

Transparent Potentials at Fixed Energy in Dimension Two. Fixed-Energy Dispersion Relations for the Fast Decaying Potentials

Piotr G. Grinevich^{1,*}, Roman G. Novikov²

¹ Landau Institute for Theoretical Physics, Kosygina 2, Moscow, 117940, Russia
e-mail: pgg@cpd.landau.free.net

² CNRS, U.R.A. 758, Département de Mathématiques, Université de Nantes,
F-44072, Nantes Cedex 03, France
e-mail: novikov@math.univ-nantes.fr

Received: 11 October 1994

Abstract: For the two-dimensional Schrödinger equation

$$[-\Delta + v(x)]\psi = E\psi, \quad x \in \mathbb{R}^2, \quad E = E_{\text{fixed}} > 0 \quad (*)$$

at a fixed positive energy with a fast decaying at infinity potential $v(x)$ dispersion relations on the scattering data are given. Under “small norm” assumption using these dispersion relations we give (without a complete proof of sufficiency) a characterization of scattering data for the potentials from the Schwartz class $S = C_{\infty}^{(\infty)}(\mathbb{R}^2)$. For the potentials with zero scattering amplitude at a fixed energy E_{fixed} (transparent potentials) we give a complete proof of this characterization. As a consequence we construct a family (parametrized by a function of one variable) of two-dimensional spherically-symmetric real potentials from the Schwartz class S transparent at a given energy. For the two-dimensional case (without assumption that the potential is small) we show that there are no nonzero real exponentially decreasing, at infinity, potentials transparent at a fixed energy. For any dimension greater or equal to 1 we prove that there are no nonzero real potentials with zero forward scattering amplitude at an energy interval. We show that KdV-type equations in dimension 2+1 related with the scattering problem (*) (the Novikov–Veselov equations) do not preserve, in general, these dispersion relations starting from the second one. As a corollary these equations do not preserve, in general, the decay rate faster than $|x|^{-3}$ for initial data from the Schwartz class.

* The main part of this work was fulfilled during the visit of one of the authors (P.G.G.) to the University of Nantes in June 1994. He is grateful to the University of Nantes for the invitation and the financial support of this visit. He was also supported by the Soros International Scientific foundation grant MD 8000 and by the Russian Foundation for Fundamental Studies grant 93-011-16087.

1. Introduction

An interesting property of the fixed-energy scattering problem for the Schrödinger equation in dimension 2,

$$L\psi = E\psi, \quad L = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + v(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad E \in \mathbb{R} \quad (E \text{ is fixed}) \tag{0.1}$$

is its deep connection with the soliton theory, i.e. the following methods can be effectively applied to this problem: the finite-gap technique, the nonlocal Riemann problem method, the $\bar{\partial}$ -problem method and this problem possesses an infinite-dimensional algebra of symmetries generated by KdV-type equations in dimension $2 + 1$ (Novikov–Veselov hierarchy). Scattering transform for Eq. (0.1) allows us to integrate these equations. Inverse scattering problem for (0.1) is closely connected also with the inverse boundary value problem (Calderon problem).

The problems mentioned above were studied in the papers [1–22] and others (some historical remarks are given in the end of this introduction). In the present paper we study the scattering transform for Eq. (0.1) for potentials with decay rate at infinity $1/|x|^{M+2+\varepsilon}$, $\varepsilon > 0$, $M = 0, 1, 2, \dots$. We show that such decay rate results in $M+1$ algebraic relations on the scattering data (we shall call them fixed-energy dispersion relations).

Let us recall the definition of the scattering data for (0.1).

We assume that

$$v(x) = \bar{v}(x), \quad v(x) \in L^\infty(\mathbb{R}^2), \quad |v(x)| < q(1 + |x|)^{-2-\varepsilon}, \quad \varepsilon > 0, \quad q > 0, \tag{0.2}$$

where $|x| = \sqrt{x_1^2 + x_2^2}$.

For $E > 0$ and any $k = (k_1, k_2) \in \mathbb{R}^2$, such that $k^2 = E$, there exists an unique bounded solution $\varphi^+(x, k)$ of Eq. (0.1) with the following asymptotics:

$$\varphi^+(x, k) = e^{ikx} - i\pi\sqrt{2\pi}e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right) \frac{e^{l|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{|x|}\right). \tag{0.3}$$

The function $f(k, l)$ in (0.3), $k \in \mathbb{R}^2$, $l \in \mathbb{R}^2$, $k^2 = l^2 = E$ is called the scattering amplitude.

Let $k \in \mathbb{C}^2$, $k^2 = E$, $\text{Im } k \neq 0$. Let, in addition $\Delta(k) \neq 0$, where Δ is the modified Fredholm determinant of the integral equation (1.3). Then there exists a unique solution of (0.1) such that

$$\psi(k, x) = e^{ikx} (1 + o(1)), \quad \text{Im } k \neq 0, \quad \text{for } |x| \rightarrow \infty. \tag{0.4}$$

It was shown in [12] that there exists a special real function $Q(|E|, \varepsilon)$ with the following properties $Q(|E|, \varepsilon) > 0$ as $E \neq 0$, $Q(|E|, \varepsilon) \rightarrow +\infty$ for fixed ε as $|E| \rightarrow \infty$ such that if a potential $v(x)$ satisfies (0.2) and

$$q < Q(|E|, \varepsilon), \tag{0.5}$$

then

1) Fredholm determinant of Eq. (1.3) $\Delta(k) \neq 0$ for all $k^2 = E$.

2) The fixed-energy scattering data for the potential $v(x)$ is “small enough” for unique solvability of the equations of inverse scattering.

The “small norm” condition (0.5) means that the potential $v(x)$ is small being compared with energy.

The solutions of the Schrödinger equation with asymptotics (0.4) were introduced to the scattering theory by L.D. Faddeev [23] as solutions of the integral equation (1.3). It can be shown [12] that for $E \in \mathbb{R}$, $\text{Im } k \neq 0$,

$$\psi(k, x) = e^{ikx} - \pi \operatorname{sgn}(\operatorname{Im} k_2 \bar{k}_1) e^{ikx} \left(\frac{a(k)}{-k_2 x_1 + k_1 x_2} + \frac{e^{-2i \operatorname{Re} k \cdot x}}{-\bar{k}_2 x_1 + \bar{k}_1 x_2} b(k) + o\left(\frac{1}{|x|}\right) \right), \tag{0.6}$$

where the function $a(k)$ and $b(k)$ are expressed through Faddeev’s scattering data by the formula (1.7). The formula similar to (0.6) can be written for any complex E .

We consider the functions $a(k)$ and $b(k)$ as additional scattering data to $f(k, l)$ for $E > 0$ and as the main scattering data for other E .

Using the results of [8, 9] it was shown in [12] that at fixed energy under conditions (0.2), (0.5) the scattering amplitude $f(k, l)$ and the function $b(k)$ uniquely determine the potential. From the inverse scattering problem it follows that in the slow decaying case $f(k, l)$ and $b(k)$ are independent at fixed energy.

For the potentials exponentially decreasing at infinity the uniqueness of the reconstruction via the fixed energy scattering amplitude was proved in [9, 12] for the two-dimensional case under the conditions (0.2), (0.5) at the fixed energy and for the three-dimensional case in [24] with and [27] without the “small norm” assumption.

The scattering amplitude at a fixed energy is insufficient, in general, to reconstruct the potential uniquely.

In the exact formulation it was shown in the series of papers [31–33] and others started by pioneering work of T. Regge [31]. In the works of this series fixed-energy inverse scattering problem is studied in the 3-dimensional spherically-symmetrical case. The existence of nonzero multidimensional potentials with zero scattering amplitude at a fixed energy (transparent at fixed energy potentials) was observed by T. Regge in [31] and with a different method it was shown by R.G. Newton in [32]. The properties of R.G. Newton’s transparent potentials were clarified by P.C. Sabatier in [33], where the one-dimensional family of transparent at a fixed energy potentials was given and it was shown that nonzero potentials from this family decrease at infinity as $|x|^{-3/2}$.

In [8] it was shown that transparent at a fixed energy two-dimensional potentials with the “small norm” assumption are parametrized by a function of two variables. From the results of the present paper it follows that constructed in [8] transparent potentials decrease, in general, as $|x|^{-2}$. The explicit real nonsingular rational two-dimensional potentials with zero scattering amplitude at a fixed energy are given in [10]. They also decrease as $|x|^{-2}$.

The central point of the present paper is a characterization of the scattering data at fixed positive energy for the real-valued potentials of the Schwartz class $S = C_\infty^{(\infty)}(\mathbb{R}^2)$. On the basis of this characterization we construct (Proposition 1, Theorem 2) real two-dimensional spherically-symmetric potentials from the

Schwartz class S with zero scattering amplitude at a fixed energy $E > 0$. The classical scattering solution $\varphi^+(x, k)$ for such potentials has the following asymptotics at infinity:

$$\varphi^+(x, k) = e^{ikx} + O(1/|x|^\infty) \quad \text{for } k^2 = E. \quad (0.7)$$

Further, (Theorem 3) we prove the following statement. Let the fixed energy scattering amplitudes of two exponentially decreasing potentials with the property (0.2) coincide and one of these potentials possesses, in addition, the property (0.5) at this fixed energy. Then these two potentials coincide. This statement improves the corresponding theorem from [9, 12]. In particular, there exists no nonzero two-dimensional exponentially decreasing real nonsingular potentials transparent at a fixed energy.

We prove that there are no nonzero real potentials transparent at an energy interval. Moreover, we prove that if the forward scattering amplitude is equal to zero at an energy interval then the real potential is equal to zero identically (Theorem 4). This result is valid without the small norm assumption in any dimension greater or equal to 1.

The most nontrivial part of our characterization theorem is the existence of additional algebraic relations on the scattering data – fixed energy dispersion relations.

Let the potential $v(x)$ satisfy (0.2), (0.5). Then the scattering amplitude $f(k, l)$ satisfies (1.29) and b satisfies (1.25). Assume now, that, in addition, the potential $v(x)$ belongs to the Schwartz class $S = C^\infty(\mathbb{R})$. Then for functions f, b we have (3.8), (3.9). In the inverse problem we may start from arbitrary functions f, b satisfying (1.29), (3.8) and (1.25), (3.9) respectively which are “small” enough for unique solvability of integral equations of the inverse problem but the corresponding potential may decrease at infinity rather slow. Necessary conditions on the scattering data for the fast decaying potentials were found in [9, 12] for the positive energy case. Another set of necessary conditions for the fast decay rate were found in [11] for the negative energy case. In the present paper we show that analogs of the necessary conditions from [11] (we call them fixed-energy dispersion relations) are valid in the positive case too. (For three-dimensional problem an analog of the first dispersion relation was used in [24].)

In the present paper we show that for real potentials from the class S under the “small norm” assumption the scattering data $f(k, l)$, $b(k)$ satisfy (1.29), (3.8), (1.25), (3.9) and $2 \cdot \infty + 2$ additional conditions from Sect. 3 corresponding to $M = \infty$ are fulfilled. Let $f(k, l)$, $b(k)$ be arbitrary functions satisfying (1.29), (3.8), (1.25), (3.9) and $2 \cdot \infty + 2$ additional conditions from Sect. 3. Assume also that $f(k, l)$, $b(k)$ are sufficiently small, so the integral equations of the inverse problem are uniquely solvable. Then our hypothesis is that the corresponding potential is from the class S . Some restriction in time gives us no possibility to carry out in the present paper a complete proof of this hypothesis. In the present paper we prove this hypothesis (Theorem 1) in the transparent case $f(k, l) \equiv 0$, $k^2 = l^2 = E$ at fixed energy E . In this case the “small norm” assumption for $b(k)$ is not necessary.

Results on inverse scattering at fixed energy for Eq. (0.1) can be applied to the solution of the Cauchy problem (and to the construction of explicit soliton

type solutions) for the KdV-type equation in dimension $2 + 1$ (Novikov–Veselov equation)

$$\begin{aligned} \frac{\partial v(x_1, x_2, t)}{\partial t} &= 2 \frac{\partial^3 v}{\partial x_1^3} - 6 \frac{\partial^3 v}{\partial x_1 \partial x_2^2} + 2 \left(\frac{\partial(Uv)}{\partial x_1} + \frac{\partial(Wv)}{\partial x_2} \right) - 2E \left(\frac{\partial U}{\partial x_1} + \frac{\partial W}{\partial x_2} \right), \\ v &= \bar{v}, \quad E \in \mathbb{R}, \quad x_1, x_2, t \in \mathbb{R}, \\ U(x_1, x_2, t) &= \frac{3}{\pi} \int_{\mathbb{R}^2} \frac{v(x'_1, x'_2, t) ((x_1 - x'_1)^2 - (x_2 - x'_2)^2)}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2)^2} dx'_1 dx'_2, \\ W(x_1, x_2, t) &= -\frac{6}{\pi} \int_{\mathbb{R}^2} \frac{v(x'_1, x'_2, t) (x_1 - x'_1)(x_2 - x'_2)}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2)^2} dx'_1 dx'_2, \end{aligned} \tag{0.8}$$

and its higher analogs. Equation (0.8) is contained implicitly in the paper of S.V. Manakov [1] as an equation possessing the following representation:

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{0.9}$$

(Manakov $L - A - B$ triple), where L is the Schrödinger operator from (0.1), A and B are suitable differential operators of the third and zero order respectively. Equation (0.8) was written in an explicit form by S.P. Novikov and A.P. Veselov in [3, 4], where higher analogs of (0.8) were also constructed.

The both Kadomtsev–Petviashvily equations can be obtained from (0.8) by considering an appropriate limit $E \rightarrow \pm\infty$ (V.E. Zakharov).

In terms of the scattering data the nonlinear equation (0.8) takes the form

$$\begin{aligned} \frac{\partial b(k, t)}{\partial t} &= 2i \left[k_1^3 + \bar{k}_1^3 - 3k_1 k_2^2 - 3\bar{k}_1 \bar{k}_2^2 \right] b(k, t), \quad k \in \mathbb{C}^2, \quad \text{Im } k \neq 0, \quad k^2 = E, \\ \frac{\partial f(k, l, t)}{\partial t} &= 2i \left[k_1^3 - 3k_1 k_2^2 - l_1^3 + 3l_1 l_2^2 \right] f(k, l, t), \quad k, l \in \mathbb{R}^2, \quad k^2 = l^2 = E. \end{aligned} \tag{0.10}$$

In the present paper (Corollary 1, Theorem 1, Theorem 5) we obtain the following result.

Let $v(x, t)$ be a solution of (0.8) with the following Cauchy data $v(x) = v(x, 0)$:

- 1) $v(x) \in C_{\infty}^{(\infty)}(\mathbb{R}^2)$,
- 2) $v(x)$ satisfies (0.5),
- 3) $v(x)$ is transparent at the energy E , i.e. $f(k, l) \equiv 0$ at the energy E ,
- 4) $v(x) \not\equiv 0$.

Then for any $t \neq 0$ $v(x, t) \in C_3^{(\infty)}(\mathbb{R}^2)$ and $v(x, t) \notin C_{3+\varepsilon}^{(0)}(\mathbb{R}^2)$ (i.e. $v(x, t)$ decreases at $|x| \rightarrow \infty$ exactly as $|x|^{-3}$).

In Theorem 5 under the “small norm” assumption we obtain, in particular, the following result. Let the Cauchy data $v(x, 0) \in C_{\infty}^{(\infty)}(\mathbb{R}^2)$ generate a solution $v(x, t)$ of (0.8) such that at a fixed $t \neq 0$ $v(x, t)$ decreases at infinity as $|x|^{-3-\varepsilon}$, $\varepsilon > 0$. Then

$$\int_{x \in \mathbb{R}^2} v(x) dx = 0. \tag{0.11}$$

We have, also, the following hypothesis. Under “small norm” assumption the Cauchy data $v(x, 0) \in C_{\infty}^{(\infty)}(\mathbb{R}^2)$ for Eq. (0.8) generates a solution $v(x, t) \in C_{2+\varepsilon}^{(\infty)}(\mathbb{R}^2)$ in x , $0 < \varepsilon < \frac{1}{2}$ for all t . This solution belongs to $C_{3+\varepsilon}^{(\infty)}(\mathbb{R}^2)$ in x if (0.11) is fulfilled. The faster decay rate for all t results in additional conditions on the Cauchy data which can be written. We think, also, that this hypothesis is true without the “small norm” assumption, but it is not clear for us how to prove it in the latter case.

Let us mention the following. The decay rate of the potentials constructed in the preceding papers was not studied carefully enough. For example, we correct Corollary 1 from paper [8] and Proposition 9.4 from [12].

Historical Remarks. The relations between the fixed-energy scattering transform for the two-dimensional Schrödinger operator and nonlinear integrable equations in dimension $2 + 1$ were observed for the first time by S.V. Manakov [1].

The methods of the soliton theory were applied for the first time to the inverse problem at fixed energy for the two-dimensional Schrödinger operator in 1976 by B.A. Dubrovin, I.M. Krichever, S.P. Novikov [2] in the quasiperiodic case.

The sufficient conditions on the finite-gap scattering data which guarantee absence of the magnetic field and reality of the potential were found by S.P. Novikov and A.P. Veselov in [3, 4].

The nonlocal Riemann problem method (Manakov [28]) together with ideas from [3, 4] were applied by the authors in [5, 6] for constructing two-dimensional Schrödinger operators with decreasing potentials and explicit solutions of the corresponding KdV-type equations in dimension $2 + 1$.

The connection between the kernel of the nonlocal Riemann problem and the scattering amplitude at the fixed energy for the corresponding potentials was found by one of the authors (R.G.N.) in [7]. As a consequence a characterisation of the fixed-energy scattering amplitude with small norm for real, smooth, decaying at infinity potentials was obtained in [7].

The scattering transform at a fixed energy for general decaying at infinity two-dimensional potentials was constructed by Manakov and one of the authors (P.G.G.) in [8]. In [8] it was shown that in this scattering transform the nonlocal Riemann problem of the type [28] and the $\bar{\partial}$ -problem of the type [29] are presented simultaneously. In [8] it was shown that the connection between the fixed-energy scattering amplitude and the nonlocal Riemann problem data found in [7] is unchanged in the presence of nontrivial $\bar{\partial}$ -problem data. Thus, the $\bar{\partial}$ -problem data parametrizes the variety of all potentials with the given fixed-energy scattering amplitude. Assuming the nonlocal Riemann problem data to be identically zero transparent at a fixed energy potentials were obtained in [8]. The “spectral transform” constructed in [8] was applied to solve the Cauchy problem for equations from the Novikov–Veselov KdV-type hierarchy for the decaying at infinity Cauchy data.

In [9] it was shown by one of the authors (R.G.N.) that the scattering data introduced in [8] can be considered as a restriction of the Faddeev scattering data [23] to a fixed energy level. The connection between the scattering amplitude at a fixed energy and the nonlocal Riemann problem data was obtained in [9] once again from the point of view of the direct problem by the technique developed in [23]. In [9] the necessary conditions on the fixed-energy scattering data corresponding to the fast decaying at the infinity potentials were found and it was shown that exponentially decreasing potentials under small norm assumption are uniquely determined by the fixed-energy scattering amplitude.

The explicit examples of real nonsingular transparent at a fixed energy potentials (rational solitons) were constructed by one of the authors (P.G.G.) in [10]. These potentials decay at infinity rather slowly (as the minus second power of distance). These potentials are constructed independently by V.E. Zakharov.

The two-dimensional scattering problem at a fixed negative energy was studied by S.P. Novikov and by one of the authors (P.G.G.) in [11]. In this case we have a pure $\bar{\partial}$ -problem. In [11] it was shown that for an arbitrary nonsingular scattering data (without the small norm assumption) satisfying the reality and the absence of magnetic field reduction the solution of the inverse problem is unique and nonsingular and the L^2 spectrum of the corresponding operator lies above our fixed energy. If it is not so the $\bar{\partial}$ -problem data is singular but rather little about inverse scattering in this case is known. For the fast decaying at infinity potentials the necessary conditions on the scattering data were found in [11]. These conditions have a different origin and structure than the necessary conditions from [9].

The further development and generalization of these papers [5–11] and some results of [23–26, 18, 30] were given in [12].

In [13] by J.-P. Francoise and one of the authors (R.G.N.) the hamiltonian systems describing dynamics of poles of the rational solitons from [10] were found.

In papers [5–12] only the case of nonzero fixed energy was studied. The zero energy level was examined by M. Boiti, J. Leon, M. Manna, F. Pempinelli [14], T.Y. Tsai [15], Z. Sun, G. Uhlmann [21], A. Nachman [22].

On the other hand the studies of the inverse problem at fixed energy ($E = 0$) for the two-dimensional Schrödinger equation (for the equation $\operatorname{div}(\gamma(x)\operatorname{grad}\psi) = 0$) in a bounded domain were stimulated by the paper [16] of A.P. Calderon. In the two-dimensional case the studies of the Calderon problem were started by R. Kohn, M. Vogelius [17], J. Sylvester and G. Uhlmann [18]. The method to apply results of the two-dimensional inverse scattering at fixed energy to the Calderon problem was given for the first time by one of the authors (R.G.N.) in [19]. Among subsequent works on the Calderon problem in dimension 2 let us mention important papers of Z. Sun, G. Uhlmann [20, 21] and A. Nachman [22].

1. The Equations of Direct Scattering

The Faddeev scattering data (see [23, 24]) $h(k, l)$, $k, l \in \mathbb{C}^2$, $k^2 = l^2 = E$, $\operatorname{Im} k = \operatorname{Im} l$ for Eq. (0.1) are defined by the formula

$$h(k, l) = \frac{1}{(2\pi)^2} \int \int_{\mathbb{R}^2} e^{-ilx} \psi(x, k) v(x) dx_1 dx_2, \tag{1.1}$$

where

$$\psi(x, k) = e^{ikx} \mu(x, k), \tag{1.2}$$

$$\mu(x, k) = 1 + \int \int_{\mathbb{R}^2} g(x - y, k) v(y) \mu(y, k) dy_1 dy_2, \tag{1.3}$$

$$g(x, k) = -\frac{1}{(2\pi)^2} \int \int_{\mathbb{R}^2} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi_1 d\xi_2, \quad \operatorname{Im} k \neq 0. \tag{1.4}$$

For $k \in \mathbb{R}^2$ the following limits exist:

$$\begin{aligned} \psi_\gamma(x, k) &= \psi(x, k + i0\gamma), \quad \mu_\gamma(x, k) = \mu(x, k + i0\gamma), \\ h_\gamma(k, l) &= h(k + i0\gamma, l + i0\gamma), \quad k, l, \gamma \in \mathbb{R}^2, \quad k^2 = l^2 = E, \quad \gamma^2 = 1. \end{aligned} \quad (1.5)$$

In addition,

$$\varphi^+(x, k) = \psi_{k/|k|}(x, k), \quad f(k, l) = h_{k/|k|}(k, l), \quad (1.6)$$

where φ^+ , f are functions from (0.3). For $k^2 = E \in \mathbb{R}$, $\text{Im } k \neq 0$ $\psi(x, k)$ is the function (0.6). For $a(k)$ and $b(k)$ the following formulas are valid

$$a(k) = h(k, k), \quad b(k) = h(k, k + \xi(k)), \quad (1.7)$$

where $\xi(k)$ is a different from zero root of the equation

$$\xi^2 + 2k\xi = 0, \quad \xi \in \mathbb{R}^2. \quad (1.8)$$

In the two-dimensional fixed-energy scattering theory it is convenient to introduce new notations

$$\begin{aligned} z &= x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \\ \lambda &= \frac{k_1 + ik_2}{\sqrt{E}}, \quad \lambda' = \frac{l_1 + il_2}{\sqrt{E}}, \quad E = k_1^2 + k_2^2 = l_1^2 + l_2^2. \end{aligned} \quad (1.9)$$

In addition,

$$k_1 = \frac{\sqrt{E}}{2} \left(\lambda + \frac{1}{\lambda} \right), \quad k_2 = \frac{i\sqrt{E}}{2} \left(\frac{1}{\lambda} - \lambda \right), \quad e^{ikx} = e^{\frac{i}{2}\sqrt{E}(\lambda\bar{z} + z/\lambda)}. \quad (1.10)$$

In new notations the Schrödinger equation (0.1) takes the form

$$L\psi = E\psi, \quad L = -4\partial_z\partial_{\bar{z}} + v(z), \quad z \in \mathbb{C}^1, \quad E \in \mathbb{R} \quad (1.11)$$

(in this paper the notation $f = f(z)$ does not mean that $\partial_{\bar{z}}f = 0$).

The functions φ^+ from (0.3), ψ, μ from (1.2), (1.3), a, b from (1.7) take the form

$$\begin{aligned} \varphi^+ &= \varphi^+(z, \lambda, E), \quad f = f(\lambda, \lambda', E), \quad \psi = \psi(z, \lambda, E), \\ \mu &= \mu(z, \lambda, E), \quad a = a(\lambda, E), \quad b = b(\lambda, E). \end{aligned} \quad (1.12)$$

Further, we shall always assume that the fixed energy

$$E = 1 \quad (1.13)$$

(the case of an arbitrary fixed positive energy may be reduced to (1.13) by scaling transformation). We shall also omit E in the further notations.

Now (1.2) is read as

$$\psi(z, \lambda) = e^{\frac{i}{2}(\lambda\bar{z} + z/\lambda)} \mu(z, \lambda). \quad (1.14)$$

For the functions $a(\lambda)$, $b(\lambda)$ we can write

$$\begin{aligned}
 a(\lambda) &= \frac{1}{(2\pi)^2} \int_{\mathfrak{C}} \int v(z) \mu(z, \lambda) dz_R dz_I, \\
 b(\lambda) &= \frac{1}{(2\pi)^2} \int_{\mathfrak{C}} \int e^{\frac{i}{2}(\lambda \bar{z} + z/\lambda + \bar{z}/\bar{\lambda})} v(z) \mu(z, \lambda) dz_R dz_I.
 \end{aligned}
 \tag{1.15}$$

For $|\lambda| = 1$ corresponding to $\text{Im } k = 0$, formulas (1.3), (1.4) make no sense without a regularization, but the boundary values

$$\mu_{\pm}(z, \lambda) = \mu(z, \lambda(1 \mp 0)), \tag{1.17}$$

are well-defined. We consider also functions

$$h_{\pm}(\lambda, \lambda') = \frac{1}{(2\pi)^2} \int_{\mathfrak{C}} \int e^{-\frac{i}{2}(\lambda' \bar{z} + z/\lambda')} v(z) \psi_{\pm}(z, \lambda) dz_R dz_I, \tag{1.18}$$

where

$$|\lambda| = |\lambda'| = 1, \quad h_{\pm}(\lambda, \lambda') = h_{\pm n_{\perp}}(k, l), \quad n_{\perp} = (-k_2, k_1)/|k|.$$

Let the potential $v(z)$ satisfy the “small norm” condition (0.5) for $E = 1$. Then the function $\psi(z, \lambda)$ has the following properties (see [8, 12]):

- 1) For all $|\lambda| \neq 1$ $\psi(z, \lambda)$ is uniquely defined by Eq. (1.3).
- 2) $\psi(z, \lambda)$ is continuous in λ outside the unit circle $|\lambda| = 1$.
- 3) There exists a function $\rho(\lambda, \lambda')$, $|\lambda| = |\lambda'| = 1$ such that the boundary values of the function $\psi(\lambda, z)$ on the unit circle $|\lambda| = 1$ satisfy

$$\psi_+(z, \lambda) = \psi_-(z, \lambda) + \oint_{|\lambda'|=1} \rho(\lambda, \lambda') \psi_-(z, \lambda') |d\lambda'|. \tag{1.19}$$

- 4) Outside the unit circle the function $\psi(z, \lambda)$ satisfies the following equation.

$$\frac{\partial \psi(z, \lambda)}{\partial \bar{\lambda}} = r(\lambda) \psi(z, -1/\bar{\lambda}), \tag{1.20}$$

where

$$r(\lambda) = \frac{\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} b(\lambda). \tag{1.21}$$

In terms of $\mu(z, \lambda)$ Eq. (1.19) and (1.20) take the form

$$\mu_+(z, \lambda) = \mu_-(z, \lambda) + \oint_{|\lambda'|=1} \rho(\lambda, \lambda', z) \mu_-(z, \lambda') |d\lambda'|, \tag{1.19'}$$

$$\frac{\partial \mu(z, \lambda)}{\partial \bar{\lambda}} = r(\lambda, z) \mu(z, -1/\bar{\lambda}), \tag{1.20'}$$

where

$$\begin{aligned}
 \rho(\lambda, \lambda', z) &= e^{-\frac{i}{2}(\lambda \bar{z} + z/\lambda - \lambda' \bar{z} - z/\lambda')} \rho(\lambda, \lambda'), \\
 r(\lambda, z) &= e^{-\frac{i}{2}(\lambda \bar{z} + z/\lambda + \bar{\lambda} z + \bar{z}/\bar{\lambda})} r(\lambda).
 \end{aligned}
 \tag{1.22}$$

5)

$$\psi(z, \lambda) = e^{\frac{1}{2}(\lambda \bar{z} + z/\lambda)} (1 + o(1)) \quad \text{as } \lambda \rightarrow 0, \infty, \quad (1.23)$$

$$\mu(z, \lambda) \rightarrow 1 \quad \text{as } \lambda \rightarrow 0, \infty. \quad (1.23')$$

The functions $\rho(\lambda, \lambda')$, $b(\lambda)$, $\psi(z, \lambda)$ have the following symmetry properties [5, 6, 8]:

$$\rho(\lambda, \lambda') + \rho(-\lambda', -\lambda) + \oint_{|\lambda''|=1} \rho(\lambda, \lambda'') \rho(-\lambda', -\lambda'') |d\lambda''| = 0 \quad (1.24a)$$

for all λ, λ' , $|\lambda| = |\lambda'| = 1$,

$$\rho(\lambda', \lambda) = \overline{\rho(\lambda, \lambda')}, \quad (1.24b)$$

$$b(1/\bar{\lambda}) = b(\lambda), \quad (1.25a)$$

$$b(-\lambda) = \overline{b(\lambda)}, \quad (1.25b)$$

$$\psi(z, -1/\bar{\lambda}) = \overline{\psi(z, \lambda)}, \quad \mu(z, -1/\bar{\lambda}) = \overline{\mu(z, \lambda)}. \quad (1.26)$$

Using (1.25) we may rewrite Eq. (1.20), (1.20') as

$$\frac{\partial \psi(z, \lambda)}{\partial \bar{\lambda}} = r(\lambda) \overline{\psi(z, \lambda)}, \quad \frac{\partial \mu(z, \lambda)}{\partial \bar{\lambda}} = r(\lambda, z) \overline{\mu(z, \lambda)}. \quad (1.27)$$

The scattering amplitude $f(\lambda, \lambda')$ and the function $\rho(\lambda, \lambda')$ are connected with $h_{\pm}(\lambda, \lambda')$ by the following equations (see [12]):

$$h_{\pm}(\lambda, \lambda') - \pi i \oint_{|\lambda''|=1} h_{\pm}(\lambda, \lambda'') \theta \left[\pm \frac{1}{i} \left(\frac{\lambda''}{\lambda} - \frac{\lambda}{\lambda''} \right) \right] f(\lambda'', \lambda') |d\lambda''| = f(\lambda, \lambda'), \quad (1.28a)$$

$$\sigma_{\pm}(\lambda, \lambda') = \theta \left[-\frac{1}{i} \left(\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'} \right) \right] h_{\pm}(\lambda, \lambda') - \theta \left[\frac{1}{i} \left(\frac{\lambda'}{\lambda} - \frac{\lambda}{\lambda'} \right) \right] h_{\mp}(\lambda, \lambda'), \quad (1.28b)$$

$$\rho(\lambda, \lambda') + \pi i \oint_{|\lambda''|=1} \rho(\lambda, \lambda'') \theta \left[\pm \frac{1}{i} \left(\frac{\lambda'}{\lambda''} - \frac{\lambda''}{\lambda'} \right) \right] \sigma_{\pm}(\lambda'', \lambda') |d\lambda''| = -\pi i \sigma_{\pm}(\lambda, \lambda'), \quad (1.28c)$$

(here $\theta(x)$ is the standard Heaviside function $\theta(x) = 0$, $x < 0$, $\theta(x) = 1$, $x \geq 0$).

It is well known that for a real sufficiently fast decreasing at infinity potential the scattering amplitude has the following properties (see, for example, [35]):

a) Reciprocity

$$f(-\lambda', -\lambda) = f(\lambda, \lambda'). \quad (1.29a)$$

b) Unitarity

$$f(\lambda, \lambda') - \overline{f(\lambda', \lambda)} + \pi i \oint_{|\lambda''|=1} f(\lambda, \lambda'') \overline{f(\lambda', \lambda'')} |d\lambda''| = 0. \quad (1.29b)$$

Due to Eqs. (1.28) the property (1.29j) implies (1.24j), where $j = a, b$ and vice versa [7, 12]. In terms of $h_+(\lambda, \lambda')$, $h_-(\lambda, \lambda')$ defined by (1.28a) the properties (1.29) take the form (see [12])

$$\begin{aligned}
 h_-(\lambda, \lambda') &= h_-(-\lambda', -\lambda) + \pi i \oint h_-(\lambda, \lambda'') h_-(-\lambda', -\lambda'') \\
 &\quad \times \left[\theta \left(-\frac{1}{i} \left(\frac{\lambda''}{\lambda} - \frac{\lambda}{\lambda''} \right) \right) - \theta \left(-\frac{1}{i} \left(\frac{\lambda''}{\lambda'} - \frac{\lambda'}{\lambda''} \right) \right) \right] |d\lambda''|, \\
 h_+(\lambda, \lambda') &= \overline{h_-(-\lambda', -\lambda)},
 \end{aligned}
 \tag{1.30}$$

where (1.28a) is also assumed to be valid.

It is well-known also that under condition (0.2) the scattering amplitude $f(\lambda, \lambda')$ is a continuous function. If (0.2) and (0.5) are valid then $b(\lambda)$ is continuous for $|\lambda| \neq 1$, and

$$r(\lambda) = \frac{\pi}{\lambda} \operatorname{sgn}(\lambda \bar{\lambda} - 1) b(\lambda) \in L_{p,2}(\mathbb{C}),
 \tag{1.31}$$

where $2 < p < 4$ (see [12]).

2. The Equations of the Inverse Scattering

Given scattering data at fixed energy $E = 1$ f and b , where $f = f(\lambda, \lambda')$, $|\lambda| = |\lambda'| = 1$ is an arbitrary continuous function satisfying (1.29) and $b(\lambda)$ is an arbitrary continuous function in the domains $D_{\pm} = \{\lambda \in \mathbb{C} \mid \pm |\lambda| \leq \pm 1\}$ satisfying (1.25) such that

$$|b(\lambda)| \leq \frac{\text{Const.}}{(|\lambda| + |1/\lambda|)^3}.
 \tag{2.1}$$

(The boundary values of $b(\lambda)$ on the unit circle $|\lambda| = 1$ in D_+ and D_- may be different.) Then the corresponding potential $v(z)$ is constructed in the following way (see [8, 12]).

1) Using Eqs. (1.28) we calculate $\rho(\lambda, \lambda')$ via $f(\lambda, \lambda')$ and define $r(\lambda)$ by (1.21).

2) We construct a function $\mu(z, \lambda)$ with the analytic properties (1.19'), (1.20'), (1.23') as a solution of the following integral equation:

$$\begin{aligned}
 \mu(z, \lambda) &= 1 + \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{d\zeta}{\zeta - \lambda} \oint_{|\lambda'|=1} \rho(\zeta, \lambda', z) \mu(z, \lambda'(1 - 0)) |d\lambda'| \\
 &\quad - \frac{1}{\pi} \iint_{\mathbb{C}} r(\zeta, z) \overline{\mu(z, \zeta)} \frac{d\zeta_R d\zeta_I}{\zeta - \lambda}.
 \end{aligned}
 \tag{2.2}$$

Here the Cauchy–Green formula was used

$$f(\lambda) = -\frac{1}{\pi} \iint_D (\partial_{\bar{\zeta}} f(\zeta)) \frac{d\zeta_R d\zeta_I}{\zeta - \lambda} + \frac{1}{2\pi i} \oint_{\partial D} f(\zeta) \frac{d\zeta}{\zeta - \lambda}.
 \tag{2.3}$$

The main case of our paper is $f(\lambda, \lambda') \equiv 0$. From (1.28) it follows that $\rho(\lambda, \lambda') \equiv 0$. In this case Eq. (2.2) is uniquely solvable in $C(\mathbb{C})$ for all z under condition (2.1) on the scattering data.

For the case of negative energy this fact was used in the paper [11] and then for the case of positive energy in [12]. Another system of integral equations for solving (1.19'), (1.20'), (1.23'), which is more convenient in the case $f(\lambda, \lambda') \neq 0$ was suggested in [12].

3) Expanding $\mu(z, \lambda)$ as $\lambda \rightarrow \infty$,

$$\mu(z, \lambda) = 1 + \frac{\mu_{-1}(z)}{\lambda} + o\left(\frac{1}{\lambda}\right) \tag{2.4}$$

(from (2.1) it follows that there is no $c(z)/\bar{\lambda}$ term in (2.4)) we define $v(z)$ by the formula

$$v(z) = 2i\partial_z\mu_{-1}(z). \tag{2.5}$$

4) It can be shown (see [8]) that

$$L\psi(z, \lambda) = \psi(z, \lambda), \tag{2.6}$$

where

$$\psi(z, \lambda) = e^{\frac{i}{2}(\lambda\bar{z}+z/\lambda)}\mu(z, \lambda), \quad L = -4\partial_z\partial_{\bar{z}} + v(z), \tag{2.7}$$

$$\overline{v(z)} = v(z). \tag{2.8}$$

Potential $v(x)$ constructed from the scattering data $f(\lambda, \lambda')$, $b(\lambda)$ with properties formulated in the beginning of this section may decay rather slowly. The necessary and sufficient conditions for decay rate at infinity faster than $|x|^{-M}$, $M > 0$ will be discussed in the next sections.

3. Fast Decaying Potentials. Necessary Conditions on the Scattering Data

Later we shall use the following notation:

$$v(x) \in C_M^{(N)}(\mathbb{R}^2) \quad \text{if} \quad \frac{\partial^{n_1+n_2}}{\partial x_1^{n_1} \partial x_2^{n_2}} v(x) \in C(\mathbb{R}^2) \quad \text{and}$$

$$\left| \frac{\partial^{n_1+n_2}}{\partial x_1^{n_1} \partial x_2^{n_2}} v(x) \right| < \frac{c_{n_1, n_2}}{(1 + |x|)^M}, \quad c_{n_1, n_2} > 0 \tag{3.1}$$

for all nonnegative integers n_1, n_2 such that $n_1 + n_2 \leq N$.

Let the potential $v(x)$ satisfy (0.2), “small norm” assumption (0.5) and

$$v(x) \in C_{M+2+i}^{(3)}(\mathbb{R}^2). \tag{3.2}$$

In this section we show that under these assumptions we have $2M + 2$ additional necessary conditions on scattering data.

Let us introduce the following functions:

$$a_m(\lambda) = \frac{1}{(2\pi)^2} \iint_{\mathbb{C}} \left[\left(\frac{\partial}{\partial \lambda} \right)^m e^{-\frac{i}{2}[\lambda\bar{z}+z/\lambda]} \right] v(z)\psi(z, \lambda) dz_R dz_I, \tag{3.3a}$$

$$b_m(\lambda) = \frac{1}{(2\pi)^2} \iint_{\mathbb{C}} \left[\left(\frac{\partial}{\partial \bar{\lambda}} \right)^m e^{\frac{i}{2}[\lambda\bar{z}+\bar{z}/\lambda]} \right] v(z)\psi(z, \lambda) dz_R dz_I. \tag{3.3b}$$

(It should be noted that $a(\lambda) = a_0(\lambda)$, $b(\lambda) = b_0(\lambda)$.)

If (0.2), (0.5), (3.2) are fulfilled then

1) For $m = 0, 1, \dots, M$ the integrals (3.3) converge, the functions $a_m(\lambda), b_m(\lambda)$ are continuous in D_- and $D_+ \setminus 0$.

2)

$$\text{as } |\lambda| \rightarrow \infty \quad a_m(\lambda) = O(1), \tag{3.4a-}$$

$$b_m(\lambda) = O\left(\frac{1}{|\lambda|^3}\right). \tag{3.4b-}$$

$$\text{as } |\lambda| \rightarrow 0 \quad a_m(\lambda) = O\left(\frac{1}{|\lambda|^{2m}}\right), \tag{3.4a+}$$

$$b_m(\lambda) = O\left(\frac{|\lambda|^3}{|\lambda|^{2m}}\right). \tag{3.4b+}$$

From (1.20) and (3.3) it follows that

3)

$$\partial_{\bar{z}} a_m(\lambda) = r(\lambda) \overline{b_m(\lambda)}, \tag{3.5a}$$

$$\partial_{\bar{z}} b_m(\lambda) = b_{m+1}(\lambda) + r(\lambda) \overline{a_m(\lambda)}, \tag{3.5b}$$

where $r(\lambda) = \pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) b(\lambda) / \bar{\lambda}$.

In the formulas (3.3) we apply the operators ∂_{λ} and $\partial_{\bar{\lambda}}$ to a holomorphic function and to an antiholomorphic function respectively. Let

$$\lambda = r e^{i\varphi}, \quad \lambda' = r' e^{i\varphi'}, \quad r, r' \in \mathbb{R}_+.$$

Then

$$\partial_{\lambda} = \frac{\bar{\lambda}}{\lambda} \partial_{\bar{\lambda}} + \frac{1}{i\lambda} \partial_{\varphi}, \quad \partial_{\bar{\lambda}} = \frac{\lambda}{\bar{\lambda}} \partial_{\lambda} - \frac{1}{i\bar{\lambda}} \partial_{\varphi}.$$

For an arbitrary holomorphic function $f(\lambda), \lambda \in \mathbb{C} \setminus 0$ we have

$$\partial_{\bar{\lambda}}^n f(\lambda) = \left(\frac{1}{i\lambda} \partial_{\varphi}\right)^n f(\lambda), \quad \partial_{\bar{\lambda}}^n \overline{f(\lambda)} = \left(-\frac{1}{i\bar{\lambda}} \partial_{\varphi}\right)^n \overline{f(\lambda)}, \quad \lambda \in \mathbb{C} \setminus 0. \tag{3.6}$$

So we can replace the operators ∂_{λ}^m and $\partial_{\bar{\lambda}}^m$ in (3.3) by $(\frac{1}{i\lambda} \partial_{\varphi})^m$ and $(-\frac{1}{i\bar{\lambda}} \partial_{\varphi})^m$ respectively.

Comparing (1.18) and (3.3) we see that

$$a_m(\lambda(1+0)) \Big|_{|\lambda|=1} = \left(\frac{1}{i\lambda'} \partial_{\varphi'}\right)^m h_-(\lambda, \lambda') \Big|_{\lambda'=\lambda}, \tag{3.7a-}$$

$$b_m(\lambda(1+0)) \Big|_{|\lambda|=1} = (-i\lambda' \partial_{\varphi'})^m h_-(\lambda, \lambda') \Big|_{\lambda'=-\lambda}, \tag{3.7b-}$$

$$a_m(\lambda(1-0)) \Big|_{|\lambda|=1} = \left(\frac{1}{i\lambda'} \partial_{\varphi'}\right)^m h_+(\lambda, \lambda') \Big|_{\lambda'=\lambda}, \tag{3.7a+}$$

$$b_m(\lambda(1-0)) \Big|_{|\lambda|=1} = (-i\lambda' \partial_{\varphi'})^m h_+(\lambda, \lambda') \Big|_{\lambda'=-\lambda}, \tag{3.7b+}$$

If (0.2), (0.5), (3.2) are fulfilled then the functions $f(\lambda, \lambda')$, $h_{\pm}(\lambda, \lambda')$ are M times continuously differentiable on the torus and $b(\lambda) \in C_3^{(M)}(D_-)$.

If $v(x) \in C_{\infty}^{(\infty)}(\mathbb{R}^2)$ then

$$f(\lambda, \lambda') \in C^{(\infty)}(T^2), \tag{3.8}$$

if, in addition, the ‘‘small norm’’ assumption (0.5) is valid then

$$b(\lambda) \in C_{\infty}^{(\infty)}(D_{\pm}). \tag{3.9}$$

The definitions (3.3) and Eqs. (3.5), (3.7), $m = 0, \dots, M$ and the property (3.4), $m = 0$ were given in [12].

Now we come up to one of the most important points of our paper. From (3.4), (3.5), (3.7) $m = 0, \dots, M$ we shall obtain $2M + 2$ additional necessary conditions on the scattering data $f(\lambda, \lambda')$ and $b(\lambda)$ for a potential $v(x)$ with properties (0.2), (0.5), (3.2). A half of these conditions was given earlier in [9, 12]. Analogs of the second half of these conditions for the case of negative energy were considered earlier in [11]. These conditions shall be written in terms of the functions $h_-(\lambda, \lambda')$, $b(\lambda)$.

We recall that the functions $h_-(\lambda, \lambda')$ and $f(\lambda, \lambda')$ are connected by (1.28a).

Let us introduce new functions

$$a_m^{\pm}(\lambda) = \theta(\pm(1 - \lambda\bar{\lambda}))a_m(\lambda), \quad b_m^{\pm}(\lambda) = \theta(\pm(1 - \lambda\bar{\lambda}))b_m(\lambda). \tag{3.10}$$

$2M + 2$ additional conditions on the scattering data will be obtained by induction.

Let $M = 0$. Equation (3.7b-) with $m = 0$ takes the form

$$b_0^-(\lambda(1 + 0))|_{|\lambda|=1} = h_-(\lambda, -\lambda), \quad b_0^-(\lambda) = \theta(\lambda\bar{\lambda} - 1)b_0(\lambda). \tag{3.11}$$

The relation (3.11) is the first additional condition on the scattering data $h_-(\lambda, \lambda')$, $b(\lambda)$. Let us calculate the function $a_0^-(\lambda)$ as a solution of the boundary value problem for Eq. (3.5a) in D_- with the boundary conditions (3.7a-) on the unit circle $|\lambda| = 1$ and (3.4a-) on $\lambda = \infty$. This boundary problem is solvable if and only if the following equality is valid

$$\left[\frac{1}{2\pi i} \oint_{\partial D_-} h_-(\xi, \xi) \frac{d\xi}{\xi - \lambda} - \frac{1}{\pi} \iint_{D_-} \frac{\pi b(\xi)\overline{b(\xi)}}{\bar{\xi}} \frac{d\xi_R d\xi_I}{\xi - \lambda} \right] \Big|_{|\lambda|=1-0} \equiv -s_0 \tag{3.12}$$

for an appropriate constant s_0 . Under condition (3.12) the function $a_0^-(\lambda)$ takes the form

$$a_0^-(\lambda) = \frac{1}{2\pi i} \oint_{\partial D_-} h_-(\xi, \xi) \frac{d\xi}{\xi - \lambda} - \frac{1}{\pi} \iint_{D_-} \frac{\pi b(\xi)\overline{b(\xi)}}{\bar{\xi}} \frac{d\xi_R d\xi_I}{\xi - \lambda} + s_0. \tag{3.13}$$

The relation (3.12) is the second additional condition on the scattering data $h_-(\lambda, \lambda')$, $b(\lambda)$.

The step of induction is the following.

Let for a fixed $M = n$ we have found $2n + 2$ additional condition on the scattering data and we have expressed the functions $a_m^-(\lambda)$, $b_m^-(\lambda)$, $m = 0, \dots, n$ via $b_0^-(\lambda)$ and $(\frac{1}{i\lambda'} \partial_{\varphi'})^m h_-(\lambda, \lambda')|_{\lambda'=\lambda}$, $m = 0, \dots, n$.

Assume now that $M = n + 1$. Then using Eqs. (3.4-), (3.5), (3.7-), $m = 0, \dots, n + 1$ and expressions for $a_n^-(\lambda), b_n^-(\lambda)$ obtained at the previous step we shall find 2 conditions more on the scattering data and we shall express $a_{n+1}^-(\lambda), b_{n+1}^-(\lambda)$ via $b_0^-(\lambda), (\frac{1}{i\lambda'}\partial_{\varphi'})^m h_-(\lambda, \lambda')|_{\lambda'=\lambda}, m = 0, \dots, n + 1$ ($C_{n+1}^{(3)}(\mathbb{R}^2) \subset C_n^{(3)}(\mathbb{R}^2)$ so all conditions found for $M = n$ are fulfilled for $M = n + 1$). Using (3.5b) we obtain $b_{n+1}^-(\lambda)$ as

$$b_{n+1}^-(\lambda) = \frac{\partial}{\partial \lambda} b_n^-(\lambda) - \frac{\pi}{\lambda} b_0^-(\lambda) \overline{a_n^-(\lambda)}. \tag{3.14}$$

The relation (3.7b-), $m = n + 1$ is the $2n + 2 + 1$ additional condition on the scattering data

$$b_{n+1}^-(\lambda(1 + 0))|_{|\lambda|=1} = (-i\lambda'\partial_{\varphi'})^{n+1} h_-(\lambda, \lambda')|_{\lambda'=-\lambda}. \tag{3.15}$$

The function $b_0^-(\lambda) \in C_3^{(n+1)}(\mathbb{C})$ so the condition (3.4b-), $m = n + 1$ is fulfilled.

Let us calculate the function $a_{n+1}^-(\lambda)$ as a solution of the boundary value problem for Eq. (3.5a) in D_- with the boundary condition (3.7a-) on the unit circle $|\lambda| = 1$ and (3.4a-) on $\lambda = \infty$. This boundary value problem is solvable if and only if the following equality is valid:

$$\left[\frac{1}{2\pi i} \oint_{\partial D_-} \left(\left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^{n+1} h(\xi, \lambda') \Big|_{\lambda'=\xi} \right) \frac{d\xi}{\xi - \lambda} - \frac{1}{\pi} \int_{D_-} \int \frac{\pi b_0^-(\xi) \overline{b_{n+1}^-(\xi)}}{\bar{\xi}} \frac{d\xi_R d\xi_I}{\xi - \lambda} \Bigg] \Bigg|_{|\lambda|=1-0} \equiv -s_{n+1} \tag{3.16}$$

for an appropriate constant s_{n+1} .

Under condition (3.16) the function $a_{n+1}^-(\lambda)$ takes the form

$$a_{n+1}^-(\lambda) = \frac{1}{2\pi i} \oint_{\partial D_-} \left(\left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^{n+1} h(\xi, \lambda') \Big|_{\lambda'=\xi} \right) \frac{d\xi}{\xi - \lambda} - \frac{1}{\pi} \int_{D_-} \int \frac{\pi b_0^-(\xi) \overline{b_{n+1}^-(\xi)}}{\bar{\xi}} \frac{d\xi_R d\xi_I}{\xi - \lambda} + s_{n+1}. \tag{3.17}$$

The relation (3.16) is the $(2n + 2 + 2)^{nd}$ additional condition on the scattering data. We recall that the function $b_{n+1}^-(\lambda)$ is expressed via $b_0^-(\lambda), (\frac{1}{i\lambda'}\partial_{\varphi'})^m h_-(\lambda, \lambda')|_{\lambda'=\lambda}, m = 0, \dots, n$. The step of induction is done.

Thus an algorithm to write $2M + 2$ additional conditions on the scattering data for a potential with the properties (0.2), (0.5), (3.2) is presented.

In the paper [12] from Eqs. (3.5), (3.7), $m = 0, \dots, M$ only ((3.4) was not used) $M + 1$ additional necessary conditions on the scattering data were derived. These conditions can be considered as a method to determine

$$\partial_{\lambda'}^m b(\lambda) \Big|_{|\lambda|=1+0}, \quad m = 0, \dots, M \tag{3.18}$$

via the function $h_-(\lambda, \lambda')$. The first of these conditions coincides with (3.11). In [12] an algorithm to write all these conditions was suggested.

Remark. Let $b(\lambda)$ be an arbitrary function such that $b(\lambda) \in C^{(M)}(D_-)$. Then the derivatives (3.18) completely determine all the derivatives

$$\partial_{\bar{\lambda}}^{n_1} \partial_{\lambda}^{n_2} b(\lambda) \Big|_{|\lambda|=1+\varepsilon} \tag{3.19}$$

for all nonnegative integers, n_1, n_2 such that $n_1 + n_2 \leq M$.

These $M + 1$ conditions on the scattering data are local for $b(\lambda)$ and almost local for $h_-(\lambda, \lambda')$. We shall call these conditions local. It is rather natural to replace $M + 1$ conditions (3.11), (3.15) in the family (3.11)–(3.17) by local conditions. It can be shown that this new collection of conditions is equivalent to the old one. The conditions (3.12)–(3.14), (3.16), (3.17) are nonlocal for $b_0^-(\lambda), h_-(\lambda, \lambda')$.

Thus for a potential with properties (0.2), (0.5), (3.2) $M + 1$ additional local conditions and $M + 1$ additional nonlocal conditions on the scattering data $b_0^-(\lambda), h_-(\lambda, \lambda')$ are constructed.

Remark. Analogs of these nonlocal conditions on the scattering data for a negative energy were constructed earlier in [11].

We have studied boundary value problems on D_- . It is rather natural to consider analogs of these conditions on D_+ . Let us show that these new conditions are equivalent to the old ones.

According to (1.30) we have

$$h_+(\lambda, \lambda') = \overline{h_-(\lambda', -\lambda)}. \tag{3.20}$$

Lemma 1. *Let $a_m(\lambda), b_m(\lambda)$ be defined by the formulas (3.3), where $m = 0, \dots, M$, $v(x)$ satisfy (0.2), (3.2). Then*

$$a_m^+(\lambda) = \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{a_k^-(1/\bar{\lambda})}, \tag{3.21a}$$

$$b_m^+(\lambda) = \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{b_k^-(1/\bar{\lambda})}, \tag{3.21b}$$

where $\beta_{mk}(\lambda)$ are defined by

$$(\lambda^2 \partial_{\lambda})^m = \sum_{k=0}^m \beta_{mk}(\lambda) \partial_{\lambda}^k. \tag{3.22}$$

The functions $\beta_{mk}(\lambda)$ have the following properties

a) $\beta_{mm}(\lambda) = \lambda^{2m}$, (3.23)

b) $\beta_{m0}(\lambda) = 0$ for $m > 0$, (3.24)

c) $\deg \beta_{mk}(\lambda) = m + k$ for $0 < k \leq m$, (3.25)

d) $\beta_{m+1,k}(\lambda) = \lambda^2 \beta_{m,k-1}(\lambda) + \lambda^2 \partial_{\lambda} b_{mk}(\lambda)$, (3.26)

where $\beta_{00} = 1, \beta_{m,-1} = 0, \beta_{m,m+1} = 0$.

The proof of Lemma 1 follows from (3.3), (1.26) and the following relations.

Let

$$F_m(\lambda) = \partial_{\lambda}^m \exp \left[-\frac{i}{2} (\lambda \bar{z} + z/\lambda) \right],$$

$$G_m(\lambda) = \partial_{\bar{\lambda}}^m \exp \left[\frac{i}{2} (\bar{\lambda} z + \bar{z}/\bar{\lambda}) \right]. \tag{3.27}$$

Then

$$\begin{aligned}
 F_m(-1/\bar{\lambda}) &= \sum_{k=0}^m \overline{\beta_{mk}(\lambda)} \overline{F_k(\lambda)}, \\
 G_m(-1/\bar{\lambda}) &= \sum_{k=0}^m \beta_{mk}(\lambda) \overline{G_k(\lambda)}.
 \end{aligned}
 \tag{3.28}$$

Lemma 2. *Let the functions $a_m^-(\lambda)$, $b_m^-(\lambda)$, $m = 0, \dots, M$ satisfy the boundary value problem (3.5), (3.4-), (3.7-). Let the functions $h_+(\lambda, \lambda')$ and $h_-(\lambda, \lambda')$ be connected by (3.20). Then the functions $a_m^+(\lambda)$, $b_m^+(\lambda)$ defined by (3.21) satisfy the boundary value problem (3.5), (3.4+), (3.7+).*

The proof of this lemma will be given at the end of this section.

Assume now that $f(\lambda, \lambda') \equiv 0$ (and $h_-(\lambda, \lambda') \equiv 0$ accordingly). Then the first $M + 1$ local conditions simply mean that on the unit circle $|\lambda| = 1$ the function $b(\lambda)$ and all the derivatives $\partial_{\lambda}^{n_1} \partial_{\lambda'}^{n_2} b(\lambda)$, $n_1 \geq 0$, $n_2 \geq 0$, $n_1 + n_2 \leq M$ are equal to zero. But the nonlocal conditions in this case are rather nontrivial. The first of them (3.12) takes the form

$$I_0(\lambda) \Big|_{|\lambda|=1} \equiv -s_0, \tag{3.29}$$

where

$$I_0(\lambda) = -\frac{1}{\pi} \int_{|\xi| \geq 1} \int \frac{\pi b(\xi) \overline{b(\bar{\xi})}}{\bar{\xi}} \frac{d\xi_R d\xi_I}{\xi - \lambda}, \tag{3.30}$$

and (3.13) takes the form

$$a_0^-(\lambda) = I_0(\lambda) + s_0. \tag{3.31}$$

From (3.3a) it follows that

$$a_0^-(\infty) = \hat{v}(0), \tag{3.32}$$

where $\hat{v}(p)$ is the Fourier transform of the potential $v(z)$.

From (3.29)–(3.32) it follows a rather interesting corollary.

Corollary 1. *Let $v(z)$ be a nonzero transparent (i.e. $f(\lambda, \lambda') \equiv 0$) at a fixed energy $E = 1$ potential satisfying (0.2), the “small norm” condition (0.5) and $v(z) \in C_{2+\epsilon}^{(3)}(\mathbb{R}^2)$. Then*

$$\hat{v}(0) > 0, \tag{3.33}$$

where

$$\hat{v}(p) = \frac{1}{(2\pi)^2} \int_{z \in \mathbb{C}} e^{-\frac{1}{2}(p\bar{z} + \bar{p}z)} v(z) dz_R dz_I, \quad p \in \mathbb{C}. \tag{3.34}$$

Proof of Corollary 1. From (3.31), (3.30) it follows that

$$\partial_{\bar{\lambda}} a_0^-(e^{i\varphi} \lambda) = \frac{\pi \theta(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} |b(e^{i\varphi} \lambda)|^2. \tag{3.35}$$

Consider the average of $a_0^-(\lambda)$ over the angle

$$\alpha_0(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} a_0^-(e^{i\varphi} \lambda) d\varphi. \tag{3.36}$$

It has the following properties

$$\partial_{\bar{\lambda}}\alpha_0(\lambda) = \frac{\pi\theta(\lambda\bar{\lambda} - 1)}{\bar{\lambda}} \frac{1}{2\pi} \int_0^{2\pi} |b(e^{i\varphi}\lambda)|^2 d\varphi, \tag{3.37}$$

$\alpha_0(\lambda) = 0$ as $|\lambda| \leq 1$, $\alpha_0(\infty) = \hat{v}(0)$.

Consider the restriction of $\alpha_0(\lambda)$ on the real axis $\text{Im}\lambda = 0$, $\text{Re}\lambda = r > 0$. Then

$$\begin{aligned} \partial_r\alpha_0(r) &= 2\partial_{\bar{\lambda}}\alpha_0(\lambda)|_{\text{Im}\lambda=0} \\ &= \frac{2\pi}{r}\theta(r^2 - 1) \int_0^{2\pi} |b(e^{i\varphi}r)|^2 d\varphi \geq 0, \end{aligned} \tag{3.38}$$

$\alpha_0(1) = 0$, $\alpha_0(\infty) = \hat{v}(0)$. Thus, $\hat{v}(0) > 0$.

From formula (3.32) and Corollary 1, Corollary 2 follows.

Corollary 2. *Let the assumptions of Corollary 1 be fulfilled. Then there exists no path connecting the points 0 and ∞ which has no intersections with the support of $b(\lambda)$.*

Proof of Corollary 2. The function $a_0^-(\lambda)$ is identically equal to 0 as $|\lambda| \leq 1$. If such a path exists then $a_0^-(\lambda)$ is holomorphic in a neighborhood of this path and as a consequence identically equal to 0 along this path so $a_0^-(\infty) = 0$. It contradicts Corollary 1.

Consider an important particular class of potentials depending only on $|z|$, $v(z) = v(|z|)$. In this case the functions $a_m(\lambda)$, $b_m(\lambda)$ possess the following symmetries.

Lemma 3. *Let the potential $v(z)$ depend only on $|z|$, i.e. $v(z) = v(|z|)$. Then*

$$a_m(e^{i\varphi}\lambda) = e^{-im\varphi}a_m(\lambda), \quad b_m(e^{i\varphi}\lambda) = e^{im\varphi}b_m(\lambda). \tag{3.39}$$

In this case all nonlocal conditions are fulfilled automatically.

Proposition 1. *Let the scattering data $b(\lambda)$ in the transparent case $f(\lambda, \lambda') \equiv 0$ have the following properties:*

- (1) $b(\lambda) \in C_\infty^{(\infty)}(\mathbb{C})$,
- (2) $\partial_{\bar{\lambda}}^m \partial_{\lambda}^n b(\lambda)|_{|\lambda|=1} = 0$ for all $m, n \geq 0$,
- (3) $b(e^{i\varphi}\lambda) = b(\lambda)$,
- (4) $b(\lambda) = \overline{b(\bar{\lambda})}$, $b(1/\lambda) = b(\lambda)$ (it follows from property 3 and (1.25)).

Then all local and nonlocal conditions on the scattering data, formulated above, are fulfilled automatically.

We shall prove Proposition 1 by induction. The function $b_0^-(\lambda) = \theta(\lambda\bar{\lambda} - 1)b(\lambda)$ is known and satisfies the first additional condition. The step of induction is the following.

Suppose that under our assumption the first $2n + 1$ additional conditions on the scattering data are fulfilled, the functions $b_n^-(\lambda)$ and for $n \geq 1$ $a_{n-1}^-(\lambda)$ are expressed in terms of $b(\lambda)$ and these functions satisfy (3.39), $b_n^-(\lambda) \in C_\infty^{(\infty)}(\mathbb{C})$ and all derivatives of $b_n^-(\lambda)$ vanish as $|\lambda| = 1$.

We will show that the two next additional conditions are fulfilled, we will express $b_{n+1}^-(\lambda)$ and $a_n^-(\lambda)$ in terms of $b(\lambda)$ and we will see that these functions satisfy (3.39), $b_{n+1}^-(\lambda) \in C_\infty^{(\infty)}(\mathbb{C})$ and all derivatives of $b_{n+1}^-(\lambda)$ vanish as $|\lambda| = 1$. We define

$$\begin{aligned} a_n^-(\lambda) &= \frac{1}{\lambda^n} [I_n(\lambda) - I_n(1)] , \\ I_n &= \partial_{\bar{\lambda}}^{-1} \left[\frac{\pi \lambda^n}{\bar{\lambda}} \theta(\lambda \bar{\lambda} - 1) b(\lambda) \overline{b_n(\lambda)} \right] \\ &= -\frac{1}{\pi} \int \int_{\mathbf{q}} \frac{\pi \xi^n}{\bar{\xi}} \theta(\xi \bar{\xi} - 1) \frac{b(\xi) \overline{b_n(\xi)}}{\xi - \lambda} d\xi_R d\xi_I . \end{aligned} \tag{3.40}$$

The function $I_n(\lambda)$ is well-defined and $I_n(\lambda) = I_n(|\lambda|)$ (it follows from (3.39)). Thus, $a_n^-(\lambda) \equiv 0$ as $|\lambda| = 1$ and it solves the boundary value problem (3.5a), (3.7a-) with $h_-(\lambda, \lambda') \equiv 0$, (3.4a-) and it satisfies (3.39). Now we can define

$$b_{n+1}^-(\lambda) = \partial_{\bar{\lambda}} b_n^-(\lambda) - \frac{\pi}{\bar{\lambda}} b(\lambda) \overline{a_n^-(\lambda)} . \tag{3.41}$$

We see that if $b_n^-(\lambda) \in C_\infty^{(\infty)}(\mathbb{C})$ and all derivatives of $b_n^-(\lambda)$ vanish as $|\lambda| = 1$ then the same is valid for $b_{n+1}^-(\lambda)$. It is the step of induction.

The proof is completed.

Proof of Lemma 2. From (3.21) and (3.5) in D_- and (3.23), (3.26) we obtain the following relations in $D_+ \setminus 0$:

$$\begin{aligned} \partial_{\bar{\lambda}} a_m^+(\lambda) &= \partial_{\bar{\lambda}} \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{a_k^-(-1/\bar{\lambda})} \\ &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{\partial_{\lambda} a_k^-(-1/\bar{\lambda})} \\ &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{[\bar{\mu}^2 \partial_{\bar{\mu}} a_k^-(\mu)]}_{\mu=-1/\bar{\lambda}} \\ &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \frac{1}{\bar{\lambda}^2} \pi \operatorname{sgn} \left(\frac{1}{\lambda \bar{\lambda}} - 1 \right) (-\bar{\lambda}) \overline{b(-1/\bar{\lambda})} \overline{b_k^-(-1/\bar{\lambda})} \\ &= \frac{\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1)}{\bar{\lambda}} b(\lambda) \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{b_k^-(-1/\bar{\lambda})} \\ &= r(\lambda) b_m^+(\lambda) , \end{aligned} \tag{3.42}$$

$$\begin{aligned} \partial_{\bar{\lambda}} b_m^+(\lambda) &= \partial_{\bar{\lambda}} \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} \\ &= \sum_{k=0}^m \beta_{m+1,k}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} - \frac{1}{\bar{\lambda}^2} \sum_{k=0}^m \beta_{m,k-1}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} \\ &\quad + \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{(\partial_{\lambda} b_k^-(-1/\bar{\lambda})})} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^m \beta_{m+1,k}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} - \frac{1}{\bar{\lambda}^2} \sum_{k=0}^m \beta_{m,k-1}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} \\
 &\quad + \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{[\bar{\mu}^2 \partial_{\bar{\mu}} b_k^-(\mu)]} \Big|_{\mu=-1/\bar{\lambda}} \\
 &= \sum_{k=0}^m \beta_{m+1,k}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} - \frac{1}{\bar{\lambda}^2} \sum_{k=0}^m \beta_{m,k-1}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} \\
 &\quad + \frac{1}{\bar{\lambda}^2} \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{b_{k+1}^-(-1/\bar{\lambda})} \\
 &\quad + \sum_{k=0}^m b_{mk}(-1/\bar{\lambda}) \frac{1}{\bar{\lambda}^2} \pi \operatorname{sgn} \left(\frac{1}{\lambda \bar{\lambda}} - 1 \right) (-\bar{\lambda}) \overline{b(-1/\bar{\lambda})} a_k^-(-1/\bar{\lambda}) \\
 &= \sum_{k=0}^m \beta_{m+1,k}(-1/\bar{\lambda}) \overline{b_k^-(-1/\bar{\lambda})} + \frac{\beta_{mm}(-1/\bar{\lambda})}{\bar{\lambda}^2} \overline{b_{m+1}^-(-1/\bar{\lambda})} + r(\lambda) \overline{a_m^+(\lambda)} \\
 &= b_{m+1}^+(\lambda) + r(\lambda) \overline{a_m^+(\lambda)}. \tag{3.43}
 \end{aligned}$$

Thus, the functions $a_m^+(\lambda)$, $b_m^+(\lambda)$ satisfy (3.5) in $D_+ \setminus 0$. The relations (3.4+) follow from (3.4-) and (3.23), (3.24), (3.25).

To prove (3.7+) we use the following identities:

$$\begin{aligned}
 \left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^m &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda}')} (-i\lambda' \partial_{\varphi'})^k, \\
 (-i\lambda' \partial_{\varphi'})^m &= \sum_{k=0}^m \beta_{mk}(\lambda') \left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^k. \tag{3.44}
 \end{aligned}$$

The relations (3.44) follow from (3.23), (3.26).

Due to (3.21a), (3.7a-), (3.20), (3.44) we have

$$\begin{aligned}
 a_m^+(\lambda(1-0)) \Big|_{|\lambda|=1} &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda})} \overline{\left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^k} h_-(-\lambda, -\lambda') \Big|_{\lambda'=\lambda} \\
 &= \sum_{k=0}^m \overline{\beta_{mk}(-1/\bar{\lambda}')} (-i\lambda' \partial_{\varphi'})^k h_+(\lambda, \lambda') \Big|_{\lambda'=\lambda} \\
 &= \left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^m h_+(\lambda, \lambda') \Big|_{\lambda=\lambda'}, \tag{3.45} \\
 b_m^+(\lambda(1-0)) \Big|_{|\lambda|=1} &= \sum_{k=0}^m \beta_{mk}(-1/\bar{\lambda}) \overline{(+i\lambda' \partial_{\varphi'})^k} h_-(-\lambda, -\lambda') \Big|_{\lambda'=-\lambda} \\
 &= \sum_{k=0}^m \beta_{mk}(\lambda') \left(\frac{1}{i\lambda'} \partial_{\varphi'} \right)^k h_+(\lambda, \lambda') \Big|_{\lambda'=\lambda} \\
 &= (-i\lambda' \partial_{\varphi'})^m h_+(\lambda, \lambda') \Big|_{\lambda'=\lambda}.
 \end{aligned}$$

Lemma 2 is proved.

4. The Construction of Potentials with Zero Scattering Amplitude at Fixed Energy

Now we are ready to formulate one of the main results of our paper.

Theorem 1. *Let $b(\lambda)$, $\lambda \in \mathbb{C}$ be an arbitrary function with the following properties:*

- 1) $b(\lambda) \in C_{\infty}^{(\infty)}(\mathbb{C})$.
- 2) $b(1/\bar{\lambda}) = b(\lambda)$, $b(-1/\bar{\lambda}) = \overline{b(\lambda)}$.
- 3) $\partial_{\lambda}^m \partial_{\bar{\lambda}}^n b(\lambda) \Big|_{|\lambda|=1} = 0$ for all $m, n \geq 0$.

4) *The function $b(\lambda)$ satisfies the first $M + 1$ nonlocal conditions on the scattering data formulated in the previous section for the case $f(\lambda, \lambda') \equiv 0$, i.e. the following $M + 1$ boundary value problems on $D_- = \{\lambda \in \mathbb{C} \mid |\lambda| \geq 1\}$*

$$\begin{aligned} \partial_{\bar{\lambda}} a_m^-(\lambda) &= r(\lambda) \overline{b_m^-(\lambda)}, \\ a_m^-(\lambda) \Big|_{|\lambda|=1} &= 0, \quad a_m^-(\infty) = O(1), \quad m = 0, 1, \dots, M \end{aligned} \tag{4.1}$$

are resolvable, where the functions $b_m^-(\lambda)$ are defined recurrently by:

$$b_{m+1}^-(\lambda) = \partial_{\bar{\lambda}} b_m^-(\lambda) - \frac{\pi}{\lambda} \theta(\lambda \bar{\lambda} - 1) b(\lambda) \overline{a_m^-(\lambda)}, \tag{4.2}$$

$$b_0^-(\lambda) = \theta(\lambda \bar{\lambda} - 1) b(\lambda). \tag{4.3}$$

Then the potential $v(z)$ constructed from the scattering data $b(\lambda)$, $f(\lambda, \lambda') \equiv 0$ by the procedure, described in Sect. 2 for $E = 1$ has the following properties:

- 1) $v(z)$ is real-valued.
- 2) $v(z) \in C_{M+3}^{(\infty)}(\mathbb{R}^2)$.
- 3) *The scattering amplitude for the two-dimensional Schrödinger equation*

$$-4\partial_z \partial_{\bar{z}} \psi(z, \lambda) + v(z) \psi(z, \lambda) = E \psi(z, \lambda), \quad E = 1 \tag{4.4}$$

is equal to zero ($f(\lambda, \lambda') \equiv 0$, $|\lambda| = |\lambda'| = 1$) at the energy level $E = 1$. Moreover, the classical scattering solutions $\varphi^+(z, \lambda)$ of (4.4) have the following asymptotics:

$$\varphi^+(z, \lambda) = e^{\frac{1}{2}(\lambda \bar{z} + z/\lambda)} + O(|z|^{-M-2}), \quad \text{where } |\lambda| = 1, \tag{4.5}$$

or in the standard notation

$$\varphi^+(k, x) = e^{ikx} + O(|x|^{-M-2}), \quad k^2 = E = 1. \tag{4.6}$$

From Theorem 1 and Proposition 1 it follows:

Theorem 2. *Let $f(\lambda, \lambda') \equiv 0$, $b(\lambda)$ satisfy the same conditions as in Proposition 1. Then the corresponding real potential $v(z) \in C_{\infty}^{(\infty)}(\mathbb{R}^2)$ and*

$$\varphi^+(k, x) = e^{ikx} + O(1/|x|^{\infty}), \quad k^2 = E = 1$$

(i.e. $\varphi^+(k, x) - e^{ikx}$ decays as $|x| \rightarrow \infty$ faster than any degree of $|x|^{-1}$).

Proof of Theorem 2. If the scattering amplitude $f(\lambda, \lambda') \equiv 0$ (and $\rho(\lambda, \lambda') \equiv 0$ accordingly) then the function $\mu(z, \lambda)$ (see Sect. 2) is defined as a solution of the equation

$$\partial_{\bar{z}}\mu(z, \lambda) = r(\lambda, z)\overline{\mu(z, \lambda)}, \tag{4.7}$$

such that

$$\mu(z, \lambda) \rightarrow 1 \text{ as } |\lambda| \rightarrow \infty, \tag{4.8}$$

where

$$r(\lambda, z) = e^{-\frac{1}{2}(\lambda\bar{z}+z/\lambda+\bar{\lambda}z+\bar{z}/\bar{\lambda})}r(\lambda), \quad r(\lambda) = \frac{\pi \operatorname{sgn}(\lambda\bar{\lambda} - 1)}{\bar{\lambda}}b(\lambda), \tag{4.9}$$

or equivalently the function $\mu(z, \lambda)$ is defined as a solution of the integral equation

$$\mu(z, \lambda) = 1 + (A_z\mu)(z, \lambda), \tag{4.10}$$

where

$$(A_z f)(\lambda) = \partial_{\bar{\lambda}}^{-1}(r(\lambda, z)\overline{f(\lambda)}) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{r(\zeta, z)\overline{f(\zeta)}}{\zeta - \lambda} d\zeta_R d\zeta_I, \tag{4.11}$$

or equivalently

$$\mu(z, \lambda) = 1 + A_z \cdot 1 + (A_z^2\mu)(z, \lambda). \tag{4.12}$$

According to the theory of generalized analytic functions (see [34]) Eqs. (4.10), (4.12) have a unique solution for all z .

This solution can be written as

$$\mu(z, \lambda) = (I - A_z^2)^{-1}(1 + A_z \cdot 1). \tag{4.13}$$

Equation (4.13) possesses a formal asymptotic expansion

$$\mu(z, \lambda) = (I + A_z^2 + A_z^4 + A_z^6 + \dots)(1 + A_z \cdot 1). \tag{4.14}$$

From (4.22) it follows that (4.14) uniformly converges for sufficiently large $|z|$.

To study (4.14) we need some estimates on A_z^2, A_z . It is convenient to write A_z^2 as

$$(A_z^2 f)(z, \lambda) = \frac{1}{\pi^2} \int \int_{\mathbb{C}} K(z, \lambda, \eta) f(\eta) d\eta_R d\eta_I, \text{ where} \tag{4.15}$$

$$K(z, \lambda, \eta) = I(\lambda, \eta, z) \exp \left[\frac{i}{2}(\eta\bar{z} + z/\eta + \bar{\eta}z + \bar{z}/\bar{\eta}) \right] \overline{r(\eta)}, \tag{4.16}$$

$$I(\lambda, \eta, z) = \int \int_{\mathbb{C}} \frac{r(\zeta)}{(\zeta - \lambda)(\bar{\eta} - \bar{\zeta})} \exp \left[-\frac{i}{2}(\zeta\bar{z} + z/\zeta + \bar{\zeta}z + \bar{z}/\bar{\zeta}) \right] d\zeta_R d\zeta_I. \tag{4.17}$$

Lemma 4. *Let $b(\lambda)$ satisfy the conditions 1), 2), 3) of Theorem 1. Then*

1)

$$|\partial_z^m \partial_{\bar{z}}^n I(z, \lambda, \eta)| \leq \frac{\alpha_{mn}^{(1)}}{(1 + |z|)} \frac{(1 + \varphi(|\lambda - \eta|))}{|\lambda - \eta|}, \tag{4.18}$$

where $\varphi(r) = \frac{\ln r}{1+r|\ln r|}$ for all $m, n \geq 0$.

2) For an arbitrary testing function $f(\lambda) \in C(\bar{\mathbb{C}})$ we have $(A_z f) \in C(\bar{\mathbb{C}})$, $(A_z^2 f) \in C(\bar{\mathbb{C}})$ and the following estimates are valid

a)

$$(A_z^2 f)(\lambda) = \frac{c_{-1}(z)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right) \text{ for } \lambda \rightarrow \infty, \text{ where} \tag{4.19}$$

$$c_{-1} = -\frac{1}{\pi^2} \int_{\mathfrak{C}} \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{r(\zeta)\overline{r(\eta)}}{\bar{\eta} - \bar{\zeta}} \exp\left[-\frac{i}{2}(\zeta\bar{z} + z/\zeta + \bar{\zeta}z + \bar{z}/\bar{\zeta})\right] \\ \times \exp\left[\frac{i}{2}(\eta\bar{z} + z/\eta + \bar{\eta}z + \bar{z}/\bar{\eta})\right] f(\eta) d\zeta_R d\zeta_I d\eta_R d\eta_I, \tag{4.20}$$

$$|\partial_z^m \partial_{\bar{z}}^n c_{-1}| \leq \frac{\beta_{mn}^{(1)}}{(1+|z|)} \|f\|_C \text{ for all } m, n \geq 0. \tag{4.21}$$

b)
$$\|(\partial_z^m \partial_{\bar{z}}^n A_z^2) f\|_C \leq \frac{\gamma_{mn}^{(1)}}{(1+|z|)} \|f\|_C \text{ for all } m, n \geq 0. \tag{4.22}$$

c)
$$\|(\partial_z^m \partial_{\bar{z}}^n A_z) f(\lambda)\|_C \leq \varepsilon_{mn} \|f\|_C \text{ for all } m, n \geq 0. \tag{4.23}$$

Equation (4.14) may be written as

$$\mu(z, \lambda) = 1 + A_z \cdot 1 + A_z^2 \cdot 1 + \dots + A_z^{2M+5} \cdot 1 + R_M, \tag{4.24}$$

where

$$R_M = \left(\sum_{k=M+3}^{\infty} A_z^{2k} \right) (1 + A_z \cdot 1). \tag{4.25}$$

From Lemma 4 we get the following estimates on R_M .

Lemma 5. Let $b(\lambda)$ satisfy conditions 1), 2), 3) of Theorem 1. Then

1)
$$|\partial_z^m \partial_{\bar{z}}^n R_M| \leq \frac{\alpha_{mn}^{(2)}}{(1+|z|)^{M+3}} \text{ for all } m, n \geq 0. \tag{4.26}$$

2)
$$R_M = \frac{q(z)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \text{ as } \lambda \rightarrow \infty, \text{ where} \tag{4.27}$$

$$|\partial_z^m \partial_{\bar{z}}^n q(z)| \leq \frac{\beta_{mn}^{(2)}}{(1+|z|)^{M+3}} \text{ for all } m, n \geq 0. \tag{4.28}$$

From (2.4), (2.5), (4.28) we see that the term R_M gives a contribution to the potential $v(z)$ and to the function $\mu(z, \lambda)$ from the functional class $C_{M+3}^{(\infty)}(\mathbb{R}^2)$.

Lemma 6. To calculate the potential $v(z)$ and the function $\mu(z, \lambda)$ up to terms of the order $O(1/|z|^{M+3})$ it is sufficient to consider only the first $2M + 6$ terms in the formula (4.24).

Let

$$\mu_M(z, \lambda) = 1 + A_z \cdot 1 + \dots + A_z^{2M+5} \cdot 1. \tag{4.29}$$

Now we give some estimates of $\mu_M(z, \lambda)$.

Lemma 7. *Let the conditions 1)–3) of Theorem 1 be valid, $f(z, \lambda)$ be a smooth function of λ, z such that all the derivatives $\partial_z^k \partial_{\bar{z}}^l \partial_\lambda^m \partial_{\bar{\lambda}}^n f(z, \lambda)$ are bounded on the λ -plane uniformly in z , i.e.*

$$|\partial_z^k \partial_{\bar{z}}^l \partial_\lambda^m \partial_{\bar{\lambda}}^n f(z, \lambda)| \leq \alpha_{klmn} \text{ for all } z, \lambda. \tag{4.30}$$

Then
1)

$$A_z * f(z, \lambda) = \frac{2i}{(z - \bar{z}/\lambda^2)} \exp \left[-\frac{i}{2}(\lambda\bar{z} + \bar{\lambda}z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \\ \times \left\{ \left[\sum_{k=0}^{\infty} (-1)^k \left(\partial_{\bar{z}} \circ \frac{2i}{z - \bar{z}/\lambda^2} \right)^k \right] * (r(\lambda) \overline{f(z, \lambda)}) + O \left(\frac{1}{|z|^\infty} \right) \right\}. \tag{4.31}$$

Here \circ denotes the product of operators and $*$ means that we apply a differential operator to the function.

2) Consider the asymptotical expansion of $A_z * f(z, \lambda)$ as $\lambda \rightarrow \infty$,

$$A_z * f(z, \lambda) = \frac{\alpha_{-1}(z)}{\lambda} + \frac{\alpha_{-2}(z)}{\lambda^2} + \dots \tag{4.32}$$

($r(\lambda)$ vanishes as $|\lambda| \rightarrow \infty$, so we have no nonholomorphic terms in (4.32).) Then all $\alpha_{-k}(z) \in C_\infty^{(\infty)}(\mathbb{R}^2)$.

3) Let $|\lambda| = 1$. Then the function $A_z * f(z, \lambda)$ decreases as $z \rightarrow \infty$ faster than any degree of $|z|$ together with all her derivatives.

4) $A_z^2 * f(z, \lambda)$ is a smooth function of z, λ such that $|z| |(\partial_z^{n_1} \partial_{\bar{z}}^{n_2} \partial_\lambda^{k_1} \partial_{\bar{\lambda}}^{k_2} f(z, \lambda))|$ are bounded in z, λ . (For $A_z * f(z, \lambda)$ it is not true.)

5)

$$A_z^2 * f(z, \lambda) = -\partial_{\bar{\lambda}}^{-1} \left\{ r(\lambda) \frac{1}{wR} \left[\sum_{k=0}^{\infty} \frac{1}{w^k} \left(\partial_\lambda \circ \frac{1}{R} \right)^k \right] * (\overline{r(\lambda)} f(\lambda, z)) \right\} + O \left(\frac{1}{|z|^\infty} \right), \tag{4.33}$$

where

$$w = \bar{z}/2i, \quad R = (1 - v/\lambda^2), \quad v = z/\bar{z}. \tag{4.34}$$

Lemma 8. *Consider the function $A_z^{2k+1} * 1$, where $k \in \mathbb{N} \cup 0$. Then*

1) For $|\lambda| = 1$,

$$A_z^{2k+1} * 1 \in C_\infty^{(\infty)}(\mathbb{R}^2) \text{ in } z.$$

2) Consider the asymptotic expansion of this function as $\lambda \rightarrow \infty$,

$$A_z^{2k+1} * 1 = \frac{\chi_{-1,2k+1}(z)}{\lambda} + O \left(\frac{1}{\lambda^2} \right).$$

Then $\chi_{-1,2k+1}(z) \in C_\infty^{(\infty)}(\mathbb{R}^2)$ in z .

So, all the terms $A_z^{2k+1} * 1, k \in \mathbb{N} \cup 0$ in expansion (4.24) give a contribution to the function $\mu(z, \lambda)$ for $|\lambda| = 1$ and to the potential $v(z)$ defined by (2.5) from

the functional class $C_\infty^{(\infty)}(\mathbb{R}^2)$ in z and in the asymptotical calculations these terms can be neglected.

This statement directly follows from statements 2–4 of Lemma 7.

We have proved that if we want to calculate the potential $v(z)$ up to terms of the order $O(1/|z|^{M+3})$ it is sufficient to approximate the function $\mu(z, \lambda)$ by

$$\mu(z, \lambda) \sim \mu_M^{\text{Appr}}(z, \lambda) = 1 + A_z^2 \cdot 1 + A_z^4 \cdot 1 + \dots + A_z^{2M+4} \cdot 1. \tag{4.35}$$

To calculate the asymptotic expansion of $\mu_M^{\text{Appr}}(z, \lambda)$ for large $|z|$ let us apply the formula (4.33) from Lemma 7.

The direct calculation with help of the formula (4.33) shows that

$$\mu_M^{\text{Appr}}(z, \lambda) = 1 + \frac{\mathcal{C}_0(\lambda, v)}{w} + \frac{\mathcal{C}_1(\lambda, v)}{w^2} + \dots + \frac{\mathcal{C}_M(\lambda, v)}{w^{M+1}} + \frac{\mathcal{C}_{M+1}(\lambda, v)}{w^{M+2}} + u_M(z, \lambda), \tag{4.36}$$

where w, R, v are defined by (4.34),

$$u_M(z, \lambda) \in C_{M+3}^{(\infty)}(\mathbb{R}^2 \setminus D) \text{ in } z \text{ for all } \lambda, \tag{4.37}$$

$$u_M(z, \lambda) = u_{M,-1}(z)/\lambda + O(1/|\lambda|^2) \text{ as } \lambda \rightarrow \infty, \quad u_{M,-1}(z) \in C_{M+3}^{(\infty)} \in (\mathbb{R}^2 \setminus D), \tag{4.38}$$

D is the unit disc $|z| < 1$, the functions $\mathcal{C}_k(\lambda, v)$ are defined recurrently by

$$\mathcal{C}_n(\lambda, v) = -\partial_{\bar{\lambda}}^{-1} \left[\frac{1}{R} r(\lambda) \sum_{k=0}^n \left(\partial_{\lambda} \circ \frac{1}{R} \right)^k * [\overline{r(\lambda)} \mathcal{C}_{n-k-1}(\lambda, v)] \right], \quad \mathcal{C}_{-1}(\lambda, v) \equiv 1. \tag{4.39}$$

Really, we have

$$\mu_{M+1}^{\text{Appr}}(z, \lambda) = 1 + A_z^2 * \mu_M^{\text{Appr}}(z, \lambda). \tag{4.40}$$

Substituting (4.33) to (4.40) and comparing expansion coefficients at $1/w^{n+1}$, $n = 0, \dots, M + 1$ in both sides we get (4.39).

Let us introduce some additional notations. Consider the differential operator $\frac{1}{R}(\partial_{\lambda} \circ \frac{1}{R})^n$. It can be written as

$$\frac{1}{R} \left(\partial_{\lambda} \circ \frac{1}{R} \right)^n = \sum_{k=0}^n f_{nk} \partial_{\lambda}^k, \text{ where } f_{nk} = f_{nk}(\lambda, v), \quad R = R(\lambda, v) = 1 - v/\lambda^2. \tag{4.41}$$

The functions $f_{nk} = f_{nk}(\lambda, v)$ have the following properties:

a)

$$f_{nn} = \frac{1}{R^{n+1}} = \frac{1}{(1 - v/\lambda^2)^{n+1}}. \tag{4.42}$$

b)

$$f_{nk} = \delta_{nk} + O(1/|\lambda|^{n-k+2}) \text{ as } \lambda \rightarrow \infty. \tag{4.43}$$

c)

$$f_{nk} = O(|\lambda|^{n+k+2}) \text{ as } \lambda \rightarrow 0. \tag{4.44}$$

d)

$$f_{n+1,k} = \frac{1}{R}(\partial_{\lambda} f_{nk} + f_{n,k-1}), \quad f_{00} = \frac{1}{R} = \frac{1}{1 - \nu/\lambda^2}, \tag{4.45}$$

$$f_{k,1} = 0, \quad f_{k,k+1} = 0.$$

e) For a fixed ν $f(\lambda, \nu)$ is meromorphic in λ with poles only in points $\lambda^2 = \nu$. In these points

$$f_{nk} = O\left(\frac{1}{\lambda^2 - \nu}\right)^{2n-k+1}. \tag{4.46}$$

Now we can formulate the main algebraic lemma of our article.

Lemma 9. *Let the scattering data $b(\lambda)$ satisfy conditions of Theorem 1. Then for $n = 1, \dots, M$ we have*

$$\mathcal{C}_n(\lambda, \nu) = \sum_{l=0}^n f_{nl} c_l(\lambda), \tag{4.47}$$

where functions $c_l(\lambda)$ satisfy the following equation:

$$\partial_{\bar{\lambda}} c_l(\lambda) = - \left[r(\lambda) \sum_{k=0}^n \partial_{\bar{\lambda}}^k (\overline{r(\lambda)} c_{n-k-1}(\lambda)) \right], \quad c_{-1}(\lambda) \equiv 1, \quad \lambda \neq 0. \tag{4.48}$$

(It can be obtained by a formal substitution $\nu = 0$ in Eq. (4.39).) These functions do not depend on ν and are defined by

$$c_n(\lambda) = -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^n \alpha_{nk}(\lambda) a_k(\lambda), \tag{4.49}$$

where $\alpha_{nk}(\lambda)$ are defined by

$$\partial_{\bar{\lambda}}^n \circ \frac{1}{\lambda} = \sum_{k=0}^n \alpha_{nk}(\lambda) \circ \partial_{\bar{\lambda}}^k, \quad \alpha_{nk}(\lambda) = (-1)^{n-k} \frac{n!}{k!} \frac{1}{\lambda^{n-k+1}} \tag{4.50}$$

and functions $a_m(\lambda)$ have the form

$$a_m(\lambda) = \theta(\lambda \bar{\lambda} - 1) a_m^-(\lambda) + \theta(1 - \lambda \bar{\lambda}) a_m^+(\lambda), \tag{4.51}$$

where $a_m^-(\lambda)$ are defined by the boundary value problems (4.1) and $a_m^+(\lambda)$ are defined by (3.21a).

The proofs of Lemmas 4, 5, 7, 9 will be given at the end of this section.

Using Lemmas 4, 5, 8, 9 we can complete the proof of Theorem 1.

The fact that the reconstruction procedure from the scattering data described in Sect. 2 gives real smooth nonsingular potentials was proved in the previous papers.

Using (4.36), (4.47), (4.49), (4.43), (4.50), (4.34) we obtain the following estimate on $\mu_M^{\text{Appr}}(z, \lambda)$ as $\lambda \rightarrow \infty$,

$$\begin{aligned} \mu_M^{\text{Appr}}(z, \lambda) = & 1 - \frac{\pi}{\lambda} \sum_{k=0}^M \left(\frac{2i}{\bar{z}}\right)^{k+1} a_k(\infty) + \frac{(2i)^{M+2} \mathcal{C}_{M+1,-1}(\nu)}{\bar{z}^{M+2}} \frac{1}{\lambda} \\ & + \frac{u_{M,-1}(z)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right). \end{aligned} \tag{4.52}$$

Here,

$$\mathcal{C}_{M+1}(\lambda, v) = \frac{\mathcal{C}_{M+1,-1}(v)}{\lambda} + O\left(\frac{1}{|\lambda|^2}\right). \tag{4.53}$$

From (4.52), (2.5), (4.34) and Lemmas 6, 8 it follows that

$$v(z) = \frac{(2i)^{M+3}}{z^{M+3}} \frac{\partial \mathcal{C}_{M+1,-1}(v)}{\partial v} + 2i\partial_z u_{M,-1}(z) + \tilde{v}_M(z), \tag{4.54}$$

where $\tilde{v}_M(z) \in C_{M+3}^{(\infty)}(\mathbb{R}^2 \setminus D)$, all the terms in (4.54) are from the functional class $C_{M+3}^{(\infty)}(\mathbb{R}^2 \setminus D)$. The function $v(z)$ is smooth, so the statement 2 is proved.

From (4.1) and property 3) of the function $b(\lambda)$ in the formulation of Theorem 1 it follows that

$$\partial_\lambda^{n_1} \partial_{\bar{\lambda}}^{n_2} a_m^-(\lambda) \Big|_{|\lambda|=1} = 0 \text{ for all } n_1, n_2 \geq 0. \tag{4.55}$$

Using (4.36), (4.47), (4.49), property e) of the function f_{nk} , (4.50), (4.55) and Lemma 8 we see that

$$\mathcal{C}_k(\lambda, v) \Big|_{|\lambda|=1} \equiv 0, \quad k = 0, \dots, M \tag{4.56}$$

and

$$\mu(z, \lambda) \Big|_{|\lambda|=1} = 1 + \frac{\mathcal{C}_{M+1}(\lambda, v)}{w^{M+2}} + u_M(z, \lambda) + \tilde{\mu}_M(z, \lambda), \tag{4.57}$$

where

$$u_M(z, \lambda) \in C_{M+3}^{(\infty)}(\mathbb{R}^2 \setminus D), \quad \tilde{\mu}_M(z, \lambda) \in C_{M+3}^{(\infty)}(\mathbb{R}^2 \setminus D).$$

From (4.57) it follows that the function

$$\varphi(z, \lambda) = e^{\frac{i}{2}(\lambda\bar{z}+z/\lambda)} \mu(z, \lambda) \Big|_{|\lambda|=1} \tag{4.58}$$

has asymptotics (4.5). Now using rather standard arguments we show that $\varphi(z, \lambda)$ coincides with the physical solution $\varphi^+(z, \lambda)$. Theorem 2 is proved.

The scheme of the proof of Lemma 4. We prove the estimate (4.18) only. The estimate (4.22) follows directly from (4.18) and conditions 1), 2), 3) of Theorem 1. The estimates (4.19), (4.21), (4.23) are rather simple.

For the sake of definiteness let us assume that $m = n = 0$. (The proof for general $m, n \geq 0$ is very similar.)

The denominator in (4.17) can be transformed in the following way:

$$\begin{aligned} \frac{1}{(\zeta - \lambda)(\bar{\eta} - \bar{\zeta})} &= -\frac{1}{(\zeta - \lambda)(\zeta - \eta)} \frac{(\eta - \zeta)}{(\bar{\eta} - \bar{\zeta})} \\ &= \frac{1}{\eta - \lambda} \left(\frac{1}{\zeta - \lambda} - \frac{1}{\zeta - \eta} \right) \frac{(\eta - \zeta)}{(\bar{\eta} - \bar{\zeta})} \\ &= \frac{1}{\eta - \lambda} \left(\frac{1}{\bar{\eta} - \bar{\zeta}} + \frac{1}{\zeta - \lambda} \frac{\eta - \zeta}{\bar{\eta} - \bar{\zeta}} \right). \end{aligned} \tag{4.59}$$

According to (4.59) we have

$$I(\lambda, \eta, z) = \frac{1}{\eta - \lambda} (I_1(\eta, z) + I_2(\lambda, \eta, z)), \tag{4.60}$$

where

$$\begin{aligned} \bar{I}_1 &= - \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{\overline{r(\zeta)}}{\zeta - \eta} e^{-iS(\zeta, z)} d\zeta_R d\zeta_I, \\ I_2 &= \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{r(\zeta)}{(\zeta - \lambda)} \frac{(\eta - \zeta)}{(\bar{\eta} - \bar{\zeta})} e^{iS(\zeta, z)} d\zeta_R d\zeta_I, \end{aligned}$$

where

$$iS(\zeta, z) = (-i/2)(\zeta\bar{z} + z/\zeta + \bar{\zeta}z + \bar{z}/\bar{\zeta}).$$

Applying the formula (4.64) from the proof of Lemma 7 we get

$$\begin{aligned} \bar{I}_1 &= \frac{\overline{\pi r(\eta)} e^{-iS(\eta, z)}}{(i/2)(z - \bar{z}/\bar{\eta}^2)} + \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{e^{-iS(\zeta, z)}}{\zeta - \eta} \partial_{\bar{\zeta}} \left(\frac{\overline{r(\zeta)}}{(i/2)(z - \bar{z}/\bar{\zeta}^2)} \right) d\zeta_R d\zeta_I, \\ I_2 &= \frac{\pi r(\lambda) e^{iS(\lambda, z)}}{(i/2)(z - \bar{z}/\bar{\lambda}^2)} \left(\frac{\eta - \lambda}{\bar{\eta} - \bar{\lambda}} \right) + \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{e^{iS(\zeta, z)}}{\zeta - \lambda} \left(\frac{\eta - \zeta}{\bar{\eta} - \bar{\zeta}} \right) \partial_{\bar{\zeta}} \left(\frac{r(\zeta)}{(i/2)(z - \bar{z}/\bar{\zeta}^2)} \right) d\zeta_R d\zeta_I \\ &\quad + \int_{\mathfrak{C}} \int_{\mathfrak{C}} \frac{e^{iS(\zeta, z)}}{\zeta - \lambda} \frac{r(\zeta)}{(i/2)(z - \bar{z}/\bar{\zeta}^2)} \partial_{\bar{\zeta}} \left(\frac{\eta - \zeta}{\bar{\eta} - \bar{\zeta}} \right) d\zeta_R d\zeta_I = I_{20} + I_{21} + I_{22}. \end{aligned} \tag{4.61}$$

The following estimate is valid

$$|I_1| + |I_{20}| + |I_{21}| \leq \frac{\text{const}}{|z|}, \tag{4.62}$$

where the constant depends only on $b(\zeta)$.

It remains to estimate I_{22} ,

$$I_{22} = \frac{1}{|z|} \frac{1}{(\eta - \lambda)} \int_{\mathfrak{C}} \int_{\mathfrak{C}} \left(\frac{1}{\zeta - \lambda} - \frac{1}{\zeta - \eta} \right) F(\zeta, \eta, z) d\zeta_R d\zeta_I,$$

where

$$F(\zeta, \eta, z) = e^{iS(\zeta, z)} \frac{|z| r(\zeta)}{(i/2)(z - \bar{z}/\bar{\zeta}^2)} \frac{(\eta - \zeta)^2}{(\bar{\eta} - \bar{\zeta})^2}$$

and $F(\zeta, \eta, z)$ is a bounded rapidly decaying function of ζ uniformly in η, z . From these properties of F it follows that

$$|I_{22}| < \frac{\text{const}}{|z|} \frac{|\ln(|\eta - \lambda|)|}{(1 + |\eta - \lambda| \ln(|\eta - \lambda|))}, \tag{4.63}$$

where the constant depends only on $b(\zeta)$.

The estimate (4.18) for $m = 0, n = 0$ follows from (4.60)–(4.63).

Proof of Lemma 7. Let us start from the following formula:

$$\begin{aligned} \partial_{\bar{\lambda}}^{-1} (e^{iS(\lambda,z)} F(\lambda,z)) &= \frac{e^{iS}}{iS_{\bar{\lambda}}} \sum_{k=0}^{N-1} (-1)^k \left(\partial_{\bar{\lambda}} \circ \frac{1}{iS_{\bar{\lambda}}} \right)^k F(\lambda,z) \\ &\quad + (-1)^N \partial_{\bar{\lambda}}^{-1} \left[e^{iS} \left(\partial_{\bar{\lambda}} \circ \frac{1}{iS_{\bar{\lambda}}} \right)^N F(\lambda,z) \right], \end{aligned} \tag{4.64}$$

where

$$(\partial_{\bar{\lambda}}^{-1} f)(\lambda) = -\frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \frac{d\zeta_R d\zeta_I}{\zeta - \lambda}. \tag{4.65}$$

This formula is formal, in general, however if all functions $(\partial_{\bar{\lambda}} \circ \frac{1}{iS_{\bar{\lambda}}})^k F(\lambda,z)$, $k = 0, \dots, N$ are continuous in λ and vanishes as $\lambda \rightarrow \infty$ sufficiently fast, then formula (4.64) is an exact identity. Applying (4.64) to $A_z * f(z, \lambda)$ we get

$$\begin{aligned} A_z * f(z, \lambda) &= \frac{2i}{(z - \bar{z}/\bar{\lambda}^2)} \exp \left[-\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \\ &\quad \times \left[\sum_{k=0}^{N-1} (-1)^k \left(\partial_{\bar{\lambda}} \circ \frac{2i}{z - \bar{z}/\bar{\lambda}^2} \right)^k \right] * (r(\lambda) \overline{f(z, \lambda)}) + R_N(z, \lambda), \end{aligned} \tag{4.66}$$

$$\begin{aligned} R_N(z, \lambda) &= (-1)^N \partial_{\bar{\lambda}}^{-1} \left[\exp \left[-\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \right. \\ &\quad \left. \times \left(\partial_{\bar{\lambda}} \circ \frac{2i}{z - \bar{z}/\bar{\lambda}^2} \right)^N * (r(\lambda) \overline{f(z, \lambda)}) \right] \\ &= \frac{1}{\bar{w}^N} \partial_{\bar{\lambda}}^{-1} \left[\exp \left[-\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \right. \\ &\quad \left. \times \left(\partial_{\bar{\lambda}} \circ \frac{1}{1 - \bar{v}/\bar{\lambda}^2} \right)^N * (r(\lambda) \overline{f(z, \lambda)}) \right]. \end{aligned} \tag{4.67}$$

The function $(\partial_{\bar{\lambda}} \circ \frac{1}{1 - \bar{v}/\bar{\lambda}^2})^N * (r(\lambda) \overline{f(z, \lambda)})$ and all their derivatives are bounded on the λ -plane uniformly in z . Thus,

$$\partial_{\bar{\lambda}}^{k_1} \partial_{\bar{\lambda}}^{k_2} \partial_z^{n_1} \partial_z^{n_2} R_N(z, \lambda) = O \left(\frac{1}{|z|^{N-k_1-k_2}} \right), \quad |z| \rightarrow \infty \text{ for all } k_1, k_2, n_1, n_2 \geq 0. \tag{4.68}$$

It proves (4.31). Here we used the following property. Let $\varphi(z, \lambda)$ be an infinitely smooth function of z, λ , $z \neq 0$ and all the derivatives $\partial_z^{n_1} \partial_z^{n_2} \varphi(z, \lambda)$, $n_1, n_2 \geq 0$ are from the Schwartz class in λ . Then

$$\partial_z^{n_1} \partial_z^{n_2} \partial_{\bar{\lambda}}^{k_1} \partial_{\bar{\lambda}}^{k_2} \partial_{\bar{\lambda}}^{-1} * \varphi(z, \lambda) = \partial_{\bar{\lambda}}^{-1} \partial_z^{n_1} \partial_z^{n_2} \partial_{\bar{\lambda}}^{k_1} \partial_{\bar{\lambda}}^{k_2} * \varphi(z, \lambda), \tag{4.69}$$

where $\partial_{\bar{\lambda}}^{-1}$ is defined by (4.65).

To calculate (4.32) consider (4.66) as $|\lambda| \rightarrow \infty$. We see that nontrivial contribution arises only from the term $R_N(\lambda, z)$,

$$\alpha_{-k}(z) = \frac{1}{\bar{w}^N} \left(\frac{1}{\pi} \right) \iint_{\mathbb{C}} \exp \left[-\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \times \left(\partial_{\bar{z}} \circ \frac{1}{1 - \bar{v}/\bar{\lambda}} \right)^N * (r(\lambda) \overline{f(z, \lambda)}) \lambda^{k-1} d\lambda_R d\lambda_I, \tag{4.70}$$

N can be chosen arbitrary large. It proves the statement 2.

Assume that $|\lambda| = 1$. Then

$$A_z * f(z, \lambda) = R_N(z, \lambda) \tag{4.71}$$

for any $N \geq 0$. It proves the statement 3.

Applying A_z to the both sides of (4.66) we get

$$A_z^2 * f(z, \lambda) = -\partial_{\bar{\lambda}}^{-1} \left\{ r(\lambda) \frac{1}{wR} \left[\sum_{k=0}^{N-1} \frac{1}{w^k} \left(\partial_{\lambda} \circ \frac{1}{R} \right)^k \right] * (\overline{r(\lambda)}) f(\lambda, z) \right\} + Q_N, \tag{4.72}$$

where $w, R(v, \lambda)$ are defined by (4.34),

$$Q_N(z, \lambda) = \partial_{\bar{\lambda}}^{-1} \left\{ r(\lambda) \exp \left[\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \overline{R_N(z, \lambda)} \right\}, \tag{4.73}$$

$R_N(z, \lambda)$ is defined by (4.67). Let $N > k_1 + k_2$. Let us apply the operator $D = \partial_z^{n_1} \partial_{\bar{z}}^{n_2} \partial_{\lambda}^{k_1} \partial_{\bar{\lambda}}^{k_2}$ to (4.72). Using (4.68), (4.69) we get

$$DA_z^2 * f(z, \lambda) = -\partial_{\bar{\lambda}}^{-1} \left\{ D * \left(\sum_{k=0}^{N-1} \left[\frac{1}{R} \frac{1}{w^{k+1}} \left(\partial_{\lambda} \circ \frac{1}{R} \right)^k \right] * (\overline{r(\lambda)}) f(\lambda, z) \right) \right\} + \partial_{\bar{\lambda}}^{-1} \left\{ D * \left(r(\lambda) \exp \left[\frac{i}{2} (\lambda \bar{z} + \bar{\lambda} z + z/\lambda + \bar{z}/\bar{\lambda}) \right] \overline{R_N(z, \lambda)} \right) \right\}. \tag{4.74}$$

Taking into account (4.30) and the properties 1)–3) from Theorem 3 we obtain that the k -term, $0 \leq k \leq N - 1$ in (4.74) is $O\left(\frac{1}{|z|^{k+1}}\right)$ as $|z| \rightarrow \infty$ uniformly in λ . Combining it with (4.67) we complete the proof of the statement 4. Using that the number N in (4.72), (4.74), (4.68) can be taken arbitrary we prove (4.33).

Proof of Lemma 9. Let us observe that functions $\mathcal{C}_n(\lambda, v)$, $n = 0, \dots, M$ satisfy (4.39) if and only if these functions satisfy the following system;

$$\begin{cases} \partial_{\bar{\lambda}} \mathcal{C}_m(\lambda, v) = \frac{1}{R} r(\lambda) \overline{\mathcal{D}_m(\lambda, v)}, & m = 0, \dots, M \\ \mathcal{D}_m(\lambda, v) = \left(\partial_{\lambda} \circ \frac{1}{R} \right) \mathcal{D}_{m-1}(\lambda, v) - r(\lambda) \overline{\mathcal{C}_{m-1}(\lambda, v)}, & m = 1, \dots, M \end{cases} \tag{4.75}$$

where

$$\mathcal{D}_0(\lambda, \nu) = -r(\lambda), \tag{4.76}$$

the functions $\mathcal{C}_m(\lambda, \nu)$ are continuous in λ for fixed ν and

$$\mathcal{C}_m(\lambda, \nu) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty. \tag{4.77}$$

Let us observe also, that functions $c_m(\lambda)$ satisfy (4.48) if and only if they satisfy the following system:

$$\begin{cases} \partial_{\bar{z}} c_m(\lambda) = r(\lambda) \overline{d_m(\lambda)}, \quad \lambda \neq 0, \quad m = 0, \dots, M \\ d_m(\lambda) = \partial_{\bar{z}} d_{m-1}(\lambda) - r(\lambda) \overline{c_{m-1}(\lambda)}, \quad \lambda \neq 0, \quad m = 1, \dots, M, \end{cases} \tag{4.78}$$

where

$$d_0(\lambda) = -r(\lambda). \tag{4.79}$$

The system (4.78) coincides with (3.5) but with a different starting function $d_0(\lambda)$ instead of $b_0(\lambda)$.

Let us prove that functions $c_m(\lambda)$ from (4.49), $m = 0, \dots, M$ and functions

$$d_m(\lambda) = -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \overline{\alpha_{mk}(\lambda)} b_k(\lambda) \tag{4.80}$$

solve (4.78). The direct calculation with help of (3.5) shows that

$$\begin{aligned} \partial_{\bar{z}} c_m(\lambda) &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \partial_{\bar{z}} (\alpha_{mk}(\lambda) a_k(\lambda)) \\ &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \alpha_{mk}(\lambda) \partial_{\bar{z}} a_k(\lambda) \\ &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \alpha_{mk}(\lambda) r(\lambda) \overline{b_k(\lambda)} \\ &= r(\lambda) \left(-\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \overline{\alpha_{mk}(\lambda)} b_k(\lambda) \right) = r(\lambda) \overline{d_m(\lambda)}, \end{aligned} \tag{4.81}$$

$$\begin{aligned} \partial_{\bar{z}} d_m(\lambda) &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \partial_{\bar{z}} (\overline{\alpha_{mk}(\lambda)} b_k(\lambda)) \\ &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \left[\sum_{k=0}^m (\partial_{\bar{z}} \overline{\alpha_{mk}(\lambda)}) b_k(\lambda) + \sum_{k=0}^m \overline{\alpha_{mk}(\lambda)} (b_{k+1}(\lambda) + r(\lambda) \overline{a_k(\lambda)}) \right] \\ &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \left[\sum_{k=0}^m \overline{\alpha_{m+1,k}(\lambda)} b_k(\lambda) - \sum_{k=1}^m \overline{\alpha_{m,k-1}(\lambda)} b_k(\lambda) \right. \\ &\quad \left. + \sum_{k=0}^m \overline{\alpha_{mk}(\lambda)} b_{k+1}(\lambda) + r(\lambda) \sum_{k=0}^m \overline{\alpha_{mk}(\lambda)} a_k(\lambda) \right] \\ &= -\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^{m+1} \overline{\alpha_{m+1,k}(\lambda)} b_k(\lambda) \\ &\quad + r(\lambda) \overline{\left(-\pi \operatorname{sgn}(\lambda \bar{\lambda} - 1) \sum_{k=0}^m \alpha_{mk}(\lambda) a_k(\lambda) \right)} \\ &= d_{m+1}(\lambda) + r(\lambda) \overline{c_m(\lambda)}. \end{aligned} \tag{4.82}$$

In these calculations and later we use the fact that the functions $a_k(\lambda), b_k(\lambda), k = 0, \dots, M$ vanish on the unit circle $|\lambda| = 1$ with all derivatives. (This property follows from conditions 1), 3), 4) of Theorem 1.) Due to this property the functions $c_n(\lambda)$ defined by (4.49) and $d_n(\lambda)$ defined by (4.80) are smooth in the neighborhood of the unit circle $|\lambda| = 1$.

Let us prove now that functions $\mathcal{C}_n(\lambda, v)$ defined by (4.47) and $\mathcal{D}_n(\lambda, v)$ defined by

$$\mathcal{D}_n(\lambda, v) = \bar{R} \sum_{l=0}^n \overline{f_{nl}(\lambda, v)} d_l(\lambda) \tag{4.83}$$

satisfy (4.75).

The direct calculation with help of (4.78), (4.45) shows that

$$\begin{aligned} \partial_{\bar{\lambda}} \mathcal{C}_m(\lambda, v) &= \partial_{\bar{\lambda}} \sum_{k=0}^m f_{mk}(\lambda, v) c_k(\lambda) \\ &= \sum_{k=0}^m f_{mk}(\lambda, v) \partial_{\bar{\lambda}} c_k(\lambda) \\ &= \sum_{k=0}^m f_{mk}(\lambda, v) r(\lambda) \overline{d_k(\lambda)} \\ &= r(\lambda) \frac{1}{R(\lambda, v)} \overline{\left(\sum_{k=0}^m R(\lambda, v) f_{mk}(\lambda, v) d_k(\lambda) \right)} \\ &= \frac{r(\lambda)}{R(\lambda, v)} \overline{\mathcal{D}_m(\lambda, v)}, \end{aligned} \tag{4.84}$$

$$\begin{aligned} \overline{\left(\partial_{\lambda} \circ \frac{1}{R(\lambda, v)} \right)} \mathcal{D}_m(\lambda, v) &= \partial_{\bar{\lambda}} \sum_{l=0}^m \overline{f_{nl}(\lambda, v)} d_l(\lambda) \\ &= \sum_{l=0}^m \overline{R(\lambda, v) f_{n+1,l}(\lambda, v)} d_l(\lambda) - \sum_{l=1}^m \overline{f_{n,l-1}(\lambda, v)} d_l(\lambda) \\ &\quad + \sum_{l=0}^m \overline{f_{nl}(\lambda, v)} d_{l+1}(\lambda) + \sum_{l=0}^m \overline{f_{nl}(\lambda, v)} r(\lambda) c_l(\lambda) \\ &= \sum_{l=0}^{m+1} \overline{R(\lambda, v) f_{n+1,l}(\lambda, v)} d_l(\lambda) + r(\lambda) \overline{\left(\sum_{l=0}^m f_{nl}(\lambda, v) c_l(\lambda) \right)} \\ &= \mathcal{D}_{m+1}(\lambda, v) + r(\lambda) \overline{\mathcal{C}_m(\lambda, v)}. \end{aligned} \tag{4.85}$$

Using the estimates (4.44), (4.50), (3.4) it is easy to show that the function $\mathcal{C}_m(\lambda, v), \mathcal{D}_m(\lambda, v), m = 0, \dots, M$ are bound in λ and $\mathcal{C}_m(\lambda, v) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Lemma 9 is proved.

5. Two Uniqueness Theorems

Definition. A measurable potential $v(z)$ will be called exponentially decreasing if there exist $\alpha > 0$ and $\beta > 0$ such that $|v(x)| < \beta e^{-\alpha|x|}$.

Theorem 3. *Let the fixed energy scattering amplitude of two exponentially decreasing potentials with the property (0.2) coincide and one of these potentials possesses, in addition, the “small norm” property (0.5) at this fixed energy. Then these two potentials coincide.*

Corollary 3. *There exist no nonzero two-dimensional exponentially decreasing real non-singular potentials transparent at a fixed energy. (There is no “small norm” assumption in Corollary 3).*

Remark. The result of Theorem 3 improves the corresponding result from [9, 12]. The proof uses, in particular, the ideas from [27].

Proof of Theorem 3. We shall use (see [12, 27]) the fact that for an exponentially decreasing potential with property (0.2) each of functions $a(\lambda), b(\lambda)$ in the domains D_+ and D_- can be written as a ratio of two real analytic functions; the Fredholm determinant $\Delta(\lambda)$ of Eq. (1.3) is real analytic in D_+ and D_- and all these three functions are uniquely determined by the scattering amplitude at fixed energy. One of the potentials satisfies the “small norm” assumption (0.5), thus $\Delta(\lambda) \neq 0$ for all $\lambda \in D_+, D_-$.

Thus for both potentials $v_1(z), v_2(z)$ the corresponding functions $\mu_1(z, \lambda), \mu_2(z, \lambda)$ satisfy Eqs. (1.19'), (1.20'), where $\rho(\lambda, \lambda')$ is defined by (1.28). But one of the potentials satisfies the “small norm” assumption (0.5) and it is shown in [12] that Eqs. (1.19'), (1.20') have unique solution, i.e. $\mu_1(z, \lambda) = \mu_2(z, \lambda)$. Thus, $v_1(z) = v_2(z)$.

Theorem 4. *Let the potential $v(z)$ satisfy (0.2) and its forward scattering amplitude $f(k, k)$ is identically zero at an energy interval $E_{\text{fix}} - \delta < k^2 < E_{\text{fix}} + \delta$. Then the potential $v(z)$ is equal to zero identically. (In this theorem we do not use the “small norm” assumption). This theorem and its proof given below are valid in any dimension $\text{dim} = 1, 2, 3, \dots$*

Proof of Theorem 4. As a consequence of the unitarity property (1.29) of the scattering operator we have the well known “optical theorem”

$$\text{Im } f(k, k) = -\frac{\pi}{2|k|} \int_{l^2=k^2} f(k, l) \overline{f(k, l)} dl. \tag{5.2}$$

From (5.2) it follows that $\text{Im } f(k, k) = 0$ if and only if $f(k, l) = 0$ for all l such that $l^2 = k^2$. So, if at the energy level E the forward scattering amplitude $f(k, k) = 0, k^2 = E$, then the whole fixed-energy scattering amplitude $f(k, l), k^2 = l^2 = E$ is equal to zero. It is well known that the forward scattering amplitude $f(s\gamma, s\gamma), \gamma \in \mathbb{R}^2, |\gamma| = 1, s \in \mathbb{R}_+$ admit a meromorphic continuation in s to the upper half plane. Thus, if the forward scattering amplitude is equal to zero on an energy interval it is equal to zero for all energies. Thus, the whole scattering amplitude at all energies is equal to zero and as a consequence the potential $v(z)$ is identically zero.

Remark. The result of Theorem 4 is valid also for the equation

$$-\Delta\psi - k^2 u(x)\psi = k^2\psi, \quad x \in \mathbb{R}^d, \quad d = 1, 2, 3, \dots, \tag{5.3}$$

where $u(x)$ is a real measurable function such that $|u(x)| < q/(1 + |x|)^{d+\varepsilon}$ and for the equation

$$-\Delta\psi + (v(x) - k^2 u(x))\psi = k^2\psi, \quad x \in \mathbb{R}^d, \quad d = 2, 3, 4, \dots, \tag{5.4}$$

where $v(x), u(x)$ are real measurable functions such that

$$|v(x)| < \frac{q_1}{(1 + |x|)^{d+\varepsilon}}, \quad |u(x)| < \frac{q_2}{(1 + |x|)^{d+\varepsilon}}.$$

For Eq. (5.4) the result that both potentials $v(x)$ and $u(x)$ are equal identically to zero if the scattering amplitude $f(k, l) = 0$ for all $k, l \in \mathbb{R}^d, d \geq 2, k^2 = l^2$ is a corollary of results obtained in [25].

6. Nonlinear Integrable Equations

In this section we discuss if the additional conditions on the scattering data studied in Sect. 3.4 are invariant under deformations, generated by nonlinear equation (0.8) and its higher analogs.

In terms of the scattering data these equations (Novikov–Veselov equations) take the form

$$\begin{aligned} \frac{\partial b(\lambda, t)}{\partial t} &= i \left(\lambda^{2l+1} + \frac{1}{\lambda^{2l+1}} + \bar{\lambda}^{2l+1} + \frac{1}{\bar{\lambda}^{2l+1}} \right) b(\lambda, t), \\ \frac{\partial f(\lambda, \lambda', t)}{\partial t} &= i \left(\lambda^{2l+1} + \frac{1}{\lambda^{2l+1}} - (\lambda')^{2l+1} - \left(\frac{1}{\lambda'} \right)^{2l+1} \right) f(\lambda, \lambda', t), \\ \frac{\partial \rho(\lambda, \lambda', t)}{\partial t} &= i \left(\lambda^{2l+1} + \frac{1}{\lambda^{2l+1}} - (\lambda')^{2l+1} - \left(\frac{1}{\lambda'} \right)^{2l+1} \right) \rho(\lambda, \lambda', t), \\ \frac{\partial h_{\pm}(\lambda, \lambda', t)}{\partial t} &= i \left(\lambda^{2l+1} + \frac{1}{\lambda^{2l+1}} - (\lambda')^{2l+1} - \left(\frac{1}{\lambda'} \right)^{2l+1} \right) h_{\pm}(\lambda, \lambda', t) \end{aligned} \quad (6.1)$$

(Eq. (0.8) corresponds to $l = 1$).

It is known that the symmetry conditions (1.25), (1.29), (1.24) (and, as a corollary, (3.20)) are invariant under the flows (6.1). For the additional conditions from Sect. 3 the situation is more interesting.

Theorem 5. (1) *Let the scattering data $b(\lambda, t), f(\lambda, \lambda', t)$ satisfy (6.1), where at $t = 0$ $b(\lambda, 0) \in C_3^{(\infty)}(D_-), f(\lambda, \lambda', 0) \in C^{(\infty)}(T^2)$ and $b(\lambda, 0), f(\lambda, \lambda', 0)$ satisfy (1.25), (1.29) and the first two additional conditions (3.11), (3.12) from Sect. 3 corresponding to $M = 0$. Then these conditions are fulfilled for all t .*

(2) *Let the scattering data $b(\lambda, t), f(\lambda, \lambda', t)$ satisfy (6.1) with $l = 1$, where at $t = 0$ $b(\lambda, 0) \in C_3^{(\infty)}(D_-), f(\lambda, \lambda', 0) \in C^{(\infty)}(T^2)$ and $b(\lambda, 0), f(\lambda, \lambda', 0)$ satisfy (1.25), (1.29) and the first 4 additional conditions (3.11), (3.12) and (3.15), (3.16) for $n = 0$ (corresponding to $M = 1$). Then these conditions are fulfilled for all t if and only if $a_0^-(\infty) = 0$. (Let us recall that for the potential $v(z)$ with the property (0.2) $a_0^-(\infty) = \hat{v}(0)$, where $\hat{v}(p)$ is the Fourier transform (3.34) of $v(z)$.)*

Proof of Theorem 5. Let $|\lambda| = 1, \lambda' = -\lambda$. Then

$$b_0^-(\lambda, t) = e^{2i(\lambda^{2l+1} + \bar{\lambda}^{2l+1})t} b_0^-(\lambda, 0), \quad (6.2)$$

$$h_-(\lambda, \lambda', t) = e^{2i(\lambda^{2l+1} + \bar{\lambda}^{2l+1})t} h_-(\lambda, \lambda', 0) \quad (6.3)$$

and (3.11) is fulfilled identically for all t if it is fulfilled for $t = 0$.

We have

$$\begin{aligned}
 h_-(\lambda, \lambda', t) &= h_-(\lambda, \lambda', 0) \quad \text{for } |\lambda| = 1, \lambda' = \lambda, \\
 b(\lambda, t)\overline{b(\lambda, t)} &= b(\lambda, 0)\overline{b(\lambda, 0)} \quad \text{for } \lambda \in D_-
 \end{aligned}
 \tag{6.4}$$

and (3.12) does not depend on t and it is fulfilled identically for all t if it is fulfilled for $t = 0$. (The first part of Theorem 5 is proved.)

Thus, for $a_0^-(\lambda, t)$ defined by (3.13) we have

$$a_0^-(\lambda, t) = a_0^-(\lambda, 0). \tag{6.5}$$

From (3.13) using the symmetries $h_-(\lambda, \lambda) = h_-(-\lambda, -\lambda)$ (it is a consequence of (1.30)) and (1.25) we get that

$$a_0^-(-\lambda, t) = a_0^-(\lambda, t). \tag{6.6}$$

From (3.14) it follows that

$$\begin{aligned}
 b_1^-(\lambda, t)|_{|\lambda|=1} &= \partial_{\bar{\lambda}} b_0^-(\lambda, t) - \frac{\pi}{\lambda} b_0^-(\lambda, t) \overline{a_0^-(\lambda, t)} \Big|_{|\lambda|=1} \\
 &= e^{2i(\lambda^{2l+1} + \bar{\lambda}^{2l+1})} b_1^-(\lambda, 0) + (2l + 1)it(\bar{\lambda}^{2l} - \lambda^{2l+2})b_0^-(\lambda, t),
 \end{aligned}
 \tag{6.7}$$

$$\begin{aligned}
 (-i\lambda' \partial_{\varphi'})h_-(\lambda, \lambda', t)|_{\substack{\lambda'=-\lambda \\ |\lambda|=1}} &= \lambda'^2 \partial_{\lambda'} h_-(\lambda, \lambda', t)|_{\substack{\lambda'=-\lambda \\ |\lambda|=1}} \\
 &= e^{2i(\lambda^{2l+1} + \bar{\lambda}^{2l+1})} \lambda'^2 \partial_{\lambda'} h_-(\lambda, \lambda', 0)|_{\lambda'=-\lambda} \\
 &\quad + (2l + 1)it(\bar{\lambda}^{2l} - \lambda^{2l+2}) h_-(\lambda, \lambda', t)|_{\lambda'=-\lambda}.
 \end{aligned}
 \tag{6.8}$$

Comparing (6.7) and (6.8) and using the fact that (3.11) is fulfilled for all t we get that if under conditions of the first part of Theorem 5 (3.15) with $n = 0$ is fulfilled for $t = 0$ then it is fulfilled for all t .

From (3.5), (6.1), (6.5) it follows that

$$\begin{aligned}
 \frac{\pi}{\lambda} b_0^-(\lambda, t) \overline{b_1^-(\lambda, t)} &= \frac{\pi}{\lambda} b_0^-(\lambda, t) \partial_{\lambda} \overline{b_0^-(\lambda, t)} - \frac{\pi^2}{\lambda \bar{\lambda}} b_0^-(\lambda, t) \overline{b_0^-(\lambda, t)} a_0^-(\lambda, t) \\
 &= -(2l + 1)it \left(\lambda^{2l} - \frac{1}{\lambda^{2l+2}} \right) b_0(\lambda, 0) \overline{b_0(\lambda, 0)} + \partial_{\bar{\lambda}} a_1^-(\lambda, 0) \\
 &= \partial_{\bar{\lambda}} a_1^-(\lambda, 0) - (2l + 1)it \partial_{\bar{\lambda}} A_1^-(\lambda),
 \end{aligned}
 \tag{6.9}$$

where

$$\partial_{\bar{\lambda}} A_1^-(\lambda) = \left(\lambda^{2l} - \frac{1}{\lambda^{2l+2}} \right) \partial_{\bar{\lambda}} a_0^-(\lambda, 0). \tag{6.10}$$

From (6.1) it follows that

$$\left. \partial_{\lambda'} h_-(\lambda, \lambda', t) \right|_{\substack{\lambda'=\lambda \\ |\lambda|=1}} = \partial_{\lambda'} h_-(\lambda, \lambda', 0) - (2l+1)it \left(\lambda^{2l} - \frac{1}{\lambda^{2l+2}} \right) h_-(\lambda, \lambda', 0) \Big|_{\substack{\lambda'=\lambda \\ |\lambda|=1}}. \quad (6.11)$$

Let $l = 1$. Using (6.9)–(6.11) we can transform the boundary value problem (3.5a), (3.4a–), (3.9a–) with $m = 1$ to the following form:

$$\partial_{\bar{\lambda}} A_1^-(\lambda) = \left(\lambda^2 - \frac{1}{\lambda^4} \right) \partial_{\bar{\lambda}} a_0^-(\lambda, 0), \quad \lambda \in D_-, \quad (6.12)$$

$$A_1^-(\lambda) = \left(\lambda^2 - \frac{1}{\lambda^4} \right) h_-(\lambda, \lambda', 0) \Big|_{\substack{|\lambda|=1 \\ \lambda'=\lambda}}, \quad (6.13)$$

$$A_1^-(\lambda) = O(1), \quad \lambda \rightarrow \infty. \quad (6.14)$$

If $b_1^-(\lambda, t)$ is expressed via $b(\lambda, t)$, $h_-(\lambda, \lambda', t)$ then this boundary value problem is equivalent to (3.16), $n = 0$.

From (6.12), (6.14), (6.6) it follows that

$$A_1^-(\lambda) = \left(\lambda^2 - \frac{1}{\lambda^4} \right) a_0^-(\lambda) - \lambda^2 a_0^-(\infty) + \varphi(\lambda) \quad (6.15)$$

for some $\varphi(\lambda)$ such that $\varphi(\lambda)$ is a bounded holomorphic function on D .

Substituting (6.15) to (6.13) and using (3.4a) with $m = 0$ we get

$$\left(\lambda^2 - \frac{1}{\lambda^4} \right) a_0^-(\infty) + \varphi(\lambda) \Big|_{|\lambda|=1} = 0. \quad (6.16)$$

The problem of finding a bounded holomorphic function $\varphi(\lambda)$ on D_- with the boundary value (6.16) is solvable if and only if $a_0^-(\infty) = 0$. Under these conditions $\varphi(\lambda) = 0$. So, under conditions of the second part of Theorem 5 the 4th additional condition is fulfilled identically in t if and only if $a_0^-(\infty) = 0$.

Theorem 5 is proved.

References

1. Manakov, S.V.: The inverse scattering method and two-dimensional evolution equations. *Uspekhi Mat. Nauk.* **31**(5), 245–246 (1976) (in Russian)
2. Dubrovin, B.A., Krichever, I.M., Novikov, S.P.: The Schrödinger equation in a periodic field and Riemann surfaces. *Dokl. Akad. Nauk. SSSR* **229**, 15–18 (1976), translation in *Sov. Math. Dokl.* **17**, 947–951 (1976)
3. Veselov, A.P., Novikov, S.P.: Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolution equations. *Dokl. Akad. Nauk. SSSR* **279**, 20–24 (1984), translation in *Sov. Math. Dokl.* **30**, 588–591 (1984)
4. Veselov, A.P., Novikov, S.P.: Finite-zone, two-dimensional Schrödinger operators. Potential operators. *Dokl. Akad. Nauk. SSSR* **279**, 784–788 (1984), translation in *Sov. Math. Dokl.* **30**, 705–708 (1984)

5. Grinevich, P.G., Novikov, R.G.: Analogues of multisoliton potentials for the two-dimensional Schrödinger operator. *Funkt. Anal. i Pril.* **19**(4), 32–42 (1985) translation in *Funct. Anal. and Appl.* **19**, 276–285 (1985)
6. Grinevich, P.G., Novikov, R.G.: Analogues of multisoliton potentials for the two-dimensional Schrödinger equations and a nonlocal Riemann problem. *Dokl. Akad. Nauk SSSR* **286**, 19–22 (1986), translation in *Sov. Math. Dokl.* **33**, 9–12 (1986)
7. Novikov, R.G.: Construction of a two-dimensional Schrödinger operator with a given scattering amplitude at fixed energy. *Teoret. Mat. Fiz.* **66**, 234–240 (1986)
8. Grinevich, P.G., Manakov, S.V.: The inverse scattering problem for the two-dimensional Schrödinger operator, the $\bar{\partial}$ -method and non-linear equations. *Funkt. Anal. i Pril.* **20**(2), 14–24 (1986), translation in *Funkt. Anal. and Appl.* **20**, 94–103 (1986)
9. Novikov, R.G.: Reconstruction of a two-dimensional Schrödinger operator from the scattering amplitude at fixed energy. *Funkt. Anal. i Pril.* **20**(3), 90–91 (1986), translation in *Funkt. Anal. and Appl.* **20**, 246–248 (1986)
10. Grinevich, P.G.: Rational solutions of the Veselov–Novikov equations—Two-dimensional potentials that are reflectionless for fixed energy. *Teoret. Mat. Fiz.* **69**(2), 307–310 (1986)
11. Grinevich, P.G., Novikov, S.P.: Two-dimensional “inverse scattering problem” for negative energies and generalized-analytic functions. I. Energies below the ground state. *Funkt. Anal. i Pril.* **22**(1), 23–33 (1988), translation in *Funkt. Anal. and Appl.* **22**, 19–27 (1988)
12. Novikov, R.G.: The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator. *J. Funkt. Anal.* **103**, 409–463 (1992)
13. François, J.-P., Novikov, R.G.: Solutions rationnelles des équations de type Korteweg-de Vries en dimension $2 + 1$ et problèmes à m corps sur la droite. *C.R. Acad. Sci. Paris* **314**, Sér. I, 109–113 (1992)
14. Boiti, M., Leon, J., Manna, M., Pempinelli, F.: On a spectral transform of a KDV-like equation related to the Schrödinger operator in the plane. *Inverse Problems* **3**, 25–36 (1987)
15. Tsai, T.Y.: The Schrödinger operator in the plane. Dissertation, Yale University (1989)
16. Calderon, A.P.: On an inverse boundary value problem. *Seminar on Numerical Analysis and its Applications to Continuum Physics*, Soc. Brasileira de Matemática, Rio de Janeiro, 65–73 (1980)
17. Kohn, R., Vogelius, M.: Determining conductivity by boundary measurements II. Interior results. *Comm. Pure Appl. Math.* **38**, 644–667 (1985)
18. Sylvester, J., Uhlmann, G.: A uniqueness theorem for an inverse boundary value problem in electrical prospection. *Comm. Pure Appl. Math.* **39**, 91–112 (1986)
19. Novikov, R.G.: Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funkt. Anal. i Pril.* **22**(4), 11–22 (1988), translation in *Funct. Anal. and Appl.* **22**, 263–272 (1988)
20. Sun, Z., Uhlmann, G.: Generic uniqueness for an inverse boundary value problem. *Duke Math. J.* **62**, 131–155 (1991)
21. Sun, Z., Uhlmann, G.: Recovery of singularities for formally determined inverse problems. *Commun. Math. Phys.* **153**, 431–445 (1993)
22. Nachman, A.I.: Global uniqueness for a two-dimensional inverse boundary value problem. Preprint (1993)
23. Faddeev, L.D.: Inverse problem of quantum scattering theory II. *Itogi Nauki i Tekhniki, Sov. Prob. Mat.* **3**, 93–180 (1974), translation in *J. Sov. Math.* **5**, 334–396 (1976)
24. Novikov, R.G., Henkin, G.M.: The $\bar{\partial}$ -equation in the multidimensional inverse scattering problem. a) Preprint 27M, Institute of Physics, Krasnoyarsk (1986), (in Russian); b) *Uspekhi Mat. Nauk.* **42**(3), 93–152 (1987), translation in *Russ. Math. Surv.* **42**(4), 109–180 (1987)
25. Novikov, R.G.: Multidimensional inverse scattering problem for acoustic equation. Preprint 32M, Institute of Physics, Krasnojarsk (1986) (in Russian)
26. Henkin, G.M., Novikov, R.G.: A multidimensional inverse problem in quantum and acoustic scattering. *Inverse Problems* **4**, 103–121 (1988)
27. Novikov, R.G.: The inverse scattering problem at fixed energy for the three-dimensional Schrödinger equation with an exponentially decreasing potential. *Commun. Math. Phys.* **161**, 569–595 (1994)
28. Manakov, S.V.: The inverse scattering transform for the time dependent Schrödinger equation and Kadomtsev–Petviashvili equation. *Physica D* **3**(1,2), 420–427 (1981)

29. Ablowitz, M.J., Bar Yaacov, D., Fokas, A.S.: On the inverse scattering transform for the Kadomtsev-Petviashvili equation. *Studies in Appl. Math.* **69**, 135–143 (1983)
30. Zakharov, V.E.: Shock waves spreading along solitons on the surface of liquid. *Izv. Vuzov Radiofiz.* **2969**, 1073–1079 (1986)
31. Regge, T.: Introduction to complex orbital moments. *Nuovo Cimento.* **14**, 951–976 (1959)
32. Newton, R.G.: Construction of potentials from the phase shifts at fixed energy. *J. Math. Phys.* **3**, 75–82 (1962)
33. Sabatier, P.C.: Asymptotic properties of the potentials in the inverse-scattering problem at fixed energy. *J. Math. Phys.* **7**, 1515–1531 (1966)
34. Vekua, I.N.: *Generalized analytic functions*. Oxford: Pergamon Press, 1962
35. Landau, L.D., Lifshitz, E.M.: *Quantum mechanics: Non-relativistic theory*. 2nd ed. Oxford: Pergamon Press, 1965

Communicated by B. Simon