Commun. Math. Phys. 175, 63–88 (1996)

Communications in Mathematical Physics

Functional Equation for Dynamical Zeta Functions of Milnor–Thurston Type

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Received: 8 April 1994

Abstract: A Milnor–Thurston type dynamical zeta function $\zeta_L(Z)$ is associated with a family of maps of the interval $(-1, 1)$. Changing the direction of time produces a new zeta function $\zeta'_L(Z)$. These zeta functions satisfy a functional equation $\zeta_L(Z)\zeta'_L(\varepsilon Z) = \zeta_0(Z)$ (where ε amounts to sign changes and, generically, $\zeta_0 \equiv 1$). The functional equation has non-trivial implications for the analytic properties of $\zeta_L(Z)$.

0. Introduction

Milnor and Thurston [2] have shown how the zeta function $\zeta(z)$ counting the periodic points of a piecewise monotone interval map f could be expressed in terms of a *kneading determinant* $D(z)$. The zeta function considered by Milnor and Thurston is closely related to the Lefschetz zeta function ζ_L, which we shall use henceforth. Baladi and Ruelle [1] have shown how to replace z in the Milnor–Thurston formula by $Z = (z_1, \ldots, z_N)$, where the interval of definition of f is cut into subintervals with different weights z_i. We shall here use a further extension of the formula $\zeta_L(Z) = D(Z)$, where f is allowed to be multivalued. The inverse f^{-1} of f is again multivalued piecewise monotone; it is associated with a zeta function $\zeta'_L(Z)$. There is a natural relation (*functional equation*)

$$\zeta_L(Z)\zeta'_L(\varepsilon Z) = \zeta_0(Z)\,,$$

where ε corresponds to some sign changes and $\zeta_0(Z)$ counts "exceptional" orbits (generically $\zeta_0(Z) = 1$). The analytic properties of $\zeta_L(Z)$ are related, via the kneading determinant $D(Z)$, to the spectral properties of a *transfer operator* $\mathscr{M}_Z$. The spectral properties needed here are a refinement of those proved in Ruelle [4]. Using these properties one shows that ζ_L is meromorphic in a certain domain, with poles only if 1 is an eigenvalue of $\mathscr{M}_Z$. Let $\mathscr{M}'_Z$ denote the transfer operator corresponding to f^{-1}; using the functional equation one shows that ζ_L can vanish only if 1 is an eigenvalue of $\mathscr{M}'_Z$.

In what follows we shall write ζ instead of ζ_L, and use a family (ψ_ω) of monotone maps, instead of the multivalued map f^{-1}. *Warning*: If the ψ_ω are the branches of the inverse of a function f, the zeta function of [1] is here denoted by $1/\zeta(\varepsilon Z)$, and the kneading determinant by $\widehat{D}(Z)$ (see Sect. 1.10) rather than $D(Z)$.

I wish to thank Viviane Baladi who carefully read the present paper, and suggested a number of improvements which have been incorporated in the manuscript.

1. Definitions and Statement of Results

1.1. Lefschetz Numbers. We shall use the notation

$$\operatorname{sgn} x = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases} \qquad \operatorname{del} x = \begin{cases} +1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}.$$

Let $a < b$, and $\psi : (a,b) \mapsto \mathbb{R}$ be continuous and strictly monotone. We let $\varepsilon = +1$ if ψ is increasing, -1 if ψ is decreasing, and we define the *Lefschetz number* $L(\psi)$ by

$$L(\psi) = L_1(\psi) + L_0(\psi),$$

$$L_1(\psi) = \frac{1}{2}[\operatorname{sgn}(\bar{\psi}(a) - a) - \operatorname{sgn}(\bar{\psi}(b) - b)],$$

$$L_0(\psi) = \frac{\varepsilon}{2}[\operatorname{del}(\bar{\psi}(a) - a) + \operatorname{del}(\bar{\psi}(b) - b)],$$

where $\bar{\psi}$ denotes the extension of ψ by continuity to $[a,b]$, so that $\bar{\psi}(a) = \lim_{x\downarrow a}\psi(a)$, $\bar{\psi}(b) = \lim_{x\uparrow b}\psi(x)$.

Therefore, when $\varepsilon = +1$ we have

$$L(\psi) = \begin{cases} 1 & \text{if } \bar{\psi}(a) \geqq a \text{ and } \bar{\psi}(b) \leqq b \\ -1 & \text{if } \bar{\psi}(a) < a \text{ and } \bar{\psi}(b) > b \end{cases},$$

when $\varepsilon = -1$ we have

$$L(\psi) = 1 \quad \text{if } \bar{\psi}(a) > a \quad \text{and} \quad \bar{\psi}(b) < b,$$

and in all other cases we have

$$L(\psi) = 0.$$

Let $\operatorname{Fix}\psi = \{x \in (a,b) \colon \psi x = x\}$. If $\operatorname{Fix}\psi$ is finite and $x \in \operatorname{Fix}\psi$ we write

$$L(x,\psi) = \frac{1}{2}\left[\lim_{y\uparrow x}\operatorname{sgn}(\psi(y) - y) - \lim_{y\downarrow x}\operatorname{sgn}(\psi(y) - y)\right].$$

Lemma (Properties of Lefschetz numbers). (a) *If a C^0-small perturbation $\tilde{\psi}$ of ψ shrinks the range (i.e., $\tilde{\psi}(a,b) \subset \psi(a,b)$) then it preserves the Lefschetz number (i.e., $L(\tilde{\psi}) = L(\psi)$).*

(b) *Consider ψ^{-1} defined on the open interval $\psi(a,b)$, then*

$$L_1(\psi^{-1}) = -\varepsilon L_1(\psi),$$

$$L_0(\psi^{-1}) = L_0(\psi).$$

(c) *Let* $\operatorname{Fix}\psi$ *be finite and $\tilde{\psi}(a) \neq a$, $\tilde{\psi}(b) \neq b$. Then*

$$L(\psi) = \sum_{x\in\operatorname{Fix}\psi} L(x,\psi).$$

Part (a) of the lemma follows from the list given above of cases when $L(\psi) = 1, -1$, or 0. Part (b) results directly from the definitions. To prove (c) notice that by assumption

$$L(\psi) = L_1(\psi) = \frac{1}{2}[\operatorname{sgn}(\bar{\psi}(a) - a) - \operatorname{sgn}(\bar{\psi}(b) - b)]$$

$$= \frac{1}{2}\left[\lim_{y \downarrow a} \operatorname{sgn}(\psi(y) - y) - \lim_{y \uparrow b} \operatorname{sgn}(\psi(y) - y)\right] = \sum_{x \in \operatorname{Fix} \psi} L(x, \psi) .$$

This concludes the proof. □

1.2. Zeta Functions. Let $(J_\omega), (\psi_\omega), (\varepsilon_\omega), (z_\omega)$ be families indexed by $\omega \in \{1, \dots, N\}$, where $J_\omega = (u_\omega, v_\omega)$ is a nonempty bounded interval of $\mathbb{R}$; $\psi_\omega : J_\omega \to \mathbb{R}$ is a strictly monotone continuous map; $\varepsilon_\omega = +1$ or -1 depending on whether ψ_ω is increasing or decreasing; and $z_\omega \in \mathbb{C}$. We write $Z = (z_\omega)$, $\varepsilon Z = (\varepsilon_\omega z_\omega)$.

It will be convenient to assume henceforth that all J_ω and $\psi_\omega J_\omega$ are contained in $(-1, +1)$; this is no restriction of generality since $\mathbb{R}$ can be mapped homeomorphically on $(-1, +1)$.

If $m \geqq 1$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in \{1, \dots, N\}^m$, we write $|\boldsymbol{\omega}| = m$, $\varepsilon(\boldsymbol{\omega}) = \prod_{k=1}^m \varepsilon_{\omega_k}$, $Z(\boldsymbol{\omega}) = \prod_{k=1}^m z_{\omega_k}$. We also let $\psi_{\boldsymbol{\omega}} : J_{\boldsymbol{\omega}} \to \mathbb{R}$ be defined by $\psi_{\boldsymbol{\omega}} = \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}$, on

$$J_{\boldsymbol{\omega}} = J_{\omega_1} \cap \psi_{\omega_1}^{-1}(J_{\omega_2} \cap \psi_{\omega_2}^{-1}(\cdots \psi_{\omega_{m-1}}^{-1} J_{\omega_m} \cdots)) .$$

If $J_{\boldsymbol{\omega}} \neq \emptyset$, we write $J_{\boldsymbol{\omega}} = (u_{\boldsymbol{\omega}}, v_{\boldsymbol{\omega}})$.

The *Lefschetz zeta function* (associated with the data $(J_\omega), (\psi_\omega)$) is the formal power series

$$\zeta(Z) = \exp \sum_{\boldsymbol{\omega}} \frac{1}{|\boldsymbol{\omega}|} L(\psi_{\boldsymbol{\omega}}) Z(\boldsymbol{\omega}) ,$$

where the sum is restricted to those $\boldsymbol{\omega}$ for which $J_{\boldsymbol{\omega}} \neq \emptyset$ (or one defines $L(\psi_{\boldsymbol{\omega}}) = 0$ when $J_{\boldsymbol{\omega}} = \emptyset$). One can write a product formula for $\zeta(Z)$ (see Appendix A) and check that $\zeta(Z), 1/\zeta(Z) \in \mathbb{Z}[[z_1, \dots, z_N]]$ (Lemma A.2). The zeta function associated with the data $(\psi_\omega J_\omega)$, (ψ_ω^{-1}) is

$$\zeta'(Z) = \exp \sum_{\boldsymbol{\omega}} \frac{1}{|\boldsymbol{\omega}|} L(\psi_{\boldsymbol{\omega}}^{-1}) Z(\boldsymbol{\omega}) ,$$

and we write

$$\widehat{\zeta}(Z) = \zeta'(\varepsilon Z) .$$

We shall also need the function

$$\zeta_0(Z) = \exp \sum_{\boldsymbol{\omega}} \frac{1}{|\boldsymbol{\omega}|} L_0(\psi_{\boldsymbol{\omega}})(1 + \varepsilon(\boldsymbol{\omega})) Z(\boldsymbol{\omega})$$

$$= \exp \sum_{\boldsymbol{\omega} : \varepsilon(\boldsymbol{\omega}) = 1} \frac{1}{|\boldsymbol{\omega}|} [\operatorname{del}(\bar{\psi}_{\boldsymbol{\omega}}(u_{\boldsymbol{\omega}}) - u_{\boldsymbol{\omega}}) + \operatorname{del}(\bar{\psi}_{\boldsymbol{\omega}}(v_{\boldsymbol{\omega}}) - v_{\boldsymbol{\omega}})] Z(\boldsymbol{\omega}) .$$

1.3. Transfer Operators and Kneading Determinant. We introduce the (generalized) *transfer operator* $\mathcal{M} = \mathcal{M}_Z$, the formal adjoint $\mathcal{M}' = \mathcal{M}'_Z$, and the associated operator $\widehat{\mathcal{M}}$ such that

$$\mathcal{M}\Phi(x) = \sum_{\omega} z_\omega \chi_\omega(x)\Phi(\psi_\omega x),$$

$$\mathcal{M}'\Phi(x) = \sum_{\omega} z_\omega \chi'_\omega(x)\Phi(\psi_\omega^{-1} x),$$

$$\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_Z = \mathcal{M}'_{\varepsilon Z},$$

where χ_ω is the characteristic function of J_ω and χ'_ω the characteristic function of $\psi_\omega J_\omega$. These operators act on the Banach space $\mathcal{B}$ of functions of bounded variation $\mathbb{R} \to \mathbb{C}$. It is also convenient to consider them as acting on the Banach space of bounded functions $\mathbb{R} \to \mathbb{C}$ (with the uniform norm $\|\cdot\|_0$).

We define $R = R(Z)$, $R' = R'(Z)$ and $\widehat{R}$ by

$$R = \lim_{m\to\infty} (\|\mathcal{M}^m\|_0)^{1/m},$$

$$R' = \lim_{m\to\infty} (\|\mathcal{M}'^m\|_0)^{1/m},$$

$$\widehat{R} = \widehat{R}(Z) = R'(\varepsilon Z).$$

The submultiplicativity of $m \mapsto \|\mathcal{M}^m\|_0, \|\mathcal{M}'^m\|_0$ guarantees the existence of the limits; R, R' and $\widehat{R}$ are in fact the spectral radii of $\mathcal{M}, \mathcal{M}'$ and $\widehat{\mathcal{M}}$ acting on bounded functions $\mathbb{R} \to \mathbb{C}$. In general $R \neq \widehat{R}$.

Let $\{a_1, \ldots, a_L\}$ contain the set of all endpoints u_ω, v_ω of the intervals J_ω, and assume that $a_1 < \cdots < a_L$. We define $\alpha_i \in \mathcal{B}$ by

$$\alpha_i(x) = \operatorname{sgn}(x - a_i)$$

for $i = 1, \ldots, L$, and write

$$D_{ij}^{(m)+} = \lim_{x\downarrow a_i} \sum_{\omega : u_\omega = a_i} z_\omega \cdot [(\mathcal{M}^{m-1}\alpha_j)(\psi_\omega x)],$$

$$D_{ij}^{(m)-} = \lim_{x\uparrow a_i} \sum_{\omega : v_\omega = a_i} z_\omega \cdot [(\mathcal{M}^{m-1}\alpha_j)(\psi_\omega x)].$$

The elements of the $L \times L$ *kneading matrix* $[D_{ij}]$ are then defined by

$$D_{ij}(Z) = \delta_{ij} + \sum_{m=1}^{\infty} \frac{1}{2} [D_{ij}^{(m)+} - D_{ij}^{(m)-}]$$

(this is an extension of the concept of kneading matrix introduced by Milnor and Thurston [2]). The determinant

$$D(Z) = \det[D_{ij}(Z)] \in \mathbb{Q}[[z_1, \ldots, z_N]]$$

is called *kneading determinant.*

1.4. Theorem A. *We have identically*

$$\zeta(Z) = D(Z).$$

This will be proved using a *homotopy argument* similar to the one used originally by Milnor and Thurston [2], and then by Baladi and Ruelle [1] in an analogous situation. This means that (for fixed families $(J_\omega), (\varepsilon_\omega), (z_\omega)$) first the formula $\zeta = D$

is checked for a special choice ψ^0 of ψ. Then, for a suitable one-parameter family (ψ^λ) with $\psi^1 = \psi$, one verifies that ζ and D are multiplied by the same factor at each bifurcation. The proof presented here is similar to that of [1], but with significant differences; we defer it to Appendix A. □

1.5. Theorem B. (a) *The spectral radius of* $\mathscr{M}$, *acting on* $\mathscr{B}$, *is* $\leqq \max(R, \widehat{R})$.

(b) *The essential spectral radius of* $\mathscr{M}$ *is* $\leqq \widehat{R}$.

This is closely related to the results of Ruelle [4] but, again, with significant differences. In Appendix B we give an improved version of the theorem of [4], which will yield Theorem B as a special case. □

1.6. Theorem C. (a) *We have identically* $\zeta(Z) \cdot \widehat{\zeta}(Z) = \zeta_0(Z)$.

(b) $\zeta(Z)$ *is holomorphic when* $R(Z) < 1$.

(c) $\zeta(Z)$ *is meromorphic when* $\widehat{R}(Z) < 1$, *with poles only when* $1 \in \operatorname{spectrum} \mathscr{M}_Z$.

(d) $\zeta_0(Z)$ *is holomorphic when* $\min\{R(Z), \widehat{R}(Z)\} < 1$.

This is proved in Sect. 2, and some strengthening of the theorem is provided by the four remarks below. □

1.7. Remark. Sharpening of Theorem C. Define

$$\mathscr{B}_\infty = \{A \in \mathscr{B}\colon \{x\colon A(x) \neq 0\} \text{ is countable}\}$$

and let

$$\mathscr{B}^\# = \mathscr{B}/\mathscr{B}_\infty$$

be the quotient Banach space. If $\Phi \in \mathscr{B}$, we may define $\Phi^\#$ by

$$\Phi^\#(x) = \frac{1}{2}\left[\lim_{y\downarrow x} \Phi(y) + \lim_{y\uparrow x} \Phi(y)\right] .$$

We have then the properties

$$\Phi = \Phi^\# + \Phi_\infty, \qquad \Phi_\infty \in \mathscr{B}_\infty ,$$

$$\Phi^\#(x) = \frac{1}{2}\left[\lim_{y\downarrow x} \Phi^\#(y) + \lim_{y\uparrow x} \Phi^\#(y)\right] .$$

If $\|[\Phi]\|^\#$ denotes the norm of the class of Φ in $\mathscr{B}/\mathscr{B}_\infty = \mathscr{B}^\#$ we have

$$\|[\Phi]\|^\# = \|\Phi^\#\|_{\mathscr{B}} .$$

Using $\|\cdot\|_0^\#$ to denote the "sup norm up to a countable set" we see that $\|\cdot\|_0^\#$ is defined on $\mathscr{B}^\#$ and that

$$\|[\Phi]\|_0^\# = \|\Phi^\#\|_0 .$$

Since $\mathscr{B}_\infty$ is stable under $\mathscr{M}, \widehat{\mathscr{M}}$ we may, by going to the quotient, define operators $\mathscr{M}^\#$, $\widehat{\mathscr{M}}^\#$ on $\mathscr{B}^\#$. We also use the notation $\mathscr{M}^\#$, $\widehat{\mathscr{M}}^\#$ for $\mathscr{M}$. $\widehat{\mathscr{M}}$ acting on bounded fucntions up to a countable set. We may then write

$$R^\# = R^\#(Z) = \lim_{m\to\infty} (\|\mathscr{M}^{\# m}\|_0^\#)^{1/m} ,$$

$$\widehat{R}^\# = \widehat{R}^\#(Z) = \lim_{m\to\infty} (\|\widehat{\mathscr{M}}^{\# m}\|_0^\#)^{1/m} .$$

The point of the above definitions is that in defining the kneading matrix $[D_{ij}]$ we may neglect countable sets, i.e., use the operator $\mathscr{M}^{\#}$ instead of $\mathscr{M}$. *As a consequence of this we may replace* $\mathscr{M}, \widehat{\mathscr{M}}, R, \widehat{R}$ *by* $\mathscr{M}^{\#}, \widehat{\mathscr{M}}^{\#}, R^{\#}, \widehat{R}^{\#}$ *in the statement of Theorem* C (b), (c), (d). We shall not give an explicit demonstration of the results thus obtained, but note that they follow by inspection of the proofs in Appendix B (#-version of Theorem B. 1) and Sect. 2. The basic fact is that the continuous linear functionals on $\mathscr{B}$ defined by $\Phi \mapsto \lim_{x\downarrow a}, \lim_{x\uparrow a} \Phi(x)$ yield continuous linear functionals on $\mathscr{B}^{\#}$ (while $\Phi \mapsto \Phi(a)$ is not defined on $\mathscr{B}^{\#}$).

The set $Z_m = X_m \cup Y_m$, with $X_m = \{u_\omega, v_\omega\colon |\omega| = m\}$, $Y_m = \{x\colon \psi_\omega(x) = \psi_{\omega'}(x)$ with $|\omega| = |\omega'| = m$ and $\varepsilon(\omega') = -\varepsilon(\omega)\}$ is finite. Given $x \notin Z_m$ there is $\delta > 0$ such that for each bounded Φ we may construct Φ_ε with $\|\Phi_\varepsilon\|_0 = \|\Phi\|_0$, and $\Phi_\varepsilon(\psi_\omega y) = \varepsilon(\omega)\Phi(\psi_\omega y)$ when $|y - x| < \delta$ and $|\omega| = m$. We have then

$$(\mathscr{M}^m_{\varepsilon Z}\Phi)(y) = (\mathscr{M}^m_Z \Phi_\varepsilon)(y) \quad \text{if } |y - x| < \delta\,,$$

hence

$$\|\mathscr{M}^{\# m}_{\varepsilon Z}\|^{\#}_0 \leqq \|\mathscr{M}^{\# m}_Z\|^{\#}_0\,,$$

hence by symmetry

$$\|\mathscr{M}^{\# m}_{\varepsilon Z}\|^{\#}_0 = \|\mathscr{M}^{\# m}_Z\|^{\#}_0\,,$$

and therefore

$$R^{\#}(\varepsilon Z) = R^{\#}(Z)\,.$$

Since $R^{\#}(Z) \leqq R(Z)$ we also have

$$R^{\#}(Z) \leqq \min\{R(Z), R(\varepsilon Z)\}\,.$$

Notice that Theorem B(b) can also be sharpened as follows: *the essential spectral radius of* $\mathscr{M}$ *is* $\leqq \widehat{R}^{\#}$. To prove this it suffices to find $\widetilde{K}_m$ of finite rank such that

$$\limsup_{m\to\infty} \|\mathscr{M}^m - \widetilde{K}_m\|^{1/m} \leqq \widehat{R}^{\#}\,. \tag{$*$}$$

We write as above $\Phi = \Phi^{\#} + \Phi_\infty$, so that

$$\|\Phi^{\#}\|_{\mathscr{B}} = \|[\Phi]\|^{\#} \quad \text{and} \quad \|\Phi_\infty\|_{\mathscr{B}} \leqq \|\Phi\|_{\mathscr{B}}\,.$$

Let χ be the characteristic function of $\bigcup_\omega\{\bar{\psi}_\omega u_\omega, \bar{\psi}_\omega v_\omega\}$ with $|\omega| = m$. The map $E : \Phi \mapsto \chi\Phi$ is of finite rank, and so is $K'_m = \mathscr{M}^m E$. Note that when $y \notin \bigcup_\omega\{\bar{\psi}_\omega u_\omega, \bar{\psi}_\omega v_\omega\}$ and $\Psi = \Psi^{\#}$, we have $(\mathscr{M}'^m\Psi)(y) = (\mathscr{M}'^m\Psi)^{\#}(y)$. We may now write

$$\begin{aligned}\operatorname{Var}(\mathscr{M}^m - K'_m)\Phi_\infty &= \operatorname{Var}\mathscr{M}^m(\Phi_\infty - \chi\Phi_\infty) = 2\sum_x |\mathscr{M}^m(\Phi_\infty - \chi\Phi_\infty)(x)|\\ &= 2\sup\left|\sum_x \Psi(x)[\mathscr{M}^m(\Phi_\infty - \chi\Phi_\infty)](x)\right|\end{aligned}$$

(where the sup is over Ψ such that $\|\Psi\|_0 = 1$ and $\Psi = \Psi^{\#}$)

$$= 2 \sup \left| \sum_y (\mathscr{M}'^m \Psi)(y) \cdot (\Phi_\infty - \chi\Phi_\infty)(y) \right|$$

$$= 2 \sup \left| \sum_y (\mathscr{M}'^m \Psi)^{\#}(y) \cdot (\Phi_\infty - \chi\Phi_\infty)(y) \right|$$

$$\leqq \|(\mathscr{M}'^{\#})^m\|_0 \|\Phi_\infty\|_{\mathscr{B}} .$$

In conclusion if $\varepsilon > 0$ we have

$$\|(\mathscr{M}^m - K'_m)\Phi_\infty\|_{\mathscr{B}} \leqq \text{const}(\widehat{R}^{\#} + \varepsilon)^m \|\Phi\|_{\mathscr{B}} .$$

By the proof of Theorem B (#-version) there is $K^{\#}_m$ of finite rank on $\mathscr{B}^{\#}$ such that

$$\|\mathscr{M}^{\# m} - K^{\#}_m\|^{\#} \leqq \text{const}(\widehat{R}^{\#} + \varepsilon)^m .$$

We choose K_m of finite rank on $\mathscr{B}$ such that K_m induces $K^{\#}_m$ on $\mathscr{B}^{\#}$ and $K_m\Phi = (K_m\Phi)^{\#}$. There is also K''_m of finite rank such that

$$\mathscr{M}^m \Phi^{\#} - K''_m \Phi^{\#} = (\mathscr{M}^m \Phi^{\#})^{\#} .$$

Therefore

$$\mathscr{M}^m \Phi^{\#} - K_m \Phi^{\#} - K''_m \Phi^{\#} = (\mathscr{M}^m \Phi^{\#} - K_m \Phi^{\#})^{\#}$$

and

$$\|(\mathscr{M}^m - K_m - K''_m)\Phi^{\#}\|_{\mathscr{B}} = \|(\mathscr{M}^{\# m} - K^{\#}_m)[\Phi]\|^{\#} \leqq \text{const}(\widehat{R}^{\#} + \varepsilon)^m \|\Phi\|_{\mathscr{B}} .$$

Defining now $\widetilde{K}_m \Phi = (K_m + K''_m)\Phi^{\#} + K'_m \Phi_\infty$ we obtain

$$\|(\mathscr{M}^m - \widetilde{K}_m)\Phi\|_{\mathscr{B}} \leqq \text{const}(\widehat{R}^{\#} + \varepsilon)^m \|\Phi\|_{\mathscr{B}} ,$$

and therefore $(*)$ holds.

One can also show that *the spectral radius of* $\mathscr{M}^{\#}$ *is* $\geqq \widehat{R}^{\#}$ (this will not be used).

If $\Phi \in \mathscr{B}_\infty$ we have

$$\text{Var}\, \mathscr{M}^m \Phi = 2 \sum_x |(\mathscr{M}^m \Phi)(x)| = 2 \sup_{\Psi : \|\Psi\|_0 = 1} \left| \sum_x \Psi(x) \cdot (\mathscr{M}^m \Phi)(x) \right|$$

$$= 2 \sup_{\Psi} \left| \sum_y (\mathscr{M}'^m \Psi)(y) \cdot \Phi(y) \right| \leqq \|\mathscr{M}'^m\|_0 \cdot \text{Var}\, \Phi$$

so that *the spectral radius of* $\mathscr{M}_Z|\mathscr{B}_\infty$ *is* $\leqq \widehat{R}(\varepsilon Z)$. In particular $\mathscr{M}_Z$ and $\mathscr{M}^{\#}_Z$ have the same eigenvalues λ with the same multiplicity when $|\lambda| > \max(\widehat{R}(Z), \widehat{R}(\varepsilon Z))$.

1.8. Remark. Further properties of ζ_0. The proof of Lemma 2.4 below shows that if $\zeta_0(Z) = 0$ and $\widehat{R}^{\#}(Z) < 1$, then 1 belongs to the spectrum of $\mathscr{M}_Z|\mathscr{B}_\infty$ or $\mathscr{M}_{\varepsilon Z}|\mathscr{B}_\infty$. Similarly, if $\zeta_0(Z) = 0$ and $R^{\#}(Z) < 1$, then 1 belongs to the spectrum of $\widehat{\mathscr{M}}_Z|\mathscr{B}_\infty$ or $\widehat{\mathscr{M}}_{\varepsilon Z}|\mathscr{B}_\infty$.

The following condition is generically satisfied.

Condition G. For all $m \geqq 1$ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ *with* $\varepsilon(\boldsymbol{\omega}) = 1$, *we have*

$$\bar{\psi}_{\boldsymbol{\omega}} u_{\omega_1} \neq u_{\omega_1} \quad \text{if } \bar{\psi}_{\boldsymbol{\omega}} u_{\omega_1} \text{ is defined},$$
$$\bar{\psi}_{\boldsymbol{\omega}} v_{\omega_1} \neq v_{\omega_1} \quad \text{if } \bar{\psi}_{\boldsymbol{\omega}} v_{\omega_1} \text{ is defined}.$$

It is clear from the definition of ζ_0 that if Condition G holds, then $\zeta_0 = 1$ identically.

1.9. Remark. Poles of $D(zZ)$. Let Z be fixed. The function $z \mapsto D(zZ)$ is meromorphic when $|z|\hat{R}^{\#}(Z) < 1$, and clearly can have a pole at λ^{-1} only if λ is an eigenvalue of $\mathscr{M}^{\#} = \mathscr{M}^{\#}_Z$. Let

$$\mathscr{D} = \{z\colon |z| < \hat{R}^{\#}(Z)^{-1} \text{ and } z^{-1} \text{ is not an eigenvalue of } \mathscr{M}_{\varepsilon Z}|B_\infty\}.$$

In particular (see the end of Remark 1.7),

$$\{z\colon |z| < \hat{R}(Z)^{-1}\} \subset \mathscr{D}.$$

We shall show that *the function $z \mapsto D(zZ)$ does not vanish in $\mathscr{D}$, and has a pole of order m at λ^{-1} precisely if λ is an eigenvalue of order m of $\mathscr{M}^{\#}$.*

The proof will be in several steps.

(i) Let us define

$$A = \{a_1, \ldots, a_L\} \cup \{\psi_{\boldsymbol{\omega}}^{-1} a_i\colon |\boldsymbol{\omega}| \geqq 1,\ 1 \leqq i \leqq L\},$$
$$\mathscr{B}^{\#}_A = \{[\Phi] \in \mathscr{B}^{\#}\colon \text{the derivative of } \Phi \text{ is an atomic measure carried by } A\}.$$

Then the generalized eigenspace of $\mathscr{M}^{\#}$ corresponding to any eigenvalue λ with $|\lambda| > \hat{R}^{\#}(Z)$ is contained in $\mathscr{B}^{\#}_A$.

We may extend the linear operator $\mathscr{M}_Z$ from bounded functions to measures by letting

$$(\mathscr{M}_Z \mu)(dx) = \sum_\omega z_\omega \chi_\omega(X) \cdot (\psi_\omega^{-1}\mu)(dx)$$

(where $\psi_\omega^{-1}\mu$ is the image of μ by ψ_ω^{-1}). We shall write

$$(\Psi, \mu) = \int \mu(dx)\Psi(x)$$

if Ψ is a continuous function. If Φ is of bounded variation, we denote by $\partial\Phi$ its derivative, which is a bounded measure. (If $\Phi \in \mathscr{B}_\infty$, then $\partial\Phi = 0$. Therefore $\partial\Phi$ only depends on the class $[\Phi] \in \mathscr{B}^{\#}$.) We also let $\mathscr{P}$ be the projection on measures μ such that $|\mu|(A) = 0$ (i.e., $\mathscr{P}$ "erases" the mass carried by A). If $X : \mathbb{R} \mapsto \{0,1\}$ is 0 on $\{a_1, \ldots, a_L\}$ and 1 elsewhere, we have

$$\mathscr{P}\partial\mathscr{M}_Z\Phi = X\mathscr{M}_{\varepsilon Z}\mathscr{P}\partial\Phi.$$

When $[\mathscr{M}\Phi] = \lambda[\Phi]$ (mod $\mathscr{B}^{\#}_A$) we have thus

$$(\Psi, \mathscr{P}\partial\Phi) = \lambda^{-m}(\Psi, \mathscr{P}\partial\mathscr{M}^m_Z\Phi) = \lambda^{-m}(\Psi, (X\mathscr{M}_Z)^m\mathscr{P}\partial\Phi)$$
$$= \lambda^{-m}((\hat{\mathscr{M}}^m_{\varepsilon Z}X)^m\Psi, \mathscr{P}\partial\Phi).$$

If $|\lambda| > \widehat{R}^{\#}(Z)$, the right-hand side must vanish, so that $\mathscr{P}\partial\Phi = 0$, i.e., $[\Phi] \in \mathscr{B}^{\#}_{A}$. By induction we see that if $[(\mathscr{M} - \lambda)^k \Phi] = 0$, i.e., if $[\Phi]$ is in the generalized eigenspace of $\mathscr{M}^{\#}$ corresponding to λ, we have $[\Phi] \in \mathscr{B}^{\#}_{A}$. □

(ii) *Let* $\lambda^{-1} \in \mathscr{D}$ *and suppose that (with* α_i *defined in Sect.* 1.3)

$$(1 - \lambda^{-1}\mathscr{M}^{\#})\Omega = 0\,,$$

$$(1 - \lambda^{-1}\mathscr{M}^{\#})\gamma_j = \alpha_j \quad \text{for } j = 1,\ldots,L\,.$$

Then if

$$(1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\left(\Omega + \sum_{j=1}^{L} c_j\gamma_j\right)$$

has no mass at $a_1,\ldots,a_L$ *we have* $\Omega = 0$ *and* $c_1 = \cdots = c_L = 0$.

Let us write

$$\Phi = \sum c_j\alpha_j, \qquad \Psi = \Omega + \sum c_j\gamma_j\,.$$

Then $(1 - \lambda^{-1}\mathscr{M}^{\#}_{Z})\Psi = \Phi$; in particular $\mathscr{M}^{\#}_{Z}\Psi = \lambda\Psi \pmod{\mathscr{B}^{\#}_{A}}$ which implies $\Psi \in \mathscr{B}^{\#}_{A}$ as we have seen in (i). Furthermore $(1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\Psi$ has no mass outside of $a_1,\ldots,a_L$, so that by assumption

$$(1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\Psi = 0\,.$$

Since $\Psi \in \mathscr{B}_A$, this is equivalent to

$$(1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\widetilde{\Psi} = 0$$

with $\widetilde{\Psi} \in \mathscr{B}_\infty$ such that $\widetilde{\Psi}(x) = (\partial\Psi)(\{x\})$, and the assumption $\lambda^{-1} \in \mathscr{D}$ implies $\widetilde{\Psi} = 0$, i.e., $\partial\Psi = 0$, i.e., $\Psi =$ constant. Therefore Φ tends to the constant Ψ at $\pm\infty$, but since $\Phi(-\infty) = -\Phi(\infty)$, we obtain $\Psi = 0$. Therefore $\Phi = 0$, so that $c_1 = \cdots = c_L = 0$, and finally also $\Omega = 0$. □

(iii) *If* $\lambda^{-1} \in \mathscr{D}$ *and* λ *is not an eigenvalue of* $\mathscr{M}^{\#}$, *then* $D(\lambda^{-1}Z) \neq 0$.

We may write $\gamma_j = (1 - \lambda^{-1}\mathscr{M}^{\#})^{-1}\alpha_j$ and define $\Phi = \sum c_j\alpha_j, \Psi = (1 - \lambda^{-1}\mathscr{M}^{\#}_{Z})^{-1}\Phi = \sum c_j\gamma_j$. Suppose there is a linear relation

$$\sum c_j D_{ij}(\lambda^{-1}Z) = 0$$

between the columns of (D_{ij}), i.e.,

$$c_i + \frac{1}{2}\lim_{x\downarrow a_i}\sum_{\omega\,:\,u_\omega = a_i}\lambda^{-1}z_\omega\Psi(\psi_\omega x) - \frac{1}{2}\lim_{x\uparrow a_i}\sum_{\omega\,:\,v_\omega = a_i}\lambda^{-1}z_\omega\Psi(\psi_\omega x) = 0\,.$$

This may be rewritten as

$$\beta_i(\Phi) + \beta_i(\lambda^{-1}\mathscr{M}_Z\Psi) - \text{correction} = 0\,,$$

where the correction corresponds to those terms $\pm\frac{1}{2}\lim_{x\to a_i}\lambda^{-1}z_\omega\Psi(\psi_\omega x)$ such that $a_i \varepsilon J_\omega$. Equivalently we may write

$$\text{mass at } a_i \text{ of } (\partial\Phi + \partial\lambda^{-1}\mathscr{M}_Z\Psi - \lambda^{-1}\mathscr{M}_{\varepsilon Z}\partial\Psi) = 0$$

or

$$\text{mass at } a_i \text{ of } (\partial\Psi - \lambda^{-1}\mathscr{M}_{\varepsilon Z}\partial\Psi) = 0\,.$$

In view of (ii) we have then $c_1 = \cdots = c_L = 0$. Therefore $D(\lambda^{-1}Z) \neq 0$. □

(iv) *If* $\lambda^{-1} \in \mathscr{D}$ *and* λ *is a simple eigenvalue of* $\mathscr{M}^{\#}$, *then* λ^{-1} *is a simple pole of* $z \mapsto D(zZ)$.

Let $\Omega \neq 0$ be chosen such that $(1 - \lambda^{-1}\mathscr{M}^{\#})\Omega = 0$.

First, we show that $\alpha_1,\ldots,\alpha_L$ cannot all be in the range of $(1 - \lambda^{-1}\mathscr{M}^{\#})$. Otherwise let $\gamma_1,\ldots,\gamma_L$ be such that

$$(1 - \lambda^{-1}\mathscr{M}^{\#})\gamma_j = \alpha_j$$

for $j = 1,\ldots,L$. In view of (ii) the L-dimensional vectors

$$\text{mass at } \{a_1,\ldots,a_L\} \text{ of } (1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\gamma_j$$

are linearly independent. Therefore we may take $c_1,\ldots,c_L$ such that

$$\text{mass at } \{a_1,\ldots,a_L\} \text{ of } (1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\left(\Omega + \sum c_j\gamma_j\right) = 0\,.$$

Using again (ii) yields $\Omega = 0$ contrary to assumption.

Let us replace $\alpha_1,\ldots,\alpha_L$ by independent linear combinations $\Phi_1,\ldots,\Phi_L$ and write,

$$\Psi_j(z) = (1 - z\mathscr{M}^{\#})^{-1}\Phi_j\,,$$

$$\Psi_{ij}(z) = \frac{1}{2}\,\text{mass at } a_i \text{ of } (\partial\Psi_j - z\mathscr{M}_{\varepsilon Z}\partial\Psi_j)\,,$$

so that

$$D(zZ) = \det(\Psi_{ij}(z))\,.$$

Since we have shown that $\alpha_1,\ldots,\alpha_L$ are not all in the range of $(1 - \lambda^{-1}\mathscr{M}^{\#})^{-1}$, we may assume that $\Psi_1(z) \sim (1 - z\lambda)^{-1}\Omega$ for z near λ^{-1}, while $\Psi_2(z),\ldots,\Psi_L(z)$ are holomorphic at λ^{-1}. To prove that λ^{-1} is a simple pole of $z \mapsto D(zZ)$, it suffices now to show that the vectors

$$\text{mass at } \{a_1,\ldots,a_L\} \text{ of } (1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\Omega$$

and

$$\text{mass at } \{a_1,\ldots,a_L\} \text{ of } (1 - \lambda^{-1}\mathscr{M}_{\varepsilon Z})\partial\Psi_j$$

for $j = 2,\ldots,L$ are linearly independent. This again results from (ii). □

(v) *If* $\lambda^{-1} \in \mathscr{D}$ *and* λ *is an eigenvalue of order* m *of* $\mathscr{M}^{\#}$, *then* λ^{-1} *is a pole of order* m *of* $z \mapsto D(zZ)$.

By extending the index set for ω from $\{1,\ldots,N\}$ to $\{1,\ldots,N^*\}$ we can obtain small perturbations $\mathscr{M}^{*\#}$ of $\mathscr{M}^{\#}$ and $\mathscr{D}^*$ of $\mathscr{D}$ such that λ is replaced by m simple eigenvalues $\lambda_1^*,\ldots,\lambda_m^*$ contained in a disk $B_{\lambda^{-1}}(\varepsilon) \subset \mathscr{D} \cap \mathscr{D}^*$ with small ε. The corresponding $D^*(zZ)$ has simple poles and no zero near λ^{-1}. Since $D^*(zZ)$ tends to $D(zZ)$ away from poles it follows that $D(zZ)$ has a pole of order m at λ^{-1}. □

1.10. Remark. Zeros of $\widehat{D}(zZ)$. Let

$$\mathscr{D}^* = \{z\colon\ |z| < \widehat{R}^{\#}(Z)^{-1} \text{ and } z^{-1} \text{ is not an eigenvalue of } \mathscr{M}_Z|\mathscr{B}_\infty \text{ or } \mathscr{M}_{\varepsilon Z}|\mathscr{B}_\infty\}\,.$$

In particular (see the end of Remark 1.7)

$$\{z\colon\ |z| < \min\{\widehat{R}(Z)^{-1},\widehat{R}(\varepsilon Z)^{-1}\}\} \subset \mathscr{D}^*\,.$$

Denote by $\widehat{D}$ the kneading determinant associated with $\widehat{\mathscr{M}}$ (so that $\widehat{D} = \widehat{\zeta}$). Then, *the function* $z \mapsto \widehat{D}(zZ)$ *is holomorphic in* $\mathscr{D}^*$ *and has a zero of order* m *at* λ^{-1} *precisely if* λ *is an eigenvalue of order* m *of* $\mathscr{M}^{\#}$ (*or equivalently* $\mathscr{M}$).

In view of Remark 1.8, the zeros of $\widehat{D}(zZ)$ are the same as the poles of $D(zZ)$, with the same multiplicity. It suffices therefore to apply Remark 1.9. (Since $(1 - \lambda^{-1}\mathscr{M}_Z)|\mathscr{B}_\infty$ is invertible when $\lambda^{-1} \in \mathscr{D}^*$, the multiplicity of λ is the same as an eigenvalue of $\mathscr{M}^\#$ or $\mathscr{M}$.) □

The function $z \mapsto \widehat{D}(zZ)$ in $\mathscr{D}^*$ is the natural generalization of the kneading determinant considered by Milnor and Thurston [2], and also in [1].

2. Proof of Theorem C

The proof results from the four lemmas below.

2.1. Lemma. *We have identically*

$$\zeta(Z)\widehat{\zeta}(Z) = \zeta_0(Z)\,.$$

Using the definitions we obtain

$$\begin{aligned}\zeta(Z)\widehat{\zeta}(Z) &= \exp\sum_\omega \frac{1}{|\omega|}[L(\psi_\omega) + L(\psi_\omega^{-1})\varepsilon(\omega)]Z(\omega)\\ &= \exp\sum_\omega \frac{1}{|\omega|}[L_1(\psi_\omega) + L_1(\psi_\omega^{-1})\varepsilon(\omega) + L_0(\psi_\omega) + L_0(\psi_\omega^{-1})\varepsilon(\omega)]Z(\omega)\\ &= \exp\sum_\omega \frac{1}{|\omega|}L_0(\psi_\omega)(1+\varepsilon(\omega))Z(\omega) = \zeta_0(Z)\,,\end{aligned}$$

which proves the lemma. □

2.2. Lemma. *$D_{ij}(Z)$ is holomorphic when $R(Z) < 1$.*

Suppose that $R(Z_0) < 1$, and let $R(Z_0) < \xi < 1$. We may then choose M such that

$$\|\mathscr{M}_{Z_0}^M\|_0 < \xi^M\,.$$

Therefore, for some $\delta > 0$, we have

$$\|\mathscr{M}_Z^M\|_0 < \xi^M \quad \text{if } |Z - Z_0| < \delta\,.$$

The polynominals $Z \mapsto D_{ij}^{(m)\pm}$ thus satisfy

$$|D_{ij}^{(m)\pm}| < C\xi^m \quad \text{if } |Z - Z_0| < \delta,\ m \geqq 0$$

for some $C > 0$. This implies that $D_{ij}(Z)$ is holomorphic for $|Z - Z_0| < \delta$, i.e., $D_{ij}(Z)$ is holomorphic when $R(Z) < 1$. □

2.3. Lemma. *$D_{ij}(Z)$ is meromorphic when $\widehat{R}(Z) < 1$, with poles only when $1 \in$ spectrum $\mathscr{M}_Z$.*

Suppose that $\widehat{R}(Z_0) < 1$. We may choose ξ such that $\widehat{R}(Z_0) < \xi < 1$ and no eigenvalue of $\mathscr{M}_{Z_0}$ has modulus ξ (cf. Theorem B(b)). There is then $\delta_0 > 0$ such that, for $|Z - Z_0| \leqq \delta_0$, we have $\widehat{R}(Z) < \xi$ and the circle $S = \{\lambda\colon |\lambda| = \xi\}$ is disjoint from the spectrum of $\mathscr{M}_Z$. We then define the projection

$$P_Z = \frac{1}{2\pi i}\oint_S \frac{d\lambda}{\lambda - \mathscr{M}_Z}\,.$$

Therefore P_Z commutes with $\mathscr{M}_Z$, and $1 - P_Z$ is finite dimensional. We may choose M such that

$$\|P_{Z_0}\mathscr{M}_{Z_0}^M\| < \xi^M .$$

For some $\delta \in (0, \delta_0)$ we also have

$$\|P_Z\mathscr{M}_Z^M\| < \xi^M \quad \text{if } |Z - Z_0| < \delta ,$$

hence, for some $C > 0$,

$$\|P_Z\mathscr{M}_Z^m\| < C\xi^m \quad \text{if } |Z - Z_0| < \delta,\ m \geqq 0 .$$

Therefore the functions

$$\lim_{x\downarrow a_i} \sum_{\omega\,:\,u_\omega=a_i} z_\omega \cdot [(P_Z(1 - \mathscr{M}_Z)^{-1}\alpha_j)(\psi_\omega x)] ,$$

$$\lim_{x\uparrow a_i} \sum_{\omega\,:\,v_\omega=a_i} z_\omega \cdot [(P_Z(1 - \mathscr{M}_Z)^{-1}\alpha_j)(\psi_\omega x)]$$

are holomorphic for $|Z - Z_0| < \delta$. The functions

$$\lim_{x\downarrow a_i} \sum_{\omega\,:\,u_\omega=a_i} z_\omega \cdot [((1 - P_Z)(1 - \mathscr{M}_Z)^{-1}\alpha_j)(\psi_\omega x)] ,$$

$$\lim_{x\uparrow a_i} \sum_{\omega\,:\,v_\omega=a_j} z_\omega \cdot [((1 - P_Z)(1 - \mathscr{M}_Z)^{-1}\alpha_j)(\psi_\omega x)]$$

are meromorphic for $|Z - Z_0| < \delta$, and in fact holomorphic if $1 \notin \text{spectrum}\,\mathscr{M}_Z$. In conclusion $D_{ij}(Z)$ is meromorphic when $\widehat{R}(Z) < 1$ and holomorphic unless $1 \in \text{spectrum}\,\mathscr{M}_Z$. □

2.4. Lemma. *$\zeta_0(Z)$ is holomorphic when* $\min\{R(Z), \widehat{R}(Z)\} < 1$.

Let $A = \{a_1-, a_1+, \ldots, a_L-, a_L+\}$. If $\xi = a_i\pm \in A$, we write $|\zeta| = a_i$, $\text{sign}\,\zeta = \pm$.

For $\zeta, \eta \in A$, $m \geqq 1$, we define $T_{\xi\eta}^{(m)}$ to be the sum of the $Z(\boldsymbol{\omega})$ over all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ such that

(a) $$\varepsilon(\boldsymbol{\omega}) = \text{sign}\,\zeta \cdot \text{sign}\,\eta ,$$

(b) $$\text{either } |\zeta| = u_{\omega_1} \text{ and } \text{sign}\,\xi = + , \quad \text{or } |\zeta| = v_{\omega_1} \text{ and } \text{sign}\,\xi = - ,$$

(c) $\bar{\psi}_{\omega_1}|\xi|$ is in the (open) interval of definition of $\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_2}$, and $\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_2}\ (\bar{\psi}_{\omega_1}|\xi|) = |\eta|$.

Denote by $T = T(Z)$ the matrix with elements

$$T_{\xi\eta} = \sum_{m\geqq 1} T_{\xi\eta}^{(m)} .$$

We shall now prove that

$$\sum_{\omega:\varepsilon(\omega)=1} \frac{Z(\omega)}{|\omega|}[\mathrm{del}(\bar{\psi}_\omega(u_\omega) - u_\omega) + \mathrm{del}(\bar{\psi}_\omega(v_\omega) - v_\omega)]$$

$$= \sum_n \frac{1}{n} \sum_{\xi_1 \cdots \xi_n} T_{\xi_1\xi_2} T_{\xi_2\xi_3} \cdots T_{\xi_{n-1}\xi_n} T_{\xi_n\xi_1} \,. \qquad (*)$$

Consider the symbol (ω, ε), where $\omega = (\omega_1, \ldots, \omega_m)$ satisfies $\varepsilon(\omega) = 1$, and $\varepsilon = \pm 1$. We write $(\omega, \varepsilon) \sim (\omega', \varepsilon')$ if $\omega' = (\omega_k, \ldots, \omega_m, \omega_1, \ldots, \omega_{k-1})$ is a circular permutation of ω and $\varepsilon' = \varepsilon\varepsilon_1 \cdots \varepsilon_{k-1}$. To a nonvanishing term $\mathrm{del}(\bar{\psi}_\omega(u_\omega) - u_\omega)$ or $\mathrm{del}(\bar{\psi}_\omega(v_\omega) - v_\omega)$ we associate the pair $(\omega, +)$ or $(\omega, -)$ respectively. The left-hand side of the formula $(*)$ may thus be rewritten as

$$\sum_{(\omega,\,\varepsilon)}^{*} \frac{1}{|\omega|} Z(\omega) \,,$$

where the sum $\sum^*$ is restricted in an obvious manner. Equivalently one can sum over equivalence classes $[(\omega, \varepsilon)]$ for the relation $\sim$, so that the above sum is

$$= \sum_{[(\omega,\,\varepsilon)]}^{**} \frac{\mathrm{card}[(\omega, \varepsilon)]}{|\omega|} Z(\omega) \,.$$

The classes $[(\omega, \varepsilon)]$ appearing in the sum correspond to "extended orbits" of the form

$$x, \ \bar{\psi}_{\omega_1} x, \ldots, (\psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1})^{-} x = x \,,$$

where $\bar{\psi}$ denotes as usual the extension of ψ by continuity to the closure of the interval of definition. Consider the values $k(i)$ (with $i = l, \ldots, n$) of k such that $1 \leqq k \leqq m$ and $(\psi_{\omega_{k-1}} \circ \cdots \circ \psi_{\omega_1})^{-} x$ is an endpoint u_{ω_k} or v_{ω_k} of J_{ω_k}. We let $k(1) < k(2) < \cdots < k(n)$ and call $\omega^{(1)}, \ldots, \omega^{(n)}$ the pieces of ω such that $\omega^{(1)} = (\omega_{\imath_{k(1)}}, \ldots, \omega_{\imath_{k(2)-1}})$ etc. We have thus $Z(\omega) = Z(\omega^{(1)}) \cdots Z(\omega^{(n)})$.

By construction, among the n circular permutations of $\{1, 2, \ldots, n\}$ generated by $1 \to 2 \to \cdots \to n \to 1$, there are $n(\omega, \varepsilon) = |\omega| / \mathrm{card}[(\omega, \varepsilon)]$ which leave

$$(\xi_1, \omega^{(1)}), (\xi_2, \omega^{(2)}), \ldots, (\xi_n, \omega^{(n)})$$

fixed, hence the number of equivalence classes of permutations is $n / n(\omega, \varepsilon)$. The sum written above is thus

$$= \sum_{[(\omega,\,\varepsilon)]}^{**} \frac{n}{n(\omega, \varepsilon)} \cdot \frac{1}{n} Z(\omega^{(1)}) \cdots Z(\omega^{(n)})$$

$$= \sum_n \frac{1}{n} \sum_{\xi_1 \cdots \xi_n} T_{\xi_1\xi_2} T_{\xi_2\xi_3} \cdots T_{\xi_{n-1}\xi_n} T_{\xi_n\xi_1}$$

proving $(*)$. Therefore

$$\zeta_0(Z) = \exp + \sum_n \frac{1}{n} \mathrm{tr}\, T^n = \exp \mathrm{tr}(-\log(1 - T)) = \det(1 - T(Z))^{-1} \,.$$

Given $\varepsilon > 0$, let χ_η^ε be the characteristic function of $(|\eta|, |\eta| + \varepsilon)$ when $\mathrm{sign}\,\eta = +$, of $(|\eta| - \varepsilon, |\eta|)$ when $\mathrm{sign}\,\eta = -$. Also write $x \to \xi$ when $\mathrm{sign}\,\xi \cdot (x - |\xi|) \downarrow 0$, and let $\sum_{\omega : \xi}$ be the sum over those ω such that $u_\omega +$ or $v_\omega -$ is ξ. Then one checks that

$$\sum_{n \geqq 1} (T^n)_{\xi\eta} = \sum_{m \geqq 1} \lim_{\varepsilon \to 0} \lim_{x \to \varepsilon} \sum_{\omega \cdot \xi} z_\omega [(\mathscr{M}^{m-1} \chi_n^\varepsilon)(\psi_\omega(x))] .$$

Therefore $\det(1 - T(Z))^{-1}$ is holomorphic when $R(Z) < 1$. By symmetry, $\zeta_0(Z)$ is holomorphic when $\min(R(Z), \widehat{R}(Z)) < 1$, proving the lemma.

Write now $\chi_{|n|}(x) = \mathrm{del}(x - |n|)$ and

$$(\mathscr{M}^{m-1})_\pm = \frac{1}{2}(\mathscr{M}_Z^{m-1} \pm \mathscr{M}_{\varepsilon Z}^{m-1}) ,$$

then we have

$$T_{\xi\eta}^{(m)} = \sum_{\omega : \xi} z_\omega [(\mathscr{M}^{m-1})_\pm \chi_{|\eta|}](\bar{\psi}_\omega |\xi|)$$

with the sign $\pm = \varepsilon_\omega \operatorname{sign} \xi \cdot \operatorname{sign} \eta$. Therefore $T_{\xi\eta}$ is a holomorphic function of Z when $\widehat{R}^\#(Z) < 1$ and 1 is not an eigenvalue of $\mathscr{M}_Z | \mathscr{B}_\infty$ or $\mathscr{M}_{\varepsilon Z} | \mathscr{B}_\infty$. This justifies Remark 1.8. □

Appendix A. Proof of Theorem A

Let $\varepsilon_\omega = \pm 1$ for $\omega = 1, \ldots, N$. Fixing (J_ω) and (ε_ω), let P be the space of families $\psi = (\psi_\omega)$ such that each $\psi_\omega \colon J_\omega \to (-1, 1)$ is continuous and strictly increasing if $\varepsilon_\omega = +1$, or strictly decreasing if $\varepsilon_\omega = -1$. We denote by $C^r(\bar{J}_\omega)$ the space of C^r functions on the closure $\bar{J}_\omega$ of J_ω, and write

$$P^1 = \Big\{ \psi \colon (\psi_\omega) \text{ extends to } (\bar{\psi}_\omega) \in \bigoplus_\omega C^1(\bar{J}_\omega) ,$$
$$\text{and the derivatives } \bar{\psi}_\omega' \text{ vanish on } \bar{J}_\omega \backslash J_\omega \Big\} ,$$

$$P^{\mathrm{pol}} = \{ \psi \in P^1 \colon \text{the } \psi_\omega \text{ are polynomials} \} .$$

We use the topology of P, P^1 induced by $\oplus C^0(\bar{J}_\omega), \oplus C^1(\bar{J}_\omega)$. In particular P^{pol} is dense in P, P^1.

For finite M we define

$$F_M = \{ \psi \colon \operatorname{Fix} \psi_{\boldsymbol{\omega}} \text{ is finite when } |\boldsymbol{\omega}| \leqq M \} ,$$

$$P_M = \{ \psi \colon \bar{\psi}_{\boldsymbol{\omega}}(u_{\boldsymbol{\omega}}) \neq u_{\boldsymbol{\omega}} \text{ and } \bar{\psi}_{\boldsymbol{\omega}}(v_{\boldsymbol{\omega}}) \neq v_{\boldsymbol{\omega}} \text{ when } |\boldsymbol{\omega}| \leqq M \text{ and } J_{\boldsymbol{\omega}} \neq \emptyset \} .$$

Equivalently we may define P_M as the set of those ψ such that $\bar{\psi}_{\boldsymbol{\omega}}(u_{\omega_1})$ (if defined) is $\neq u_{\omega_1}$, and $\bar{\psi}_{\boldsymbol{\omega}}(v_{\omega_1})$ (if defined) is $\neq v_{\omega_1}$, when $|\boldsymbol{\omega}| \leqq M$. We also write

$$F_\infty = \bigcap_M F_M , \qquad P_\infty = \bigcap_M P_M .$$

Note that P_M is open in P.

A.1. Lemma. *If $\psi \in F_M \cap P_M$ and $|\boldsymbol{\omega}| \leqq M$, we have*

$$L(\psi_{\boldsymbol{\omega}}) = \sum_{x \in \operatorname{Fix} \psi_{\boldsymbol{\omega}}} L(x, \psi_{\boldsymbol{\omega}}) .$$

This follows from part (c) of the lemma of Sect. 1.1. □

Let $|\omega|$ be the class of ω under circular permutations, and say that $[\omega]$ is prime if ω is not the periodic repetition of n copies of a sequence ω' with $|\omega'| < |\omega|$. Then we have the *product formula*

$$\zeta(Z) = \prod_{[\omega]\text{ prime}} G_{[\omega]}(Z),$$

where

$$G_{[\omega]}(Z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} L(\psi_\omega^n) Z(\omega)^n .$$

The following possibilities exist

$$\begin{array}{lllllll}
(0) & \varepsilon(\omega) = \pm 1, & L(\psi_\omega) & = & 0, & \text{then } G_{[\omega]}(Z) & = 1, \\
(1) & \varepsilon(\omega) = +1, & L(\psi_\omega) & = & -1, & \text{then } G_{[\omega]}(Z) & = 1 - Z(\omega), \\
(2) & \varepsilon(\omega) = +1, & L(\psi_\omega) & = & 1, & \text{then } G_{[\omega]}(Z) & = (1 - Z(\omega))^{-1}, \\
(3) & \varepsilon(\omega) = -1, & L(\psi_\omega \circ \psi_\omega) & = & 1, & \text{then } G_{[\omega]}(Z) & = (1 - Z(\omega))^{-1}, \\
(4) & \varepsilon(\omega) = -1, & L(\psi_\omega \circ \psi_\omega) & = & -1, & \text{then } G_{[\omega]}(Z) & = 1 + Z(\omega).
\end{array}$$

A.2. Lemma. *$\zeta(Z)$ and $1/\zeta(Z) \in \mathbb{Z}[[z_1,\ldots,z_N]]$. If $\mathfrak{J}_{M+1}$ is the ideal of elements of order $\geqq M+1$ in $\mathbb{Q}[[z_1,\ldots,z_N]]$, then $\zeta(Z) \pmod{\mathfrak{J}_{M+1}}$ is locally constant on P_M.*

This follows from the product formula given above and the definition of P_M.□

A.3. Lemma. *If ψ satisfies $\psi_\omega J_\omega > a_L$ for $\omega = 1,\ldots,N$, we have*

$$\zeta = D = 1 .$$

Clearly $\psi \in F_\infty \cap P_\infty$. In fact $\operatorname{Fix}\psi_\omega = \emptyset$ for all ω, hence $\zeta(Z) = 1$.
In the present situation $\mathcal{M}^m = 0$ for $m > 1$. We have thus

$$D_{ij} = \delta_{ij} + A_i ,$$

$$A_i = \frac{1}{2}\left[\sum_{\omega\,:\,u_\omega = a_i} z_\omega - \sum_{\omega\,:\,v_\omega = a_i} z_\omega\right],$$

i.e., the kneading matrix $[D_{ij}]$ is the sum of the unit matrix $[\delta_{ij}]$ and a matrix of rank $\leqq 1$. Therefore

$$D = \det[D_{ij}] = 1 + \sum_i A_i = 1 + \frac{1}{2}\left(\sum_\omega z_\omega - \sum_\omega z_\omega\right) = 1 ,$$

which concludes the proof. □

A.4. Lemma. *Let $\tilde{J}_\omega, \tilde{\zeta}, \tilde{D}$ correspond to J_ω, ζ, D when $\tilde{\psi}$ replaces ψ. Given $M \geqq 1$, we assume that $\tilde{\psi}$ is sufficiently close to ψ in P (in particular $\tilde{J}_\omega = J_\omega$), and that*

$$\tilde{J}_\omega \supset J_\omega, \qquad \tilde{\psi}_\omega \tilde{J}_\omega \subset \psi_\omega J_\omega , \tag{1}$$

$$J_\omega \cap \psi_\omega J_\omega = \emptyset \Rightarrow \tilde{J}_\omega \cap \tilde{\psi}_\omega \tilde{J}_\omega = \emptyset \tag{2}$$

for $|\omega| \leqq M$. Then

$$\tilde{\zeta}(Z) = \zeta(Z) \pmod{\mathfrak{J}_{M+1}},$$

$$\tilde{D}(Z) = D(Z) \pmod{\mathfrak{J}_{M+1}}.$$

Furthermore, if

$$\bar{\tilde{\psi}}_\omega(u_\omega) \neq \tilde{\psi}_\omega(u_\omega), \qquad \bar{\tilde{\psi}}_\omega(v_\omega) \neq \tilde{\psi}_\omega(v_\omega) \tag{3}$$

for $|\omega| \leqq M$, *we may assume that*

$$\bar{\tilde{\psi}}_\omega(u_\omega), \bar{\tilde{\psi}}_\omega(v_\omega) \notin \{a_1, \ldots, a_L\}$$

when $|\omega| \leqq M$ *(in particular* $\tilde{\psi} \in P_M$*).*

First note that if $J_\omega = \emptyset$, then $\tilde{J}_\omega = \emptyset$ (because (1) gives $\tilde{\psi}_\omega \tilde{J}_\omega \subset \psi_\omega J_\omega = \emptyset$). If $J_\omega \neq \emptyset$, the set $\tilde{J}_\omega$ is close to J_ω and the set $\tilde{\psi}_\omega \tilde{J}_\omega$ is close to $\psi_\omega J_\omega$; then (2) and the inclusions (1) imply that $L(\tilde{\psi}_\omega) = L(\psi_\omega)$ for $|\omega| \leqq M$ (the argument is the same as for part (a) of the lemma in Sect. 1.1: check the list of cases when $L(\psi_\omega) = 1, -1$, or 0). This implies $\tilde{\zeta}(Z) = \zeta(Z) \pmod{\mathfrak{J}_{M+1}}$.

Suppose that $u_\omega = u_{\omega_1} = a_i$. When $\tilde{\psi} \to \psi$, then

$$\bar{\tilde{\psi}}_\omega(a_i) \to \bar{\psi}_\omega(a_i)\,,$$

and the inclusion (1) implies that the above limit is reached on the same side as the limit

$$\psi_\omega(x) \to \bar{\psi}_\omega(a_i)$$

when $x \downarrow a_i$. Therefore (for $\tilde{\psi}$ close to ψ)

$$\tilde{D}_{ij}^{(m)+} = D_{ij}^{(m)+}\,,$$

and similarly

$$\tilde{D}_{ij}^{(m)-} = D_{ij}^{(m)-}\,.$$

This means that

$$\tilde{D}_{ij} = D_{ij} \pmod{\mathfrak{J}_{M+1}}\,,$$

hence

$$\tilde{D}(Z) = D(Z) \pmod{\mathfrak{J}_{M+1}}\,.$$

The last statement of the lemma follows from the fact that the numbers

$$|\bar{\tilde{\psi}}_\omega(u_\omega) - \bar{\psi}_\omega(u_\omega)|, \qquad |\bar{\tilde{\psi}}_\omega(v_\omega) - \bar{\psi}_\omega(v_\omega)|$$

are in an arbitrarily small interval $(0, \delta)$. □

A.5. Proof of the Theorem. It will suffice to prove Theorem A $\pmod{\mathfrak{J}_{M+1}}$ for all integers $M \geqq 1$. We fix M for the rest of the argument.

For small $\delta > 0$, let the homeomorphism $\varphi_\omega : (u_\omega, v_\omega) \to (u_\omega + \delta, v_\omega - \delta)$ be the identity on $[u_\omega + 2\delta, v_\omega - 2\delta]$ and a contraction on $(u_\omega, u_\omega + 2\delta)$ and $(v_\omega - 2\delta, v_\omega)$. We define $\tilde{\psi}_\omega = \psi_\omega \circ \varphi_\omega$ for $\omega = 1, \ldots, N$. Writing

$$\omega = (\omega_1, \ldots, \omega_m), \qquad \omega' = (\omega_1, \ldots, \omega_{m-1})\,,$$

we may assume that the length of $J_{\omega_m} \cap \psi_{\omega'} J_{\omega'}$ is $\geqq a > 0$ whenever $|\omega| \leqq 2M$ and $J_\omega \neq \emptyset$. If δ is sufficiently small we may also assume that the length of $J_{\omega_m} \cap \psi_{\omega'} J_{\omega'}$ is $\geqq a > 0$ whenever $|\omega| \leqq 2M$ and $J_\omega \neq \emptyset$. If δ is sufficiently small we may also assume that the length of $J_{\omega_m} \cap \tilde{\psi}_{\omega'} \tilde{J}_{\omega'}$ is $\geqq b = \frac{a}{2}$. Note that

$$\tilde{\psi}_\omega \tilde{J}_\omega = \tilde{\psi}_{\omega_m}(J_{\omega_m} \cap \tilde{\psi}_{\omega'} \tilde{J}_{\omega'})\,.$$

Assuming now that $2\delta < b$, we see by induction on $|\omega|$ that

$$\widetilde{\psi}_\omega \tilde{J}_\omega \subset \psi_\omega J_\omega .$$

Writing $\omega(k) = (\omega_1, \dots, \omega_k)$ we also see (by induction on k, and assuming δ small enough) that

$$\widetilde{\psi}_{\omega(k)} J_\omega \subset \psi_{\omega(k)} J_\omega .$$

In particular $\widetilde{\psi}_\omega$ is defined on J_ω, i.e.,

$$\tilde{J}_\omega \supset J_\omega .$$

The condition (1) of Lemma A.4 is thus satisfied when $|\omega| \leqq 2M$. Writing $(\omega_1, \dots, \omega_m, \omega_1, \dots, \omega_m) = 2\omega$, we have

$$J_\omega \cap \psi_\omega J_\omega = \psi_\omega J_{2\omega} .$$

Therefore (2) for $|\omega| \leqq M$ follows from the implication $\psi_{2\omega} J_{2\omega} = \emptyset \Rightarrow \widetilde{\psi}_{2\omega} \tilde{J}_{2\omega} = \emptyset$ (which follows from (1)). By induction on m we see that (3) also holds.

We may now approximate $\widetilde{\psi}$ in P^0 by $\psi^1 \in P^{\text{pol}}$ while respecting the conditions (1), (2), and (3). Lemma A.4 thus shows that, to prove Theorem A, it suffices to prove that

$$\zeta^1(Z) = D^1(Z) \pmod{\mathfrak{J}_{M+1}} ,$$

where $\zeta^1(Z)$ and $D^1(Z)$ are constructed with $\psi^1 \in P^{\text{pol}}$ such that

$$\tilde{\psi}^1_\omega(u_\omega),\ \tilde{\psi}^1_\omega(v_\omega) \notin \{a_1, \dots, a_L\}$$

for $|\omega| \leqq M$.

Let $\psi^0 \in P^{\text{pol}}$ be defined as in Lemma A.3, and $\psi^\lambda = (1-\lambda)\psi^0 + \lambda\psi^1$. By definition, $\psi^\lambda = (\psi^\lambda_1, \dots, \psi^\lambda_N)$ is an N-tuple of polynomials, none of which is affine [$\bar{\psi}^\lambda_\omega$ is non-constant, with derivatives vanishing at u_ω, v_ω]; in particular $\psi^\lambda \in F_\infty$. Note that the functions $(x, \lambda) \mapsto \psi^\lambda_\omega(x), \psi^\lambda_\omega$ are polynomials, and extend therefore naturally to $\mathbb{R}^2$. Until further notice we shall use these extended definitions. The polynomials $\lambda \mapsto \psi^\lambda_\omega(a_i) - a_j$ (defined for all $\omega = (\omega_1, \dots, \omega_m)$ with $1 \leqq m \leqq M$ and $i, j \in \{1, \dots, L\}$) may be assumed not to vanish at $\lambda = 0, 1$. Therefore there is a finite set $\Lambda \subset (0, 1)$ of values of λ such that

$$\psi^\lambda_\omega(a_i) = a_j$$

for some i, j, and ω. If ζ^λ and D^λ denote ζ and D computed with ψ^λ, we see that $\zeta^\lambda \pmod{\mathfrak{J}_{M+1}}$ remains constant in each interval of $[0,1]\backslash\Lambda$ [see Lemma A.2] and the same is true for $D^\lambda \pmod{\mathfrak{J}_{M+1}}$ [because the $D^{(m)\pm}_{ij}$ are constant].

In view of Lemma A.3, in order to prove Theorem A it suffices to show that ζ^λ and D^λ are multiplied by the same factor $\pmod{\mathfrak{J}_{M+1}}$ whenever λ crosses a point of Λ.

The changes of sign of the $\psi^\lambda_\omega(a_i) - a_j$ when λ crosses an element of Λ may be complicated. We shall make them simpler by modifying (ψ^λ) to obtain a family $(\widetilde{\psi}^\lambda)$ with nonlinear dependence on λ.

Let us assume that $(x, \lambda) \mapsto \widetilde{\psi}^\lambda_\omega(x)$, defined on $\mathbb{R}^2$, is C^∞ close to $(x, \lambda) \mapsto \psi^\lambda_\omega(x)$, for $\omega = 1, \dots, N$, and construct $\widetilde{\psi}^\lambda_\omega = \widetilde{\psi}^\lambda_{\omega_m} \circ \dots \circ \widetilde{\psi}^\lambda_{\omega_1}$. In particular the functions

$\lambda \mapsto \tilde{\psi}^\lambda_\omega(a_i) - a_j$ are C^∞ close to the polynomials $\lambda \mapsto \psi^\lambda_\omega(a_i) - a_j$ and may be assumed not to vanish at $\lambda = 0, 1$. Let $\tilde{\Lambda}$ be the set of all $\lambda \in (0,1)$ for which $\tilde{\psi}^\lambda_\omega(a_i) = a_j$ for some i, j, and some ω with $|\omega| \leqq M$. Then card $\tilde{\Lambda}$ is bounded by the sum (over i, j, ω) of the degrees of the polynomials $\lambda \mapsto \psi^\lambda_\omega(a_i) - a_j$, hence uniformly in $(\tilde{\psi}^\lambda)$ for $(\tilde{\psi}^\lambda)$ in a suitable C^∞ neighborhood of (ψ^λ). We shall use the uniformity of this bound in a moment.

Given $\lambda_0 \in \Lambda$ we construct an oriented graph Γ as follows. The set of vertices of Γ is

$$X = \{\xi \in \mathbb{R}\colon \text{ there exist } \boldsymbol{\omega} = (\omega_1, \ldots, \omega_m) \text{ with } 1 \leqq m \leqq M,\ i, j \in \{1, \ldots, L\}$$
$$\text{and } k \in \{0, \ldots, m\} \text{ such that } \psi^{\lambda_0}_{\omega_k} \circ \cdots \circ \psi^{\lambda_0}_{\omega_1} a_i = \xi, \psi^{\lambda_0}_{\omega_m} \circ \cdots \circ \psi^{\lambda_0}_{\omega_{k+1}} \xi = a_j\} .$$

The set of arrows is

$$\{(\xi, \omega)\colon 1 \leqq \omega \leqq N,\ \xi \in X \text{ and } \psi^{\lambda_0}_\omega \xi \in X\} .$$

The arrow (ξ, ω) starts at ξ and goes to $\eta = \psi^{\lambda_0}_\omega \xi$; there may thus be several arrows $\xi \Rightarrow \eta$. An arrow (ξ, ω): $\xi \Rightarrow \eta$ may be removed from the graph corresponding to λ_0 by a C^∞ small change of $(x, y) \mapsto \psi^\lambda_\omega(x)$ near (ξ, λ_0). Repeating this operation, we can arrange that (ψ^λ) is replaced by $(\tilde{\psi}^\lambda)$ such that the graph corresponding to λ_0 consists of a simple arc $a_i \Rightarrow \xi \Rightarrow \eta \Rightarrow \cdots \Rightarrow a_j$ (where a_j may be equal to a_i) and $\xi, \eta, \ldots \notin \{a_i, \ldots, a_L\}$. This means that there are unique i, j, and $\boldsymbol{\omega}^*$ with $|\boldsymbol{\omega}^*| \leqq M$ such that $\tilde{\psi}^{\lambda_0}_{\boldsymbol{\omega}^*}(a_i) = a_j$ and $\tilde{\psi}^{\lambda_0}_{\omega^*_k} \circ \cdots \circ \tilde{\psi}^{\lambda_0}_{\omega^*_1} a_i \notin \{a_1, \ldots, a_L\}$ for $k < |\boldsymbol{\omega}^*|$. By a small change of $(\tilde{\psi}^\lambda)$ near $\lambda = \lambda_0$ we may further achieve that Fix $\tilde{\psi}^{\lambda_0}_\omega$ is finite when $|\boldsymbol{\omega}| \leqq M$, and that the fixed points are not degenerate (i.e., the derivative of $\tilde{\psi}^{\lambda_0}_\omega$ at $\xi \in \text{Fix}\, \tilde{\psi}^{\lambda_0}_\omega$ is $\neq 1$). Note that the families (ψ^λ) and $(\tilde{\psi}^\lambda)$ coincide outside of a small neighborhood of λ_0; to obtain $\tilde{\Lambda}$ from Λ we have replaced λ_0 by a finite set $\{\lambda_0, \lambda'_0, \ldots\}$.

We may now start again the above process with a new element $\tilde{\lambda}_0$ of $\tilde{\Lambda}$ (being careful to leave $(\tilde{\psi}^{\lambda_0})$ unchanged). Since the cardinality of the sets $\Lambda, \tilde{\Lambda}, \ldots$ is uniformly bounded, after a finite number of steps the family (ψ^λ) is replaced by (Ψ^λ) with the following properties.

(a) *$\Psi^\lambda \in P^1$, $(x, \lambda) \mapsto \Psi^\lambda(x)$ is C^∞, and $\Psi^0 = \psi^0$, $\Psi^1 = \psi^1$.*

(b) *For λ outside of a finite set Λ^*,*

$$\Psi^\lambda_\omega(a_i) \neq a_j$$

if $i, j \in \{1, \ldots, L\}$ and $|\boldsymbol{\omega}| \leqq M$.

(c) *If $\lambda \in \Lambda^*$ there are unique $i, j \in \{1, \ldots, L\}$ and $\boldsymbol{\omega}^*$ with $|\boldsymbol{\omega}^*| \leqq M$ such that*

$$\Psi^\lambda_{\boldsymbol{\omega}^*}(a_i) = a_j ,$$

and $\Psi^\lambda_{\omega^*_k} \circ \cdots \circ \Psi^\lambda_{\omega^*_1} a_i \notin \{a_1, \ldots, a_L\}$ if $k < |\boldsymbol{\omega}^*|$.

(d) *If $\lambda \in \Lambda^*$, and $|\boldsymbol{\omega}| \leqq M$, then* Fix Ψ^λ_ω *is finite and the fixed points $\xi \in$* Fix Ψ^λ_ω *are nondegenerate, i.e., $(\Psi^\lambda_\omega)'(\xi) \neq 1$.*

To prove the theorem it suffices therefore to check (under the conditions (a), (b), (c), (d)) that the zeta function ζ and the kneading determinant D associated with (Ψ^λ) are multiplied by the same factor $(\bmod \mathfrak{J}_{M+1})$ when λ crosses a point of Λ^*. This is done in the following lemma. □

We return now to the standard notation where ψ_ω is defined only on J_ω and $\bar{\psi}_\omega$ is the extension of ψ_ω by continuity to the closure $\bar{J}_\omega$; similarly for $\psi_\omega, \bar{\psi}_\omega$.

A.6. Lemma. *Let $\psi \in P^1$ be such that there are unique $i, j \in \{1, \ldots, L\}$ and ω^* with $|\omega^*| \leqq M$ such that*

$$\bar{\psi}_{\omega^*}(a_i) = a_j$$

and $\bar{\psi}_{\omega_k^*} \circ \cdots \circ \bar{\psi}_{\omega_1^*} a_i \notin \{a_1, \ldots, a_L\}$ if $k < |\omega^*|$. We further assume that whenever $|\omega| \leqq M$ the set $\mathrm{Fix}\, \psi_\omega$ is finite and consists of nondegenerate fixed points ξ, i.e., $\psi'_\omega(\xi) \neq 1$.

Then if $\psi^>$, $\psi^<$ are sufficiently close to ψ in P^1 and such that $\psi^>_{\omega^*}(a_i) > a_j$, $\psi^<_{\omega^*}(a_i) < a_j$ we have

$$\zeta^>/\zeta^< = D^>/D^< \quad (\bmod \mathfrak{J}_{M+1}),$$

where $\zeta^{\gtrless}$, $D^{\gtrless}$ denote ζ, D computed from $\psi^{\gtrless}$

We first observe that $\zeta^>/\zeta^< = D^>/D^< = 1 \ (\bmod \mathfrak{J}_{M+1})$ unless a_i is one of the endpoints $u_{\omega_1^*}$ or $v_{\omega_1^*}$ of $J_{\omega_1^*}$. Using the symmetry $x \to -x$ of $\mathbb{R}$ we see that it suffices to consider the situation where $u_{\omega_1^*} = a_i$. In this case we claim that we have $(\bmod \mathfrak{J}_{M+1})$

$$\zeta^> = \zeta^<, \qquad D^> = D^< \qquad \text{if } j \neq i,$$

$$\zeta^> = \zeta^< \cdot (1 - Z(\omega^*))^{-1}, \qquad D^> = D^< \cdot (1 - Z(\omega^*))^{-1} \quad \text{if } j = i.$$

We first discuss the easy proof of the formulas for the zeta function. If $j \neq i$, then $\zeta \ (\bmod \mathfrak{J}_{M+1})$ is locally constant at ψ (Lemma A.2), hence $\zeta^> = \zeta^<$.

Let $j = i$. We have $\bar{\psi}_{\omega^*} a_i = a_i$. The point a_i bifurcates into an attracting fixed point for $\psi^>_{\omega^*}$, absent for $\psi^<_{\omega^*}$ (see the figure). Apart from the periodic orbit thus created, the periodic orbits for $\psi, \psi^>, \psi^<$ correspond to each other, with the same weight, up to order $\geqq M+1$, if $\psi^>$ and $\psi^<$ are sufficiently close to ψ in P^1. Therefore

$$\zeta^> = \zeta^<(1 - Z(\omega^*))^{-1}$$

as announced.

Graph of ψ_ω. The graph of $\psi^>_{\omega^*}$ (resp. $\psi^<_{\omega^*}$) is obtained by pushing the graph of ψ_{ω^*} upwards (resp. downwards).

We consider now the changes for D. Let δD denote the jump of D from $\psi^<$ to $\psi^>$ and similarly for $\delta D_{ik}, \ldots$. We have $\delta D_{ik}^{(m)-} = 0$, hence

$$\delta D_{ik} = \frac{1}{2} \sum_{m=1}^{\infty} \delta D_{ik}^{(m)+} = \frac{1}{2} \sum_{m \geqq 1} \lim_{x \downarrow a_i} [((\mathscr{M}^>)^m \alpha_k)(x) - ((\mathscr{M}^<)^m \alpha_k)(x)]$$

$$= \frac{1}{2} Z(\omega^*) \sum_{n \geqq 0} \lim_{x \downarrow a_i} [((\mathscr{M}^>)^n \alpha_k)(\psi^>_{\omega^*} x) - ((\mathscr{M}^<)^n \alpha_k)(\psi^<_{\omega^*} x)]$$

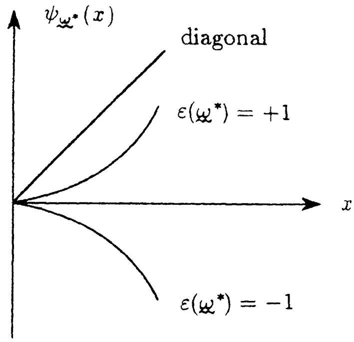

Fig. 1.

with obvious notation. Let Φ denote a function which is locally constant on $\mathbb{R}$ outside of $\{a_1,\dots,a_L\}$, like χ_ω or α_k. If $|\omega^*| + |\boldsymbol{\omega}| \leqq M$ we have

$$\lim_{x\downarrow a_i} (\Phi \circ \psi^>_\omega \circ \psi^>_{\omega^*})(x) = \lim_{x\downarrow a_j} (\Phi \circ \psi^>_\omega)(x) ,$$

$$\lim_{x\downarrow a_i} (\Phi \circ \psi^<_\omega \circ \psi^<_{\omega^*})(x) = \lim_{x\uparrow a_j} (\Phi \circ \psi^<_\omega)(x) = \lim_{x\uparrow a_j} (\Phi \circ \psi^>_\omega)(x) ,$$

when $\psi^<$ and $\psi^>$ are sufficiently close to ψ in P^1. Therefore (mod $\mathfrak{J}_{M+1}$)

$$\delta D_{ik} = Z(\boldsymbol{\omega}^*)\sum_{n\geqq 0} \frac{1}{2}\left[\lim_{x\downarrow a_i} ((M^>)^n\alpha_k)(x) - \lim_{x\uparrow a_i} ((M^>)^n\alpha_k)(x)\right] = Z(\boldsymbol{\omega}^*)D^>_{jk} .$$

If $i \neq j$, we have $D^>_{jk} = D_{jk}$. Therefore in $\delta[D_{ik}]$ the i^{th} and j^{th} line are proportional, giving $\delta D = 0$, i.e., $D^< = D^>$.

If $i = j$, we have

$$D^> - D^< = \delta D = Z(\boldsymbol{\omega}^*)D^> ,$$

hence

$$D^> = D^< \cdot (1 - Z(\boldsymbol{\omega}^*))^{-1}$$

as announced. □

Appendix B. Generalized Transfer Operators

As before, $\mathscr{B}$ denotes the Banach space of functions $\Phi : \mathbb{R} \to \mathbb{C}$ of bounded variation. We use on $\mathscr{B}$ the norm Var defined by

$$\operatorname{Var} \Phi = \lim \left[|\Phi(a_0)| + \sum_{i=1}^{n} |\Phi(a_i) - \Phi(a_{i-1})| + |\Phi(a_n)|\right] ,$$

where the limit is taken over finite sets $\{a_0,\dots,a_n\}$ (with $a_0 < a_1 < \cdots < a_n$) ordered by inclusion.

We also write

$$\mathscr{B}_\infty = \{\Phi \in \mathscr{B}\colon \{x\colon \Phi(x) \neq 0\} \text{ is countable}\}$$

and let $\|\cdot\|^{\#}$ denote the quotient norm on $\mathscr{B}^{\#} = \mathscr{B}/\mathscr{B}_{\infty}$. We have then

$$\|[\Phi]\|^{\#} = \mathrm{Var}^{\#}\Phi ,$$

where $\mathrm{Var}^{\#}$ is defined like Var, but with $\{a_0, \ldots, a_n\}$ ranging over the finite subsets of a generic dense set $\mathbf{R}$. By this we mean that the closure of $\mathbf{R}$ is $\mathbb{R}$, and that $\mathbf{R}$ is disjoint from any countable set given in advance. (For the definition of $\mathrm{Var}^{\#}\Phi$, the set to avoid is that of discontinuities of Φ.) Using $\mathrm{Var}^{\#}$ it is easy to implement Remark 1.7, and obtain a #-version of Theorem B.1 below.

We let Ω be a countable set and for each $\omega \in \Omega$ we suppose that

Λ_ω is an interval of $\mathbb{R}$ (not necessarily open or closed).

$\psi_\omega : \Lambda_\omega \to \mathbb{R}$ is continuous and strictly monotone (i.e. $\psi_\omega : \Lambda_\omega \to \psi_\omega\Lambda_\omega$ is a homeomorphism).

$\varphi_\omega : \Lambda_\omega \to \mathbb{C}$ has bounded variation. We also assume that

$$V = \sum_{\omega\in\Omega} \mathrm{Var}\varphi_\omega < \infty .$$

[In order to define $\mathrm{Var}\,\varphi_\omega$, we extend φ_ω to be 0 on $\mathbb{R}\backslash\Lambda_\omega$.]

We write $\varepsilon_\omega = +1$ if ψ_ω is increasing, -1 if ψ_ω is decreasing [we make an arbitrary choice if Λ_ω is reduced to one point or empty].

On the Banach space $\mathscr{B}$ we define the operators $\mathscr{M}$ and $\widehat{\mathscr{M}}$ such that

$$\mathscr{M}\Phi(x) = \sum_\omega \varphi_\omega(x)\Phi(\psi_\omega x) ,$$

$$\widehat{\mathscr{M}}\Phi(x) = \sum_\omega \varepsilon_\omega\varphi_\omega(\psi_\omega^{-1}x)\Phi(\psi_\omega^{-1}x) .$$

[We let $\varphi_\omega(x)\Phi(\psi_\omega x) = 0$ if $x \notin \Lambda_\omega$ and $\varphi_\omega(\psi_\omega^{-1}x)\Phi(\psi_\omega^{-1}x) = 0$ if $x \notin \psi_\omega\Lambda_\omega$.] The operators $\mathscr{M}$ and $\widehat{\mathscr{M}}$ are bounded. If we denote by $\|M\|$ the norm of the operator M acting on $\mathscr{B}$ (with the Var norm) and by $\|M\|_0$ the norm of the operator M acting on bounded function (with the uniform norm $\|\cdot\|_0$) we have

$$\|\mathscr{M}\|,\ \|\widehat{\mathscr{M}}\|,\ \|\mathscr{M}\|_0, \|\widehat{\mathscr{M}}\|_0 < V .$$

We write

$$R = \lim_{m\to\infty} (\|\mathscr{M}^m\|_0)^{1/m} ,$$

$$\widehat{R} = \lim_{m\to\infty} (\|\widehat{\mathscr{M}}^m\|_0)^{1/m} .$$

The submultiplicativity of $m \mapsto \|\mathscr{M}^m\|_0$, $\|\widehat{\mathscr{M}}^m\|_0$ guarantees the existence of the limits; R and $\widehat{R}$ are in fact the spectral radii of $\mathscr{M}, \widehat{\mathscr{M}}$ acting on bounded functions $X \to \mathbb{C}$. In general $R \neq \widehat{R}$.

B.1. Theorem.[1] (a) *The spectral radius of* $\mathscr{M}$ *acting on* $\mathscr{B}$ *is* $\leqq \max(R,\widehat{R})$ *and* $\geqq \widehat{R}$.

(b) *The essential spectral radius of* $\mathscr{M}$ *is* $\leqq \widehat{R}$.

(c) *If* $\varphi_\omega \geqq 0$ *for all* ω, *the spectral radius of* $\mathscr{M}$ *is* $\geqq R$. *If furthermore* $\widehat{R} < R$, *then* R *is an eigenvalue of* $\mathscr{M}$, *and there is a corresponding eigenfunction* $\Phi_R \geqq 0$.

[1] This is an improved version of the theorem of [4].

Note that $\mathscr{M}, \widehat{\mathscr{M}}$ play symmetric roles: $\mathscr{M}$ may be replaced by $\widehat{\mathscr{M}}$ in the theorem if $R, \widehat{R}$ are interchanged.

It will be convenient to assume that all Λ_ω and $\psi_\omega \Lambda_\omega$ are contained in $(-1,+1)$. This can be achieved by the embedding $\mathbb{R} \to (-1,+1)$ given by $x \to x(1+x^2)^{-1/2}$. We can then also extend the ψ_ω to homeomorphisms $\mathbb{R} \to \mathbb{R}$, and take $\varphi_\omega | (\mathbb{R} \backslash \Lambda_\omega) = 0$.

The proof of the theorem will use bilinear forms on $\mathscr{B}$ which we now introduce. If $\Phi, \Psi : \mathbb{R} \to \mathbb{C}$ are of bounded variation we may define

$$\langle \Psi, \Phi \rangle_+ = \lim \sum_{i=1}^{n} \Psi(a_i)(\Phi(a_i) - \Phi(a_{i-1})) ,$$

$$\langle \Psi, \Phi \rangle_- = \lim \sum_{i=1}^{n} \Psi(a_{i-1})(\Phi(a_i) - \Phi(a_{i-1})) ,$$

$$\begin{aligned}\langle \Psi, \Phi \rangle &= \frac{1}{2}\langle \Psi, \Phi \rangle_+ + \frac{1}{2}\langle \Psi, \Phi \rangle_- \\ &= \lim \sum_{i=0}^{n} \frac{\Psi(a_i) + \Psi(a_{i-1})}{2} (\Phi(a_i) - \Phi(a_{i-1})) .\end{aligned}$$

The limits are taken over finite sets $\{a_0, \ldots, a_n\}$ (with $a_0 < a_1 < \cdots < a_n$) ordered by inclusion. The limits for $\langle \Psi, \Phi \rangle_\pm$ exist by monotonicity if Φ, Ψ are real monotone and Φ is constant on $(\infty, a]$ and $[b, \infty)$. Therefore (using linear combinations and density) the limits exist in general.

Note that $\langle \Psi, \Phi \rangle$ depends only on the restriction of Ψ to a small neighborhood of the support of Φ. Also

$$|\langle \Psi, \Phi \rangle| \leqq \|\Psi\|_0 \operatorname{Var} \Phi .$$

Let $\mathscr{B}_0 = \{\Phi \in \mathscr{B}\colon \lim_{|x| \to \infty} \Phi(x) = 0\}$ and denote by Ψ_x the characteristic function of $(-\infty, x)$. Using the linear form

$$\Psi \mapsto \alpha(\Psi) = \langle \Psi, \Phi \rangle ,$$

we define

$$\Phi_\alpha(x) = 2\alpha(\Psi_x) - \lim_{y \nearrow x} \alpha(\Psi_y) .$$

When $\Phi \in \mathscr{B}_0$, it is easily checked that $\Phi_\alpha = \Phi$. More generally if $\alpha : \mathscr{B} \to \mathbb{C}$ is linear and satisfies

$$|\alpha(\Psi)| \leqq C_\alpha \|\Psi\|_0 ,$$

the function $x \mapsto \alpha(\Psi_x)$ has Var $\leqq 2C_\alpha$ and

$$\operatorname{Var} \Phi_\alpha \leqq 6C_\alpha .$$

[Φ_α is thus in $\mathscr{B}$, but not necessarily in $\mathscr{B}_0$. Furthermore it is not claimed that $\langle \Psi, \Phi_\alpha \rangle = \alpha(\Psi)$.]

B.2. Proof of part (a). Using the notation

$$\varphi_{\omega_1 \cdots \omega_m}(x) = \varphi_{\omega_1}(x) \cdots \varphi_{\omega_m}(\psi_{\omega_{m-1}} \cdots \psi_{\omega_1} x) ,$$

we have

$$\langle \Psi, \mathscr{M}^m \Phi \rangle = \sum_{\omega_1 \cdots \omega_m} \langle \Psi, \varphi_{\omega_1 \cdots \omega_m} \cdot (\Phi \circ \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1}) \rangle .$$

We may write

$$
\begin{aligned}
&\langle \Psi, \varphi_{\omega_1\cdots\omega_m} \cdot (\Phi \circ \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_1})\rangle \\
&= \sum_{k=1}^{m} \lim \sum_{i=1}^{m} \frac{1}{2}\{[\varepsilon_{\omega_1} \cdots \varepsilon_{\omega_{k-1}} \cdot (\varphi_{\omega_1\cdots\omega_{k-1}} \cdot \Psi) \circ \psi_{\omega_1}^{-1} \circ \cdots \circ \psi_{\omega_{k-1}}^{-1}](a_i) \\
&\quad \cdot [\varphi_{\omega_{k+1}\cdots\omega_m} \cdot (\Phi \circ \psi_{\omega_m} \circ \cdots \circ \psi_{\omega_{k+1}})](\psi_{\omega_k} a_{i-1} + \mathrm{sym}\} \\
&\quad \cdot [\varphi_{\omega_k}(a_i) - \varphi_{\omega_k})(a_{i-1})] \\
&\quad + \lim \sum_{i=1}^{n} \frac{1}{2}\{[\varepsilon_{\omega_1} \cdots \varepsilon_{\omega_m} \cdot (\varphi_{\omega_1\cdots\omega_m} \cdot \Psi) \circ \psi_{\omega_1}^{-1} \circ \cdots \circ \psi_{\omega_m}^{-1}](a_i) + \mathrm{sym}\} \\
&\quad \cdot [\Phi(a_i) - \Phi(a_{i-1})] \, ,
\end{aligned}
$$

where the "sym" terms are obtained by exchanging a_i and a_{i-1}. Note that when the function $\psi_{\omega_{k-1}} \circ \cdots \circ \psi_{\omega_1}$ is decreasing, the change of variables that it defines interchanges "symmetric" terms and produces a negative sign (this is reflected in the factor $\varepsilon_{\omega_1} \cdots \varepsilon_{\omega_{k-1}}$ of the formula). We have thus

$$
|\langle \Psi, \mathscr{M}^m \Phi\rangle| \leqq \sum_{k=1}^{m} \|\widehat{\mathscr{M}}^{k-1}\Psi\|_0 \, \|\mathscr{M}^{m-k}\Phi\|_0 V + \|\widehat{\mathscr{M}}^m \Psi\|_0 \operatorname{Var} \Phi \, .
$$

Therefore if $\xi > \max(R, \widehat{R})$, there is $C > 0$ such that

$$
\begin{aligned}
|\langle \Psi, \mathscr{M}^m \Phi\rangle| &\leqq C(m\xi^m \|\Psi\|_0 \|\Phi\|_0 + \xi^m \|\Psi\|_0 \operatorname{Var} \Phi) \\
&\leqq (m+1)C\xi^m \|\Psi\|_0 \operatorname{Var} \Phi \, ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\operatorname{Var} \mathscr{M}^m \Phi &\leqq 6(m+1)C\xi^m \operatorname{Var} \Phi \, , \\
\|\mathscr{M}^m\| &\leqq 6(m+1)C\xi^m \, ,
\end{aligned}
$$

and finally

$$
\text{spectral radius } \mathscr{M} \leqq \max(R, \widehat{R}). \qquad \square
$$

B.3. Proof of part (b). If (K_m) is a sequence of operators of finite rank we have the general formula[2]

$$
\text{essential spectral radius of } \mathscr{M} \leqq \limsup_{m\to\infty} (\|\mathscr{M}^m - K_m\|)^{1/m} \, .
$$

Let $\xi > \widehat{R}$; there is thus $C > 0$ such that

$$
\|\widehat{\mathscr{M}}^m\|_0 \leqq C\xi^m
$$

for all m. To prove (b) we will show that (for suitable K_m) we have

$$
\|\mathscr{M}^m - K_m\| \leqq P(m) \cdot \xi^m \, ,
$$

where $P(m)$ is a polynomial (of degree 1) in m.

[2] This is a relatively elementary fact, which constitutes the "easy" part of Nussbaum's essential spectral radius formula (Nussbaum [3]).

We can choose a finite set $\Omega^* \subset \Omega$ so that the operator $\mathscr{M}^*$ defined by

$$(\mathscr{M}^* \Phi)(x) = \sum_{\omega \in \Omega^*} \varphi_\omega(x) \Phi(\psi_\omega x)$$

is arbitrarily close to $\mathscr{M}$. We have indeed

$$\|\mathscr{M} - \mathscr{M}^*\|, \|\widehat{\mathscr{M}} - \widehat{\mathscr{M}}^*\| \leqq \sum_{\omega \in \Omega \setminus \Omega^*} \operatorname{Var} \varphi_\omega \,,$$

$$\|\mathscr{M}^*\|, \|\widehat{\mathscr{M}}^*\| < V \,.$$

We may thus take Ω^* (depending on m) such that

$$\|\mathscr{M}^k - \mathscr{M}^{*k}\|, \|\widehat{\mathscr{M}}^k - \widehat{\mathscr{M}}^{*k}\| \leqq \xi^k$$

for $k = 1, \ldots, m$. The same estimates may be assumed to hold for the $\| \;\|_0$ operator norms; in particular we obtain

$$\|\widehat{\mathscr{M}}^{*k}\|_0 \leqq (C+1)\xi^k$$

for $k = 1, \ldots, m$.

For each $\omega \in \Omega^*$ we decompose Λ_ω into finitely many intervals $\Lambda_{(\omega, i)}$ and define a function $\overline{\varphi}_\omega$ with constant value $\varphi(\omega, i)$ in $\Lambda_{(\omega, i)}$. Taking $\varphi(\omega, i) \in \varphi_\omega \Lambda_{(\omega, i)}$ we have

$$\operatorname{Var} \overline{\varphi}_\omega \leqq \operatorname{Var} \varphi_\omega \,.$$

Given $\delta > 0$ we may also assume that the $\Lambda_{(\omega, i)}$ are such that

$$\|\varphi_\omega - \overline{\varphi}_\omega\|_0 < \delta / \operatorname{card} \Omega^* \,.$$

We define the operator $\bar{\mathscr{M}}$ by

$$(\bar{\mathscr{M}} \Phi)(x) = \sum_{\omega \in \Omega^*} \overline{\varphi}_\omega(x) \Phi(\psi_\omega x) \,,$$

and obtain thus

$$\|\mathscr{M}^* - \bar{\mathscr{M}}\|_0, \|\widehat{\mathscr{M}}^* - \widehat{\bar{\mathscr{M}}}\|_0 \leqq \delta \,,$$

$$\|\bar{\mathscr{M}}\|, \|\widehat{\bar{\mathscr{M}}}\|, \|\bar{\mathscr{M}}\|_0, \widehat{\bar{\mathscr{M}}} \leqq V \,.$$

We may thus choose δ sufficiently small that

$$\|\mathscr{M}^{*k} - \bar{\mathscr{M}}^k\|_0, \|\widehat{\mathscr{M}}^{*k} - \widehat{\bar{\mathscr{M}}}^k\|_0 \leqq \xi^k$$

for $k = 1, \ldots, m$. In particular

$$\|\widehat{\bar{\mathscr{M}}}^k\|_0 \leqq (C+2)\xi^k$$

for $k = 1, \ldots, m$.

We note that the linear form associated with $\mathscr{M}^{*m}\Phi$ is

$$\Psi \mapsto \langle \Psi, \mathscr{M}^{*m}\Phi \rangle = \sum_{k=1}^{m} \lim \sum_{i=1}^{n} \sum_{\omega_k} \frac{1}{2} \{[(\widehat{\mathscr{M}}^{*k-1}\Psi)(a_i)] \cdot [(\mathscr{M}^{*m-k}\Phi)(\psi_{\omega_k} a_{i-1})]$$

$$+ \text{sym}\} \cdot [\varphi_{\omega_k}(a_i) - \varphi_{\omega_k}(a_{i-1})]$$

$$+ \lim \sum_{i=1}^{n} \frac{1}{2} \{(\widehat{\mathscr{M}}^{*m}\Psi)(a_i) + \text{sym}\} \cdot [\Phi(a_i) - \Phi(a_{i-1})] \,.$$

This expression will be used in a moment.

Let us denote by $\psi_{(\omega,i)}$ the restriction of ψ_ω to $\Lambda_{(\omega,i)}$. For fixed k the intervals of definition of the $\psi_{(\omega_m,i_m)} \circ \cdots \circ \psi_{(\omega_{k+1},i_{k+1})}$ generate a partition of $\mathbb{R}$ into a finite set $\mathfrak{J}_{m-k}$ of intervals. Let $\mathfrak{J}'_{m-k}$ be the set of interval endpoints, and $\mathfrak{J}''_{m-k}$, the set of interval interiors (this is a finite set of open intervals). For each $I \in \mathfrak{J}''_{m-k}$, choose $x_I \in I$ and define the operator $\mathcal{N}_{m-k}$ by

$$(\mathcal{N}_{m-k}\Phi)(x) = \begin{cases} (\bar{\mathcal{M}}^{m-k}\Phi)(x) & \text{if } x \in \mathfrak{J}'_{m-k} \\ \langle \Psi_{x_I}, \bar{\mathcal{M}}^{m-k}\Phi\rangle & \text{if } x \in I \in \mathfrak{J}''_{m-k} \end{cases} .$$

Finally we define the operator K_m by

$$K_m\Phi = \Phi_\alpha ,$$

where $\Phi_\alpha \in \mathcal{B}$ is the function associated with the linear form α:

$$\Psi \mapsto \alpha(\Psi) = \sum_{k=1}^{m} \lim \sum_{i=1}^{n} \sum_{\omega_k} \frac{1}{2}\{[(\widehat{\mathcal{M}}^{*k-1}\Psi)(a_i)] \cdot [(\mathcal{N}_{m-k}\Phi)(\psi_{\omega_k}a_{i-1})] + \text{sym}\}$$
$$\cdot [\varphi_{\omega_k}(a_i) - \varphi_{\omega_k}(a_{i-1})] .$$

Therefore K_m is of finite rank.

The values of $\bar{\mathcal{M}}^{m-k}\Phi - \mathcal{N}_{m-k}\Phi$ on the open interval $I \in \mathfrak{J}''_{m-k}$ are determined by

$$\bar{\mathcal{M}}^{m-k}\Phi(x) - \mathcal{N}_{m-k}\Phi(x) = 2\widetilde{\Phi}(x) - \lim_{y \nearrow x} \widetilde{\Phi}(y) ,$$

where

$$\widetilde{\Phi}(x) = \langle \Psi_x - \Psi_{x_I}, \bar{\mathcal{M}}^{m-k}\Phi\rangle = \langle \widehat{\bar{\mathcal{M}}}^{m-k}(\Psi_x - \Psi_{x_I}), \Phi\rangle ,$$

so that

$$|\widetilde{\Phi}(x)| \leqq \|\widehat{\bar{\mathcal{M}}}^{m-k}\|_0 \cdot \operatorname{Var}\Phi$$

and

$$\|\bar{\mathcal{M}}^{m-k}\Phi - \mathcal{N}_{m-k}\Phi\|_0 \leqq 3\|\widehat{\bar{\mathcal{M}}}^{m-k}\|_0 \operatorname{Var}\Phi \leqq 3(C+2)\xi^{m-k} \operatorname{Var}\Phi .$$

Since we also have

$$\|\mathcal{M}^{*m-k}\Phi - \bar{\mathcal{M}}^{m-k}\Phi\|_0 \leqq \xi^{m-k}\|\Phi\|_0 ,$$

we find

$$\|\mathcal{M}^{*m-k}\Phi - \mathcal{N}_{m-k}\Phi\|_0 \leqq (3C+7)\xi^{m-k} \operatorname{Var}\Phi .$$

By definition of K_m, we find that $\mathcal{M}^{*m}\Phi - K_m\Phi$ is the function associated with the linear form $\Psi \mapsto \langle \Psi, \mathcal{M}^{*m}\Phi\rangle - \alpha(\Psi)$. We have the estimate

$$|\langle \Psi, \mathcal{M}^{*m}\Phi\rangle - \alpha(\Psi)| \leqq \sum_{k=1} \Bigg| \lim \sum_{i=1}^{n} \sum_{\omega_k} \frac{1}{2}\{[(\widehat{\mathcal{M}}^{*k-1}\Psi)(a_i)]$$
$$\times[(\mathcal{M}^{*m-k}\Phi - \mathcal{N}_{m-k}\Phi)(\psi_{\omega_k}a_{i-1})] + \text{sym}\} \cdot [\varphi_{\omega_k}(a_i) - \varphi_{\omega_k}(a_{i-1})]\Bigg|$$
$$+ \left|\lim \sum_{i=1}^{n} \frac{1}{2}\{(\widehat{\mathcal{M}}^{*m}\Psi)(a_i) + \text{sym}\} \cdot [\Phi(a_i) - \Phi(a_{i-1})]\right|$$
$$\leqq \sum_{k=1}^{m} (C+1)\xi^{k-1}\|\Psi\|_0 \cdot (3C+7)\xi^{m-k} \operatorname{Var}\Phi \cdot V + (C+1)\xi^m\|\Psi\|_0 \cdot \operatorname{Var}\Phi$$
$$= (mC' + C + 1)\xi^m\|\Psi\|_0 \operatorname{Var}\Phi$$

and

$$\|\mathscr{M}^m - K_m\| \leqq \xi^m + 6(mC' + C + 1)\xi^m = P(m) \cdot \xi^m ,$$

with $P(m) = 6mC' + 6C + 7$, of degree 1 in m as announced. □

B.4. Proof of part (c). We refer to [4] where a similar result is proved. The proof given in [4] also applies here, with inessential modifications.

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Communicated by J.-P. Eckmann