in the supercritical case.

Proof. It is easy to see that in this case for $c > 0$ and $\alpha > 1, \forall t > 0$,

$$y(t) \geqq \int_0^t (y^\alpha(s) - H_0)ds\,,$$

where

$$y(t) = \left[\left(\int |\nabla u|^2(t)\right)^{1/2} + \left(\int n^2(t)\right)^{1/2} \left(\int v^2(t)\right)^{1/2}\right],$$

which implies blow-up in finite time for $y(t)$ and thus for $|u, n, v|_{H_1}(t)$. This concludes the proof of Proposition 5.1.

Acknowledgement. The author thanks the Courant Institute and Rutgers University where part of this work was done.

References

1. Added, H., Added, S.: Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2. C.R. Acad. Sci. Paris **200**, 551–554 (1984)
2. Added, H., Added, S.: Equations of Langmuir turbulence and nonlinear Schrödinger equations: Smoothness and approximation. J. Funct. Anal. **79**, 183–210 (1988)
3. Berestycki, H., Cazenave, T.: Instabilité des états stationnaires dans les équations de Schrö dinger et de Klein–Gordon non linéaires. C. R. Acad. Sci. Paris **293**, 489–492 (1981)

4. Ginibre, J., Velo, G.: On a class of nonlinear Schrödinger equations I, II. The Cauchy problem, general case. J. Funct. Anal. **32**, 1–71 (1979)
5. Glangetas, L., Merle, F.: Existence of self-similar blow-up solution for Zakharov equation in dimension two. Commun. Math. Phys. **160**, 173–215 (1994)
6. Glangetas, L., Merle, F.: Concentration properties of blow-up solutions and instability results for Zakharov equation in dimension two. Part II. Commun. Math. Phys. **160**, 349–389 (1994)
7. Glassey, R.T.: On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. J. Math. Phys. **18**, 1794–1797 (1977)
8. Kato, T.: On nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Physique Théorique **49**, 113–129 (1987)
9. Landman, M., Papanicolaou, G.C., Sulem, C., Sulem, P.L., Wang, X.P.: Stability of isotropic self-similar dynamics for scalar collapse. Phys. Rev. A **46**, 7869–7876 (1992)
10. Merle, F.: Determination of blow-up solutions with minimal mass for Schrödinger equation with critical power. Duke J. **69**, n°2, 427–454 (1993)
11. Merle, F.: On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass. Commun. Pure Appl. Math. **45**, 203–254 (1992)
12. Ozawa, T., Tsutsumi, Y.: The nonlinear Schrödinger limit and the initial layer of the Zakharov equations. Preprint
13. Ozawa, T., Tsutsumi, Y.: Existence of smoothing effect of solutions for the Zakharov equations. Preprint
14. Ozawa, T., Tsutsumi, Y.: Blow-up of H^1 solution for the nonlinear Schrödinger equation. J. Diff. Eq. **92**, 317–330 (1991)
15. Ozawa, T., Tsutsumi, Y.: Blow-up of H^1 solutions for the one-dimension nonlinear Schrödinger equation with critical power nonlinearity. Proc. of A.M.S. **111**, 486–496 (1991)
16. Papanicolaou, G.C., Sulem, C., Sulem, P.L., Wang, X.P.: Singular solutions of the Zakharov equations for Langmuir turbulence. Phys. Fluids **B3**, 969–980 (1991)
17. Sobolev, V.V., Synach, V.S., Zakharov, V.E.: Character of the singularity and stochastic phenomena in self-focusing. Zh. Eksp. Theor. Fiz, Pis'ma Red **14**, 390–393 (1971)
18. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)
19. Sulem, C., Sulem, P.L.: Quelques résultats de régularité pour les équations de la turbulence de Langmuir. C.R. Acad. Sci. Paris **289**, 173–176 (1979)

Communicated by A. Kupiainen