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Spectra, Eigenvectors and Overlap Functions for Representation Operators of q-Deformed Algebras

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Abstract: Operators of representations corresponding to symmetric elements of the q-deformed algebras $U_q(\mathrm{su}_{1,1}), U_q(\mathrm{so}_{2,1}), U_q(\mathrm{so}_{3,1}), U_q(\mathrm{so}_n)$ and representable by Jacobi matrices are studied. Closures of unbounded symmetric operators of representations of the algebras $U_q(\mathrm{su}_{1,1})$ and $U_q(\mathrm{so}_{2,1})$ are not selfadjoint operators. For representations of the discrete series their deficiency indices are $(1, 1)$. Bounded symmetric operators of these representations are trace class operators or have continuous simple spectra. Eigenvectors of some operators of representations are evaluated explicitly. Coefficients of transition to eigenvectors (overlap coefficients) are given in terms of q-orthogonal polynomials. It is shown how results on eigenvectors and overlap coefficients can be used for obtaining new results in representation theory of q-deformed algebras.

1. Introduction

There is a connection between representations of a semisimple Lie group and representations of its Lie algebra [1]. To noncompact generators I there correspond unbounded operators in infinite dimensional irreducible representations T of a semisimple Lie algebra g. To every such representation T of g there corresponds an irreducible representation T of the Lie group G with the Lie algebra g. Operators of a representation T of G are bounded. If a representation T of G is unitary, then to noncompact generators I from g, multipled by $\mathrm{i} = \sqrt{-1}$, there correspond symmetric operators on a Hilbert space. Unitarity of a representation T of G means that closures of these symmetric operators are selfadjoint operators. Properties of self-adjointness for operators corresponding to symmetric elements of the universal enveloping algebra $U(g)$ of g are also well known (see, [1], Chapter 11).

The corresponding theory is absent for infinite dimensional representations of quantum algebras. Moreover, simple examples show that the situation for quantum algebras is unlike that which we have in the classical case.

Quantum algebras are q-deformed universal enveloping algebras $U_q(g)$ corresponding to simple Lie algebras (we do not consider here quantum algebras corresponding to affine Lie algebras). Such q-deformations are constructed for all complex simple Lie algebras [2, 3]. Real forms A of these quantum algebras are constructred with the help of involutions, which are antilinear antiautomorphisms (that is, if $\alpha \in \mathbf{C}$ and $a, b \in A$, then $(\alpha a)^* = \bar{\alpha} a^*$ and $(ab)^* = b^* a^*$). The real form A is called noncompact if it corresponds to a noncompact Lie group. In the other case the real form is called compact. A representation T of a real form A will be called infinitesimally unitary if operators $T(a)$ are symmetric for elements $a \in A$ such that $a^* = a$. For $q = 1$ (that is, for Lie algebras) these representations correspond to unitary representations of Lie groups. Dealing with infinitesimally unitary representations we shall omit the word "infinitesimally."

In Sect. 3 we consider operators of the representations T_l^+ of the discrete series of the quantum algebra $U_q(\mathrm{su}_{1,1})$ and show that closures of unbounded operators (for example, of the operator $T_l^+(E_+ - E_-)$) are not selfadjoint operators, as is the case for the Lie algebra $\mathrm{su}_{1,1}$. Deficiency indices of these closures are equal to $(1, 1)$, that is, deficiency subspaces are one-dimensional and these closures have selfadjoint extensions (there exist infinitely many extensions). Coordinates of basis vectors of these subspaces are expressed in terms of q-orthogonal polynomials.

A distinction of the case of the quantum algebra $U_q(\mathrm{su}_{1,1})$ is that the operators $T_l^+(I_2)$ of the representations T_l^+ of the q-deformed algebra $U_q(\mathrm{so}_{2,1})$ (which are analogues of the operators $T_l^+(E_+ - E_-)$ for $U_q(\mathrm{su}_{1,1})$) have bounded closures. At $q \to 1$, their spectra are expanding and these closures tend to unbounded operators.

It is very important to have a general theory of representation operators for noncompact quantum algebras analogous to the corresponding theory for classical Lie algebras. However, it is a difficult problem to construct this theory. Especially, if we take into account that infinite dimensional irreducible representations of quantum algebras are not satisfactorily studied. Irreducible infinite dimensional representations are constructed only for some special types of q-deformed algebras. In this connection, it is of a great importance to construct the spectral theory of representation operators for simplest q-deformed algebras and for simplest representations of complicated quantum algebras, especially if they are interesting for physics.

In this paper we study those representation operators which are given by Jacobi matrices with respect to some bases. The theory of Jacobi matrices reduces the spectral theory of such operators to studying three-term recurrence relations, the corresponding orthogonal polynomials and orthogonality relations for them. Solutions of these relations in our cases are q-orthogonal polynomials. Actually, values of these polynomials are coefficients of the transition from a certain orthonormal basis to another one (for infinite dimensional representations the second basis can be continual, that is, of the type of the basis $\{e^{i\lambda x}\}$ of the Hilbert space $L^2(\mathbf{R})$). Sometimes these transition coefficients can be evaluated explicitly. They allow us to find spectra and spectral measures for the corresponding operators. The transition coefficients under considerations are also called overlap coefficients or overlap functions for the corresponding bases of the carrier space.

Overlap coefficients for two bases of carrier spaces of irreducible representations of Lie groups and Lie algebras are of great importance for physics. If we interpret infinitesimal operators as physical observables, then overlap coefficients are connected with probabilities of observable values. Overlap coefficients for representations of quantum algebras can be also used in physics.

Overlap coefficients for operators of representations of quantum algebras can also be applied for studying q-special functions. Various applications of overlap coefficients for the case of Lie groups can be found in [4].

In this paper we are not interested in a structure of a Hopf algebra (a comultiplication, a counit, an antipode) for quantum algebras because we do not need this structure for our investigation. Quantum algebras and q-deformed algebras are considered here as associative algebras generated by a finite number of elements. Everywhere below we suppose that $0 < q < 1$.

In Sect. 2 we give some information on difference operators of the second order which is used below. Section 3 is devoted to studying representation operators for the quantum algebra $U_q(\mathrm{su}_{1,1})$. Consideration here is more detailed than in the next sections. In Sects. 4 and 5 we study representation operators for the q-deformed algebras $U_q(\mathrm{so}_{2,1})$ and $U_q(\mathrm{so}_{3,1})$. In Sect. 6 we deal with representation operators of the compact q-deformed algebra $U_q(\mathrm{so}_n)$. The aim of Sect. 7 is to show how results on eigenvectors and overlap coefficients can be used for obtaining new results in representation theory of q-deformed algebras. In this section we construct infinite dimensional representations of the algebra $U_q(\mathrm{so}_{r,2})$. Most of these representations are irreducible.

2. Difference Operators of the Second Order

We denote by V the Hilbert space with the orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$. Let L be a linear operator on V acting upon basis elements as

$$L|n\rangle = a_n|n+1\rangle + b_n|n\rangle + c_n|n-1\rangle \, , \tag{1}$$

and let

$$|z\rangle = \sum_{n=0}^{\infty} p_n(z)|n\rangle \tag{2}$$

be an eigenvector of L with an eigenvalue $z : L|z\rangle = z|z\rangle$. Then

$$L|z\rangle = \sum_{n=0}^{\infty}(p_n(z)a_n|n+1\rangle + p_n(z)b_n|n\rangle + p_n(z)c_n|n-1\rangle) = z\sum_{n=0}^{\infty} p_n(z)|n\rangle \, .$$

Equating coefficients at the vector $|n\rangle$ we have the recurrence relation for the coefficients from (2):

$$c_{n+1}p_{n+1}(z) + b_n p_n(z) + a_{n-1}p_{n-1}(z) = zp_n(z) \, . \tag{3}$$

Since $p_{-1}(z) \equiv 0$ then setting $p_0(z) \equiv 1$, we see that this relation completely determines the coefficients $p_n(z)$. Moreover, $p_n(z)$ are polynomials in z of degree n.

Sometimes, vectors $\mathbf{v} = \sum_{n=0}^{\infty} v_n|n\rangle$ of V are written down as numerical sequences $(v_0, v_1, \ldots)$. In this case formula (1) can be represented as

$$(L\mathbf{v})_n = a_{n-1}v_{n-1} + b_n v_n + c_{n+1}v_{n+1} \, .$$

Because of this, such operators L are called second order difference operators.

Now let L be a symmetric operator. Then formula (1) is written as

$$L|n\rangle = a_n|n+1\rangle + b_n|n\rangle + a_{n-1}|n-1\rangle \tag{4}$$

and Eq. (3), determining eigenvectors, is reduced to the recurrence relation

$$a_n p_{n+1}(z) + b_n p_n(z) + a_{n-1}p_{n-1}(z) = zp_n(z) \, . \tag{5}$$

One says that the operator L is representable by a Jacobi matrix. If the coefficients a_n and b_n in (5) are real, then all coefficients of the polynomials $p_n(z)$ are real.

We suppose that a_n and b_n are real and $a_n > 0$. If the operator L is unbounded, then we denote the closure of L by $\bar{L}$. The operator $\bar{L}$ may be not selfadjoint. In this case $\bar{L}$ has nonzero deficiency indices (m,k) which determine dimensions of deficiency subspaces. (The definitions of deficiency indices and deficiency subspaces, as well as their properties, can be found in [5].) To every complex number z, Im $z \neq 0$, there corresponds its deficiency subspace N_z. The following statements are valid [6]:

(A) Deficiency indices of the operator $\bar{L}$ are coinciding. Moreover, these indices are (0,0) or (1,1). In the first case the operator L is selfadjoint. In the second case $\bar{L}$ is not selfadjoint, however it has selfadjoint extensions.

(B) Deficiency indices of $\bar{L}$ are (0,0) if and only if the series $\sum_{n=0}^{\infty}|p_n(z)|^2$ diverges for all complex z, Im $z \neq 0$, where $p_n(z)$ are polynomials from (5). If deficiency indices are (1,1), then this series converges for all complex z, Im $z \neq 0$.

(C) If deficiency indices are (1,1), then deficiency subspaces are one-dimensional. The deficiency subspace $N_{\bar{z}}$ corresponding to a complex number $\bar{z}$ is spanned by the vector $\sum_{n=0}^{\infty} p_n(z)|n\rangle$, where $p_n(z)$ are taken from formula (5).

Hamburger's moment problem is related to the operator L given by formula (4) [7]. Moreover, if deficiency indices of $\bar{L}$ are (0,0), then it corresponds to a determined moment problem. If deficiency indices of $\bar{L}$ are (1,1), then we deal with an undetermined moment problem.

To find whether or not the operator $\bar{L}$ is selfadjoint, one may use the following statements [6]:

(a) If the coefficients a_n and b_n from (4) are bounded, then the operator $\bar{L}$ is bounded and, therefore, selfadjoint. Therefore, the corresponding moment problem is determined.

(b) If b_n are any real numbers and a_n are such that

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty ,$$

then the operator $\bar{L}$ is selfadjoint (Carleman's criterion). In this case the moment problem is determined.

(c) Let $|b_n| \leqq C$, $n = 0, 1, 2, \ldots$, and for some positive integer j we have $a_{n-1}a_{n+1} \leqq a_n^2$, $n \geqq j$. If

$$\sum_{n=0}^{\infty} \frac{1}{a_n} < \infty , \tag{6}$$

then the operator $\bar{L}$ is not selfadjoint and an undetermined moment problem corresponds to it.

If $\bar{L}$ is not a selfadjoint operator, then it has selfadjoint extensions. There are infinitely many selfadjoint extensions of $\bar{L}$. We refer the reader to the books [5, 6] for a more detailed discussion of selfadjoint extensions.

Using the terminology of [6], we can say the following about the operator $\bar{L}$. Let B be either the operator $\bar{L}$ if it is selfadjoint or its selfadjoint extension if it is not selfadjoint. Let $E(\Delta)$ be the decomposition of unity of the operator B. Then

$$E(\Delta) = \int_{\Delta} P(\lambda) d\rho(\lambda) ,$$

where $P(\lambda)$ are operators of generalized projections acting from the space $l^2([0,\infty), d_n)$ into the space $l^2([0,\infty), d_n^{-1})$. Here $d_n \geqq 0$ and such that $\sum_{j=0}^{\infty} d_j^{-1} < \infty$. Note that $l^2([0,\infty), d_n)$ is the space of sequences $(a_0, a_1, a_2, \ldots)$ such that $\sum_{n=0}^{\infty} |a_n|^2 d_n < \infty$. The operators $P(\lambda)$ are matrix operators with positive definite matrices $(\Phi_{jk}(\lambda))_{j,k=0}^{\infty}$ satisfying the condition [6]

$$\sum_{j,k=0}^{\infty} |\Phi_{jk}(\lambda)|^2 (d_j d_k)^{-1} \leqq 1 .$$

Moreover, we have

$$\Phi_{jk}(\lambda) = p_j(\lambda) p_k(\lambda) \Phi_{00}(\lambda), \quad j,k = 0,1,2,\ldots ,$$

where p_n are the polynomials from (5). Let $d\sigma(\lambda) = \Phi_{00}(\lambda) d\rho(\lambda)$. It is shown in [6] that

$$\int_{-\infty}^{\infty} p_j(\lambda) p_k(\lambda) d\sigma(\lambda) = \delta_{jk}, \quad j,k = 0,1,2,\ldots . \tag{7}$$

If the operator L is bounded and selfadjoint then we can set $d_n = 1$, $n = 0,1,2,\ldots$, and polynomials $p_j(\lambda)$ from (7) are overlap coefficients for the corresponding bases. In this case $d\sigma(\lambda)$ is the spectral measure of L and (7) is the orthogonality relation for polynomials p_n.

Remark that if the operator L is selfadjoint (that is, the corresponding moment problem is determined), then there exists a unique orthogonality relation for the polynomials p_n which are solutions of recurrence relation (5). If the operator $\bar{L}$ is not selfadjoint (and we have an undetermined moment problem), then there exist infinitely many selfadjoint extensions of $\bar{L}$ and to every extension there corresponds an orthogonality relation for the polynomials p_n. Thus, in the last case there exist infinitely many orthogonality relations for $p_n, n = 0,1,2,\ldots$.

In the general case, it is difficult to explicitly evaluate the polynomials $p_n(z)$. There are different methods of their evaluation: by using the corresponding generating function, by using the recurrence relation, and so on. For many representation operators of type (4) corresponding to infinite dimensional representations of simplest Lie groups (for the groups $SL(2,\mathbf{R})$, $SO_0(3,1)$) and to infinite dimensional class 1 representations of high dimension Lie groups they are evaluated by means of matrix elements of representations (see [4] and references therein). In this paper we evaluate them for certain operators of representations of q-deformed algebras. They are expressed in terms of q-orthogonal polynomials.

3. Representation Operators of the Quantum Algebra $U_q(\mathrm{su}_{1,1})$

The quantum algebra $U_q(\mathrm{sl}_2)$ is the associative algebra generated by the elements E_+, E_-, H that satisfy the commutation relations

$$[H, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}} = \frac{\sinh hH}{\sinh(h/2)} .$$

Introducing into $U_q(\mathrm{sl}_2)$ the involution defined by the relations $E_\pm^* = -E_\mp$, $H^* = H$, we obtain the real quantum algebra $U_q(\mathrm{su}_{1,1})$.

The representations T_l^+ of the discrete series of the algebra $U_q(\mathrm{su}_{1,1})$ are given by a positive number l and act on the Hilbert spaces V_l with the orthonormal bases $|m\rangle$, $m = l+1, l+2, \ldots$. The operators $T_l^+(E_\pm)$ and $T_l^+(H)$ act upon basis elements $|m\rangle$ by the formulas

$$T_l^+(H)|m\rangle = m|m\rangle, \qquad T_l^+(E_+)|m\rangle = ([-l+m][l+m+1])^{1/2}|m+1\rangle\,, \tag{8}$$

$$T_l^+(E_-)|m\rangle = -([-l+m-1][l+m])^{1/2}|m-1\rangle\,, \tag{9}$$

where $[a]$ is a q-number defined as

$$[a] = (q^{a/2} - q^{-a/2})/(q^{1/2} - q^{-1/2})\,.$$

Simplest elements of $U_q(\mathrm{su}_{1;1})$, symmetric with respect to the involution, are of the form $H, E_+ - E_-, i(E_+ + E_-)$. It is seen from formula (8) that the operator $T_l^+(H)$ is unbounded. Since it is diagonal with respect to the basis $\{|m\rangle\}$, then its closure is a selfadjoint operator. It follows from formulas (8) and (9) that the operators $L' = T_l^+(\mathrm{i}E_+ + \mathrm{i}E_-)$ and $L = T_l^+(E_+ - E_-)$ are also unbounded. It is easy to show that when passing from the basis $|m\rangle$, $m = l+1, l+2, \ldots$, to the basis $|m\rangle' = \mathrm{i}^m|m\rangle$, $m = l+1, l+2, \ldots$, $\mathrm{i} = \sqrt{-1}$, we go over from the matrix of the operator L' to the matrix of the operator L. Because of this fact, the closures $\bar{L}$ and $\bar{L}'$ of the operators L and L' are simultaneously selfadjoint or not selfadjoint and their deficiency indices are coinciding. For this reason, we shall deal only with the operator L.

We shall study symmetric operators of the representations T_l^+ representable with respect to the basis $\{|m\rangle\}$ by a Jacobi matrix. Such natural operators are

$$T_l^+(q^{pH}(E_+ - E_-)q^{pH}) + cq^{rH}, \quad p, c, r \in \mathbf{R}\,.$$

Let us first consider the operators

$$B_p = T_l^+(q^{pH/4}(E_+ - E_-)q^{pH/4}), \quad p \in \mathbf{R}\,.$$

It follows from formulas (8) and (9) that

$$B_p|m\rangle = b_{pm}|m+1\rangle + b_{p,m-1}|m-1\rangle\,, \tag{10}$$

$$b_{pm} = q^{p(k+l)/2} q^{3p/4} \sqrt{[k+1][k+2l+1]}$$
$$= \frac{q^{(k+l+1)(p-1)/2} q^{p/4}}{q^{-1/2} - q^{1/2}} \sqrt{(1-q^{k+1})(1-q^{k+2l+2})}\,,$$

where $k = m - l - 1$. We remark that flipping q to q^{-1} corresponds to flipping p to $-p$ for the operator B_p.

Proposition 1. *If $p \geqq 1$ then the operator B_p is bounded and has a unique selfadjoint extension coinciding with its closure $\overline{B_p}$. If $p < 1$ then the deficiency indices of the operator $\overline{B_p}$ are $(1,1)$ and the operator $\overline{B_p}$ has infinitely many selfadjoint extensions.*

Proof. For the numbers b_{pm} from formula (10) we have

$$b_{p,m+1}/b_{pm} \to q^{(p-1)/2} \quad \text{when } m \to +\infty\,. \tag{11}$$

Since $0 < q < 1$, then the operator B_p is bounded for $p \geqq 1$. This proves the first part of our proposition. It follows from (11) that the operator $\overline{B_p}$ is unbounded if $p < 1$. To prove the second part of the proposition we note that the inequality

$$(1-q^{n-1})(1-q^{n+1}) \leqq (1-q^n)^2 \tag{12}$$

is valid. In fact, removing the parentheses we obtain $q+q^{-1} \geqq 2$. This inequality is correct for all real q and the equality is achieved at $q=1$. It follows from (12) that

$$b_{p,m-1}b_{p,m+1} \leqq b_{p,m}^2 \quad \text{for all } m > l\,.$$

Besides, in this case we have $\sum_{m=l+1}^{\infty} b_{pm}^{-1} < \infty$ since

$$b_{pm}/b_{p,m+1} \to q^{-(p-1)/2} < 1 \quad \text{if } m \to +\infty\,.$$

Therefore, according to criterion (c) from Sect. 2 the deficiency indices of the operator $\overline{B_p}$, $p < 1$, are $(1,1)$. This proves our proposition.

Let us investigate the spectrum of the operator $\overline{B_p}$ for different values of p. If $p < 1$ then the operator $\overline{B_p}$ has deficiency indices $(1,1)$. In this case any selfadjoint extension $B_p^{\rm ext}$ of $\overline{B_p}$, constructed without coming from the carrier Hilbert space V_l, has a purely discrete simple spectrum [6]. Moreover, there exists a function $f(z)$ from the space $\mathbf{U}$ such that the spectrum of the operator $B_p^{\rm ext}$ coincides with the set of zeros λ_j $(j=1,2,3,\ldots)$ of $f(z)$ and jumps

$$\mu_j = \sigma(\lambda_j+0) - \sigma(\lambda_j), \quad j=1,2,3,\ldots\,,$$

of the spectral function $\sigma(u)$ of $B_p^{\rm ext}$ are such that the following conditions are fulfilled:

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j(1+\lambda_j^2)(f'(\lambda_j))^2} < \infty\,, \qquad \sum_{j=1}^{\infty} \frac{1}{\mu_j(f'(\lambda_j))^2} = \infty$$

(see [7], chapter 4), where $f'(z) = \frac{d}{dx}f(z)$. Here $\mathbf{U}$ is the space of entire real functions on $\mathbf{C}$ such that the following conditions are fulfilled: (a) all zeros λ_j of f are real; (b) the absolute convergent expansion

$$\frac{1}{f(z)} = \sum_{j=1}^{\infty} \frac{1}{f'(\lambda_j)(z-\lambda_j)}$$

has a place; (c) all series $\sum_{j=1}^{\infty} \lambda_j^m/f'(\lambda_j)$, $m=0,1,2,\ldots$, are convergent.

It follows from these assertions that the discrete spectrum of the operator $\overline{B_p}$ has the infinite point as the only point of accumulation.

If $p > 1$ then $\overline{B_p}$ is an operator of trace class. In fact, in this case all matrix elements of $\overline{B_p}$ with respect to the basis $\{|m\rangle\}$ are nonnegative, and due to formula (11) we have

$$\sum_{m=l+1}^{\infty} b_{pm} < \infty\,.$$

Since $\overline{B_p}$ is a trace class operator, it has a purely discrete spectrum with zero as the only point of accumulation. It follows from the results of papers [8,9] (see also [10]) that the spectrum is symmetric with respect to the point $x=0$ which also belongs to the spectrum. It follows from Theorem 3 in [9] that the transcendental meromorphic function $G(z)$ exists with the expansion

$$G(z) = -A + \sum_{n=1}^{\infty} \frac{2A_n}{z^2 - \alpha_n^2} ,$$

where $-\sum_n A_n \alpha_n^{-2} < \infty$, $A \leqq 0$ and $A_n < 0, n = 1,2,\ldots$, such that the spectrum of $\overline{B_p}$ coincides with the set of points $x = 0$ and $x = \pm 1/\alpha_n$, $n = 1,2,\ldots$. Clearly, the points $x = \pm 1/\alpha_n$, $n = 1,2,\ldots$, are poles of the function $G(1/z)$. Jumps of the spectral measure $\sigma(x)$ of $\overline{B_p}$ at these poles are equal to residues of the function $z^{-1}G(1/z)$ at these points. They coincide with

$$\sigma(x+0) - \sigma(x-0) = -A_n \alpha_n^{-2} \quad (\text{for the poles } \pm \alpha_n^{-1})$$

and $\sigma(+0) - \sigma(-0) = -A$. Of course, the function $G(z)$ is determined by the coefficients b_{pm} from (10). However, the expression for $G(z)$ in terms of b_{pm} is very complicated (see formula (2.7) in [9]).

We considered spectra of the operators $\overline{B_p}$ for $p > 1$ and $p < 1$. Now we have to consider the spectrum of the operator

$$B_1 = T_l{}^+(q^{H/4}(E_+ - E_-)q^{H/4}) .$$

We have

$$B_1|m\rangle = b_m|m+1\rangle + b_{m-1}|m-1\rangle, \quad b_m = ([k+1][k+2l+2])^{1/2} q^{m/2} q^{1/4} ,$$

where $k = m - l - 1$. The operator B_1 is bounded. Therefore, its closure is a bounded selfadjoint operator. A generalized vector

$$|y\rangle = \sum_{k=0}^{\infty} P_k(y)|k+l+1\rangle \tag{13}$$

is an eigenvector of $\overline{B_1}$ corresponding to an eigenvalue y if $P_k(y)$, $k = 0,1,2,\ldots$, satisfy the recurrence relation

$$q^{k/2} q^{1/4} a_k P_{k+1}(y) + q^{k/2} q^{-1/4} a_{k-1} P_{k-1}(y) = y P_k(y) q^{-(l+1)/2} ,$$

$$a_k = ([k+1][k+2l+2])^{1/2} ,$$

and the initial conditions $P_{-1}(y) = 0$, $P_0(y) = 1$. The substitution

$$P_k(y) = \{(q;q)_k (q^{2l+2};q)_k\}^{-1/2} P_k'(y) ,$$

where $(d;q)_k = (1-q)(1-dq)\cdots(1-dq^{k-1})$, reduces this relation to the form

$$P_{k+1}'(y) + (1-q^k)(1-q^{k+2l+1}) P_{k-1}'(y) = y(q^{-1/2} - q^{1/2}) P_k'(y) .$$

Replacing $q^{-1/2}(1-q)y$ by $2x$ and $P_k'(2x(q^{-1/2} - q^{1/2})^{-1})$ by $P_k''(x)$, we obtain the relation

$$P_{k+1}''(x) + (1-q^k)(1-q^{k+2l+1}) P_{k-1}''(x) = 2x P_k''(x) . \tag{14}$$

To solve this recurrence relation we consider the orthogonal polynomials [11]

$$p_n(\cos(\theta+\phi); a, c|q) = a^{-n}e^{-\mathrm{i}n\phi}(ace^{2\mathrm{i}\phi}; q)_n(a^2; q)_n(ac; q)_n \times {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1}a^2c^2, ae^{2\mathrm{i}\phi}e^{\mathrm{i}\theta}, ae^{-\mathrm{i}\theta} \\ ace^{2\mathrm{i}\phi}, a^2, ac \end{matrix} ;\; q, q\right), \qquad (15)$$

where ${}_4\varphi_3$ is the basic hypergeometric function (see [12] for the definition and properties of this function). The recurrence relation for these polynomials is of the form [11]

$$2xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)\,, \qquad (16)$$

where $x = \cos(\theta+\phi)$ and

$$A_n = (1-a^2c^2q^{n-1})(1-a^2c^2q^{2n-1})^{-1}(1-a^2c^2q^{2n})^{-1}\,,$$

$$B_n = \frac{2q^{n-1}(a+c)\cos\phi\{(1+a^2c^2q^{2n-1})(q+ac) - q^{n-1}(1+q)a^2c^2(1+acq)\}}{(1-a^2c^2q^{2n-2})(1-a^2c^2q^{2n})}\,,$$

$$C_n = \frac{(1-q^n)(1-acq^{n-1})(1-a^2q^{n-1})(1-c^2q^{n-1})(1-2acq^{n-1}\cos 2\phi + a^2c^2q^{2n-2})}{(1-a^2c^2q^{2n-1})(1+acq^{n-1})}\,.$$

If a and c are real and $|a| < 1, |c| < 1$, then the orthogonality relation for these polynomials is

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} p_n(\cos(\theta+\phi);\, a, c|q)p_m(\cos(\theta+\phi);\, a, c|q)w(\theta)d\theta = \delta_{mn}h_n\,, \qquad (17)$$

where

$$w(\theta) = \left|\frac{(e^{2\mathrm{i}(\theta+\phi)}; q)_\infty}{(ae^{\mathrm{i}\theta}; q)_\infty(ce^{\mathrm{i}\theta}; q)_\infty(ae^{\mathrm{i}(\theta+2\phi)}; q)_\infty(ce^{\mathrm{i}(\theta+2\phi)}; q)_\infty}\right|^2$$

(the explicit expression for the constants h_n see in [11]). A direct evaluation shows that relation (14) coincides with recurrence relation (16) for $\phi = \pi/2$, $c = 0$, $a = q^{l+1}, x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. Therefore,

$$P_k''(x) = p_k\left(\cos\left(\theta+\frac{\pi}{2}\right); q^{l+1}, 0|q\right) = (iq^{l+1})^{-k}(q^{2l+2}; q)_k \times {}_3\varphi_2\left(\begin{matrix} q^{-k}, e^{-\mathrm{i}\theta}q^{l+1}, -e^{\mathrm{i}\theta}q^{l+1} \\ q^{2l+2}, 0 \end{matrix} ; q, q\right).$$

Passing on the polynomials $P_k(y)$, normalized by the condition $P_0(y) = 1$, we obtain that in formula (13) we have

$$P_k(y) = (\mathrm{i}q^{l+1})^{-k}\left(\frac{(q^{2l+1}; q)_k}{(q; q)_k}\right)^{1/2} {}_3\varphi_2\left(\begin{matrix} q^{-k}, e^{-\mathrm{i}\theta}q^{l+1}, -e^{\mathrm{i}\theta}q^{l+1} \\ q^{2l+2}, 0 \end{matrix} ; q, q\right), \qquad (18)$$

where

$$y = \frac{2}{q^{-1/2} - q^{1/2}} \cos\left(\theta + \frac{\pi}{2}\right) = -\mathrm{i}\frac{e^{-\mathrm{i}\theta} - e^{\mathrm{i}\theta}}{q^{-1/2} - q^{1/2}} . \tag{19}$$

We derive from formula (17) that the orthogonality relation for these polynomials is of the form

$$\int_{-b}^{b} P_n(y)P_k(y)w(y)dy = \delta_{kn} ,$$

where $b = 2/(q^{-1/2} - q^{1/2})$ and

$$w(y) = \frac{q^{-1/2} - q^{1/2}}{4\pi \cos\theta}(q;q)_\infty (q^{2l+2};q)_\infty \left| \frac{(-e^{2\mathrm{i}\theta};q)_\infty}{(e^{2\mathrm{i}\theta}q^{2l+2};q^2)_\infty} \right|^2 .$$

This relation means that the spectrum of the operator $\overline{B_1}$ is simple and covers exactly the interval $(-2/(q^{-1/2} - q^{1/2}),\ 2/(q^{-1/2} - q^{1/2}))$. The spectral measure of this operator coincides with the measure $d\sigma(y) = w(y)dy$. When $q \to 1$ then the spectrum turns into the real line and polynomials $P_n(y)$ tend to the corresponding Meixner–Pollaczek polynomials. This agrees with results for the classical Lie group $SU(1,1)$ (see [4], Chapter 7, and [13]). Now we can formulate the following theorem:

Theorem 2. *If $p > 1$ then the operator $\overline{B_p}$ has a discrete simple spectrum with zero as the only point of accumulation. If $p < 1$ then all selfadjoint extensions of the operator $\overline{B_p}$ (without coming from the Hilbert space V_l) have discrete simple spectra with infinity as the only point of accumulation. The operator $\overline{B_1}$ has a continuous simple spectrum which covers the interval $(-b,b), b = 2/(q^{-1/2} - q^{1/2})$.*

Let us remark that overlap functions (18) depend in a complicated way on eigenvalues y of the operator $\overline{B_1}$. And utilization of the theory of q-orthogonal polynomials make it possible to find the overlap coefficients explicitly. It is interesting to note that the results of this theory, used in our paper, were discovered quite recently. In fact, they were discovered almost simultaneously with the discovery of quantum groups.

The operator B_1 can be represented in the form

$$B_1|k\rangle = a_k|k+1\rangle + a_{k-1}|k-1\rangle ,$$

$$a_k = \frac{(1-q^{k+1})^{1/2}(1-q^{k+2l+2})^{1/2}}{q^{-1/2} - q^{1/2}} ,$$

where the basis vectors are labelled by k instead of $m = k + l + 1$. Clearly, B_1 depends on l. Taking the limit $l \to +\infty$ we obtain the operator $Q = \lim_{l\to\infty} B_1$ such that

$$(q^{-1/2} - q^{1/2})Q|k\rangle = (1-q^{k+1})^{1/2}|k+1\rangle + (1-q^k)^{1/2}|k-1\rangle .$$

So, we see that the operator Q, up to a constant, coincides with the operator of the canonical coordinate in the q-oscillator algebra introduced by Macfarlane [14]. The spectrum, the spectral measure and the corresponding overlap polynomials p_n for the operator Q are found in [15]. The polynomials p_n are expressed in terms of the continuous q-Hermite polynomials from [16]. Thus, the polynomials p_n are obtained at the limit $l \to \infty$ from polynomials (18).

Let us consider the operators

$$A_{pbr} = T_l^+(q^{pH/4}(E_+ - E_-)q^{pH/4}) + bq^{rH}, \quad p, b, r \in \mathbf{R}. \tag{20}$$

Such an operator is given by formula (10) (with the same coefficients b_{pm}) in which the summand $bq^{rm}|m\rangle$ is added. Using the criteria of selfadjointness or non-selfadjointness of symmetric operators from Sect. 2 and the previous results on operators B_p we can formulate the following proposition:

Proposition 3. *The operator $\overline{A_{pbr}}$ is selfadjoint for all values of p, $p \geqq 1$. If $p < 1$ then the operator $\overline{A_{pbr}}$ has the deficiency indices $(1, 1)$ if $b \in \mathbf{R}$ and $r \geqq 0$. In the first case $\overline{A_{pbr}}$ is a trace class operator if $p > 1$ and $r > 0$, and therefore has a discrete simple spectrum with zero as the only point of accumulation. In the second case all selfadjoint extensions of $\overline{A_{pbr}}$ (without coming from the Hilbert space V_l) have discrete simple spectra with infinity as the only point of accumulation.*

Proposition 3 describes spectra of operators $\overline{A_{pbr}}$ for $p > 1$ and $p < 1$. We have to consider the operators $\overline{A_{pbr}}$ at $p = 1$. We do not study all such operators but the operators

$$A(\phi) = T_l^+(q^{H/4}(E_+ - E_-)q^{H/4}) + bq^H, \quad b = \frac{2\cos\phi}{q^{-1/2} - q^{1/2}} \tag{21}$$

with b depending on ϕ. The recurrence relation corresponds to this operator,

$$a_k P_{k+1}(z) + a_{k-1} P_{k-1}(z) = \{(q^{-1/2} - q^{1/2})z - bq^{l+k+1}\} P_k(z),$$

$$a_k = (1 - q^{k+1})^{1/2}(1 - q^{k+2l+2})^{1/2}.$$

Making the substitution

$$P_k(z) = \{(q;q)_k (q^{2l+2};q)_k\}^{-1/2} P'_k(z),$$

we transform it into relation (16) with

$$a = q^{l+1}, \qquad c = 0, \qquad z = 2x(q^{-1/2} - q^{1/2}).$$

As a result, we obtain that

$$\begin{aligned} P_k(z) &= \{(q;q)_k (q^{2l+2};q)_k\}^{-1/2} p_k(\cos(\theta + \phi); q^{l+1}, 0|q) \\ &= \frac{(q^{2l+2};q)_k}{(q^{l+1}e^{i\phi})^k} {}_3\varphi_2(q^{-k}, q^{l+1}e^{2i\phi + i\theta}, q^{l+1}e^{-i\theta}; q^{2l+2}, 0; q, q), \end{aligned}$$

where $z = 2\cos(\theta + \phi)/(q^{-1/2} - q^{1/2})$. It follows from (17) that

$$\int_{-d}^{d} P_m(z) P_n(z) W(z) dz = \delta_{mn},$$

where $d = (q^{-1/2} - q^{1/2})/4\pi \cos(\theta + \phi)$ and

$$W(z) = q^{2l+2}(q;q)_\infty (q^{2l+2};q)_\infty \left| \frac{(e^{2i(\theta+\phi)};q)_\infty}{(q^{l+1}e^{i\theta};q)_\infty (q^{l+1}e^{i(\theta+2\phi)};q)_\infty} \right|^2.$$

This formula shows that the spectra of all operators (21) coincide with the interval $(-d,d)$, $d=2/(q^{-1/2}-q^{1/2})$. The spectral measure $d\sigma$ of the operator $A(\phi)$ is $d\sigma = W(z)dz$.

We found explicit expressions for overlap polynomials $P_n(z)$ and the corresponding orthogonality relations for the operators (21). Unfortunately, we could not find an explicit expression for these polynomials in the case of the operator A_{pbr} for arbitrary p,b,r. But it is possible to find polynomials $P_n(z)$ for different particular values of these parameters. For example, if $p=-1, b=0$, then for the polynomials $P_n(z)$ corresponding to the operator $A_{-1,0,r}$ we obtain the recurrence relation which, after the appropriate substitution, reduces to the following one:

$$(1-q^{n+1})P'_{n+1}(z)+q(1-q^{n+2l+1})P'_{n-1}(z)=zdq^nP'_n(z)\,, \tag{22}$$

where $d=q^{l+1}q^{1/4}(q^{-1/2}-q^{1/2})$. This recurrence relation can be solved by the method of a generating function used in the theory of q-orthogonal polynomials (see, for example, [16]). Namely, we set

$$f(r,z)=\sum_{n=0}^{\infty}P'_n(z)r^n\,. \tag{23}$$

So, $f(r,z)$ is a generating function for the polynomials $P'_n(z)$. Multiplying both sides of (22) by r^{n+1} and summing over n we obtain

$$f(r,z)-f(rq,z)+r^2qf(r,z)-r^2qq^{2l+2}f(rq,z)=zdrf(rq,z)\,.$$

Therefore,

$$f(r,z)=\frac{1+zdr+r^2qq^{2l+2}}{1+r^2q}f(qr,z)\,. \tag{24}$$

Setting $zd=-2q^{l+3/2}\cosh t$ and iterating (24) one has

$$f(r,z)=\frac{(r'q^{l+1}e^t;q)_n(r'q^{l+1}e^{-t};q)_n}{(\mathrm{i}r';q)_n(-\mathrm{i}r';q)_n}f(q^nr,z)\,,$$

where $r'=rq^{1/2}$. Taking the limit $n\to\infty$ we obtain the explicit expression for the generating function $f(r,z)$:

$$f(r,z)=\frac{(r'q^{l+1}e^t;q)_\infty(r'q^{l+1}e^{-t};q)_\infty}{(\mathrm{i}r';q)_\infty(-\mathrm{i}r';q)_\infty}\,.$$

Applying here the q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1,\ |q|<1$$

(see, for example, [17], Theorem 2.1) and comparing the obtained expression with formula (23), we obtain the expression for $P'_m(z)$,

$$P'_m(z)=(-\mathrm{i})^mq^{m/2}\sum_{n=0}^{m}\frac{(-1)^n(-\mathrm{i}q^{l+1}e^t;q)_n(\mathrm{i}q^{l+1}e^{-t};q)_{m-n}}{(q;q)_n(q;q)_{m-n}}\,.$$

It is a polynomial in e^t+e^{-t}. However, we could not separate e^t+e^{-t} in this expression. This expression can be represented in terms of basic hypergeometric functions. In fact, taking into account the relation

$$(a;q)_{m-n} = \frac{(a;q)_m q^{n(n+1)/2}}{(-aq^m)^n (q^{1-m}a^{-1};q)_n},$$

we derive

$$P'_m(z) = c\sum_{k=0}^{m} \frac{(-iq^{l+1}e^t;q)_n (q^{-m};q)_n}{(q;q)_n (-iq^{-m-l}e^t;q)_n} (iq^{-l}e^t)^n$$

$$= c\, {}_2\varphi_1(-iq^{l+1}e^t, q^{-m}; -iq^{-m-l}e^t; q, iq^{-l}e^t)\,,$$

where $c = (-iq^{1/2})^m (iq^{l+1}e^{-t};q)_m/(q;q)_m$. By making use of the transformation formula for basic hypergeometric functions (see Apendix III in [12]) we can represent $P'_m(z)$ in the different form,

$$P'_m(z) = c'e^{-mt}\, {}_3\varphi_2(q^{-m}, -iq^{l+1}e^t, iq^{l+1}e^t; q^{2l+2}, 0; q, q)$$

$$= c'e^{-mt}\, {}_3\varphi_1(q^{-m}, -iq^{l+1}e^t, iq^{l+1}e^{-t};\ q^{2l+2};\ q, q^m)\,,$$

where $c' = (q^{2l+2};q)_m q^{-(l+1/2)m}/(q;q)_m$. Here an explicit dependence on $e^t + e^{-t}$ is also absent. These polynomials of z have many orthogonality relations. It is difficult to find them explicitly.

In a similar way one can find polynomials $P_n(z)$ for the operators $A_{-1,b,0}$ and $A_{-1,b,-1}, b \in \mathbf{R}$. They have similar expressions in terms of basic hypergeometric functions.

We remark that we can also find the corresponding polynomials for some particular cases of the operators of the type (20) if $p > 1$. For example, for the operator

$$T_l^+(q^{3H/4}(E_+ - E_-)q^{3H/4}) + ([H-l-1]q^{l/2} + [H+l+1]q^{-l/2})q^{3H/2} \qquad (25)$$

the overlap polynomials $P_k(z)$ are

$$P_k(z) = \left(\frac{q^{-(2l+2)k}(q^{2l+2};q)_k}{(q;q)_k}\right)^{1/2} p_k(q^y; q^{2l+1}, 0|q)\,, \qquad (26)$$

where $z = q^y/(1-q^{-1})$ and p_k are the little q-Jacobi polynomials (see [12] for the definition of these polynomials). Due to the orthogonality relation for the polynomials p_k [12] we obtain

$$\sum_{y=0}^{\infty} P_m(q^y/(1-q^{-1}))P_n(q^y/(1-q^{-1}))W(y) = \delta_{mn}\,,$$

where $W(y) = q^{(2l+2)y}(q^{2l+2};q)_\infty (q;q)_y^{-1}$. Therefore, the spectrum of the operator (25) consists of the negative numbers

$$q^n/(1-q^{-1}), \quad n = 0,1,2,\dots\,.$$

Note that the little q-Jacobi polynomials $p_k(x;a,b|q)$ at $b=0$ turn (after changing variables and renormalization) into the Wall polynomials $W_k(x;a,q)$ and into the generalized Stieltjes–Wigert polynomials $S_k(x;p,q)$ which are q-analogues of Laguerre polynomials [12]. This result agrees with the results of [13] for the case $q=1$.

Spectral properties of operators of representations of the negative discrete series of the quantum algebra $U_q(\mathrm{su}_{1,1})$ can be considered completely in the same way. The quantum algebra $U_q(\mathrm{su}_{1,1})$ has other series of unitary representations (the principal unitary series, the supplementary series, the strange series; see, for example, [18]). The operators

$$L_p = T(q^{pH/4}(E_+ - E_-)q^{pH/4}), \quad p \in \mathbf{R},$$

in these representations are given by formula (10), where coefficients $b_{p,m}$ have other expressions (they can be easily derived from the results of [18]) and m changes from $-\infty$ to $+\infty$. It is easy to verify that the operator $\overline{L_p}$ is symmetric and unbounded for all these unitary representations. It is well known (see, for example, [19]) that deficiency indices of the operator $\overline{L_p}$ can be found in the following way. We divide the operator $\overline{L_p}$ into two operators $L_{p,1}$ and $L_{p,2}$: one acting upon basis vectors $|m\rangle$, where $m > 0$, and the second acting upon $|m\rangle$ with $m < 0$. The deficiency indices of $\overline{L_p}$ are equal to the sum of the corresponding deficiency indices of $L_{p,1}$ and $L_{p,2}$. As in the case of operators of the representations T_l^+, it is shown that the deficiency indices of the operators $L_{p,1}$, $p \geqq 1$, and $L_{p,2}$, $p \leqq -1$, are (0,0). The deficiency indices of the operators $L_{p,1}$, $p < 1$, and $L_{p,2}$, $p > -1$, are (1,1). Therefore, *the deficiency indices of the operator* $\overline{L_p}$ *in any unitary irreducible representation with m changing from* $-\infty$ *to* $+\infty$ *are* (1,1) *if* $p \leqq -1$ *or if* $p \geqq 1$ *and* (2,2) *if* $-1 < p < 1$. Thus, this operator has selfadjoint extensions. The detailed studying of diagonalization of the operator $\overline{L_p}$ is awkward and will be given in a separate paper.

4. Representation Operators for the Discrete Series of the q-Deformed Algebra $U_q(\mathrm{so}_{2,1})$

The q-deformed algebra $U_q(\mathrm{so}(3,\mathbf{C}))$ is the associative algebra generated by the elements I_1, I_2, I_3 which satisfy the commutation relations [20]

$$[I_1, I_2]_q \equiv q^{1/4} I_1 I_2 - q^{-1/4} I_2 I_1 = I_3,$$

$$[I_3, I_1]_q = I_2, \qquad [I_2, I_3]_q = I_1.$$

The first relation shows that $U_q(\mathrm{so}(3,\mathbf{C}))$ is generated by two elements I_1 and I_2. These elements satisfy relations of Serre's type:

$$I_1^2 I_2 - (q^{1/2} + q^{-1/2}) I_1 I_2 I_1 + I_2 I_1^2 = -I_2,$$

$$I_1 I_2^2 - (q^{1/2} + q^{-1/2}) I_2 I_1 I_2 + I_2^2 I_1 = -I_1.$$

The q-deformed algebra $U_q(\mathrm{so}_{2,1})$ is obtained from $U_q(\mathrm{so}(3,\mathbf{C}))$ by introducing the involution which is uniquely determined by the formulas $I_1^* = -I_1$, $I_2^* = I_2$.

Remark that the Lie groups $SU(1,1)$ and $SO_0(2,1)$ are locally isomorphic and their Lie algebras are isomorphic. The q-deformed algebras $U_q(\mathrm{su}_{1,1})$ and $U_q(\mathrm{so}_{2,1})$ are not isomorphic. Moreover, they have non-coinciding sets of unitary irreducible representations [20].

Representations T_l^+ of the discrete series of the algebra $U_q(\mathrm{so}_{2,1})$ are given by a positive number l and act on the Hilbert space V_l with the orthonormal basis $|m\rangle$, $m = l+1, l+2, \ldots$. The operators $T_l^+(I_1)$ and $T_l^+(I_2)$ act upon the basis elements $|m\rangle$ as follows [20]

$$T_l^+(I_1)|m\rangle = \mathrm{i}[m]|m\rangle, \quad T_l^+(I_2)|m\rangle = a_m|m+1\rangle + a_{m-1}|m-1\rangle\,,$$

$$a_m = \left(\frac{[m][m+1]}{[2m][2m+2]}\right)^{1/2} ([-l+m][l+m+1])^{1/2}\,, \tag{27}$$

where, as before, $[a]$ denotes a q-number.

The closure of $\mathrm{i}T_l^+(I_1)$ is an unbounded selfadjoint operator. Since

$$\lim_{m\to\infty} a_m = (q^{-1/2} - q^{1/2})^{-1} q^{-1/4}\,,$$

then the operator $T_l^+(I_2)$ is bounded. Therefore, its closure is a selfadjoint operator. Remark that boundedness of the operator $T_l^+(I_2)$ is the property of the q-deformed algebra $U_q(\mathrm{so}_{2,1})$, $q \neq 1$. When $q \to 1$ then $(q^{-1/2} - q^{1/2})^{-1} \to \infty$ and the operator $T_l^+(I_2)$ becomes unbounded.

We study symmetric operators of the representations T_l^+ representable with respect to the basis $\{|m\rangle\}$ by Jacobi matrices. Such natural operators are

$$A_{pbr} = T_l^+((\mathrm{i}I_1)^p I_2 (\mathrm{i}I_1)^p + b(\mathrm{i}I_1)^r)\,, \quad p, b, r \in \mathbf{R}\,.$$

Let us consider the operators

$$B_p = T_l^+(\mathrm{i}I_1)^p I_2 (\mathrm{i}I_1)^p), \quad p \in \mathbf{R}\,.$$

It follows from (27) that

$$B_p|m\rangle = b_{p,m}|m+1\rangle + b_{p,m-1}|m-1\rangle\,,$$

$$b_{p,m} = q^{-p/2} \frac{(1-q^m)^p(1-q^{m+1})^p}{q^{pm}(q^{-1/2}-q^{1/2})^{2p+1}} \times \left(\frac{(1-q^m)(1-q^{m+1})(1-q^{m-l})(1-q^{l+m+1})}{(1-q^{2m})(1-q^{2m+2})}\right)^{1/2}.$$

In the same way as in the case of operators of the representations T_l^+ of $U_q(\mathrm{su}_{1,1})$, we prove that at $p > 0$ the operator $\overline{B_p}$ is unbounded and has the deficiency indices (1,1). If $p < 0$ then $\overline{B_p}$ is a trace class operator. So, at $p > 0$ we can say that the operator $\overline{B_p}$ has the same properties as in the case of the algebra $U_q(\mathrm{su}_{1,1})$. We have to research the case $p = 0$. Let us find eigenvectors

$$|y\rangle = \sum_{k=0}^{\infty} P_k(y)|l+k+1\rangle \tag{28}$$

of the operator $B_0 = T_l^+(I_2)$ and its spectrum. The arguments of Sect. 2 show that $P_k(y)$ must be orthogonal polynomials in y satisfying the recurrence relation

$$a_k P_{k+1}(y) + a_{k-1}(y) = y P_k(y)\,,$$

$$a_k = \left(\frac{[l+k+1][l+k+2]}{[2l+2k+2][2l+2k+4]}\right)^{1/2} ([k+1][2l+k+2])^{1/2}\,,$$

where $k = 0, 1, 2, \ldots$. Making the substitution

$$P_k(y) = P'_k(y)\left(\frac{(1-q^{2l+2k+2})(q^{2l+2};q)_k(q^{2l+3};q^2)_k^{-2}}{(1-q^{2l+2})(q;q)_k(q^{l+1};q)_k(q^{l+2};q)_k}\right)^{1/2},$$

we obtain the relation

$$\frac{1-q^{2l+k+2}}{(1-q^{2l+2k+3})(1-q^{2l+2k+2})}P'_{k+1}(y) + \frac{(1-q^{l+k})(1-q^{l+k+1})(1-q^k)}{(1-q^{2l+2k+2})(1-q^{2l+2k+1})^{-1}}P'_{k-1}(y)$$

$$= (q^{-1/2} - q^{1/2})yP'_k(y)\,. \tag{29}$$

Setting $\phi = \pi/2$, $a^2 = q^{l+1}$, $c^2 = q^{l+2}$, $2x = (q^{-1/2} - q^{1/2})y$ into (16) we obtain relation (29). This means that the polynomials $P_n(y)$ from (28) are of the form

$$P_n(y) = \left(\frac{(q^{2l+2};q)_n(q^{2l+3};q^2)_n^{-2}(1-q^{2n+2l+2})}{(q;q)_n(q^{l+1};q)_n(q^{l+2};q)_n(1-q^{2l+2})}\right)^{1/2}$$

$$\times\, p_n(\cos(\theta+\pi/2);\; q^{(l+1)/2}, q^{(l+2)/2}|q)\,, \tag{30}$$

where $p_n(x)$ are q-orthogonal polynomials from (15) and

$$y = 2\cos\left(\theta + \frac{\pi}{2}\right)(q^{-1/2} - q^{1/2})^{-1}\,.$$

In the same way as in the case of polynomials (18), we find that the orthogonality relation for polynomials (30) is of the form

$$\int_{-b}^{b} P_n(y)P_k(y)w(y)dy = \delta_{kn}\,, \tag{31}$$

where $b = 2(q^{-1/2} - q^{1/2})^{-1}$ and

$$w(y) = \frac{q^{-1/2} - q^{1/2}}{4\pi\cos\theta}\left|\frac{(q^{l+1};q)_\infty^{1/2}(q^{l+1};q)_{l+1}^{1/2}(q^{2l+3};q^2)_\infty(-e^{2i\theta};q)_\infty}{(q;q)_\infty^{-1/2}(e^{2i\theta}q^{l+1};q^2)_\infty(e^{2i\theta}q^{l+2};q^2)_\infty}\right|^2\,.$$

Formula (31) means that the operator $T_l^+(I_2)$ has a simple spectrum and this spectrum covers exactly the interval $(-b, b)$, $b = 2(q^{-1/2} - q^{1/2})^{-1}$. The spectral measure of the operator $T_l^+(I_2)$ is $w(y)dy$. Now we can formulate the following theorem:

Theorem 4. *If $p > 0$ then the operator $\overline{B_p}$ has the deficiency indices $(1,1)$ and all its selfadjoint extensions (without coming from the Hilbert space V_l) have discrete simple spectra with infinity as the only point of accumulation. If $p < 0$ then $\overline{B_p}$ is a trace class operator and has a discrete simple spectrum with zero as the only point of accumulation. The operator $\overline{B_0}$ has the continuous simple spectrum which covers the interval $(-b, b)$, $b = 2/(q^{-1/2} - q^{1/2})$.*

Using assertions of Sect. 2 we can formulate for the operators A_{pbr} the statement similar to Proposition 3. As in the case of the algebra $U_q(\mathrm{su}_{1,1})$, it is possible to find explicit form of polynomials $P_n(z)$ corresponding to the operators A_{pbr} for many particular values of p, b and r.

5. Representation Operators for the q-Deformed Lorentz Algebra $U_q(\mathrm{so}_{3,1})$

There exist several definitions of the quantum algebra $U_q(\mathrm{so}(n,\mathbf{C}))$. We use the definition which allows the reduction from $U_q(\mathrm{so}(n,\mathbf{C}))$ to $U_q(\mathrm{so}(n-1,\mathbf{C}))$ (see [21]) and does not coincide with the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ defined by Drinfeld [2] and Jimbo [3]. Namely, we use the q-deformed algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ which is the associative algebra generated by the elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}I_{i+1,i}^2 - (q^{1/2}+q^{-1/2})I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1} ,$$

$$I_{i,i-1}^2 I_{i+1,i} - (q^{1/2}+q^{-1/2})I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i} ,$$

$$[I_{i,i-1}, I_{j,j-1}] = 0, \quad |i-j| > 1 ,$$

where $[\,\cdot\,,\,\cdot\,]$ is the usual commutator. The elements $I_{i,i-1}$ are q-analogues of the standard elements $I_{i,i-1}$ of the Lie algebra $\mathrm{so}(n,\mathbf{C})$.

We introduce into $U_q(\mathrm{so}(n,\mathbf{C}))$ the involution defined uniquely by the relations $I_{i,i-1}^* = -I_{i,i-1}$, $i = 2,3,\ldots,n$, and obtain the compact q-deformed algebra $U_q(\mathrm{so}_n)$. The formulas $I_{i,i-1}^* = -I_{i,i-1}$, $i \neq r+1$, $I_{r+1,r}^* = I_{r+1,r}$ determine the q-deformed algebra $U_q(\mathrm{so}_{r,n-r})$. This algebra contains the subalgebras $U_q(\mathrm{so}_r)$ and $U_q(\mathrm{so}_{n-r})$. In particular, in this way we obtain the q-deformed algebras $U_q(\mathrm{so}_{n,1})$. We remark that the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$, defined here, can be embedded into $U_q(\mathrm{gl}(n,\mathbf{C}))$ and is important for construction of a q-analogue of the symmetric Riemannian space $U(n)/SO(n)$ [22].

As in the case of the algebra $U_q(\mathrm{so}(3,\mathbf{C}))$, we can define three additional elements of the algebra $U_q(\mathrm{so}(4,\mathbf{C}))$:

$$I_{31} = q^{1/4}I_{21}I_{32} - q^{-1/4}I_{32}I_{21} ,$$

$$I_{42} = q^{1/4}I_{32}I_{43} - q^{-1/4}I_{43}I_{32} ,$$

$$I_{41} = q^{1/4}I_{21}I_{42} - q^{-1/4}I_{42}I_{21} .$$

The elements I_{kr}, $1 \leqq r < k \leqq 4$, satisfy the relations of the type

$$[I_{21}, I_{32}]_q \equiv q^{1/4}I_{21}I_{32} - q^{-1/4}I_{32}I_{21} = I_{31} .$$

The involution in $U_q(\mathrm{so}(4,\mathbf{C}))$, defined uniquely by the formulas

$$I_{21}^* = -I_{21}, \qquad I_{32}^* = -I_{32}, \qquad I_{43}^* = I_{43} ,$$

determines the q-deformed Lorentz algebra $U_q(\mathrm{so}_{3,1})$. Irreducible representations [20] of $U_q(\mathrm{so}_{3,1})$ are given in a similar way as in the case of the Lorentz group $SO_0(3,1)$. These representations $T_{\sigma s}$ are defined by a complex number σ and by an integral or half-integral number s. In order to give these representations it is sufficient to have the operators $T_{\sigma s}(I_{21})$, $T_{\sigma s}(I_{32})$, $T_{\sigma s}(I_{43})$. Without loss of generality we may assume [20] that $s \geqq 0$. The representation $T_{\sigma s}$ acts on the Hilbert space V_s with the orthonormal basis

$$|l,m\rangle, \quad l = s, s+1, s+2, \ldots, \quad m = -l, -l+1, \ldots, l .$$

On the subspace V_{ls} spanned by the basis elements $|l,m\rangle$ with fixed l the irreducible representation T_l of the subalgebra $U_q(\mathrm{so}_3)$ acts. These representations of $U_q(\mathrm{so}_3)$

are described in [20]. The operator $T_{\sigma s}(I_{43})$ is given by the formula [20]

$$T_{\sigma s}(I_{43})|l, m\rangle = -\mathrm{i}\frac{[\sigma][s][m]}{[l][l+1]}|l, m\rangle + a_l|l+1, m\rangle + a_{l-1}|l-1, m\rangle \,,$$

$$a_{l-1} = ([l+\sigma][l-\sigma])^{1/2}\left(\frac{[s+l][l-s][l+m][l-m]}{[l]^2[2l-1][2l+1]}\right)^{1/2} . \tag{32}$$

There exist equivalence relations in the class of representations $T_{\sigma s}$. The following representations $T_{\sigma s}$ are unitary [20]:

(a) the representations $T_{\sigma s}$, $\sigma = \mathrm{i}\rho$, $\rho > 0$ (the principal unitary series);
(b) the representations $T_{\sigma s}$, $s = 0$, $0 < \sigma < 1$ (the supplementary series);
(c) the representations $T_{\sigma s}$, $\mathrm{Im}\ \sigma = \pi/h$, $\mathrm{Re}\ \sigma > 0$, where $q = \exp\ h$ (the strange series).

For all representations $T_{\sigma s}$, the set of matrix elements of the operator $T_{\sigma s}(I_{43})$ from (32) is bounded when l runs over values from s to ∞. Therefore, this operator is bounded for all these representations. For unitary representations the closure $L \equiv \overline{T_{\sigma s}(I_{43})}$ of $T_{\sigma s}(I_{43})$ is a selfadjoint operator.

Let us sketch how generalized eigenvectors

$$|x, m\rangle = \sum_{l=s}^{\infty} P_{l-s}(x)|l, m\rangle, \quad s \geqq m \,, \tag{33}$$

of the operator $L = \overline{T_{\sigma s}(I_{43})}$ corresponding to an eigenvalue x are evaluated. We remark that the condition $s \geqq m$ does not restrict a generality. In fact, it is seen from formula (32) that the matrix elements of the operator $T_{\sigma s}(I_{43})$ are symmetric with respect to permutation of s and m. If $s < m$ then we would consider the representation $T_{\sigma s}$ and permute s and m in (32).

The polynomials $P_{l-s}(z)$ from (33) satisfy the recurrence relation which is derived from formula (32). After the replacement

$$P_n(x) = \left(\frac{(-q^{s+1};q)_n^{-2}(q^{2s+2};q)_{n-1}(1-q^{2n+2s+1})(q;q)_n^{-1}}{(q^{\sigma+s+1};q)_n(q^{-\sigma+s+1};q)_n(q^{s+m+1};q)_n(q^{s-m+1};q)_n}\right)^{1/2} P_n'(x)$$

and some computation we obtain the recurrence relation for $P_n'(x)$ (we do not give it here). Comparing it with the recurrence relation (1.24) of the paper [11] for the Askey–Wilson polynomials, we derive that

$$P_n(x) = a^{-n}\left(\frac{(q^{\sigma+s+1};q)_n(q^{s+m+1};q)_n(q^{2s+1};q)_n(1-q^{2n+2s+1})}{(q;q)_n(q^{-\sigma+s+1};q)_n(q^{s-m+1};q)_n(1-q^{2s+1})}\right)^{1/2}$$

$$\times {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n+2s+1}, e^{\mathrm{i}\theta}a, e^{-\mathrm{i}\theta}a \\ q^{\sigma+s+1}, \quad q^{s+m+1}, \quad -q^{s+1} \end{matrix}; \; q, q\right) , \tag{34}$$

where $x = 2(q^{-1/2} - q^{1/2})^{-1}\cos\theta$ and $a = -\mathrm{i}q^{(\sigma+m+s+1)/2}$.

Using the orthogonality relation for Askey–Wilson polynomials (Theorem 2.2 in [11]) we derive that for polynomials (34):

$$\int_{-b}^{b} P_n(x)P_k(x)w(x)dx = \delta_{nk} \,, \tag{35}$$

where $b = 2(q^{-1/2} - q^{1/2})^{-1}$. The weight function $w(x)$ is given by the formula

$$w(x) = \frac{1}{4\pi} \frac{q^{-1/2} - q^{1/2}}{\sin\theta} \frac{h(x;1)h(x;-1)h(x;q^{1/2})h(x;-q^{1/2})}{h(x;a)h(x;b)h(x;c)h(x;d)}$$
$$\times (q;q)_{2s+1}(q^{\sigma+s+1};q)_\infty(q^{-\sigma+s+1};q)_\infty$$
$$\times (q^{s+m+1};q)_\infty(q^{s-m+1};q)_\infty(-q^{s+1};q)^2_\infty ,$$

where $x = 2(q^{-1/2} - q^{1/2})^{-1}\cos\theta$ and

$$a = -\mathrm{i}q^{(\sigma+m+s+1)/2}, \quad b - \mathrm{i}q^{(\sigma-m+s+1)/2}, \quad c = \mathrm{i}q^{(m-\sigma+s+1)/2}, \quad d = -\mathrm{i}q^{(s-\sigma-m+1)/2} ,$$

$$h(x;a) = (ae^{\mathrm{i}\theta};q)_\infty(ae^{-\mathrm{i}\theta};q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k(q^{-1/2} - q^{1/2})x + a^2q^{2k}) .$$

The space V_s of the representation $T_{\sigma s}$ can be decomposed into the orthogonal sum

$$V_s = \sum_{m=-\infty}^{\infty} \oplus V_{sm} ,$$

where V_{sm} is spanned by the basis vectors $|l,m\rangle$, $l = |m|, |m|+1, |m|+2, \ldots$. It follows from formula (35) that the operator $T_{\sigma s}(I_{43})$ has a simple spectrum on each subspace V_{sm} and this spectrum covers exactly the interval $(-b,b)$, $b = 2/(q^{-1/2} - q^{1/2})$. The spectral measure of $T_{\sigma s}(I_{43})$ on V_{sm} is determined by the measure $w(x)dx$ from (35). The spectrum of the operator $T_{\sigma s}(I_{43})$ on the space V_s is the same, however now spectral points are of infinite multiplicity.

6. Representation Operators for the Algebra $U_q(\mathrm{so}_n)$

We consider irreducible finite dimensional representations of the q-deformed algebra $U_q(\mathrm{so}_n)$ which are of class 1 with respect to $U_q(\mathrm{so}_{n-1})$. As in the case of the Lie group $SO(n)$, these representations T_l of $U_q(\mathrm{so}_n)$ are given by a positive integer l and act on the same spaces V_l with the same orthonormal bases as the representations T_l of the Lie group $SO(n)$ (see Sect. 9 in [4]). The basis elements are labelled by

$$|l,m,j,k,\ldots,r\rangle, \quad l \geqq m \geqq j \geqq k \geqq \cdots \geqq |r| , \tag{36}$$

where $j,m,k,\ldots,r$ are integers. The operator $T_l(I_{n,n-1})$ acts upon these basis vectors as [20]

$$T_l(I_{n,n-1})|m,j,\ldots,r\rangle = b_m|m+1,j,\ldots,r\rangle + b_{m-1}|m-1,j,\ldots,r\rangle , \tag{37}$$

$$b_m = \mathrm{i}\left(\frac{[l+m+n-2][m+j+n-3][m-j+1]}{[l-m]^{-1}[2m+n-1][2m+n-3]}\right)^{1/2} .$$

As in the classical case, the operators $T_l(I_{i,i-1})$, $i < n$, act by the same formula upon the corresponding parts of the pattern $|m,j,\ldots,r\rangle$.

The space V_l can be decomposed into the orthogonal sum

$$V_l = \sum_M \oplus V_M, \quad M = (j,\ldots,r) .$$

The operator $T_l(I_{n,n-1})$ leaves the subspaces V_M invariant. The operator $L_l \equiv T_l(-\mathrm{i}I_{n,n-1})$ is selfadjoint. We shall evaluate the spectrum and eigenvectors

$$|x,j,\dots,r\rangle = \sum_{m=j}^{l} P_{m-j}(x)|m,j,\dots,r\rangle \tag{38}$$

of the operator L_l on the subspace V_M. We suppose that

$$L_l|x,j,\dots,r\rangle = [x]|x,j,\dots,r\rangle .$$

The vector (38) is an eigenvector for the operator L_l with the eigenvalue $[x]$ if the polynomials $P_{l-j}(x)$ satisfy the recurrence relation

$$a_k P_{k+1}(x) + a_{k-1}P_{k-1}(x) = [x]P_k(x)\,, \tag{39}$$

$$a_k = ([N+k+n'-1][N-k])^{1/2}\left(\frac{[k+n'-2][k+1]}{[2k+n'][2k+n'-2]}\right)^{1/2},$$

where $l-j=N$, $m-j=k$, $n+2j-1=n'$. We multiply both sides of (39) by $-q^{-N/2}$ and set

$$P_k(x) = \left(\frac{[k+n'-3]![2k+n'-2]}{[N-k]![N+k+n'-2]![k]!}\right)^{1/2} P'_k(x)\,.$$

After some transformations we obtain the relation

$$-q^{-N/2}(q^{-1/2}-q^{1/2})[x]P'_k(x) = \frac{(1-q^{k-N})(1-q^{k+n'-2})}{1-q^{2k+n'-2}}P'_{k+1}(x)$$
$$-\,q^{-N}\frac{(1-q^{N+k+n'-2})(1-q^k)}{1-q^{2k+n'-2}}P'_{k-1}(x)\,.$$

Comparing this relation with the recurrence relation (7.2.1) in [12] for the q-Racah polynomials

$$R_n(\mu(y);\alpha,\beta,\gamma,\delta|q) = {}_4\varphi_3\left(\begin{matrix} q^{-y}, q^{y-1}\gamma\delta, q^{-n}, q^{n+1}\alpha\beta \\ \alpha q, \beta\delta q, \gamma q\end{matrix}\;;q,q\right)$$

where $\mu(y) = q^{-y} + q^{y+1}\gamma\delta$ and one of the numbers αq, $\beta\delta q$, γq in the function ${}_4\varphi_3$ is equal to q^{-N}, $N \in \mathbf{Z}_+$, we see that

$$P'_k(x) = R_k(\mu(y);\alpha,\beta,\gamma,\delta|q)\,,$$

where

$$\alpha=\beta=-q^{(n'-3)/2},\quad \gamma=q^{(n'-3)/2},\quad \delta=-q^{-N-(n'-1)/2}\,,$$

$$[x] = q^{N/2}(q^{1/2}-q^{-1/2})^{-1}\mu(y),\quad \mu(y) = q^{-y}+\gamma\delta y^{y+1} = q^{-y}-q^{y-N}$$

Thus, the solution of recurrence relation (39) normalized by the condition $P_0(x)\equiv 1$ is of the form

$$P_k(x) = \left(\frac{[N]![N+n'-2]![2k+n'-2][k+n'-3]!}{[n'-2]![k]![N-k]![N+k+n'-2]!}\right)^{1/2}$$
$$\times R_k(\mu(y);-q^{(n'-3)/2},-q^{(n'-3)/2},q^{(n'-3)/2},-q^{-N-(n'-1)/2}|q)\,, \tag{40}$$

where $x = N - 2y$. It follows from the orthogonality relation for q-Racah polynomials (formula (7.2.18) in [12]) that y runs over the values $0, 1, 2, \ldots, N$. Therefore, the spectrum of the operator L_l on the subspace V_M consists of the points

$$[-N], [-N+2], [-N+4], \ldots, [N], \quad N = l - j\,.$$

The corresponding eigenvectors are determined by formulas (38) and (40). The orthogonality relation for the polynomials $P_n(x)$ are of the form

$$\sum_{y=0}^{N} P_n(x) P_m(x) W(x) = \delta_{mn}\,, \tag{41}$$

$$W(x) = \frac{1}{2}\,\frac{[N]![n'-2]!![2N-2y+n'-3]!![2y+n'-3]!![4y-2N]}{[N+n'-2]![n'-3]!![2N-2y]!![2y]!![2y-N]}\,,$$

where $[s]!! = [s][s-2][s-4]\cdots[1]$ (or $[0]$).

Joining spectra of the operator L_l on all subspaces V_M, we obtain the spectrum of L_l on the carrier space V_l of the representation T_l.

7. Representations of the Algebra $U_q(\mathrm{so}_{r,2})$

The aim of this section is to show how results on eigenvectors and overlap coefficients can be used for obtaining new results in representation theory of q-deformed algebras.

In the previous section we constructed the new orthonormal basis $\{|x, j, \ldots, r\rangle\}$ of the carrier space V_l of the representation T_l of the algebra $U_q(\mathrm{so}_n)$. Clearly, the vectors

$$|x, j, \ldots, r\rangle = \sum_{m=j}^{l} P''_{m-j}(x)|m, j, \ldots, r\rangle\,, \tag{42}$$

where $P''_{m-j}(x) = W^{1/2}(x)P_{m-j}(x)$, are orthogonal. These vectors differ from vectors (38) by the multiplier $W^{1/2}(x)$ determined by formula (41). We derived in the previous section that the operator $T_l(I_{n,n-1})$ acts upon the basis $\{|x, j, \ldots, r\rangle\}$ as

$$T_l(I_{n,n-1})|x, j, \ldots, r\rangle = \mathrm{i}[x]|x, j, \ldots, r\rangle\,.$$

The operators $T_l(I_{i,i-1})$, $i = n-2, n-3, \ldots, 2$, act upon this basis by the same formulas as upon the basis $\{|m, j, \ldots, r\rangle\}$. Thus, to determine the representation T_l in the basis $\{|x, j, \ldots, r\rangle\}$ we have to find how the operator $T_l(I_{n-1,n-2})$ acts upon this basis. Due to formulas (37) and (42),

$$\begin{aligned} T_l(I_{n,n-1})|x, j, \ldots, r\rangle = &\sum_{m=j}^{l} P''_{m-j}(x) b'_j |m, j+1, \ldots, r\rangle \\ &+ \sum_{m=j}^{l} P''_{m-j}(x) b'_{j-1} |m, j-1, \ldots, r\rangle\,, \end{aligned} \tag{43}$$

where b'_j are coefficients b_m from formula (37) taken for the subalgebra $U_q(\mathrm{so}_{n-1})$. Now it is necessary to go from the basis elements $\{|m, j \pm 1, \ldots, r\rangle\}$ to basis elements (42). To fulfill this transition we apply to the basic hypergeometric function ${}_4\varphi_3$ contained in the expression for $P''_{m-j}(x)$ the recurrence formula (7.2.13) from [12]. As a result, we express $P''_{m-j}(x)$ as the linear combination of $P''_{m-j-1}(x+1)$ and $P''_{m-j-1}(x-1)$. We substitute this expression for $P''_{m-j}(x)$ into the first sum on the right-hand side of (43). It turns out that in this sum we can separate two sums (42) in which j is replaced by $j+1$ and x is replaced respectively by $x+1$ and $x-1$. Thus, the first sum on the right-hand side of (43) is a linear combination of the vectors $|x \pm 1, j+1, \ldots, r\rangle$ with certain coefficients. These coefficients are evaluated under fulfillment of the procedure just described. Now we fulfill the same procedure with the second sum of the right-hand side of (43) using formula (7.3.12) (instead of formula (7.3.13)) from [12]. As a result, this sum is expressed as a linear combination of the vectors $|x \pm 1, j-1, \ldots, r\rangle$ with certain coefficients. This leads to the following formula for the operator $T_l(I_{n-1,n-2})$:

$$
\begin{aligned}
&T_l(I_{n-1,n-2})|x,j,k,\ldots,r\rangle \\
&\quad = \mathrm{i}K_jL_x([l-j-x][l+j+x+n-2])^{1/2}|x+1,j+1,k,\ldots,r\rangle \\
&\qquad - \mathrm{i}K_jL_{x-1}([l-j+x][l+j-x+n-2])^{1/2}|x-1,j+1,k,\ldots,r\rangle \\
&\qquad - \mathrm{i}K_{j-1}L_x([l-j+x+2][l+j-x+n-4])^{1/2}|x+1,j-1,k,\ldots,r\rangle \\
&\qquad + \mathrm{i}K_{j-1}L_{x-1}([l-j-x+2][l+j+x+n-4])^{1/2}|x-1,j-1,k,\ldots,r\rangle\,,
\end{aligned}
$$

where

$$
K_j = \left(\frac{[j+k+n-4][j-k+1]}{[2j+n-2][2j+n-4]}\right)^{1/2}, \qquad L_x = \left(\frac{[x][x+1]}{[2x][2x+2]}\right)^{1/2}. \tag{44}
$$

Thus, we obtain formulas of actions of operators of the representations T_l upon the basis $\{|x,j,k,\ldots,r\rangle\}$. This basis differs from the Gel'fand–Tsetlin basis $\{|m,j,k,\ldots,r\rangle\}$ and corresponds to the reduction from $U_q(\mathrm{so}_n)$ onto the subalgebra $U_q(\mathrm{so}_2+\mathrm{so}_{n-2}) \equiv U(\mathrm{so}_2)\times U_q(\mathrm{so}_{n-2})$.

As in the case of representations of compact and noncompact real Lie groups, by making use of an analytical continuation in the parameter giving representations we can obtain infinite dimensional representations of the q-deformed algebra $U_q(\mathrm{so}_{n-2,2})$ from the representations T_l of $U_q(\mathrm{so}_n)$. In this way, we obtain the representations $T_{\sigma,\varepsilon}, \sigma \in \mathbf{C},\ \varepsilon \in \{0,1\}$, of $U_q(\mathrm{so}_{n-2,2})$ which act on the Hilbert spaces H_ε with the orthonormal basis

$$
|x,j,k,\ldots,r\rangle, \quad x+j \equiv \varepsilon (\mathrm{mod}\, 2), \quad j \geqq k \geqq \cdots \geqq |r|\,, \tag{45}
$$

where x runs over integers and j runs over nonnegative integers. The operators $T_{\sigma,\varepsilon}(I_{n,n-1})$ and $T_{\sigma,\varepsilon}(I_{i,i-1}), i=2,3,\ldots,n-2$ (these operators correspond to elements from the sub-algebra $U_q(\mathrm{so}_2+\mathrm{so}_{n-2})$), act upon basis vectors (45) by the same formulas as in the case of the representations T_l, and for the operator $T_{\sigma,\varepsilon}(I_{i,i-1})$ we have

$$
\begin{aligned}
&T_{\sigma,\varepsilon}(I_{n-1,n-2})|x,j,k,\ldots,r\rangle \\
&\quad = K_jL_x([\sigma+j+x][-\sigma+j+x+n-2])^{1/2}|x+1,j+1,k,\ldots,r\rangle \\
&\qquad + K_jL_{x-1}([\sigma+j-x][-\sigma+j-x+n-2])^{1/2}|x-1,j+1,k,\ldots,r\rangle \\
&\qquad - K_{j-1}L_x([\sigma+j-x-2][-\sigma+j-x+n-4])^{1/2}|x+1,j-1,k,\ldots,r\rangle \\
&\qquad - K_{j-1}L_{x-1}([\sigma+j+x-2][-\sigma+j+x+n-4])^{1/2}|x-1,j-1,k,\ldots,r\rangle\,,
\end{aligned} \tag{46}
$$

where K_j and L_x are given by formulas (44). Substituting these expressions for the operators $T_{\sigma,\varepsilon}(I_{i,i-1})$ into the defining relations for the algebra $U_q(\mathrm{so}(n,\mathbf{C}))$ from Sect. 5 we make sure that they really determine a representation of $U_q(\mathrm{so}_{n-2,2})$.

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