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Meromorphic Zeta Functions for Analytic Flows

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Dedicated to Steve Smale

Abstract: We extend to hyperbolic flows in all dimensions Rugh's results on the meromorphic continuation of dynamical zeta functions. In particular we show that the Ruelle zeta function of a negatively curved real analytic manifold extends to a meromorphic function on the complex plane.

In this paper we address a problem Smale poses in his survey article on dynamical systems ([Sm], II.4): given an isolated, compact, hyperbolic set Ω for a flow ϕ on a manifold M, find a meromorphic function on **C** which admits the product expansion

$$R(z) = \prod_{\gamma}(1 - e^{-z\ell(\gamma)})$$

for $\operatorname{Re} z \gg 0$, where γ runs over the periodic trajectories in Ω of multiplicity 1 and $\ell(\gamma)$ denotes the period of γ. The theorem in Sect. 7 does this for ϕ a C^ω (real analytic) flow.

In fact we meromorphically extend any "zeta function in one variable for (Ω, ϕ)" to **C**. A precise definition is given at the end of Sect. 6, with a discussion which shows how it includes all the usual examples. In particular we treat Selberg's zeta function $S(z)$ for a cocompact Fuchsian group Γ, where $\Omega = \Gamma\backslash PSl(2, \mathbf{R})$ and ϕ_t is given by right multiplication by the one parameter group diag $(e^{t/2}, e^{-t/2})$, and where $S(z)$ is defined as

$$S(z) = R(z)R(z+1)R(z+2)\ldots .$$

In [Se], Selberg uses his trace formula to meromorphically extend $S(z)$. Hence $R(z) = S(z)/S(z+1)$ also has a meromorphic extension in this case, which motivated Smale's problem.

This paper combines the methods of a seminal paper of Ruelle with an innovative idea of Rugh. Ruelle's paper [R1] concerns the case where the stable and unstable bundles E^s, E^u of $\phi_t|\Omega$ extend to C^ω bundles on a neighborhood of Ω.

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He employs the Markov partitions of Sinai and Bowen and a combinatorial argument of Manning and Bowen to produce a finite sequence of trace class operators $L_m(z)$, $m = 0, 1, 2, \ldots, z \in \mathbf{C}$, such that

$$\prod_{m \geqq 0} \det(I - L_m(z))^{(-1)^m} = \exp - \sum_{\gamma} \frac{1}{\mu(\gamma)} \frac{1}{\det(I - S_\gamma)} e^{-z\ell(\gamma)}, \quad \operatorname{Re} z \gg 0 ,$$

where γ runs over all periodic trajectories in Ω, S_γ is the stable summand of the linear Poincaré map of γ and $\mu(\gamma)$ is the multiplicity of γ. Ruelle's *transfer operators* $L_m(z)$ are defined by the action of real analytic contraction maps on spaces of holomorphic functions associated to the partition. This product formula furnishes the meromorphic extension of the right-hand side, which we define to be the *Selberg function* of (Ω, ϕ). For (Ω, ϕ) as in the preceding paragraph, $S_\gamma = e^{-\ell(\gamma)}$, $1/\det(I - S_\gamma) = \sum_{k=0}^{\infty} e^{-k\ell(\gamma)}$, and $\exp - \sum_\gamma \frac{1}{\mu(\gamma)} e^{-(z+k)\ell(\gamma)} = R(z+k)$ so the Selberg function is just $S(z)$. In this way Ruelle gives a dynamical construction of the meromorphic extension of $S(z)$, independent of Selberg's trace formula.

Unfortunately the analyticity condition on E^s, E^u holds in rather few examples. For instance, our theorem applies to the geodesic flow on the unit tangent bundle $\Omega = UQ$ of any connected, closed C^ω Riemannian manifold Q of negative curvature. These bundles are only known to be C^ω when Q is locally symmetric, that is the quotient of F-hyperbolic n-space by a discrete group of isometries, with $n \geqq 2$, $F = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $n = 2$, $F = \mathbf{O}$. For Q of dimension 2 or 3 the only known examples with E^s, E^u analytic have constant sectional curvature.

A way around a similar problem is due to Rugh. Given a hyperbolic analytic map on a rectangle $I_1 \times I_2$, he introduces a space of holomorphic functions of 2 variables, one in a neighborhood of I_2 in $\mathbf{C}$, and the other in the *exterior* of a neighborhood of I_1 in $\mathbf{C}$ and a corresponding transfer operator [Ru]. Since the points of geometric interest lie outside the domain of these functions, the significance of the eigenfunctions of his transfer operator is obscured.

Recall, however, that there is a pairing between forms $f(z)dz$ holomorphic on the disc $|z| \leqq 1$ and functions $g(z)$ holomorphic on the disc $|z| \geqq 1$ in the Riemann sphere $\mathbf{CP}^1$, given by

$$\langle g(z) | f(z)dz \rangle = \frac{1}{2\pi i} \oint g(z) f(z) dz .$$

For $i \geqq 0$, $j \geqq 0$ we have $\langle z^{-j-1} | z^i dz \rangle = \delta_{ij}$. This shows that Rugh's function space combines a function space in one variable with part of the *dual* of a space of holomorphic forms in the other variable.

In Sect. 2 we consider 2 regions (closures of bounded open sets) in complex space $W \subset \mathbf{C}^u$ and $Z \subset \mathbf{C}^s$. We define $A(Z)$ as the uniform limits of polynomials on Z and $V(W)$ as the forms $\phi(w)dw_1 \wedge \cdots \wedge dw_u$ with $\phi(w) \in A(W)$. Then we define $K(W, Z)$ as the Banach space of linear operators from $V(W)$ to $A(Z)$ that are norm limits of finite rank operators. This space of "kernels" replaces the holomorphic function spaces of Ruelle and Rugh.

In Sect. 1 we formalize the notion of hyperbolic correspondence f by parametrizing the correspondence as the graph of a certain *cross map* c. If f goes from $X \times Y$ to $X' \times Y'$, c takes $X' \times Y$ to $X \times Y'$. For f hyperbolic, c is a contraction. Passing to small complex neighborhoods, we extend a C^ω c in Sect. 2 to a holomorphic contraction $C : W' \times Z \to W \times Z'$ that is the cross map of a hyperbolic correspondence F from $W \times Z$ to $W' \times Z'$. Then F defines a *kernel transfer* $L_F : K(W', Z') \to K(W, Z)$.

Given r, a $d \times d$ matrix over $K(W', Z)$, there is an analogous operator $L_{r,F}$ on d-tuples of kernels. In the split case, where C is the Cartesian product of a contraction $C_1: W' \to W$ and a contraction $C_2: Z \to Z'$, $L_{r,F}$ pulls back a d-tuple of kernels to $K(W', Z)$ using C_2, multiplies it by r, and then pushes this d-tuple of kernels forward to $K(W, Z)$ using C_1. In general, $L_{r,F}$ is constructed as the partial adjoint in the first factor of an operator on d-tuples of holomorphic forms that involves pullback by C. $L_{r,F}$ is well approximated in norm by finite rank operators, hence of trace class.

These linear operators are dual to the nonlinear *graph transform* operators of hyperbolic dynamics. For provided we identify $V(W)$ with $A(W)$, a function $f: W \to Z$ with $f(W) \subset\subset Z$ and f a uniform limit of polynomials defines a linear functional on $K(W, Z)$ by $k \mapsto Tr(kf^*)$, where $kf^*: A(Z) \to A(Z)$.

In Sect. 3 we show that kernel transfer behaves well under composition of correspondences. For $W' = W$ and $Z' = Z$, the trace of $L_{r,F}$ is calculated in Sect. 4. In these 2 sections we use the holomorphic fixed point formula of Atiyah–Bott instead of the explicit contour integrations of [Ru]. In Sect. 5 we calculate the Fredholm determinant $\det(I - L)$ for L a block operator whose entries are of the form $L_{r,F}$, in terms of the periodic points of a finite system of hyperbolic correspondences.

Given a complex vector bundle ξ over Ω, the various lifts $\psi_t : \xi \to \xi$ of $\phi_t|\Omega$ to flows of bundle morphisms form a parameter space. For ψ sufficiently contractive we use the trace $\chi_\gamma(\psi)$ of the holonomy of ψ around closed orbits γ to define the zeta function of this lift:

$$\zeta(\psi) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \frac{\chi_\gamma(\psi)}{|\det(I - \mathbf{P}_\gamma)|} = \exp - \int_0^\infty \frac{1}{t} d\nu(t) ,$$

where $\nu = Tr^\flat \psi_t^*$ is the *flat-trace* of the pullback operator ψ_t^* on sections of ξ. Then we show that each example in the literature of a zeta function of the variable z associated to (Ω, ϕ) is a ratio of 2 functions of the special form $\zeta(\psi^{za})$. Here ψ^{za} is a curve in our parameter space, parametrized by $z \in \mathbf{C}$:

$$\psi_t^{za}(v) = \exp[-z \int_0^t a(\psi_s v) ds] \psi_t(v) ,$$

where a is a given function on ξ, constant on fibers, with positive real part. For $a \equiv 1$, we define $\zeta(\psi^z) = T_\psi^\flat(z)$ to be the *flat-trace function* of the lift ψ. Taking the ratio of pairs of flat-trace functions, we obtain any Ruelle or Selberg function and also any of the torsion functions $Z_\alpha(z)$ that arise in Lefschetz formulas for (Ω, ϕ).

We show in Sect. 7 that $\zeta(\psi^{za})$ is a finite alternating product of Fredholm determinants, where each factor $\det I - L_m(z)$ involves a holomorphic family $L_m(z)$ of operators associated to a system of holomorphic correspondences. This gives a meromorphic extension of $\zeta(\psi^{za})$ to $\mathbf{C}$.

In Sect. 7 we formulate 2 conjectures, one concerning Lefschetz formulas for geodesic flows on negatively curved manifolds and the other concerning the analyticity of $\zeta(\psi^{za})$.

Our combination of holomorphic function spaces and their duals is seen most clearly in a naive example, given as I. in Sect. 8, where we treat hyperbolic toral automorphisms without Markov partitions. The particular operators used in Sect. 5 are illustrated in Sect. 8, II, where we give a complete spectral analysis of a certain example with $u = s = 1$.

In Sect. 9 we study the length spectrum of a C^ω basic set in terms of the divisor of its Ruelle function. This leads to an asymptotic statement with a sharp error term for the averaged length distribution.

In Sect. 10 we derive a *Lefschetz formula* for a nonsingular Smale flow (i.e. an Axiom A-No Cycles C^ω flow with one-dimensional nonwandering set and no stationary points), relating the local properties of its closed orbits to the ambient topology. This takes the form of finding R-torsion as a special value of a certain zeta function.

Consider any C^ω metric g of negative curvature on $\Gamma\backslash H^2$, where Γ is a cocompact Fuchsian group. As in Sect. 6, the resulting geodesic flow ϕ on $\Gamma\backslash UH^2 = \Gamma\backslash PSl(2,\mathbf{R})$ has a Selberg function $S^g(z)$ that equals $S(z)$ for the metric g_0 induced from H^2. Since the unstable bundle for g is not C^ω in general we cannot expect to meromorphically continue $S^g(z)$. However, we can write $S(z) = T^\flat_{\psi^+}(z)/T^\flat_{\psi^-}(z)$, where ψ^+ and ψ^- are certain lifts, and this ratio defines a meromorphic function $\tilde{S}^g(z)$ on $\mathbf{C}$ for any g. The order of $\tilde{S}^g(z)$ is at most 3, by the theorem in Sect. 7. However Selberg showed that the order of $S(z)$ is 2, and we seem to have an *explosion of order* when going from $S(z) = \tilde{S}^{g_0}(z)$ to $\tilde{S}^g(z)$. It would be interesting to verify this phenomenon numerically. We expect the instant creation of many nearby pairs of zeroes and poles for g near g_0.

Since this paper was written, Rugh has extended his results to hyperbolic flows in dimension 3 [Ru2] and Kitaev has made progress on the case of nonanalytic diffeomorphisms [K].

We are pleased to dedicate this paper to Steve Smale, whose bold suggestions have stimulated a generation of researchers in dynamical systems.

Contents:

Section 1. The Iterates of a Hyperbolic Correspondence

Let X, X', Y and Y' be complete metric spaces and $f \subset (X \times Y) \times (X' \times Y')$ a closed correspondence from $X \times Y$ to $X' \times Y'$. We write $f(x, y)$ for the projection to $X' \times Y'$ of $f \cap (\{(x, y)\} \times (X' \times Y'))$. If for each $x' \in X'$ and $y \in Y$ there is a unique $x \in X$ and $y' \in Y$ with $(x', y') \in f(x, y)$, we say f admits the *cross map* c: $X' \times Y \to X \times Y'$ with $c(x', y) = (x, y')$. Then this map c determines the correspondence f by the rule

$$(x', y') \in f(x, y) \Leftrightarrow (x, y') = c(x', y) .$$

Here the projection $f \to Y \times X'$ is a homeomorphism and f is parametrized by $(x', y) \mapsto (c_1(x', y), y, x', c_2(x', y))$.

If the components $c_1 : X' \times Y \to X$ and $c_2 : X' \times Y \to Y'$ are uniformly contracting in x' for y fixed and in y for x' fixed we say f is a *hyperbolic correspondence*. In most applications, f is (the graph of) a homeomorphism from a subset of $X \times Y$ to a subset of $X' \times Y'$ which expands in the first factor and contracts in the second. However we allow other situations, such as when c is constant. In this case, if (x_0, y_0') is the value of c then

$$f(x_0, y) = X' \times \{y_0'\}, \quad f(x, y) = \emptyset \quad \text{for} \quad x \neq x_0 .$$

This correspondence is the acme of hyperbolicity since it contracts infinitely in y and "expands infinitely" in x.

In [Ru] the case where X, X', Y and Y' are compact intervals and c is real analytic was discussed. There the components c_1 and c_2 were called "pinning coordinates." There are advantages to combining them into one map, as will be clear in Sect. 2.

To find the inverse correspondence f^{-1} from $X' \times Y'$ to $X \times Y$ we switch 2 pairs of coordinates in f, whereas the graph of c is found by switching only 1 pair. Hence c is a "partial inverse" of f. When c_1 depends on x' alone and c_2 depends on y alone we say f (or c) is *split*. In this case, $f(x, y) = c_1^{-1}\{x\} \times \{c_2(y)\}$ so f is the Cartesian product of c_1^{-1} and c_2.

The composition $h = g \circ f$ of 2 hyperbolic correspondences $f : X \times Y \to X' \times Y'$ and $g: X' \times Y' \to X'' \times Y''$ admits a cross map and is frequently a hyperbolic correspondence. Namely $(x'', y'') \in h(x, y)$ if for some $(x', y') \in X' \times Y'$,

$$c(x', y) = (x, y') \quad \text{and} \quad d(x'', y') = (x', y'') ,$$

where c and d are the cross maps for f and g, respectively. Thus

$$x' = d_1(x'', c_2(x', y)) \quad \text{and} \quad y' = c_2(d_1(x'', y'), y) .$$

Since $d_1(x'', -)$ and $c_2(-, y)$ are contraction mappings for x'' and y fixed, x' and y' are uniquely determined by these equations. On the other hand, as X' and Y' are complete we can use the contraction mapping theorem to define x', y' in terms of x'', y by these equations. We obtain a cross map e for h, with

$$e(x'', y) = (c_1(x', y), d_2(x'', y')) .$$

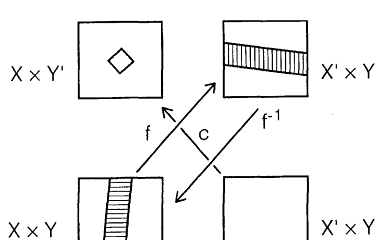

Fig. 1. A hyperbolic correspondence f and its cross map c.

To check whether h is a hyperbolic correspondence, we introduce contraction constants α_c for $c_1(-, y)$, β_c for $c_1(x', -), \gamma_c$ for $c_2(-, y)$ and δ_c for $c_2(x', -)$, good for all $x' \in X'$, $y \in Y$. We refer to β_c, γ_c as *minor constants* and α_c, δ_c as *major constants*. Likewise we define contraction constants α_d, β_d, γ_d, δ_d and Lipschitz constants α_e, β_e, γ_e, δ_e. We estimate the latter from above, as follows.

Lemma 1.

$$\begin{pmatrix} \alpha_e & \beta_e \\ \gamma_e & \delta_e \end{pmatrix} \leqq \begin{pmatrix} \alpha_c\alpha_d(1-\beta_d\gamma_c)^{-1} & \beta_c + \alpha_c\beta_d\delta_c(1-\beta_d\gamma_c)^{-1} \\ \gamma_d + \alpha_d\gamma_c\delta_d(1-\beta_d\gamma_c)^{-1} & \delta_c\delta_d(1-\beta_d\gamma_c)^{-1} \end{pmatrix}.$$

Proof. We will bound γ_e, the other cases being handled similarly. Suppose $x'', x_0'' \in X''$ and $y \in Y$. Then

$$\left| e_2(-, y)\Big|_{x_0''}^{x''} \right| = \left| d_2(x'', y') - d_2(x_0'', y_0') \right| ,$$

where y' is as above and $y_0' = c_2(d_1(x_0'', y_0'), y)$. (To simplify notation, we assume our metric spaces are isometrically embedded as subspaces of some normed vector space.) We have

$$\begin{aligned} |y' - y_0'| &\leqq \gamma_c |d_1(x'', y') - d_1(x_0'', y_0')| \\ &\leqq \gamma_c(|d_1(x'', y') - d_1(x'', y_0')| + |d_1(x'', y_0') - d_1(x_0'', y_0')|) \\ &\leqq \gamma_c(\beta_d |y' - y_0'| + \alpha_d |x'' - x_0''|) , \end{aligned}$$

so that

$$|y' - y_0'| \leqq (1 - \beta_d\gamma_c)^{-1}\alpha_d\gamma_c |x'' - x_0''| .$$

Thus

$$\begin{aligned} \left| e_2(-, y)\Big|_{x_0''}^{x''} \right| &\leqq |d_2(x'', y') - d_2(x_0'', y')| + |d_2(x_0'', y') - d_2(x_0'', y_0')| \\ &\leqq \gamma_d |x'' - x_0''| + \delta_d |y' - y_0'| . \end{aligned}$$

Using our bound on $|y' - y_0'|$ we obtain the desired estimate on γ_e.

Clearly h is a hyperbolic correspondence whenever the 4 matrix entries on the right in this lemma are less than 1. To apply this criterion to a composition of a large number of hyperbolic correspondences, we will suppose that they are nearly split as follows.

Suppose that for $i = 0, 1, 2, \ldots$ we have complete metric spaces $X^{(i)}$, $Y^{(i)}$ and hyperbolic correspondences $f^{(i)}: X^{(i)} \times Y^{(i)} \to X^{(i+1)} \times Y^{(i+1)}$ with cross maps $C^{(i)}$: $X^{(i+1)} \times Y^{(i)} \to X^{(i)} \times Y^{(i+1)}$ and contraction constants $\alpha, \beta, \gamma, \delta$ that are independent of i. Then if the minor constants β and γ are small compared to the major constants α and δ, the following proposition asserts that each composition $f^{(m-1)} \circ \cdots \circ f^{(0)}$ will be hyperbolic. For motivation, note that the composition of split hyperbolic correspondences is obviously split hyperbolic and split is equivalent to $\beta = 0 = \gamma$.

Proposition 1. *The composition of a sequence of hyperbolic correspondences is hyperbolic provided the major constants are bounded by some constant* $\sigma < 1$ *and the minor constants are bounded by some* $\rho < 1$, *with* ρ *a positive function of* σ.

Proof. Let $\rho = \max\{\beta, \gamma\}, \sigma = \max\{\alpha, \delta\}$ and choose r positive with $r^2 < 1 - \sigma$. We define ρ_i, σ_i for $i \geqq 0$ by $\sigma_0 = \sigma$, $\sigma_{i+1} = \sigma_i^2(1-r^2)^{-1}$, $\rho_0 = \rho$, $\rho_{i+1} = \rho_i(1+\sigma_{i+1})$. Our choice of r shows that σ_i is a decreasing sequence and the series $\sum_i \sigma_i$ is summable. Clearly ρ_i is an increasing sequence. Assuming

$$(*) \qquad \rho < r/\prod_{i=1}^{\infty}(1+\sigma_i)\,,$$

we find that $\rho_i < r$ for all i. In particular $\sigma_i < 1$ and $\rho_i < 1$ for all i.

Now we use $(*)$ to show by induction on $k = 0, 1, 2, \ldots$ that any composition $f^{(i+m-1)} \circ \cdots \circ f^{(i)}$, $1 \leqq m \leqq 2^k$, $i \geqq 0$ is hyperbolic, and that for $i \geqq 0$ and $2^{k-1} \leqq m \leqq 2^k$ the minor constants are no more than ρ_k and the major constants are no more than σ_k. For $k = 0$, this is our hypothesis on the $f^{(i)}$ combined with the definition of σ_0, ρ_0. Assume it holds for $k \geqq 0$. Then if $2^k \leqq m \leqq 2^{k+1}$, we write $m = m_1 + m_2$ with $2^{k-1} \leqq m_1, m_2 \leqq 2^k$ and apply Lemma 1 to $g = f^{(i+m-1)} \circ \cdots \circ f^{(i+m_2)}$ and $f = f^{(i+m_2-1)} \circ \cdots \circ f^{(i)}$. Using the induction hypothesis to estimate the constants on the right-hand side of Lemma 1 gives

$$\alpha_e, \delta_e \leqq \sigma_k^2(1-\rho_k^2)^{-1} < \sigma_{k+1}\,,$$

$$\beta_e, \gamma_e \leqq \rho_k + \sigma_k^2\rho_k(1-\rho_k^2) < \rho_{k+1}\,,$$

since $\rho_k < r$. The proposition follows.

Section 2. The Transfer Operator for a C^ω Hyperbolic Correspondence

Let u, s be nonnegative integers and let X, X' (respectively Y, Y') be regions in $\mathbf{R}^u$ (respectively $\mathbf{R}^s$), where by *region* we mean a compact set with dense interior. Suppose f is a hyperbolic correspondence from $X \times Y$ to $X' \times Y'$ such that the cross map c is real analytic, i.e. given by a convergent power series near every point of $X' \times Y$. We will attach Banach spaces to $X \times Y$ and $X' \times Y'$ such that f defines a compact operator between them.

Consider first the case $u = 0$, treated first in [R1]. Then $f = c$ is a contraction mapping from Y to Y'. We choose regions Z and Z' in $\mathbf{C}^s$ that are neighborhoods of Y and Y' such that the power series defining c gives a contraction mapping $C: Z \to Z'$ with values in $\mathrm{int}(Z')$, that is $C(Z) \subset\subset Z'$. Let $A = A(Z)$ denote the Banach subalgebra of $\mathscr{C}(Z)$ generated by the coordinate functions $z_1, \ldots, z_s$, that is the uniform closure of the polynomial functions on Z. If the component functions $z_i \circ C$, $i = 1, \ldots, s$, are in $A(Z)$ we define $C^*: A(Z') \to A(Z)$ by $C^*a = a \circ C$.

Provided Z is sufficiently small each $z_i \circ C$ will lie in $A(Z)$, so C^* is defined. Indeed we have

Lemma 2. *If Y is a compact subset of $\mathbf{R}^s$ and h is holomorphic near Y, then on any sufficiently small neighborhood Z of Y in $\mathbf{C}^s$, $h|Z$ is a uniform limit of polynomials on Z.*

Proof. Y is *polynomially convex* in $\mathbf{C}^s$, that is for each $z \in \mathbf{C}^s$, $z \notin Y$ there is a polynomial p with $|p(Y)| \leqq 1$, $|p(z)| > 1$ (for $z \in \mathbf{R}^s$ one uses the Weierstrass approximation theorem, for $z_i \notin \mathbf{R}$ one uses a polynomial in the i^{th} variable).

It follows ([H], 2.7.4) that every neighborhood of Y contains a *polynomial polyhedron* (also called a *Weil domain*) D, where for some polynomials $p_1, \ldots, p_n$ in $z_1, \ldots, z_s$,

$$D = \{(z_1, \ldots, z_s) : |p_i(z)| \leqq 1, \ i = 1, \ldots, n\} .$$

But D is a *Runge domain*, i.e. every holomorphic function on D is a uniform limit of polynomials ([H], 2.7.7), so the lemma follows.

Now choose Z_0 a compact neighborhood of $c(Z)$ in $\operatorname{int}(Z')$. Then the restriction operator $A(Z') \to A(Z_0)$ is *s-compact* in the sense of [F2], i.e. it can be approximated by operators of rank $\leqq n^s$ with an error whose norm decreases exponentially in n. This is shown by truncating power series expansions in $A(Z')$ after n terms and estimating the error in terms of the gap between Z_0 and $\operatorname{int}(Z')$. Since C^* factors through this restriction operator, C^* is also *s-compact*. This is the result for $u = 0$ that we must generalize.

We review some functional analysis that helps to handle functions on a Cartesian product space. Note that for compact regions W in $\mathbf{C}^u$, Z in $\mathbf{C}^s$ the image in $A(W \times Z)$ of the tensor product $A(W) \otimes A(Z)$ is dense, since it contains all the coordinate functions. This defines a certain norm on $A(W) \otimes A(Z)$ whose completion is $A(W \times Z)$.

One has, in fact, a norm on any tensor product $B_1 \otimes B_2$ of Banach spaces that generalizes this example and leads to a Banach space completion B_1B_2 of $B_1 \otimes B_2$. In the literature, where various other norms are used, this norm on $B_1 \otimes B_2$ is called the *injective norm* and B_1B_2 the *injective completion* (it is variously denoted $B_1 \hat{\hat{\otimes}} B_2$, $B_1 \hat{\otimes}_\varepsilon B_2$ or $B_1 \check{\otimes} B_2$). Quite simply, we regard an element $b_1 \otimes b_2$ as defining a bounded continuous function $\phi(b_1 \otimes b_2)$ on the product $U_1 \times U_2$ of the unit balls U_i in the dual Banach spaces B_i^*, i.e. $\phi(b_1 \otimes b_2)(u_1, u_2) = \langle u_1 | b_1 \rangle \langle u_2 | b_2 \rangle$. Then ϕ extends to a linear embedding $B_1 \otimes B_2 \to \mathscr{C}(U_1 \times U_2)$ which defines our norm and completion $B_1B_2 \subset \mathscr{C}(U_1 \times U_2)$.

We have $\mathscr{C}(W)\mathscr{C}(Z) = \mathscr{C}(W \times Z)$, indeed this holds for any pair of compact metric spaces ([Tr], Ex. 44.2). Also if $\alpha_i : A_i \to B_i$ are bounded linear operators then there is a bounded operator $\alpha_1\alpha_2 : A_1A_2 \to B_1B_2$ extending $\alpha_1 \otimes \alpha_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$. Such an operator from A_1A_2 to B_1B_2 is called *decomposable*. If α_1 and α_2 are embeddings onto a closed subspace, so is $\alpha_1\alpha_2$ (hence the name "injective completion"). In particular if A_i is a closed subspace of B_i then $A_1A_2 \subset B_1B_2$ is a closed subspace ([Tr], Prop. 43.7 and Cor.). From these facts follow the relation

$$A(W \times Z) = A(W)A(Z) ,$$

which we will use to separate variables.

We note, however, that the evaluation functional $A \otimes A^* \to \mathbf{C}$, $a \otimes a^* \mapsto \langle a | a^* \rangle$, does not extend continuously to a map $AA^* \to \mathbf{C}$ for A a Banach space of infinite dimension. For if $k \in BA^*$ then the operator $a \mapsto b = \langle k | a \rangle$, $A \to B$ is a norm limit of finite rank mappings. Indeed $\|k\|$ is the operator norm of $a \mapsto b$ and any operator in the norm closure of finite rank is determined by such a kernel k. Thus the evaluation $A \otimes A^* \to \mathbf{C}$ corresponds to the trace on finite rank operators, which is not norm continuous since $\dim A = \infty$.

An operator $\lambda : B \to A$ is *integral* if there is a bounded functional $BA^* \to \mathbf{C}$, called the *valuation functional relative to* λ, with $b \otimes a^* \mapsto \langle \lambda(b) | a^* \rangle$. If λ is n-compact for some $n > 0$ then λ is integral (using [Tr], Prop. 49.5).

Now we make our constructions for all u. With X, X' (respectively Y, Y') as above we choose compact neighborhoods W, W' of X, X' in $\mathbf{C}^u$ (respectively Z, Z' of Y, Y' in $\mathbf{C}^s$) such that c extends to a holomorphic map $C: W' \times Z \to W \times Z'$ with values in the interior of $W \times Z'$ and such that the components C_1, C_2 of C are uniformly contracting in each variable. Thus C defines a hyperbolic correspondence F from $W \times Z$ to $W' \times Z'$.

We first define the action $\tilde{C}$ of C on *horizontal volume forms*

$$\tilde{C}(\phi(w, z)dw_1 \wedge \cdots \wedge dw_u) = J(w, z)\phi(C(w, z))dw_1 \wedge \cdots \wedge dw_u ,$$

where $J = \det\left(\frac{\partial}{\partial w'}\right) C_1$ is the partial Jacobian of C in the horizontal variables. Note that $\tilde{C}$ is the usual pullback of forms followed by the projection that deletes all components containing a dz_j. We assume that W' and Z are chosen so that $w_i \circ C$ and $z_j \circ C$ belong to $A(W' \times Z)$ for $i = 1, \ldots, u$, and $j = 1, \ldots, s$. Then we find $\phi \in A(W \times Z') \Rightarrow J(C^*\phi) \in A(W' \times Z)$. Separating variables, and passing to the space $V(W)$ of volume forms $\phi(w)dw_1 \wedge \cdots \wedge dw_u$, $\phi \in A(W)$ we obtain a bounded operator

$$\tilde{C}: V(W)A(Z') \to V(W')A(Z) .$$

As $C(W' \times Z) \subset \operatorname{int}(W \times Z')$, we can factor $\tilde{C}$ through a restriction operator to show it is $(u + s)$-compact.

Our transfer operator L_F will be obtained as the *partial adjoint of* $\tilde{C}$ *in the first factor* (or in the horizontal variables), that is

$$L_F: V^*(W')A(Z') \to V^*(W)A(Z)$$

will satisfy the partial adjoint formula

$$\langle v|L_F(v^* \otimes a)\rangle = \langle v^*|\tilde{C}(v \otimes a)\rangle \in A(Z)$$

for each $v^* \in V^*(W')$, $a \in A(Z')$ and $v \in V(W)$. Clearly this formula defines L_F on $V^*(W') \otimes A(Z')$ and we need only verify continuity. This can be done simply if $\tilde{C}$ can be expanded as an absolutely convergent series $\sum_i \xi_i\eta_i$ of decomposable operators. Then $\|\xi_i\eta_i\| = \|\xi_i\|\|\eta_i\| = \|\xi_i^*\eta_i\|$ so $\sum_i \xi_i^*\eta_i$ is bounded, and defines L_F.

In general we produce L_F as follows. We choose compact regions W_0 and W_0' in $\mathbf{C}^u$ with $W_0 \subset\subset W$ and $W' \subset\subset W_0'$ such that C analytically continues to $W_0' \times Z$ with values in int $(W_0 \times Z')$. We denote this extension by C as well. The restriction operator $V(W) \to V(W_0)$ is u-compact, hence can be represented by a kernel $k \in V(W_0)V^*(W)$. Consider the composite operator L

$$\begin{aligned} V^*(W')A(Z') &\to V^*(W')A(Z')V(W_0)V^*(W) \\ &\to V^*(W')V(W_0')A(Z)V^*(W) \to V^*(W)A(Z) \end{aligned}$$

in which (aside from permutations of the factors) the first arrow corresponds to multiplication by k, the second arrow corresponds to $I\tilde{C}I$, and the third arrow to the evaluation functional relative to the restriction map $V(W_0') \to V(W')$. Then L is clearly continuous and

$$(*) \qquad L(v^* \otimes a) = \sum_i \langle v^*|\tilde{C}(v_i \otimes a) \otimes v_i^*\rangle$$

for $k = \sum v_i \otimes v_i^*$, $\sum \|v_i\| \|v_i^*\| < \infty$. Thus

$$\langle v | L(v^* \otimes a) \rangle = \langle v^* | \tilde{C}(\sum_i \langle v | v_i^* \rangle v_i \otimes a) \rangle = \langle v^* | \tilde{C}(v \otimes a) \rangle ,$$

since $\sum_i \langle v | v_i^* \rangle v_i$ is the restriction of v to W_0. This shows $L = L_F$ on $V^*(W') \otimes A(Z')$, hence that $(*)$ defines L_F.

We denote $V^*(W)A(Z)$ by $K(W,Z)$. As elements of $K(W,Z)$ are kernels (of finite rank operators $V(W) \to A(Z)$ and their norm limits) we call $L_F: K(W',Z') \to K(W,Z)$ the *kernel transfer* of F. Choosing $Z_0' \subset\subset Z'$ so that C extends to a map from $W_0' \times Z$ to $\operatorname{int}(W \times Z_0')$, we can factor L_F through a natural operator on kernels

$$\varepsilon\rho : K(W',Z') \to K(W_0',Z_0') .$$

Here $\varepsilon\rho$ decomposes into a u-compact extension operator $\varepsilon: V^*(W') \to V^*(W_0')$ and an s-compact restriction operator $\rho: A(Z') \to A(Z_0')$. Clearly $\varepsilon\rho$ is $(u+s)$-compact, hence L_F is also $(u+s)$-compact

Now suppose $r \in A(W' \times Z)$ and consider the operator $r\tilde{C}: V(W)A(Z') \to V(W')A(Z)$,

$$(r\tilde{C})(\phi(w,z)dw_1 \wedge \cdots \wedge dw_u) = (Jr)(w,z)\phi(C(w,z))dw_1 \wedge \cdots \wedge dw_u .$$

Then we can as above form the partial adjoint of $r\tilde{C}$ in the first factor, which we denote

$$L_{r,F}: K(W',Z') \to K(W,Z) .$$

Finally suppose given a positive integer d and a $d \times d$ matrix r over $A(W' \times Z)$. Then we can combine the transfer operators for the entries of r into a block operator on d-tuples of kernels that we denote

$$L_{r,F}: K^d(W',Z') \to K^d(W,Z) .$$

Section 3. Functoriality of Transfer

Suppose f (respectively f') is a hyperbolic correspondence from $X \times Y$ to $X' \times Y'$ (respectively from $X' \times Y' \to X'' \times Y''$) for X,X',X'' regions in $\mathbf{R}^u$ and Y,Y',Y'' regions in $\mathbf{R}^s$. Suppose the cross maps c and c' are real analytic and that we have complex neighborhoods W,W',W'' of X,X',X'' in $\mathbf{C}^u$ and Z,Z',Z'' of Y,Y',Y'' in $\mathbf{C}^s$ such that c and c' extend to maps

$$C: W' \times Z \to W \times Z', \qquad C': W'' \times Z' \to W' \times Z'' ,$$

which are uniform limits of polynomials, whose components contract in each variable and with $C^{(i)}(W^{(i+1)} \times Z^{(i)}) \subset\subset W^{(i)} \times Z^{(i+1)}$, $i = 0,1$. Then as in Sect. 2, the corresponding hyperbolic correspondences $F^{(i)}$ from $W^{(i)} \times Z^{(i)}$ to $W^{(i+1)} \times Z^{(i+1)}$ define operators $L_{F^{(i)}}: K(W^{(i+1)},Z^{(i+1)}) \to K(W^{(i)},Z^{(i)})$. Also $H = F' \circ F$ admits a cross map $E: W'' \times Z \to W \times Z''$ as in Sect. 1, with $E(W'' \times Z) \subset\subset W \times Z''$. We can then define $L_H: K(W'',Z'') \to K(W,Z)$. We will prove

Proposition 2. $L_H = L_F \circ L_{F'}$.

First we introduce $W_0^{(i)} \subset\subset W^{(i)}$ a neighborhood of $X^{(i)}$ such that our cross maps take values in $W_0^{(i)} \times Z^{(j)}$, where $(i,j) = (0,1),(1,2)$ and $(0,2)$ for F, F' and H respectively. We set $V_0^{(i)} = V(W_0^{(i)})$ and $A^{(i)} = A(Z^{(i)})$ and define operators $\tilde{C}, \tilde{C}', \tilde{E}$ from $V_0^{(i)}A^{(j)}$ to $V_0^{(j)}A^{(i)}$ as in Sect. 2.

The proposition will hold if $L_F \circ L_{F'}$ is the partial adjoint of $\tilde{E}$ in the first factor. Using formula $(*)$ of Sect. 2, we let $\Sigma_k v'_k \otimes v^*_k \in V'_0 V^*(W')$ be the kernel of the u-compact restriction operator $V(W') \to V'_0$ and we must show that for any $v_0 \in V_0$, $a'' \in A''$ and $v^* \in (V''_0)^*$,

$$\langle v^* | \tilde{E}(v_0 \otimes a'')\rangle = \sum_k \langle v^*_k | \tilde{C}(v_0 \otimes a'_k)\rangle \,,$$

where $a'_k = \langle v^* | \tilde{C}'(v'_k \otimes a'')\rangle$.

We next eliminate v^* from this formula. Switching the factors in the domain of $\tilde{C}, \tilde{C}', \tilde{E}$ gives operators

$$\hat{C}^{(i)}: A^{(i+1)} V_0^{(i)} \to V_0^{(i+1)} A^{(i)}, \qquad \hat{E}: A'' V_0 \to V''_0 A \,.$$

Then the proposition reduces to

$$\hat{E}(a'' \otimes v_0) = \sum_k \langle (I\hat{C} \circ \hat{C}'I)(a \otimes v'_k \otimes v_0) | v'_k\rangle \,,$$

where $I\hat{C}: V''_0 A' V_0 \to V''_0 V'_0 A$ and $\hat{C}'I : A'' V'_0 V_0 \to V''_0 A' V_0$. Note that V'_0 is the middle factor in the domain and range of $I\hat{C} \circ \hat{C}'I$. This version of the proposition can be described as follows:

$$\hat{E} \text{ is the partial trace of } I\hat{C} \circ \hat{C}'I \text{ in the middle factor}\,.$$

We now calculate this partial trace. We pass from volume forms $\phi dw_1 \wedge \cdots \wedge dw_u$ to functions ϕ so as to replace our Banach spaces by Banach algebras of the form $A(R)$. Then with these identifications

$$\hat{C}'I: A(W_0 \times W'_0 \times Z'') \to A(W_0 \times Z' \times W''_0),$$

$$\phi(w, w', z'') \mapsto \det\left(\frac{\partial}{\partial w''} C'_1\right) \phi(w, C'(w'', z'))\,,$$

$$I\hat{C}: A(W_0 \times Z' \times W''_0) \to A(Z \times W'_0 \times W''_0),$$

$$\psi(w, z', w'') \mapsto \det\left(\frac{\partial}{\partial w'} C_1\right) \psi(C(w', z), w'')\,.$$

Thus $I\hat{C} \circ \hat{C}'I = \gamma\mu^*$, where γ is the product of the 2 Jacobian factors and $\mu : Z \times W'_0 \times W''_0 \to W_0 \times W'_0 \times Z''$ is the holomorphic map

$$(z, w', w'') \mapsto (C_1(w', z), C'_1(w'', C_2(w', z)), C'_2(w'', C_2(w', z)))\,.$$

Note that $\mu_2(z,-,w'')$ is the composition of $C_2(-,z)$ and $C_1'(w'',-)$, hence defines a holomorphic contraction map of W_0' with values in int (W_0').

We are now in the following general situation. We are given a holomorphic map $v: W \times Z_1 \to W \times Z_2$ with values in int $(W) \times Z_2$, where W, Z_1 and Z_2 are regions in complex space and v is uniformly approximable by polynomials. Then for any function $\delta \in A(W \times Z_1)$ the operator

$$\delta v^*: A(W \times Z_2) \to A(W \times Z_1)$$

has a partial trace in the factor W, which is an operator $Tr_W(\delta v^*): A(Z_2) \to A(Z_1)$. We are given, moreover, that $v_1(-,z)$ is a contraction of W for all $z \in Z_1$. The fixed point $p(z)$ of this contraction gives a map $p: Z_1 \to W$ that is uniformly approximable by polynomials. Then we have the following formula for the partial trace, in terms of $(p, id): Z_1 \to W \times Z_1$ and the components of v:

$$Tr_W(\delta v^*) = (p, id)^* \left(\frac{\delta v_2^*}{\mathrm{Det}(I - \frac{\partial}{\partial w} v_1)} \right) .$$

Note that when Z_2 is a point, this is just the Atiyah–Bott fixed point formula ([AB], see also [R1, F1]) for a family of holomorphic contraction maps with parameter space Z_1. In general we can reduce to this case by evaluating both sides on a fixed element of $A(Z_2)$.

Now we apply this partial trace formula to our example. We have $p: W_0'' \times Z \to W_0'$ with

$$C_1'(w'', C_2(w', z)) = w' \quad \text{for } w' = p(w'', z) .$$

The function $(p, id)^*(\gamma/\mathrm{Det}(I - \frac{\partial}{\partial w'}\mu_2))$ is just $\det\left(\frac{\partial}{\partial w''}E_1\right)$, as follows from

Lemma 3. *Given $w'' \in W_0''$ and $z \in Z$,*

$$\det\left(\frac{\partial E_1}{\partial w''}\bigg|_{(w'',z)} \right) = \frac{\det\left(\frac{\partial C_1'}{\partial w''}\bigg|_{(w'',z')} \right) \det\left(\frac{\partial C_1}{\partial w'}\bigg|_{(w',z)} \right)}{\det\left(I - \frac{\partial C_1'}{\partial z'}\bigg|_{(w'',z')} \frac{\partial C_2}{\partial w'}\bigg|_{(w',z)} \right)} ,$$

where $w' = C_1'(w'', C_2(w', z))$ and $z' = C_2(C_1'(w'', z'), z)$.

Proof. By definition, $E_1(w'', z) = C_1(w', z)$. Thus $\frac{\partial}{\partial w''}E_1 = \left(\frac{\partial}{\partial w'}C_1\right)\left(\frac{\partial}{\partial w''}w'\right)$. We differentiate w' implicitly with respect to w'' to get

$$\frac{\partial w'}{\partial w''} = \frac{\partial C_1'}{\partial w''} + \left(\frac{\partial C_1'}{\partial z'}\right)\left(\frac{\partial C_2}{\partial w'}\right)\left(\frac{\partial w'}{\partial w''}\right) .$$

Solving for $\frac{\partial}{\partial w''}w'$ gives

$$\frac{\partial E_1}{\partial w''} = \frac{\partial C_1}{\partial w'}\left(I - \frac{\partial C_1'}{\partial z'}\frac{\partial C_2}{\partial w'}\right)^{-1}\frac{\partial C_1'}{\partial w''} ,$$

and the lemma follows.

Finally we see that $E(w'',z) = (\mu_1,\mu_3)(z, p(w'',z), w'')$, that is $E = v_2 \circ (p, id)$. Substituting into the partial trace formula gives

$$Tr_{W_0'}(I\hat{C} \circ \hat{C}'I) = \hat{E} ,$$

which implies the proposition.

Now we suppose given $r \in A(W' \times Z)$ and $r' \in A(W'' \times Z')$. We define $t \in A(W'' \times Z)$ by

$$t(w'',z) = r(w',z)r'(w'',z') ,$$

where w' and z' are as in Lemma 2. Then the following result generalizes Proposition 2.

Proposition 3. $L_{t,H} = L_{r,F} \circ L_{r',F'}$.

The proof is quite close to that just given, with $\hat{C}, \hat{C'}, \hat{E}$ replaced by $r\hat{C}, r'\hat{C'}, t\hat{E}$ throughout. The function γ used above is multiplied by the factor $r(w',z)r'(w'', C_2(w',z))$. If we call this factor $\tau(z, w', w''), (p, id)^*\gamma$ is multiplied by $(p, id)^*\tau = t$, and the proposition follows.

Finally, we note that when r and r' are $d \times d$ matrices over $A(W' \times Z)$ and $A(W'' \times Z')$ then the formula for t defines an $d \times d$ matrix over $A(W'' \times Z)$. The resulting transfer operators on d-tuples of kernels still satisfy Proposition 3.

Section 4. The Trace of the Kernel Transfer

Suppose in Sect. 2 that $W' = W$ and $Z' = Z$. Then for any square matrix r over $A(W \times Z)$ the kernel transfer $L_{r,F}$ is $(s+u)$-compact and so has a trace. The cross map C contracts $W \times Z$ and so has a unique fixed point p. Thus p is also the unique fixed point of F.

Proposition 4. *$Tr(L_{r,F}) = (-1)^u Tr(r(p))/\mathrm{Det}(I - D_pF)$ when F is a holomorphic map near p. Otherwise $Tr(L_{r,F}) = 0$.*

Clearly this reduces to the case $d = 1$, $r \in A(W \times Z)$. Since $L_{r,F}$ is the partial adjoint of $r\tilde{C}$, it follows that $Tr(L_{r,F}) = Tr(r\tilde{C})$. By the fixed point formula of Atiyah–Bott [AB], cited in Sect. 3, $Tr(r\tilde{C}) = J(p)r(p)/\mathrm{Det}(I - D_pC)$, where $J = \frac{\partial}{\partial w'}C_1$. Thus the proposition reduces to

Lemma 4. *$(-1)^u\mathrm{Det}(I - D_pC) = J(p)\,\mathrm{Det}(I - D_pF)$ when F is a holomorphic map near p. Otherwise $J(p) = 0$.*

Proof. From $F(C_1(w',z),z) = (w', C_2(w',z))$, the chain rule gives

$$DF\begin{pmatrix} \dfrac{\partial C_1}{\partial w'} & \dfrac{\partial C_1}{\partial z} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ \dfrac{\partial C_2}{\partial w'} & \dfrac{\partial C_2}{\partial z} \end{pmatrix} \quad \text{so } (I - DF)\begin{pmatrix} \dfrac{\partial C_1}{\partial w'} & \dfrac{\partial C_1}{\partial z} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}(I - DC) ,$$

and we take determinants to get the desired equation. If $J(p) \neq 0$, then D_pF is determined by the above formula so, by the holomorphic inverse function theorem, F is a holomorphic map near p.

Section 5. Zeta Functions for a C^ω System of Hyperbolic Correspondences

We suppose given integers $u \geqq 0$ and $s \geqq 0$, regions X_j in $\mathbf{R}^u$ and $Y_j \in \mathbf{R}^s$, and hyperbolic correspondences f_i from $X_{\alpha(i)} \times Y_{\alpha(i)}$ to $X_{\omega(i)} \times Y_{\omega(i)}$ whose cross maps c_i are C^ω, where the *vertices* j and the *arrows* i run over finite index sets V and A and $\alpha, \omega : A \to K$. We choose $\varepsilon > 0$ and let W_j (respectively Z_j) be the closed ε-neighborhood of X_j (respectively Y_j) in $\mathbf{C}^u$ (respectively $\mathbf{C}^s$). For ε sufficiently small, c_i extends to a holomorphic map $C_i : W_{\omega(i)} \times Z_{\alpha(i)} \to W_{\alpha(i)} \times Z_{\omega(i)}$ with values in int $(W_{\alpha(i)} \times Z_{\omega(i)})$ whose components C_{i1}, C_{i2} contract uniformly in each factor. Thus C_i is the cross map of a hyperbolic correspondence F_i from $W_{\alpha(i)} \times Z_{\alpha(i)}$ to $W_{\omega(i)} \times Z_{\omega(i)}$.

In order to iterate this system of correspondences, we suppose that the c_i satisfy the hypotheses of Proposition 1. For ε sufficiently small, the C_i will also satisfy these hypotheses so $F_{i_n} \circ \cdots \circ F_{i_1}$ is hyperbolic whenever $\omega(i_1) = \alpha(i_2), \ldots, \omega(i_{n-1}) = \alpha(i_n)$.

Fix $d \geqq 1$. Suppose for each i that we are given a $d \times d$ matrix r_i of C^ω functions on $W_{\omega(i)} \times Z_{\alpha(i)}$. For ε sufficiently small, each r_i extends to a $d \times d$ matrix $r_i \in M_d(A(W_{\omega(i)} \times Z_{\alpha(i)}))$, as shown in Lemma 2. Then for each i we have the i^{th} transfer operator

$$L_{r_i, F_i} : K^d(W_{\omega(i)}, Z_{\omega(i)}) \to K^d(W_{\alpha(i)}, Z_{\alpha(i)}) .$$

We set $K = \oplus_j K^d(W_j, Z_j)$ with the sup norm. Then for each i we let L_i be the operator on K which has the block form consisting of the i^{th} transfer operator in the block at $(\alpha(i), \omega(i))$ with other blocks zero. We define $L = \sum_{i \in A} L_i$ to be the *kernel transfer for the system of correspondences* F_i *and weights* r_i. Clearly L depends linearly on these weights and L is n-compact, $n = u + s$. Accordingly L has a Fredholm determinant, which we can estimate using [F1], Lemma 6:

$$\log|\det(I - L)| \leqq c(1 + \log_+ C)^{n+1} ,$$

where

$$L = \sum_{i=1}^{\infty} x_i x_i^*, |\langle x_i | x_i^* \rangle| \leqq C \exp(\beta \sqrt[n]{i}) ,$$

and c depends only on β and n. Factoring our L through a restriction operator, we see that we can take β independent of the weights r_i and C a constant times the sup norm of these r_i, so we have shown

Lemma 5. $\log|\det(I - L)| \leqq c(1 + \log_+ \sup_i \|r_i\|)^{u+s+1}$ *for some positive constant* c *independent of the* r_i.

We can calculate this Fredholm determinant $\det(I - L)$ provided $\|L\| < 1$, using

$$\det(I - L) = \exp - \sum_{n=1}^{\infty} \frac{1}{n} Tr(L^n), \qquad Tr(L^n) = \sum_{A^n} Tr(L_{i_1} \circ \cdots \circ L_{i_n}) .$$

We suppose the "arrows" $i_1, \ldots, i_n$ form an *n-loop*, that is $\omega(i_1) = \alpha(i_2), \ldots,$ $\omega(i_{n-1}) = \alpha(i_n)$ and $\omega(i_n) = \alpha(i_1)$, since the trace term vanishes otherwise. We then use Propositions 1 and 3 to write $L_{i_1} \circ \cdots \circ L_{i_n} = L_{t,H}$, where $H = H(i_1, \ldots, i_n)$ the hyperbolic correspondence $F_{i_n} \circ \cdots \circ F_{i_1}$ and t is a certain $d \times d$ matrix over $A(W_j, Z_j)$, $j = \alpha(i_1) = \omega(i_n)$. Then by Proposition 4,

$$Tr(L_{i_1} \circ \cdots \circ L_{i_n}) = (-1)^u Tr\, t(p)/\mathrm{Det}(I - D_p H)$$

if H is a holomorphic map near its fixed point $p = p(i_1, \dots, i_n)$, and $= 0$ otherwise. We make the convention that the determinant is ∞ when H is not a holomorphic map near p, so this formula continues to hold.

We calculate $t(p)$ as follows. Let $t_k = r_{i_k}(p(i_k, \dots, i_n, i_1, \dots, i_{k-1}))$ for $k = 1, \dots, n$. Then from the definition of t in Proposition 3, we find

$$t(p) = t_1 \circ \cdots \circ t_n \, .$$

We let Λ_n, $n = 1, 2, \dots$, consist of the n-loops $(i_1, \dots, i_n) \in \Lambda^n$. We define the multiplicity μ of such an n-loop to be the largest divisor of n such that $i_k = i_\ell$ for $k \equiv \ell (\mathrm{mod} \frac{n}{\mu})$. Thus an n-loop of multiplicity μ is the "μ^{th} power" of an (n/μ)-loop of multiplicity 1. We let $\Lambda'_n \subset \Lambda_n$ be the *prime* n-loops, i.e. the n-loops of multiplicity 1.

Now we consider a prime n-loop $\vec{i} = (i_1, \dots, i_n)$ and its cyclic permutations and their powers and assess their contribution $Z_{\vec{i}}$ to det $(I - L)$. We obtain

$$\det(I - L) = \prod_{n=1}^{\infty} \prod_{\vec{i} \in \Lambda''_n} Z_{\vec{i}} \, ,$$

where $\Lambda''_n \subset \Lambda'_n$ and every prime n-loop is a cyclic permutation of just one element of Λ''_n. Then

$$Z_{\vec{i}} = \exp - \sum_{m=1}^{\infty} \frac{(-1)^u}{m} Tr(t(p))^m / \mathrm{Det}\,(I - (D_p H)^m)$$

with p and $t(p) = t_1 \circ \cdots \circ t_n$ as above. In deriving this expression we use that $\mathrm{Tr}(t_k \circ \cdots \circ t_n \circ t_1 \circ \cdots \circ t_{k-1}) = \mathrm{Tr}(t(p))^m$ for all k, and a similar observation for the determinantal factor. The cyclic permutations of the m^{th} power of $\vec{i}$ contribute then n equal terms, and the factor $\frac{1}{m}$ arises as $n(\frac{1}{nm})$. We summarize this as

Proposition 5. *For weights r_i with $\|L\| < 1$,*

$$\det(I - L) = \prod_{n=1}^{\infty} \prod_{\vec{i} \in \Lambda''_n} \exp - \sum_{m=1}^{\infty} \frac{(-1)^u}{m} \frac{Tr(t(p))^m}{\mathrm{Det}\,(I - (D_p H)^m)} \, .$$

Observe that the product on the right is absolutely convergent for $\|L\| < 1$. Note that the r_i's only enter linearly into L, so the left-hand side is an entire function on this space of weights. So we have analytically continued this product to all choices of weights $(r_i) \in \oplus_{i \in \Lambda} M_d(A(W_{\omega(i)} \times Z_{\alpha(i)}))$. Note also that the terms on the right are independent of our choice of complex neighborhoods.

Section 6. Zeta Functions for a Hyperbolic Set of a Flow

Suppose that M is a manifold with a C^1 flow $\phi_t : M \to M$, and $\Omega \subset M$ is a compact invariant set. If ϕ has no stationary points in Ω, there is a line subbundle $E^c \subset T_\Omega M$ invariant by the lifted flow $D\phi_t : T_\Omega M \to T_\Omega M$, such that a section of E^c is a multiple of the vectorfield that generates ϕ on Ω. We say Ω is a *hyperbolic set* for

ϕ if $T_\Omega M/E^c$ is the direct sum of 2 continuous invariant subbundles, one contracted by $D\phi_t$ for $t \gg 0$ and the other contracted by $D\phi_t$ for $t \ll 0$. Then we find a continuous invariant splitting $T_\Omega M = E^u \oplus E^s \oplus E^c$, where E^s is contracted by $D\phi_t$ for $t \gg 0$ and E^u is contracted by $D\phi_t$ for $t \ll 0$. Given $\delta > 0$, the metric on $T_\Omega M$ can be chosen (take any metric and average it over a long time interval $[-\tau, \tau]$) so we have uniform contraction for $t \geqq \delta$ or $t \leqq -\delta$, respectively. We set $u = \dim E^u$ and $s = \dim E^s$, so $u + s + 1 = \dim(M)$. One calls E^c, E^u and E^s the center, unstable and stable bundles in $T_\Omega M$.

Suppose that we are given a (real or complex) vector bundle ξ over Ω with projection $\pi: \xi \to \Omega$. A continuous flow $\psi_t: \xi \to \xi$ with $\pi \circ \psi_t = \phi_t \circ \pi$ such that $\psi_t: \pi^{-1}(q) \to \pi^{-1}(\phi_t q)$ is a linear map for all $q \in \Omega$ and $t \in \mathbf{R}$ is called a *lift* of ϕ. We call a pair of lifts (ψ^+, ψ^-), relative to a pair of vector bundles (ξ^+, ξ^-) over Ω, a *virtual lift* $\psi^\pm$ of ϕ to the *virtual bundle* $\xi^\pm$. Setting $\xi^- = 0, \xi^+ = \xi$ and $\psi^+ = \psi$ we identify any lift ψ with a corresponding virtual lift $(\psi, 0)$.

Now we consider any *periodic orbit* γ of ϕ in Ω, determined by a point $q \in \Omega$ and a positive number ℓ with $\phi_\ell(q) = q$. Of course (q, ℓ) and $(\phi_t(q), \ell)$ determine the same γ, for all $t \in \mathbf{R}$. If the smallest $t > 0$ with $\phi_t(q) = q$ is ℓ/μ we say γ has *period* $\ell = \ell(\gamma)$ and *multiplicity* $\mu = \mu(\gamma)$. If $\mu(\gamma) = 1$ we say γ is *prime*.

Given a lift ψ of $\phi|\Omega$ we define the *holonomy* of ψ over γ to be the similarity class of the transformation $\psi_{\ell(\gamma)}: \xi_q \to \xi_q$, and the *$\gamma$-character* $\chi_\gamma(\psi)$ to be the trace of this transformation. Clearly this trace does not change when q is replaced by $\phi_t q$, $t \in \mathbf{R}$, so χ_γ depends only on γ. Given a virtual lift $\psi^\pm$, we define $\chi_\gamma(\psi^\pm) = \chi_\gamma(\psi^+) - \chi_\gamma(\psi^-)$. The *linear Poincaré map* $\mathbf{P}_\gamma$ is the holonomy of the natural lift of ϕ to $T_\Omega M/E^c \cong E^u \oplus E^s$. Clearly $\mathbf{P}_\gamma = U_\gamma \oplus S_\gamma$, where U_γ is a linear expansion of E^u and S_γ is a linear contraction of E^s.

Consider the series

$$\sum_\gamma \frac{1}{\mu(\gamma)} \frac{\chi_\gamma(\psi^\pm)}{|\mathrm{Det}\,(I - \mathbf{P}_\gamma)|},$$

where γ runs over all periodic orbits of $\phi|\Omega$. Provided $\psi_t^\pm$ are sufficiently contractive as $t \to +\infty$, this series is absolutely convergent for the following reason. The number $N(t)$ of γ with $\ell(\gamma) \leqq t$ satisfies $\lim\sup_{t\to\infty} \frac{1}{t} \log N(t) \leqq h(\phi_1|\Omega) < \infty$, where h denotes topological entropy. The *growth rate* of a lift ψ is $\mathrm{gr}(\psi) = \lim_{t\to\infty} \sup \frac{1}{t} \log \|\psi_t\| < \infty$, where the choice of fiber metric on ξ is irrelevant since Ω is compact. To estimate the denominator, we write

$$|\mathrm{Det}\,(I - \mathbf{P}_\gamma)| = |\mathrm{Det}\,(I - S_\gamma)||\mathrm{Det}\,(I - U_\gamma)| = |\mathrm{Det}\,(U_\gamma)|(1 + o(1))$$

and note that $\lim_\gamma \inf \frac{1}{\ell(\gamma)} \log|\det U_\gamma| \geqq |g|$, where $g \leqq 0$ is the growth rate of the natural lift $\Lambda^u(D\phi_{-t}|E^u)$ of the time reversed flow ϕ_{-t} to the line bundle $\Lambda^u(E^u)$. If $\mathrm{gr}(\psi) < |g| - h(\phi_1|\Omega)$ then, breaking the sum over γ up into a sum over $n \in \mathbf{Z}$ and over those γ with $e^{n-1} \leqq \ell(\gamma) < e^n$, we see this series is absolutely convergent.

For $\psi^\pm$ with $\mathrm{gr}\,(\psi^\pm) < |g| - h(\phi_1|\Omega)$ we define

$$\zeta(\psi^\pm) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \frac{\chi_\gamma(\psi^\pm)}{|\mathrm{Det}\,(I - \mathbf{P}_\gamma)|}.$$

We regard Ω, ϕ, and $\xi^\pm$ as fixed but the virtual lift $\psi^\pm$ as a parameter. In particular, for ξ a complex vector bundle any lift ψ of ϕ to ξ determines a family ψ^z

of lifts involving one complex parameter z with $\psi_t^z = e^{-tz}\psi_t$. As $\mathrm{gr}(\psi^z) \leqq \mathrm{gr}(\psi) - \mathrm{Re}(z)$, ψ^z lies in the domain of ζ if $\mathrm{Re}\, z$ is sufficiently large. We define the *flat-trace function*

$$T^\flat(z) = \zeta(\psi^{\pm z}) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \frac{\chi_\gamma(\psi^\pm)}{|\mathrm{Det}(I - \mathbf{P}_\gamma)|} e^{-z\ell(\gamma)},$$

where on any compact subset of the halfplane

$$\mathrm{Re}\, z > gr(\psi^\pm) + h(\phi_1|\Omega) + gr(\Lambda^u(D\phi_{-t}|E^u))$$

the series in γ is uniformly convergent. Thus $T^\flat$ is holomorphic on this halfplane. Clearly $T^\flat(z) = T_+^\flat(z)/T_-^\flat(z)$, where $T_+^\flat(z)$ and $T_-^\flat(z)$ are defined using $\chi_\gamma(\psi^+)$ and $\chi_\gamma(\psi^-)$ in place of $\chi_\gamma(\psi^\pm)$. If necessary, we write $T_\psi^\flat(z)$ or $T_{\psi^\pm}^\flat(z)$ for $T^\flat(z)$.

We define the *flat-trace* of the operator ψ_t^* of ψ_t on continuous sections of ξ to be the following atomic measure on $(0,\infty)$ concentrated on the lengths of periodic orbits

$$Tr^\flat \psi_t^* = \sum_\gamma \frac{\ell(\gamma)}{\mu(\gamma)} \frac{\chi_\gamma(\psi)}{|\mathrm{Det}(I - \mathbf{P}_\gamma)|} \delta(x - \ell(\gamma))$$

(cf. [Gu, GS]). To understand the sense in which this measure is a "trace", at least for C^∞ flows and lifts, see [GS], Chapter 6. In brief, even though the averaged operators $\int_0^\infty \alpha(t)\psi_t^* dt$ for $\alpha(t)$ continuous with compact support do not have continuous kernels, one may introduce a many-parameter deformation of ϕ_t, ψ_t to obtain averaged operators with continuous kernels. Their traces define a measure on the parameter space which restricts to the flat-trace.

The relation of $T_\psi^\flat(z)$ to the flat-trace of ψ_t^* is clear (cf. [F3], Sect. 5):

$$T_\psi^\flat(z) = \exp - \mathscr{L}\left(\frac{1}{t} Tr^\flat \psi_t^*\right)(z),$$

where $\mathscr{L}(v): z \mapsto \int_0^\infty e^{-tz} dv(t)$ denotes the Laplace transform of a measure v on the ray $t > 0$. In some formal sense, $T_\psi^\flat(z)$ is a Fredholm determinant for the infinitesimal generator X_ψ of ψ_t^*. Indeed consider the definition for the zeta-regularized determinant of an elliptic operator Δ with positive symbol and no zero eigenvalues on a closed manifold:

$$\det{}_\zeta \Delta = \exp - \frac{d}{ds}\bigg|_{s=0} \mathrm{Tr}\, \Delta^{-s},$$

where

$$\Delta^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\Delta} dt, \quad \mathrm{Re}\ s \ll 0,$$

and where $\mathrm{Tr}\ \Delta^{-s}$ is analytically continued to $s = 0$. If the underlying manifold has odd dimension then $\mathrm{Tr}\ \Delta^{-s}|_{s=0} = 0$, so

$$\det{}_\zeta \Delta = \exp - \int_0^\infty t^{s-1} \mathrm{Tr}\ e^{-t\Delta} dt \bigg|_{s=0}.$$

Replacing the generator $-\Delta$ of $e^{-t\Delta}$ by the generator $X_\psi - z$ of $(\psi_t^z)^*$, and trace by flat-trace, we make the definition

$$\det_\zeta(z - X_\psi) = T_\psi^\flat(z) .$$

More precisely, the generator of $(\psi_t^z)^*$ is $X_\psi - zE$, where E denotes the Euler vector field on ξ which is tangent to the fibers and generates the dilation flow $v \mapsto e^t v$ on ξ, so one should write here $\det_\zeta(zE - X_\psi)$.

For a virtual lift, we let $Tr^\flat \psi_t^{\pm,*} = Tr^\flat \psi_t^{+,*} - Tr^\flat \psi_t^{-,*}$, so $T^\flat(z) = \exp - \mathscr{L}(t^{-1} Tr^\flat \psi_t^{\pm,*})$. Most of the dynamical zeta functions in one variable that one associates to ϕ_t and its hyperbolic set Ω are of the form $T^\flat(z)$ for some choice of $\xi^\pm$ and $\psi_t^\pm$. Consider, for example, the *Ruelle function* and the *Selberg function*

$$R(z) = \prod_{\gamma : \mu(\gamma) = 1} (1 - e^{-z\ell(\gamma)}), \qquad S(z) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \frac{e^{-z\ell(\gamma)}}{\operatorname{Det}(I - S_\gamma)} ,$$

where γ runs over the periodic orbits of $\phi_t|\Omega$. To write $S(z) = T^\flat(z)$ we find $\xi^\pm, \psi^\pm$ so that for any γ

$$|\operatorname{Det}(I - \mathbf{P}_\gamma)| = \operatorname{Det}(I - S_\gamma)\chi_\gamma(\psi^\pm) .$$

Let $\varepsilon_\gamma = sgn \det(I - \mathbf{P}_\gamma)$. Then this equation reduces to

$$\chi_\gamma(\psi^\pm) = \varepsilon_\gamma \det(I - U_\gamma) = \varepsilon_\gamma \sum_{j=0}^\infty (-1)^j Tr(\Lambda^j U_\gamma) .$$

The natural lift $\Lambda^j(D\phi_t|E^u)$ of $\phi_t|\Omega$ to $\Lambda^j E^u$ has holonomy $\operatorname{Tr}(\Lambda^j U_\gamma)$. On the other hand $\varepsilon_\gamma = sgn \det(-U_\gamma)$, since $(I - U_\gamma) = -U_\gamma(I - U_\gamma^{-1})$ and $\det(I - U_\gamma^{-1}) > 0$. Thus $(-1)^u \varepsilon_\gamma = sgn \det U_\gamma$ is the holonomy of the flat *unstable orientation line bundle* w of E^u for its flat lift. We choose

$$\xi^\pm = (\Lambda^{\pm(-1)^u} E^u) \otimes w ,$$

where $\Lambda^+\xi$ and $\Lambda^-\xi$ denote the direct sum of the exterior powers $\Lambda^j\xi$ over j even and j odd. If $\psi^\pm$ is the tensor product of the lifts on each factor then $\chi_\gamma(\psi^\pm)$ has the desired form, so $S(z) = T_{\psi^\pm}^\flat(z)$.

Next we note $-\log R(z) = \sum_\gamma \frac{1}{\mu(\gamma)} e^{-z\ell(\gamma)}$. To arrange $R(z) = T^\flat(z)$ we find $\xi^\pm, \psi^\pm$ so that for any γ

$$|\operatorname{Det}(I - \mathbf{P}_\gamma)| = \chi_\gamma(\psi^\pm) .$$

We find that

$$\xi^\pm = (\Lambda^{\pm(-1)^u}(TM/E^c)) \otimes w$$

with the natural lifts $\psi^\pm$ does the trick.

If we are given a lift ρ_t of ϕ_t to a bundle η, we define the corresponding Ruelle and Selberg functions

$$R_\rho(z) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} e^{-z\ell(\gamma)} \chi_\gamma(\rho) ,$$

$$S_\rho(z) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} e^{-z\ell(\gamma)} \frac{\chi_\gamma(\rho)}{\det(I - S_\gamma)} .$$

To choose $\xi^{\pm}$ and $\psi^{\pm}$ for which $R_\rho(z) = T^\flat(z)$ or $S_\rho(z) = T^\flat(z)$, we tensor the bundles and lifts used for $R(z)$ or $S(z)$ with η and ρ. In particular, we are interested in $R_\rho(z)$ for η a flat bundle of degree d on M with flat lift ρ and holonomy given by $\alpha : \pi_1 M \to Gl(d, \mathbf{C})$. Then

$$R_\rho(z) = \prod_{\gamma:\mu(\gamma)=1} \det\left(I - e^{-z\ell(\gamma)}\alpha(\gamma)\right) .$$

The *torsion function* of such a representation α is

$$Z_\alpha(z) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} e^{-z\ell(\gamma)}\varepsilon_\gamma Tr(\alpha(\gamma)) .$$

Here we choose simply $\xi^{\pm} = \eta \otimes \Lambda^{\pm}(TM/E^c)$, with the natural lifts $\psi^{\pm}$ (flat on the η factor and induced by $D\phi_t$ on the other). These are the dynamical zeta functions that carry topological information about the flow. It is so named because of the cases where the value of $Z_\alpha(z)$ at $z = 0$ can be defined by analytic continuation and identified with the *Reidemeister torsion* $\tau_\alpha(N, N_-)$ for a certain pair of spaces (N, N_-) (here Ω is isolated, N an isolating block and N_- its exit set, see [F3, F4, MS, S]).

The relationship between Ruelle and Selberg functions was worked out in [F1], Proposition 2:

$$R_\rho(z) = S_{\rho^+}(z)/S_{\rho^-}(z) ,$$

where $\rho^{\pm} = \rho \otimes \Lambda^{\pm}(D\phi_t|E^s)$. The above expansions of R_ρ and S_ρ as $T^\flat_{\psi^+}(z)/T^\flat_{\psi^-}(z)$ are quite analogous. We tend to regard the flat-trace functions $T^\flat_\psi(z)$ as building blocks and functions such as $R(z)$ and $Z_\alpha(z)$ as the zeta functions of primary interest. However, even the case $\xi = \Omega \times \mathbf{C}$, ψ_t trivial, leads to the *correlation zeta function*

$$d(z) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \frac{1}{|\mathrm{Det}(I - \mathbf{P}_\gamma)|} e^{-z\ell(\gamma)}$$

which is of special interest in statistical physics [Ru].

The class of zeta functions $T^\flat(z)$ can be widened when one is given a continuous weight function a on Ω with $\mathrm{Re}(a) > 0$. Then a given lift ψ can be deformed to ψ^{za} with

$$\psi^{za}(v) = \exp\left(-z \int_0^t a(\phi_s p)\, ds\right) \psi v ,$$

where $p = \pi(v)$, $v \in \xi$. One has

$$\chi_\gamma(\psi^a) = \exp\left(- \int_0^{\ell(\gamma)} a(\phi_s p)\, ds\right) \chi_\gamma(\psi) ,$$

where γ passes through p.

When given a virtual lift $\psi^{\pm}$ we set

$$\zeta(\psi^{\pm,za}) = \zeta(\psi^{+,za})/\zeta(\psi^{-,za}) .$$

Of course for $a = 1$ we recover the function $T^\flat(z)$. For $a = \log|J_u|$, where J_u is the unstable expansion of ϕ_t relative to some fiber metric on $T_\Omega M$, we obtain the *differential zeta function* of Parry [Pa].

While the zeta functions we know of in the literature are all of this form $\zeta(\psi^{\pm,za})$, one can define a more general class, as follows. A *zeta function of one variable for* (Ω,ϕ) is the ratio $\zeta(\psi^+(z))/\zeta(\psi^-(z))$, where $\psi^+(z)$ and $\psi^-(z)$ are lifts of $\phi|\Omega$ depending holomorphically on a parameter $z \in \mathbf{C}$ such that $\|\psi_1^+(x)\|$, $\|\psi_1^-(x)\| \to 0$ as $x \to +\infty$, $x \in \mathbf{R}$. Then this zeta function is holomorphic on some ray $[c,\infty)$.

Section 7. Meromorphic Extension of $T^\flat(z)$ and $\zeta(\psi)$

There are two sorts of obstructions to extending $T^\flat(z)$ to a meromorphic function on $\mathbf{C}$.

The first is topological. Suppose ϕ is the suspension flow of the Smale horseshoe $f: S^2 \to S^2$ with return time 1. Let Ω be an invariant set in the suspended invarient cantor set so $u = s = 1$. Then the corresponding Ruelle function

$$R(z) = \prod_{\gamma:\mu(\gamma)=1} (1 - x^{\ell(\gamma)})\big|_{x=e^{-z}} = \zeta_\Omega(e^{-z})^{-1} ,$$

where the power series $\zeta_\Omega(x)$ is the Artin–Mazur generating function for $f|(\Omega \cap S^2)$. It is possible to find invariant sets in the horseshoe whose Artin–Mazur function is a rather general subproduct of $\zeta_\Omega(x)$, hence does not extend meromorphically beyond some radius (e.g. $|x| = 1$ in [BL], p. 47, as follows from the Fabry gap theorem, cf. [B]II, p. 296).

We avoid this difficulty by restricting to the case when Ω is *isolated*, i.e. $\Omega = \bigcap_{-\infty}^{+\infty} \phi_t(U)$ for some open set $U \subset M$. An isolated hyperbolic set if a *basic set* in Smale's terminology.

For Ω basic, there is still an obstruction – the smoothness of the flow. If ϕ is not C^r for some $r < \infty$ then there is sometimes no meromorphic extension to $\mathbf{C}$ [Ga, P, PP]. Examples can be constructed on suspension flows of horseshoes with variable return time. Accordingly we must assume ψ, ϕ are C^∞. By analogy with the well-understood case of expanding maps, this should be enough ([T, R2, F5]) but the case when they are C^ω should be simpler ([R1, F1]).

We now fix a C^ω flow ϕ and a compact basic set Ω for ϕ. We assume also that $(\xi^\pm, \psi^\pm)$ is a C^ω virtual lift, with $\xi^\pm$ complex vector bundles over Ω.

Theorem. *$T^\flat(z)$ extends to a meromorphic function on $\mathbf{C}$ of order at most* $\dim(M)$. *Moreover $\zeta(\psi^+)/\zeta(\psi^-)$ is a meromorphic function of $\psi_t^\pm$ and every zeta function of one variable for (Ω,ϕ) extends meromorphically to $\mathbf{C}$.*

Referring to the examples of Sect. 6, we see that for ξ, ψ of class C^ω, the Ruelle function $R_\psi(z)$ extends to a meromorphic function on $\mathbf{C}$. (On the other hand, we cannot continue the Selberg function $S_\psi(z)$, or even $S(z)$, using this theorem unless the stable bundle is C^ω.) We apply this to the case when $M = UQ$ is the unit tangent bundle of a closed manifold Q with a C^ω Riemannian metric of negative curvature, ϕ is the geodesic flow on M and $\Omega = M$. In this case we obtain a result in which no flow is mentioned.

Corollary. *For a closed Riemannian manifold Q of negative curvature, the product*

$$R(z) = \prod_\gamma 1 - e^{-z\ell(\gamma)}, \quad \operatorname{Re} z \gg 0 ,$$

where γ ranges over all prime closed oriented geodesics on Q, extends to a meromorphic function on $\mathbf{C}$. *Similarly, the Ruelle and torsion functions of a matrix representation* $\alpha : \pi_1(UQ) \to Gl(d;\mathbf{C})$

$$R_\alpha(z) = \prod_\gamma \det(I - e^{-z\ell(\gamma)}\alpha(\gamma))$$

and

$$Z_\alpha(z) = \prod_\gamma \det(I - \sigma_\gamma e^{-z\ell(\gamma)}\alpha(\gamma))^\varepsilon, \quad \varepsilon = (-1)^{\dim Q - 1}$$

have meromorphic extensions to $\mathbf{C}$. *Here* $\sigma_\gamma = 1$ *or* -1 *as γ preserves or reverses the orientation of Q. These functions have order at most* $2(\dim Q) - 1$.

This corollary opens up the possibility of exploring the special values of these functions, which were previously only known to exist for Q locally symmetric. In particular, if there are no α-twisted harmonic forms on UQ (i.e. no nonzero d-tuple of harmonic forms on the universal cover of UQ is α-equivariant) we conjecture

$$Z_\alpha(0) = \tau_\alpha(UQ)\,,$$

where τ_α is the Reidemeister torsion of the manifold UQ. This is a sort of "Lefschetz formula" relating the *periodic orbits* of the geodesic flow over Q to the *topology* of Q [F3]. It is known when Q is $\mathbf{R}$-hyperbolic, i.e. of constant negative curvature, and α unitary ([F6, F7]). Note that our condition on harmonic forms implies that α is acyclic but the converse only holds for unitary α – see [F4] for some pathological situations where this distinction seems to matter.

In other examples, the special value of a ratio of Selberg functions is of *geometric* interest. For Q hyperbolic with $\dim Q = 2k+1$, Millson expressed $e^{\pi i\eta}$, η the eta invariant of Q, as the value $S_{\rho^+}(z)/S_{\rho^-}(z)|_{z=k}$, where $\rho^\pm$ are the natural lifts of the geodesic flow on UQ to the spin bundles $\xi^\pm$ with $\xi_v^+ \oplus \xi_v^- = \Lambda^k(T_pQ/\mathbf{R}_v)$ for $v \in UQ$, $p = \pi(v)$ [M]. For Q $\mathbf{C}$-hyperbolic (i.e. Kähler with constant negative *holomorphic* sectional curvature) the holomorphic torsion invariants of Ray–Singer are expressible as such special values of Selberg functions [F8]. One may hope to generalize these results to variable negative curvature using our corollary.

Proof of the theorem. Clearly we may reduce to the case of a C^ω lift (ξ,ψ). For some small neighborhood U of Ω, ξ extends to a C^ω bundle on U and ψ to a C^ω family of bundle maps $\xi|_{U\cap\phi_{-t}U} \to \xi|_{U\cap\phi_tU}$, which we will also denote (ξ,ψ).

Since ϕ has no stationary points in Ω and Ω is compact, we can choose a finite number of small transverse compact disjoint C^ω discs D_k of codimension 1, such that for every $p \in \Omega$ there is a $t \in (0,1)$ with $\phi_t p \in \bigcup_k \operatorname{int}(D_k)$. We choose a basis of C^ω sections for $\xi|D_k$, so $\xi|D_k \cong D_k \times \mathbf{C}^d$, $d = \operatorname{rank}(\xi)$.

Now we fix $\delta > 0$ so that if $p, \phi_t p \in \bigcup_k D_k$ and $t > 0$ then $t > \delta$. We choose a C^ω Riemannian metric on U and a constant $\sigma < 1$ so that

$$\|D\phi_\delta|E^s(\Omega)\| < \sigma, \qquad \|D\phi_{-\delta}|E^u(\Omega)\| < \sigma\,.$$

Consider a point $p \in \Omega \cap \operatorname{int}(D_k)$ and the C^ω coordinates $(x_1,\dots,x_{u+s})$, $\|x\| < 1$, on D_k. Choose an ordered orthonormal basis for $T_pD_k \cap (E_p^c \oplus E_p^u)$ and $T_pD_k \cap (E_p^c \oplus E_p^s)$ and use an affine coordinate change that takes p to 0 and these bases to the standard bases for $\mathbf{R}^u \oplus 0$ and $0 \oplus \mathbf{R}^s$. When $t_0 > 0$ is chosen with $\phi_{t_0}(p) \in \mathrm{int}(D_{k'})$ and $p' \in D_{k'}$ is near $\phi_{t_0}(p)$ we choose coordinates in this same way on $D_{k'}$. Let $r > 0$ be a constant small relative to the distances from p to the boundary of D_k. If p' is sufficiently near $\phi_{t_0}(p)$, the return map for ϕ_t gives a hyperbolic correspondence in local coordinates on the product $B^u_r \times B^s_r$ of balls of radius r, more precisely from $B^u_r(p) \times B^s_r(p)$ to $B^u_r(p') \times B^s_r(p')$. We can arrange that the major constants of this correspondence are $< \sigma$ and the minor constants are smaller than any given positive ρ, by choosing r small enough and p' near enough to $\phi_{t_0} p$.

Now we use [B1] to find a fine Markov family of section R_j with each $R_j \subset \Omega \cap \mathrm{int}(D_k)$ for some $k = k(j)$. If the R_j are sufficiently small then for any choice of $p_j \in R_j$ and any Markov transition i from $\alpha(i)$ to $\omega(i)$, the Markov correspondence from $R_{\alpha(i)}$ to $R_{\omega(i)}$ extends to a hyperbolic correspondence f_i from $X_{\alpha(i)} \times Y_{\alpha(i)}$ to $X_{\omega(i)} \times Y_{\omega(i)}$ with $f_i = f_{p,t_0}$ for $p = p_{\alpha(i)}$ and $X_j = B^u_r(p_j)$, $Y_j = B^s_r(p_j)$. We arrange that the major constants for the cross maps c_i are $< \sigma$ and the minor constants are $< \rho$, where ρ is the positive function of σ given in Proposition 1.

Next we associate a system of weights r_i to our lift ψ_t. Fix i and choose $x' \in X_{\omega(i)}$ and $y \in Y_{\alpha(i)}$ and let $(x, y') = c_i(x', y)$. Then $(x', y') = \phi_t(x, y)$ for $t = t_i(x', y)$ a positive C^ω function, the *return time* for the i^{th} transition. Then $\psi_t\colon \xi_{(x,y)} \to \xi_{(x',y')}$ is represented, using our trivialization of $\xi|(\cup_k D_k)$, by an invertible $d \times d$ matrix $r_i(x', y)$.

Thus our C^ω product neighborhoods $X_j \times Y_j$ of the Markov section $\mathbf{R}_j$ and our C^ω trivializations of $\xi|D_k$ define a system of C^ω hyperbolic correspondences f_i and C^ω weights r_i, indexed by the Markov transitions. As in Sect. 5, this system defines a transfer operator L. Provided ψ_t is sufficiently contractive as $t \to +\infty$, so that the series in $\zeta(\psi)$ coverges absolutely and $\|L\| < 1$, we will compare $\zeta(\psi)$ to $\det(I - L)$.

Suppose $\vec{i} = (i_1, \ldots, i_n) \in \Lambda_n$ is an n-loop with multiplicity $\mu(\vec{i})$. There are $n/\mu(\vec{i})$ distinct elements of Λ_n obtained by cyclically permuting $\vec{i}$ and altogether these contribute a factor

$$\exp - \left[\frac{1}{\mu(\vec{i})} \frac{(-1)^u Tr(t_1 \circ \cdots \circ t_n)(p)}{\mathrm{Det}(I - D_p(F_{i_n} \circ \cdots \circ F_{i_1}))} \right] .$$

Since $F_{i_n} \circ \cdots \circ F_{i_1}$ has a fixed point, p is real and

$$\mathrm{Det}(I - D_p(F_{i_n} \circ \cdots \circ F_{i_1})) = \mathrm{Det}(I - D_p(f_{i_n} \circ \cdots \circ f_{i_1})) = \mathrm{Det}(I - \mathbf{P}_\gamma) ,$$

where γ is the periodic orbit through p determined by $\vec{i}$. Moreover $(t_1 \circ \cdots \circ t_n)(p)$ is the holonomy of ψ around γ, so

$$Tr(t_1 \circ \cdots \circ t_n)(p) = \chi_\gamma(\psi) .$$

The terms in $-\log\det(I - L)$ and $-\log\zeta(\psi)$ corresponding to $\vec{i}$ are then

$$\frac{1}{\mu(\vec{i})} \frac{(-1)^u \chi_\gamma(\psi)}{\mathrm{Det}(I - \mathbf{P}_\gamma)} \quad \text{and} \quad \frac{1}{\mu(\gamma)} \frac{\chi_\gamma(\psi)}{|\mathrm{Det}(I - \mathbf{P}_\gamma)|} .$$

Here $\mu(\vec{i})$ divides $\mu(\gamma)$ and the second factors differ a sign $(-1)^u \varepsilon_\gamma = sgn \det U_\gamma$, i.e. the holonomy of the unstable orientation line bundle ω for the flat lift.

For simplicity we consider the case dim $\Omega = 1$, such as holds for a suspended horseshoe. Then we may choose the R_j's disjoint so that $\mu(\vec{i}) = \mu(\gamma)$ for all $\vec{i}$ and γ's correspond 1–1 to loops modulo cyclic permutations. Then we have shown

$$\det(I - L) = \zeta(\psi \otimes w) .$$

Now we vary ψ over some family of lifts with a compact parameter space. We choose the complex extensions $W_j \times Z_j$ so that our transfer operator is defined for all parameter values. For a family ψ^{za} for instance, with a C^ω, we require that the function

$$\int_0^{t_i(x',y)} g(\phi_s p) ds, \; p = (c_i(x',y), y)$$

on $X_{\omega(i)} \times Y_{\alpha(i)}$ extends to $W_{\omega(i)} \times Z_{\alpha(i)}$ (and we can take all of $\mathbf{C}$ for our parameter space). Then the expression det $(I - L)$ defines an analytic function of these parameters, so $\zeta(\psi \otimes w)$ is an analytic function of ψ. As $(\psi \otimes w) \otimes w = \psi$ we deduce that $\zeta(\psi)$ is an analytic function of ψ. Moreover $\zeta(\psi^{za})$ is an entire function of z. In particular $T_\psi^\flat(z)$ is entire. Also any zeta function of one variable for (Ω, ϕ) is meromorphic.

For general Ω, we follow Bowen's form of the Manning counting argument for a Markov partition ([Ma, B2]) as adapted to flows in [F1]. For $m \geqq 0$ there is a directed graph with vertex set V_m and arrow set A_m such that a loop in A_m defines a periodic orbit γ which passes at all times through at least $m + 1$ Markov flowboxes. One has the inclusion–exclusion formula

$$\frac{1}{\mu(\gamma)} = \sum_m (-1)^m \sum_{\vec{i}} \frac{\varepsilon(\vec{i})}{\mu(\vec{i})} ,$$

where $\vec{i}$ runs over loops in A_m representing γ and $\varepsilon(\vec{i}) = \pm 1$, see [F1], Proposition 1. For $m = 0$, V_0 and A_0 consist of the Markov vertices j and transitions i respectively, and $\varepsilon(\vec{i}) = 1$ for all $\vec{i}$. We set

$$\zeta_m(\psi) = \exp - \sum_{\vec{i}} \frac{\varepsilon(\vec{i})}{\mu(\vec{i})} \frac{(-1)^u \chi_\gamma(\psi)}{\mathrm{Det}\,(I - \mathbf{P}_\gamma)} ,$$

where $\vec{i}$ runs over loops in A_m and ψ_t is sufficiently contracting as $t \to +\infty$. Then $\zeta(\psi)$ is a *finite* alternating product

$$\zeta(\psi) = \prod_m \zeta_m(\psi)^{(-1)^m} .$$

As above, $\zeta_0(\psi \otimes w) =$ det $(I - L)$. By the construction of the A_m and V_m in [F1], one can easily imitate the construction of L for any $m \geqq 0$ to obtain kernel transfers L_m, $m = 0, 1, 2, \ldots$ with $L_0 = L$ and

$$\zeta_m(\psi \otimes w) = \det(I - L_m) .$$

The L_m depend linearly on $d \times d$ weight matrices that in turn depend holomorphically on ψ_t, so this formula extends $\zeta_m(\psi \otimes w)$ to an analytic function of ψ. Taking the alternating product over $m, \zeta(\psi \otimes w)$ is a meromorphic function of ψ. Hence $\zeta(\psi)$ is meromorphic in ψ, and we have the required meromorphic extensions of zeta functions of one variable.

The weight functions for ψ^z are $\exp(-zt_i)r_i$ with $\delta < t_i < 1$ on $X_{\omega(i)} \times Y_{\alpha(i)}$. We may assume the W_j and Z_j are chosen so that $|t_i| < 1$ on $W_{\omega(i)} \times Z_{\alpha(i)}$. Then

$$\| \exp(-zt_i)r_i \| \leqq \|r_i\| e^{|z|}$$

and Lemma 5, applied to $L_0, L_1, L_2, \ldots$ shows that for some positive C

$$\log |\det I - L_m(\psi^z)| \leqq C|z|^{\dim M},$$

proving the theorem.

Note that the same bound holds for $\log|\zeta(\psi^{za})|$, so $\zeta(\psi^{za})$ also has order $\leqq \dim M$ for any C^ω function a.

We wish to correct a misstatement in [F1]. The inequality on p. 507, 6 lines from the bottom should use $e^{a|z|}$ instead of $e^{-a\mathrm{Re}(z)}$. All the assertions concerning *order* of functions are unchanged but those concerning *right order* (i.e. bounds on halfplanes $\mathrm{Re}\, z \geqq \sigma$) are in doubt. Theorems 3 and 4 should be amended, with "right order" replaced by "order."

We showed above that $\zeta(\psi)$ is analytic in ψ when $\dim \Omega = 1$. This implies that $T^b(z)$ is expressible as a ratio of entire functions independent of the choice of Markov sections:

$$T^b(z) = T_{\psi^+}(z)/T_{\psi^-}(z).$$

We conjecture that $\zeta(\psi)$ is analytic for $\dim \Omega > 1$ as well as suggested by the formalism $\zeta(\psi) = \det_\zeta(-X_\psi)$. One approach to this would be analytic sheaf cohomology, cf [R2] and example II in Sect. 8. Another would be to develop a C^∞ theory of kernel transfer, comparable to the results of [R2, F5] for expanding maps. For $u = s = 1$ Rugh has recently proven that $\zeta(\psi^{za})$ is entire for ξ the trivial bundle and ψ the trivial lift.

Section 8. Examples

I. We first illustrate our use of generalized functions in an ad hoc example. Let $G = T^n$ be the n-torus and $\hat{G} = \mathbf{Z}^n$. Given any positive weights w_χ, $\chi \in \mathbf{Z}^n$, we consider the *weighted Hilbert space* $H(w_\chi)$ of Fourier series $\Sigma_\chi c_\chi \chi$ with $\Sigma |c_\chi|^2 w_\chi < \infty$. We first let $w_\chi = \exp(2\varepsilon \sum_{i=1}^n |\chi_i|)$ to obtain a space H_ε. Clearly $H_0 = L^2(T^n)$. For $\varepsilon > 0$, H_ε consists of functions holomorphic on the n-fold product of annuli $R_\varepsilon = \{(z_1, \ldots, z_n)\colon |z_i|, |z_i^{-1}| < \exp(\varepsilon)\}$ with L^2 boundary values. For $\varepsilon < 0$, H_ε consists of generalized functions in the dual of $H_{|\varepsilon|}$.

Suppose $A\colon T^n \to T^n$ is a hyperbolic toral automorphism with dual automorphism $\hat{A} \in Gl(n, \mathbf{Z})$. Provided $w_{\chi \circ A}/w_\chi$ is bounded above, A acts on $H(w_\chi)$ by a bounded operator A^*. We introduce the splitting of $\hat{G} \otimes \mathbf{R} = \mathbf{R}^n = U \oplus S$ and norms on U, S that are contracted by $\hat{A}^{-1}$ and $\hat{A}$, respectively. We let H denote the Hilbert space $H(w_\chi)$ with

$$w_\chi = \exp(\|\chi_s\| - \|\chi_u\|), \quad \text{for } \chi = \chi_u + \chi_s, \chi_u \in U, \chi_x \in S.$$

If we take $\beta > 1$ with $\|\hat{A}|S\| \leqq 1 - \log \beta$ and $\|\hat{A}^{-1}|U\| \leqq (1 + \log \beta)^{-1}$, then with $\|\chi\| = \|\chi_u\| + \|\chi_s\|$,

$$w_{\chi \circ A}/w_\chi \leqq \beta^{-\|\chi\|}.$$

It follows that for the orthogonal projection π_r to χ's with $\|\chi\| \leqq r$, $\pi_r : H \to H$,

$$\|A^* - A^*\pi_r\| = O(\beta^{-r}).$$

Since rank $\pi_r = O(r^n)$ we see that A^* is an n-compact operator on H.

For some $\varepsilon > 0$, $H_\varepsilon \subset H \subset H_{-\varepsilon}$. However H is neither a space of analytic functions nor the dual of such a space but rather some mixture of the two.

Considering the action of A^* on x's, it is clear that $\mathrm{Tr}(A^*) = 1$ and the only nonzero eigenfunctions of A^* are constants.

II. Suppose $n = 2$ and $\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the Fibonacci automorphism of $\mathbf{Z}^2$. Then $A : T^2 \to T^2$ admits a Markov partition with 2 rectangles [AW], which we parametrize by $R_1 = [0,1]^2$ and $R_\rho = [0,\rho] \times [-\rho, 0]$, $\rho = \frac{\sqrt{5}-1}{2}$ such that the Markov correspondences are

$$f_{1\rho}(x,y) = (\rho^{-1}x, -\rho y) \text{ from } R_1 \text{ to } R_\rho,\ 0 \leqq x \leqq \rho^2, 0 \leqq y \leqq 1,$$

$$f_{\rho 1}(x,y) = (\rho^{-1}x, -\rho y) \text{ from } R_\rho \text{ to } R_1,$$

and

$$f_{11}(x,y) = (\rho^{-1}x - \rho, 1 - \rho y) \text{ from } R_1 \text{ to } R_1,\ \rho^2 \leqq x \leqq 1, 0 \leqq y \leqq 1.$$

These correspondences are C^ω and split. Their cross maps $c_{1\rho}, c_{\rho 1}, c_{11}$ are affine maps so we will ignore the choice of W_j and Z_j, $j = 1, \rho$. With $L = L_{1\rho} + L_{\rho 1} + L_{11}$ as in Sect. 5 we find

$$Tr\ L^q = N_q \frac{-1}{(1-(-\rho)^q)(1-\rho^{-q})},$$

where N_q is the number of period q points of the Markov correspondence. Since $\hat{A}$ is the transition matrix for this correspondence, $N_q = Tr(\hat{A}^q) = \rho^{-q} + (-\rho)^q$. Thus

$$Tr\ L^q = \frac{1 + (-\rho^2)^q}{(1-(-\rho)^q)(1-\rho^q)} = \sum_{m \geqq 0, n \geqq 0} (\rho^m(-\rho)^n)^q + (\rho^{m+1}(-\rho)^{n+1})^q.$$

It follows that the nonzero spectrum of L consists of 2 double sequences

$$\rho^m(-\rho)^n \quad \text{and} \quad \rho^{m+1}(-\rho)^{n+1}, \quad (m,n) \geqq (0,0).$$

This can be seen more concretely as follows. Suppose $f(x,y)$ is an eigenkernel for this transfer with eigenvalue λ. Then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (if they are nonzero) are eigenkernels with eigenvalues $\rho\lambda$ and $(-1/\rho)\lambda$. By inspection, L fixes the kernel α with $\alpha_j(x,y) = 1$, $0 \leqq x \leqq j$, $= 0$ for other x ($j = 1, \rho$). The first double sequence, then, consists of generalized functions which are polynomial of degree n in y whose n^{th} y derivative is $\left(\frac{\partial}{\partial x}\right)^m \alpha$. Note that $\frac{\partial \alpha}{\partial x}$ is a delta function at $x = 0$ minus a delta function at $x = j$, for $j = 1, \rho$.

The second double sequence has a similar interpretation starting from an eigenkernel β with eigenvalue $-\rho^2$. Here too, β depends only on x and is supported on $[0,1] \coprod [0,\rho]$. This can be seen by considering the hyperbolic correspondence on this pair of intervals given by the first coordinate of $f_{1\rho}, f_{\rho 1}$ and f_{11}. For this correspondence, $u = 1$, $s = 0$, and

$$Tr\ L^q = N_q \frac{-1}{1-\rho^{-q}} = (1 + \rho^q + \rho^{2q} + \cdots)(1 + (-\rho^2)^q),$$

so the spectrum is $\rho^m, (-\rho^2)\rho^m$. This corresponds to the m^{th} derivatives of α and β.

Unlike α, there is no simple formula for the generalized function $\beta(x)$. The component $\beta_1 = \beta|[0,1]$ satisfies a simple functional equation that characterizes it (and hence $\beta_\rho = \beta|[0,\rho]$) uniquely up to a constant factor:

$$-\rho^2\beta_1(x) = -\rho^{-2}\beta_1(\rho^{-2}x) + \beta_1(-\rho + \rho^{-1}x).$$

Using the results of [R2] (see also [F5]) one can see that $\beta(x)$ is in the dual of $C^{2+\varepsilon}$, for any $\varepsilon > 0$.

Consider the time 1 suspension flow ϕ_t of A and its zeta function $T^\flat(z)$ for the trivial bundle and lift:

$$T^\flat(z) = \exp - \sum_{q=1}^{\infty} \frac{1}{q} \tilde{N}_q \frac{1}{(1-(-\rho)^q)(\rho^{-q}-1)} e^{-zq},$$

where $\tilde{N}_q = \#\ \mathrm{Fix}(A^q)$. By the Lefschetz fixed point formula, or an elementary count,

$$-\tilde{N}_q = 1 - Tr\ \hat{A}^q + (-1)^q = (1-(-\rho)^q)(1-\rho^{-q}),$$

$$T^\flat(z) = \exp - \sum_{q=1}^{\infty} \frac{1}{q} e^{-zq} = 1 - e^{-z}.$$

Thus the transfer operators $L_0(z), L_1(z), \ldots$ satisfy

$$\prod_{m \geqq 0} \det(I - L_m(z))^{(-1)^m} = 1 - e^{-z}.$$

This example suggests we interpret the alternating product formula for $T^\flat(z)$ in terms of a sort of analytic sheaf cohomology. We see that L has a spectrum consisting of 2 double sequences, but only the eigenkernel α for the eigenvalue 1 gives rise to an A-invariant kernel on T^2. The latter is just the constant function 1 on T^2, as in example I. All the other eigenkernels for L are either concentrated on the unstable boundary of R_1 and R_ρ and project to zero in T^2 or are incompatible along the stable boundary, and do not fit together to form an eigenkernel on T^2. The sheafs involved here are based on morphisms that mix restriction and extension, like the operator $\varepsilon\rho$ of Sect. 2.

Section 9. The Explicit Formula for a C^ω Basic Set

Let μ^ϕ be the atomic measure on $(0,\infty)$ supported on the length spectrum of $\phi|\Omega$, $\mu^\phi = \sum_\gamma \frac{1}{\mu(\gamma)} \delta(x - \ell(\gamma))$. We call μ^ϕ the *length distribution* of $\phi|\Omega$. Then if $N(T)$ is the number of closed orbits of length $\leqq T$ and $N_p(T)$ is the number of prime closed orbits of length $\leqq T$ we can compute any of μ^ϕ, N and N_p using the formulas

$$\mu^\phi(0,T] = N_p(T) + \frac{1}{2}N_p(T/2) + \frac{1}{3}N_p(T/3) + \cdots,$$
$$N(T) = N_p(T) + N_p(T/2) + N_p(T/3) + \cdots.$$

The sharpest asymptotic statements concerning the length spectrum involve certain means of μ^ϕ. Define $M_i(T)$ for $i \geqq 0$, $T > 1$ inductively by

$$M_0(T) = \int_0^{\ln T} t\,d\mu^\phi(t) = \sum_{\gamma:\ell(\gamma)\leqq \ln T} \ell(\gamma)/\mu(\gamma),\; M_{i+1}(T) = \int_1^T M_i(t)dt\,.$$

By the remarks following the proof of Theorem 5 in [F2], we obtain an analogue of the Weil explicit formula for prime numbers, valid for any basic set Ω for a C^ω flow on a smooth manifold M. Take $k \geqq \dim M$ and $T > 1$. Then $M_k(T)$ is a "power series" with complex coefficients and complex exponents:

$$T^{-k}M_k(T) = \sum_\rho c_\rho T^\rho\,.$$

Here either $\rho \in \{0, -1, \ldots, -k\}$ and c_ρ depends on the first 2 terms in the Laurent expansion of $R'(s)/R(s)$ or ρ is a zero/pole of $R(s)$ of order n_ρ and $c_\rho = n_\rho/\rho(\rho+1)\cdots(\rho+k)$. We have the following direct expression for $M_k(T)$ in terms of periodic orbits γ:

$$T^{-k}M_k(T) = \sum_\gamma \frac{\ell(\gamma)}{\mu(\gamma)}(1 - T^{-1}e^{\ell(\gamma)})_+^k\,,$$

where x_+^k is x^k for $x \geqq 0$ and 0 for $x < 0$.

For the geodesic flow over a closed hyperbolic surface, this formula was proven for $k \geqq 2$ by Randol using the Selberg trace formula [Ra]. In case E^u is C^ω one can choose any $k \geqq 1 + \dim E^u$ [F2]. Also similar expansions hold for μ^ϕ replaced by $\mu^\psi = \sum_\gamma \frac{1}{\mu(\gamma)}\chi_\gamma(\psi)\delta(x - \ell(\gamma))$ for any C^ω lift ψ.

From knowledge of the divisor of $R(z)$ one can convert the exact formula to an asymptotic formula. Suppose $\rho_1, \ldots, \rho_N$ are the only zeroes/poles of $R(z)$ with Re $z > a$ for some $a > 0$. Then for any $b > a$,

$$T^{-k}M_k(T) = c_{\rho_1}T^{\rho_1} + \cdots + c_{\rho_N}T^{\rho_N} + O(T^b)\,.$$

Consider for instance a closed oriented hyperbolic manifold X of dimension d, with unit sphere bundle $M = \Omega$ and geodesic flow $\phi_t : M \to M$. As E^u is analytic and dim $E^u = d - 1$, we may take $k \geqq d$. By [F6]

$$R(z) = \prod_{j=0}^{d-1} S_j(j+z)^{(-1)^j}\,,$$

where S_j is a certain Selberg function. Thus

$$n_\rho = \sum_{j=0}^{d-1}(-1)^j \operatorname{ord}_{\rho+j}(S_j)\,.$$

For w with Re $w > 0$, $\operatorname{ord}_w S_j$ is the multiplicity of the eigenvalue λ for the Laplacian Δ_j on coclosed j-forms, where $\lambda = (n-j)^2 - (w-n)^2$, $n = \frac{d-1}{2}$ ([F6], p. 538). Thus an eigenvalue of Δ_j of the form $\lambda = \rho(d-1-2j-\rho)$ contributes $(-1)^j$ to n_ρ for Re $\rho > 0$. For Re $\rho > n$, we use $\lambda \geqq 0$ to find that $\rho \in \mathbf{R}$ and $\rho \leqq d - 1 - 2j$.

Say $d \geqq 6$. It follows that if $j = 1, \lambda = 0$ (corresponding to a harmonic one form on X) there is a pole of $R(z)$ at $z = d - 3$ (unless it is canceled by an eigenfunction of Δ_0 with $\lambda = 2(d-3)$). Thus the cohomology of X affects the coefficient of T^{d-3} in the asymptotic behavior of the averaged length distribution $M_d(T)$.

Section 10. Lefschetz Formulas for Smale Flows

This section uses our analytic continuation result to extend results of [F3], to which we refer for a more extended discussion of the notions that follow. Suppose M is a closed C^ω manifold and $\phi_t: M \to M$ a C^ω flow such that the chain recurrent set Ω is hyperbolic. If dim $\Omega = 1$ we say ϕ_t is a *Smale flow* (this is a more general definition than that first given by Zeeman, who required the strong transversality property). Then Ω is isolated and so admits an *isolating block* N. N is a manifold with corners whose boundary is the union of 2 manifolds with boundary N_-, N_+ that meet transversely in their common boundary: $N_- \cap N_+ = \partial N_- = \partial N_+, N_- \cup N_+ = \partial N$. The flow ϕ_t is transverse to $N_\pm$ (inward on N_+, outward on N_-) and $\cup_{t=-\infty}^{\infty} \phi_t N = \Omega$. The simple homotopy type of the pair N, N_-) is an invariant of ϕ.

Suppose E is a flat bundle over M whose holonomy α has $|\det \alpha| = 1$ and such that $H^*(N, N_-; E) = 0$. It follows that $H^*(M; E) = 0$ and that the Reidemeister torsions are equal: $\tau_E(M) = \tau_E(N, N')$. The latter may be calculated by choosing a cross-section K for $\phi|N$, with return map $r: K \cup N_- \to K \cup N_-$, where r fixes N_- and $r(x), x \in K - N_-$, is the first point of $K \cup N_-$ on the forward trajectory from x. Let r_i^* be the action of r on $H^i(K \cup N_-, N_-; E)$ and define the Lefschetz zeta function of $(\phi|\Omega, E)$ as

$$\tilde{\zeta}_E(t) = \prod_{i \geqq 0} (\det(I - tr_i^*))^{(-1)^{i+1}}.$$

Then $\tilde{\zeta}_E$ is a rational function. Following Milnor's calculation of the torsion of an infinite cyclic covering, one shows $|\tilde{\zeta}_E(1)|^{-1} = \tau_E(N, N_-)$ ([F3], Theorem 3.3).

To describe $\tilde{\zeta}_E$ in terms of periodic orbits, we choose a Markov family of sections whose union is $K \cap \Omega$. We set

$$Z_{E,K}(z,s) = \exp - \sum_\gamma \frac{1}{\mu(\gamma)} \varepsilon_\gamma \chi_\gamma(\psi_E) e^{-z\ell(\gamma) - sn(\gamma)},$$

where ψ_E denotes the flat lift of ϕ to E and $n(\gamma)$ is the number of t, $0 < t \leqq \ell(\gamma)$, where $\phi_t p \in K$ for p in γ. Clearly $Z_{E,K}(z,s)$ is defined for Re $s \geqq 0$, Re $z \gg 0$ or for Re $z \geqq 0$, Re $s \gg 0$. Then Sect. 7 gives that $Z_{E,K}(z,s)$ is an entire function of $(z,s) \in \mathbf{C}^2$. Here we deform ψ_E to ψ_E^{z+sa}, where $a > 0$ is chosen so that $\int_0^{t(x)} a(\phi(x,u))du = 1$ for $x \in K \cap \Omega$ and $t(x)$ the first return time of x. Then

$$Z_{E,K}(0,s) = \tilde{\zeta}_E(e^{-s})^{-1}$$

by the Lefschetz fixed point formula with coefficients in E for the iterates of r. Also

$$Z_{E,K}(z,0) = Z_\alpha(z)$$

is the torsion function of $\phi|\Omega$ for the flat bundle E with holonomy α. We find

$$|Z_\alpha(0)| = |\tilde{\zeta}_E(1)|^{-1} = \tau_E(M).$$

Thus R-torsion is obtained from the periodic orbits of ϕ using the Lefschetz formula. We summarize this result as follows:

Theorem 2. *If ϕ is a C^ω Smale flow on M and E is a flat bundle on M which is acyclic for a pair (N, N_-), with N an isolating block for the chain recurrent set of ϕ, then ϕ is Lefschetz at E, that is $|Z_E(0)| = \tau_E(M)$.*

The Fuller index of γ is $\mathrm{ind}_F(\gamma) = \frac{1}{\mu(\gamma)}\varepsilon_\gamma$. If α denotes the holonomy of E then the preceding formula may be paraphrased

$$\sum_\gamma \mathrm{ind}_F(\gamma)\ \mathrm{Tr}\ \alpha(\gamma) = -\log \tau_E(M)\,,$$

where the divergent series on the left is regularized by the analytic continuation result of Sect. 7.

References

[AW] Adler, R., Weiss, B.: Similarity of automorphisms of the torus. Memoirs AMS **98**, (1970)

[AB] Atiyah, M., Bott, R.: Notes on the Lefschetz fixed point theorem for elliptic complexes. Harvard, 1964

[B] Bieberbach, L.: Lehrbuch der Funktionentheorie. Teubner, B.G., 1927

[B1] Bowen, R.: Symbolic dynamics for hyperbolic flows. Am. J. Math. **95**, 429–460 (1973)

[B2] Bowen, R.: On Axiom A diffeomorphisms. CBMS Reg. Conf. **35**, Providence, RI: AMS, 1978

[BL] Bowen, R., Lanford, O.: Zeta functions of restrictions of the shift transformation. In: Global Analysis, Proceedings of 1968 Berkeley conference. Proc. Symp. Pure Math. **XIV**, Providence, RI: AMS, 1970

[F1] Fried, D.: Zeta functions of Ruelle and Selberg I. Ann. Sci. E.N.S. **19**, 491–517 (1986)

[F2] Fried, D.: Symbolic dynamics for triangle groups. To appear in Inv. Math.

[F3] Fried, D.: Lefschetz formulas for flows. In: The Lefschetz Centennial Conference Part III, Proceedings of 1984 conference, in Mexico City, Contemporary Mathematics **58**, III (1987), pp. 19–69

[F4] Fried, D.: Counting circles. In: Dynamical Systems, Proceedings of 1986–87 Special Year conference at Univ. of Maryland. Springer LNM **1342**, 196–215 (1988)

[F5] Fried, D.: Flat-trace asymptotics of a uniform system of contractions. To appear in: Erg. Th. and Dyn. Syst.

[F6] Fried, D.: Analytic torsion and closed geodesics on hyperbolic manifolds. Inv. Math. **84**, 523–540 (1986)

[F7] Fried, D.: Fuchsian groups and Reidemeister torsion. In: The Selberg Trace Formula and Related Topics, Proceedings of the 1984 conference at Bowdoin College. Contemp. Math. **53**, 141–163 (1986)

[F8] Fried, D.: Torsion and closed geodesics on complex hyperbolic manifolds. Inv. Math. **91**, 31–51 (1988)

[Ga] Gallavotti, G.: Funzioni zeta ed insiemi basilas. Accad. Lincei, Rend. Sc. fismat. e nat. **61**, 309–317 (1976)

[G] Grothendieck, A.: La theorie de Fredholm. Bull. Soc. Math. France **84**, 319–384 (1956)

[GS] Guillemin, V., Sternberg, S.: Geometric Asymptotics. AMS Math Surveys **14**, 1978

[Gu] Guillemin, V.: Lectures on the spectral theory of elliptic operators. Duke Math. J. **44**, 485–517 (1977)

[H] Hormander, L.: An Introduction to Complex Analysis in Several Variables. Amsterdam: North-Holland, 1973

[K] Kitaev, A. Yu.: Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness. Preprint

[Ma] Manning, A.: Axiom A diffeomorphisms have rational zeta functions. Bull. London Math. Soc. **3**, 215–220 (1971)

[M] Millson, J.: Closed geodesics and the η-invariant. Annals of Math. **108**, 1–39 (1978)
[MS] Stanton, R., Moscovici, H.: R-torsion and zeta functions for locally symmetric manifolds. Inv. Math. **105**, 185–216 (1991)
[Pa] Parry, W.: Synchronisation of canonical measures for hyperbolic attractors. Commun. Math. Phys. **106**, 267–275 (1986)
[PP] Parry, W., Pollicott, M.: Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics. Asterisque **187–188**, Soc. Math. de France, 1990
[P] Pollicott, M.: Meromorphic extension for generalized zeta functions. Inv. Math. **85** (1986)
[Ra] Randol, B.: On the asymptotic distribution of closed geodesics on compact Riemann surfaces. Trans. AMS **233**, 241–247 (1977)
[R1] Ruelle, D.: Zeta functions for expanding maps and Anosov flows. Inv. Math. **34**, 231–242 (1976)
[R2] Ruelle, D.: An extension of the theory of Fredholm determinants. Publ. Math. I.H.E.S. **72**, 311–333 (1990)
[Ru] Rugh, H.H.: The correlation spectrum for hyperbolic analytic maps. Nonlinearity **5**, 1237–1263 (1992)
[Ru] Rugh, H.H.: Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems. Preprint
[S] Sanchez-Mergado, H.: Lefschetz formulas for Anosov flows on 3-manifolds. Erg. Th. and Dyn. Syst. **13**, 335–347 (1993)
[Se] Selberg, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Ind. Math. Soc. **20**, 47–87 (1956)
[Sm] Smale, S.: Differentiable dynamical systems. BAMS **73**, 747–817 (1967)
[T] Tangerman, F.: Meromorphic continuation of Ruelle zeta functions. Thesis, Boston University, 1986
[Tr] Treves, F.: Topological Vector Spaces, Distributions and Kernels. New York: Academic Press, 1967

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