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The Effective Gauge Field Action of a System of Non-Relativistic Electrons

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Abstract: We consider a system of free, non-relativistic electrons at zero temperature and positive density, coupled to an arbitrary, external electromagnetic vector potential, A. By integrating out the electron degrees of freedom we obtain the effective action for A. We show that, in the scaling limit, this effective action is quadratic in A and can be viewed as an integral over the Fermi sphere of effective actions of $(1+1)$-dimensional, chiral Schwinger models. We use this result to elucidate Luther–Haldane bosonization of systems of non-relativistic electrons. We also study systems of weakly coupled interacting electrons for which the BCS channel is turned off. Using the quadratic dependence of the effective action on A, we show that, in the scaling limit, the RPA yields the dominant contribution.

0. Introduction

Non-relativistic electrons are described by two-component Pauli spinors $\psi(x) = \binom{\psi_\uparrow}{\psi_\downarrow}(x)$. The invariance of the system under global phase transformations of the form $\psi(x) \to \psi'(x) = e^{-i\chi}\psi(x)$, with $\chi \in \mathbf{R}$, turns to a local symmetry, with $\chi(x)$ a real-valued function, in the presence of (real-valued) gauge fields $A_\rho(x)$, $\rho = 0, 1, \ldots, d$, with the transformation property $A_\rho(x) \to A_\rho(x) + \partial_\rho\chi(x)$. The gauge fields are coupled to the matter system by replacing derivatives ∂_ρ by covariant ones, $D_\rho(A) = \partial_\rho + iA_\rho$. Electromagnetism provides a realization of this gauge symmetry. The electromagnetic potential A_ρ couples to the charge current of the electron system.

We analyse the coupling of an external gauge field A to a system of non-interacting electrons. In this paper, we neglect the magnetic moment of the electron; the coupling of the electromagnetic field to the spin currents is a higher relativistic correction (cf. [1]). For this system, we show that the leading term in the effective gauge field action in a regime of large distance scales and low frequencies (the so-called scaling limit) is quadratic in the gauge field. This follows from the observation that, in the scaling limit, the effective action, as a functional of the external electromagnetic vector potential A, for a system of free, non-relativistic electrons at a positive density and zero temperature is given by an integral of effective actions of $1+1$ *dimensional, chiral relativistic fermions* describing the low-energy degrees of freedom corresponding to the different directions on the Fermi sphere. The effective actions of these $1+1$ dimensional, relativistic systems are quadratic in the external gauge potential, as originally noted by Schwinger, and this property is inherited by the effective action of the non-relativistic electron gas.

We then study the effect of perturbing this system by repulsive two-body interactions. Using our result on the scaling limit of a system of free electrons and an assumption on the renormalization properties of an interacting electron system, we find that the RPA approximation describes the scaling limit of certain interacting electron systems with two-body interactions. In a subsequent paper [3], we shall use this result to show that, in dimension $d \geqq 2$ and assuming that the Cooper channel is turned off, systems of electrons interacting through repulsive density–density interactions (including long-range interactions) renormalize to the free-fermion fixed point, the Landau liquid, in the scaling limit. We also derive Luther–Haldane bosonization from our results on the effective action. Applications to two-dimensional Luttinger liquids will appear in [3].

Next, we summarize the contents of the various sections of this paper.

In Sect. 1, we define the effective action of the gauge field, $S^{\mathrm{eff}}(A)$. It is the generating function of the $U(1)$-current Green functions. We shall calculate this effective action in the scaling limit which characterises the universal, large scale properties of the system.

In Sect. 2, we clarify what we mean by the scaling limit, and how one can determine it.

In Sect. 3, we present the explicit calculation of $S^{\mathrm{eff}}(A)$ in the scaling limit, for a system of free electrons at positive density and zero temperature. It is shown that, to leading order in the scale parameter, the $d+1$ dimensional system decomposes into independent $1+1$ dimensional systems, one along each direction $[\omega]$ of $\mathbf{R}^d$. The $1+1$ dimensional systems correspond to relativistic fermions moving along the direction $[\omega]$ with velocity $\pm v_F$. By dimensional reduction, the calculation of $S^{\mathrm{eff}}(A)$ of the original system is reduced to the calculation of the effective gauge field action for a family of independent Schwinger models which are well known to be quadratic in A.

In Sect. 4, we extend our analysis to electron systems perturbed by pair interactions. Under the assumption that the two procedures of adding the perturbation and of taking the scaling limit commute, we calculate the effective gauge field action in the scaling limit of the interacting system. This limit turns out to be equivalent to the RPA result. By using diagrammatic perturbation theory, we attempt to characterize those interactions for which our assumption is likely to hold.

Finally, in Sect. 5, we discuss the connection between the dimensional reduction of the free system, gauge invariance and the Luther–Haldane bosonization formulas for this theory [3]. Along the way, we establish the relationship between Luther–Haldane bosonization and the general bosonization formulas proposed by the authors in [2].

Our bosonization formulas enable us to explicitly calculate the electron propagators of interacting electron liquids (within the approximation provided by bosonization) [3]. This enables us to distinguish systems that renormalize to a Landau liquid from systems renormalizing to a higher dimensional analogue of the Luttinger liquid, as the scaling limit is taken. We find that only two-dimensional systems with long-range, transverse current–current interactions renormalize to Luttinger liquids [3]. Quantum Hall fluids at filling factor $\nu = \frac{1}{2}, \frac{1}{4}$ are realizations of such systems [5].

Our results in Sects. 3 and 5 are mathematically precise, while the discussion in Sect. 4 is on a heuristic level.

1. The Effective Action $S^{\text{eff}}(A)$

Our goal is to determine the effective action $S^{\text{eff}}(A)$ of the abelian gauge field A_ρ, $\rho = 0, 1, \ldots, d$, coupled to a system of *non-relativistic* electrons at finite density n. It is defined by

$$e^{\frac{i}{\hbar}S^{\text{eff}}(A)} = \frac{Z(A)}{Z(0)} = \frac{\int \mathscr{D}(\psi^*, \psi) e^{\frac{i}{\hbar}S_{(\mu)}(\psi^*, \psi; A)}}{\int \mathscr{D}(\psi^*, \psi) e^{\frac{i}{\hbar}S_{(\mu)}(\psi^*, \psi; 0)}} \,. \tag{1.1}$$

The functional $S_{(\mu)}(\psi^*, \psi; A)$ is the action of the system in the presence of the external gauge field A, for a given value of the chemical potential μ. For non-interacting electrons it takes the form:

$$\begin{aligned} S^{\circ}_{(\mu)}(\psi^*, \psi; A) =& \int dt \int d^d x \Bigg[i\hbar c \psi^*(x) D_0(A)\psi(x) - \mu\psi^*(x)\psi(x) \\ & - \frac{\hbar^2}{2m} \sum_{k=1}^{d} (D_k(A)\psi(x))^* D_k(A)\psi(x) \Bigg] \\ =& \int dt \int d^d x \Bigg[i\hbar c \psi^*(x)\partial_0\psi(x) - \frac{\hbar^2}{2m} \sum_{k=1}^{d} (\partial_k \psi(x))^* \partial_k\psi(x) \\ & - \mu\psi^*(x)\psi(x) - \sum_{\rho=0}^{d} A_\rho(x) j^\rho(\psi^*, \psi; A) \Bigg] \,. \end{aligned} \tag{1.2}$$

The gauge field A couples to the current densities

$$\begin{aligned} j^0(\psi^*, \psi) &:= -e\psi^*(x)\psi(x), \quad x := (t, \mathbf{x}) \,, \\ j^k(\psi^*, \psi; A) &:= \frac{-e\hbar}{i2mc}[\psi^*(x)D_k(A)\psi(x) - (D_k(A)\psi(x))^*\psi(x)], \quad k = 1, \ldots, d \,, \end{aligned} \tag{1.3}$$

where $D_\rho(A) := \partial_\rho - i\frac{e}{\hbar c}A_\rho$, $\rho = 0, 1, \ldots, d$, are the covariant derivatives and e denotes the (absolute value of the) elementary charge of an electron. The space–time indices $\rho = 0, 1, \ldots, d$ of vectors are raised or lowered by using the Minkowski metric $g^{\rho\sigma} = g_{\rho\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$.

The electron system is quantized by functional integrals. The integration variables $\psi^*(x) = (\psi^*_\uparrow(x), \psi^*_\downarrow(x))$ and $\psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}$ are independent, two-component Grassmann fields. (For details concerning the description of nonrelativistic electrons cf. [1].)

The Gaussian Berezin–Grassmann integral is determined by the free electron propagator G°. The generating functional, $W(u^*, u)$, of the connected Green functions for the fields ψ^*, ψ is given by

$$e^{\frac{i}{\hbar}W(u^*,u)} := \int \frac{\mathscr{D}(\psi^*, \psi)}{Z(0)} e^{\frac{i}{\hbar}S^\circ(\psi^*,\psi)} e^{i\int d^{d+1}x(u^*(x)\psi(x)+\psi^*(x)u(x))}$$
$$= e^{-i\hbar\int d^{d+1}x\int d^{d+1}y\, u^*(x)G^\circ(x-y)u(y)}\,, \tag{1.4}$$

where u^* and u are Grassmann sources, and

$$G^\circ_{\alpha,\beta}(x-y) := -i\langle(\psi_\alpha(x)\psi^*_\beta(y))\rangle_\mu$$
$$= \delta_{\alpha,\beta}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{e^{i(k_0(t-s)-\mathbf{k}(\mathbf{x}-\mathbf{y}))}}{\omega - \varepsilon_\mathbf{k} + i\delta\,\mathrm{sgn}\,(\varepsilon_\mathbf{k})} \tag{1.5}$$

with

$$\varepsilon_\mathbf{k} := \frac{(\hbar\mathbf{k})^2}{2m} - \mu, \quad \alpha,\beta : \text{Spin indices}, \quad \text{and} \quad \delta \to 0^+\,.$$

The symbol $\langle\cdots\rangle_\mu$ denotes an expectation in the ground state of the electron system, for a given value of the chemical potential μ. The chemical potential determines the mean electron density

$$n := \frac{1}{V}\left\langle \int_V d^dx\; \psi^*(x)\psi(x)\right\rangle_\mu\,.$$

As a consequence of the Pauli principle, all one particle states with momentum $|\mathbf{k}| \leqq \hbar k_F$ of a system of free electrons with average density n are occupied in the ground state, so that

$$n = 2\int \frac{d^dk}{(2\pi)^d}\,\theta(k_F - |\mathbf{k}|)\,.$$

The Fermi wave number k_F is related to the chemical potential by $\mu = \frac{\hbar^2}{2m}k_F^2$. The wave vectors $\mathbf{k}$, with $|\mathbf{k}| = k_F$, are forming the Fermi sphere $S^{d-1}_{k_F}$ of radius k_F. Analytically, one encounters this surface as the points, where the propagator $G^\circ(k_0,\mathbf{k}) = \int dt \int d^dx\, e^{-i(k_0t-\mathbf{kx})}G^\circ(t,\mathbf{x})$ in momentum space becomes singular for $k_0 \to 0$. In the following analysis, the existence of the Fermi surface plays a central role.

The effective action $S^{\text{eff}}(A)$ of the gauge field A is the generating function of the connected $U(1)$-current Green functions and encodes transport properties of the underlying electron system. At non-coinciding arguments, one has that

$$\prod_{i=1}^n \frac{\delta}{\delta A_{\rho_i}(x_i)} S^{\text{eff}}(A)\Big|_{A_{cl}} = -\left(\frac{-i}{\hbar}\right)^{n-1} \left\langle \prod_{i=1}^n j^{\rho_i}(\psi^*,\psi;A;x_i)\right\rangle^{\text{con}}_{A_{cl}}\,, \tag{1.6}$$

where A_{cl} denotes a background field. Since the currents of non-relativistic electrons depend on the gauge field, one has to add local Schwinger terms at coinciding arguments.

One can take advantage of gauge invariance to determine $S^{\text{eff}}(A)$. Gauge invariance implies that $S^{\text{eff}}(A_\rho + \partial_\rho\chi) = S^{\text{eff}}(A_\rho)$, for an arbitrary function $\chi(x)$. This identity summarizes Ward identities which will be very useful (cf. Sect. 2). Contrary to the background field A_{cl} which describes a physical situation, we think of

A as a small perturbation field that does not change the essential characteristics of the system.

We shall make a specific "scaling ansatz" for the perturbing field A that guarantees that its amplitude is small on large scales (cf. Sect. 2). In order to calculate $S^{\rm eff}(A)$ we expand this functional in A (requiring A to belong to a suitable function space):

$$e^{\frac{i}{\hbar}S^{\rm eff}(A)} = \int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)} e^{\frac{i}{\hbar}S^\circ(\psi^*,\psi)} \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{i}{\hbar} \int d^{d+1}x A_\rho(x) j^\rho(\psi^*,\psi;A;x) \right]^n$$
$$= \int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)} e^{\frac{i}{\hbar}S^\circ(\psi^*,\psi)} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{i}{\hbar} \int d^{d+1}x \left\{ A^0(x) j^0(\psi^*,\psi;x) \right.\right.$$
$$\left.\left. -\sum_{k=1}^{d} A^k(x) j^k(\psi^*,\psi;A\equiv 0;x) + \sum_{k=1}^{d} \frac{e^2}{mc^2} A^k(x)A^k(x)\psi^*(x)\psi(x) \right\} \right]^n . \tag{1.7}$$

The terms on the R.S. of (1.7) can be evaluated by using Wick's theorem:

$$\int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)} e^{\frac{i}{\hbar}S^\circ(\psi*,\psi)} \prod_{i=1}^{n} \psi^*(x_i)\psi(x_i) = (-i\hbar)^n \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n} G^\circ(x_i - x_{\sigma(i)}) \tag{1.8}$$

with the propagator G° defined in (1.5).

2. The Scaling Limit of the Effective Gauge Field Action

We are interested in universal large-scale and low-energy properties of systems of electrons. In the context of electron systems with pair interactions, the scaling limit is constructed by using renormalization-group techniques, where one gradually integrates out the modes of the electron field ψ corresponding to wave vectors far from the Fermi surface, in order to derive an effective Hamiltonian (or action) for the modes close to the Fermi surface [6–9]. In 1+1 dimensions, the classification of the possible relevant, marginal and irrelevant pair interactions has been known for some time [10]. Recently, there has been much progress to extend this program to higher dimensions [8, 11, 4]. At weak coupling, there emerges naturally a large N phenomenon [11] that allows one to determine some fixed points of the renormalization group transformations.

For a noninteracting system coupled to an external gauge field, we use another method to get the desired information about the scaling limit of $S^{\rm eff}(A)$. This method has been developed in [1], where it has been applied to "incompressible" systems – i.e., systems with a Hamiltonian that has a gap in the excitation spectrum – such as QH fluids or insulators.

As noted in Sect. 1, the coefficients of the Taylor expansion of $S^{\rm eff}(A)$ are given – at non-coinciding arguments – by the current Green functions:

$$S^{\rm eff}(A_{cl}+A) - S^{\rm eff}(A_{cl}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} dx_i^{d+1} C^{\rho_1,\dots,\rho_n}(x_1,\dots,x_n)\Big|_{A_{cl}} A_{\rho_1}(x_1)\cdots A_{\rho_n}(x_n),$$

where

$$C^{\rho_1 \dots \rho_n}(x_1, \dots, x_n)|_{A_{cl}} = -\left(\frac{-i}{\hbar}\right)^{n-1} \langle j^{\rho_1}(x_1) \cdots j^{\rho_n}(x_n) \rangle_{A_{cl}}^{\text{con}}, \quad \text{for } x_i \neq x_j . \tag{2.1}$$

We introduce a rescaling map from the physical system in a space–time region $\Omega^{(\lambda)}$ to a reference system in the region $\Omega^{(1)}$, where $\lambda > 1$ is a scale parameter. Thus, we consider a family of ever larger physical systems ($\lambda \to \infty$), keeping the chemical potential μ fixed. Under the rescaling map, the coordinates transform as $x = \lambda\xi \in \Omega^{(\lambda)} \longmapsto \xi \in \Omega^{(1)}$. The essential idea of our construction of the "scaling limit," $S^*(A)$, of $S^{\text{eff}}_{(\lambda)}\left(A^{(\lambda)}\right)$ is to extract the large-distance information needed to determine $S^*(A)$ by studying the asymptotic form of the current Green functions in the limit $\lambda \to \infty$ and plugging the result into Eq. (2.1).

For incompressible systems, one can take advantage of the fact that, in the limit $\lambda \to \infty$, the current Green functions are converging to local distributions; (for details, cf. [1]). As a consequence, one can show by using power counting that, for such systems, only finitely many terms in (2.1) are relevant, the ones neglected being of order $0\left(\frac{1}{\lambda}\right)$.

In our system, the low-energy properties are dominated by the existence of a Fermi surface, and there is no gap in the excitation spectrum. There is no simple power counting argument permitting us to truncate the expansion (1.5) at a finite order. Nevertheless, we are able to show that – independently of the space dimension – the leading order of $S^{\text{eff}}_{(\lambda)}(A^{(\lambda)})$ is given by the quadratic term in the gauge field $A^{(\lambda)}$. To show this (cf. Sect. 3), we consider the current correlation functions at non-coinciding arguments, let the difference in the arguments increase with λ and determine the expansion coefficients in (2.1) to leading order in $\frac{1}{\lambda}$.

The scaling procedure generates ultraviolet divergencies in the perturbative reconstruction of the leading order of $S^{\text{eff}}_{(\lambda)}(A^{(\lambda)})$. To fix a resulting ambiguity, we use the Ward identities implied by gauge invariance, i.e.,

$$\frac{\partial}{\partial x_i^{\rho_i}} C^{\rho_1,\dots,\rho_i,\dots,\rho_n}(x_1, \dots, x_i, \dots, x_n) = 0 \tag{2.2}$$

(cf. Appendix D).

Before proceeding in our calculation, we recapitulate the precise definition of what we understand by the scaling limit, $S^*(A)$, of the effective action $S^{\text{eff}}_{(\lambda)}(A^{(\lambda)})$. First, we require that under the rescaling map from the physical system in $\Omega^{(\lambda)}$ to the reference system in $\Omega^{(1)}$, the gauge field $A^{(\lambda)}_\rho(x)$ transforms as

$$A^{(\lambda)}_\rho(x) \longmapsto \frac{1}{\lambda} a_\rho\left(\xi = \frac{x}{\lambda}\right), \tag{2.3}$$

where $a_\rho(\xi)$ is an arbitrary, but fixed function on $\Omega^{(1)}$. Consequently, the gauge field has the same scaling dimension as the momentum operator:

$$D^{(\lambda)}_\rho(A^{(\lambda)}) = \frac{\partial}{\partial x^\rho} - i\frac{e\hbar}{c} A^{(\lambda)}_\rho(x) \longrightarrow \frac{1}{\lambda}\left(\frac{\partial}{\partial \xi^\rho} - i\frac{e\hbar}{c} a_\rho(\xi)\right).$$

Then, using the asymptotic form of the current Green functions for large λ, we order the expansion of $S^{\text{eff}}_{(\lambda)}(A^{(\lambda)}(\lambda\xi))$ in powers of λ:

$$S^{\text{eff}}_{(\lambda)}(A^{(\lambda)}(\lambda\xi)) = \sum_{n=-n_0}^{\infty} S^{\text{eff}}_n(a(\xi)) \cdot \lambda^{-n} . \tag{2.4}$$

In our definition of the "scaling limit" $S^*(a)$, we retain only those terms that diverge as $\lambda \to \infty$, i.e., the terms proportional to a positive power of λ. In fact, in Sect. 3, we determine only the most relevant (i.e., the most divergent) term.

3. Calculation of $S^*(a)$ by "Dimensional Reduction"

In this section, we determine the "scaling limit", $S^*(a)$, of the effective action $S^{\text{eff}}(A)$, as defined in Sect. 2, for a system of non-interacting electrons in d space dimensions. We shall see that the calculation in $d > 1$ space dimensions reduces to the one in one dimension. The "scaling limit" of the one-dimensional system is equivalent to the Schwinger model for which one can calculate the effective gauge field action exactly.

To simplify our notation, we set $c \equiv \hbar \equiv 1$, throughout the rest of this paper. Furthermore, we analyse the system in the euclidean region, reached by analytic continuation in the time t to the halfplane $\mathrm{Im}\, t > 0$ and setting $\tau = it$ (Wick-rotation). The euclidean versions, $(1.1)^E$–$(1.8)^E$, of the corresponding formulae (1.1)–(1.8) can be obtained by using the following substitutions:

$$\begin{aligned} x_0 &\to x_0^E = ix_0 & x_k &\to x_k^E = x_k & \\ \partial_0 &\to \partial_0^E = -i\partial_0 & \partial_k &\to \partial_k^E = \partial_k \qquad k = 1,\dots,d\,. \\ A_0 &\to A_0^E = -iA_0 & A_k &\to A_k^E = A_k & \end{aligned} \tag{3.1}$$

$$g_{\rho\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \longrightarrow g^E_{\rho\sigma} = \delta_{\rho\sigma}\,.$$

In the rest of the paper, *we use the euclidean notation without changing the symbols,* as there is no danger of confusion.

Our task consists in evaluating the following perturbation expansion:

$$\begin{aligned} e^{-S^{\text{eff}}(A)} = \int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)}\, e^{-S^{\circ}_{(\mu)}(\psi^*,\psi)} \Bigg\{ 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \Bigg[- \int d^{d+1}x \\ \cdot \Bigg(\Bigg(-ieA_0(x) + \sum_{k=1}^{d} \frac{e^2}{m} A_k(x)A_k(x) \Bigg) \psi^*(x)\psi(x) \\ + \sum_{k=1}^{d} \frac{-e}{i2m} A_k(x)\, \psi^*(x)\partial_k\psi(x) - (\partial_k\psi^*(x))\psi(x) \Bigg) \Bigg]^n \Bigg\} . \end{aligned} \tag{3.2}$$

As explained, we evaluate the expansion coefficients only at non-coinciding arguments, i.e. self-contractions are omitted. The local terms will be determined by requiring gauge invariance.

First, we use Wick's theorem $(1.8)^E$:

$$e^{-S^{\mathrm{eff}}(A)} = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \prod_{i=1}^{n} \sum_{s_i=\pm 1} \int d^{d+1}x_i \sum_{\substack{\sigma\in S_n \\ \sigma(i)\neq i}} \mathrm{sgn}\,(\sigma) f(A;x_i) G^{\circ}_{s_i,s_{\sigma(i)}}(x_i - x_{\sigma(i)})\,, \tag{3.3}$$

where

$$f(A;x_i) := -ieA_0(x_i) + \sum_{k=1}^{d} \left(\frac{e^2}{m} A_k(x_i)A_k(x_i) - \frac{e}{m} A_k(x_i)\frac{1}{i}\partial_k^{(i)} \right)$$

and

$$G^{\circ}_{s,s'}(x) = \delta_{s,s'} \int \frac{dk_0}{2\pi} \int \frac{d^dk}{(2\pi)^d} \frac{e^{-i(k_0\tau+\mathbf{kx})}}{ik_0 - \varepsilon_{\mathbf{k}}} = \delta_{s,s'} G^{\circ}(x)\,,$$

see (1.5).

By some elementary transformations (cf. Appendix A), this expression can be rewritten as

$$e^{-S^{\mathrm{eff}}(A)} = \exp\left\{ -2\sum_{l=2}^{\infty} \frac{1}{l} \prod_{j=1}^{l} \int d^{d+1}x_j\, f(A;x_j) G^{\circ}(x_j - x_{j+1}) \right\}. \tag{3.4}$$

Here, the products of propagators are taken along loops, i.e., $x_{l+1} \equiv x_1$ is understood.

To determine the scaling limit, we must find the asymptotic form of the propagator $G^{\circ}_{(\lambda)}(\lambda(\xi_i - \xi_j))$, as $\lambda \to \infty$, at non-coinciding points. In Appendix B, we show that it is given by:

$$G^{\circ}_{(\lambda)}(\lambda\xi_0, \lambda\boldsymbol{\xi}) \approx \frac{1}{\lambda} \left(\frac{k_F}{2\pi}\right)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega\, e^{-ik_F\lambda\boldsymbol{\omega\xi}}\, G^{\circ}_{\boldsymbol{\omega}}(\xi_0, \xi_{\|})\,, \tag{3.5}$$

with

$$G^{\circ}_{\omega}(\xi_0, \xi_{\|}) = \int_{\mathbf{R}} \frac{dk_0}{2\pi} \int_{\mathbf{R}} \frac{dk_{\|}}{2\pi} \frac{e^{-i(k_0\xi_0 + k_{\|}\xi_{\|})}}{ik_0 - \frac{k_F}{m} k_{\|}}$$

and

$$\xi_{\|} := \boldsymbol{\omega\xi} \quad \boldsymbol{\xi}, \boldsymbol{\omega} \in \mathbf{R}^d \quad S_1^{d-1} = \{\boldsymbol{\omega} \in \mathbf{R}^d; (\boldsymbol{\omega})^2 = 1\}\,.$$

In one dimension, the vector $\boldsymbol{\omega}$ takes only the values ± 1, and the integral $\int_{S_1^{d-1}} d^{d-1}\omega$ has to be replaced by $\sum_{\omega=\pm 1}$. The symbol "$\approx$" means "equal up to leading order in $\frac{1}{\lambda}$".

Furthermore,

$$\frac{\partial}{\partial(\lambda\boldsymbol{\xi})} G^{\circ}_{(\lambda)}(\lambda\xi) \approx \frac{1}{\lambda} \left(\frac{k_F}{2\pi}\right)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega\, e^{-ik_F\lambda\boldsymbol{\omega\xi}} (-ik_F\boldsymbol{\omega})\, G^{\circ}_{\omega}(\xi)\,. \tag{3.6}$$

We notice that the $(d+1)$-dimensional propagator G° is an integral of $(1+1)$-dimensional ones: $G^{\circ}_{\boldsymbol{\omega}}$ is the propagator of relativistic electrons which move in the direction $\boldsymbol{\omega}$ with velocity $v_F = \frac{k_F}{m}$. For gauge fields $A_\rho^{(\lambda)}$ with the scaling property

(2.3), one derives from (3.5) that

$$S^{\text{eff}}_{(\lambda)}(A^{(\lambda)}(\lambda\xi)) \approx 2 \cdot \sum_{l=2}^{\infty} \frac{1}{l} \prod_{j=1}^{l} \left(\frac{\lambda k_F}{2\pi}\right)^{d-1} \int d^{d+1}\xi_j \int_{S_1^{d-1}} d^{d-1}\omega_j \, e^{-ik_F\lambda\omega_j(\xi_j-\xi_{j+1})} \cdot \bar{f}_{\omega_j}(a;\xi_j) G^{\circ}_{\omega_j}(\xi^0_j - \xi^0_{j+1}, \xi^{\|}_j - \xi^{\|}_{j+1}) , \tag{3.7}$$

where

$$\bar{f}_{\omega_j}(a;\xi_j) = -iea_0(\xi_j) + ev_F \sum_{k=1}^{d} \omega_{j,k} a_k(\xi_j) .$$

We note that only the projection of the gauge field onto the direction ω enters the expression for $\bar{f}_\omega(a;\xi)$. The quadratic (diamagnetic) term (cf. (3.3)) is left out, because it is of lower order in λ.

With

$$\prod_{j=1}^{l} e^{-ik_F\lambda\omega_j(\xi_j-\xi_{j+1})} = \prod_{j=1}^{l-1} e^{-ik_F\lambda(\omega_j-\omega_l)(\xi_j-\xi_{j+1})}$$

we arrive at

$$S^{\text{eff}}_{(\lambda)}(A^{(\lambda)}(\lambda\xi)) \approx 2\sum_{l=2}^{\infty} \frac{1}{l} \left(\lambda\frac{k_F}{2\pi}\right)^{d-1} \int d^{d+1}\xi_l \int_{S_1^{d-1}} d^{d-1}\omega_l \, \bar{f}_{\omega_l}(a;\xi_l) \cdot G^{\circ}_{\omega_l}(\xi^0_l - \xi^0_1, \xi^{\|}_l - \xi^{\|}_1) \prod_{j=1}^{l-1} \left(\frac{\lambda k_F}{2\pi}\right)^{d-1} \int d^{d+1}\xi_j \int_{S_1^{d-1}} d^{d-1}\omega_j \cdot e^{-ik_F\lambda(\omega_j-\omega_l)(\xi_j-\xi_{j+1})} \bar{f}_{\omega_j}(a;\xi_j) G^{\circ}_{\omega_j}(\xi^0_j - \xi^0_{j+1}, \xi^{\|}_j - \xi^{\|}_{j+1}) . \tag{3.8}$$

To proceed in our calculation, we have to distinguish between $d > 1$ and $d = 1$.

- For $d > 1$, we apply the following lemma, proven in Appendix C, to Eq. (3.8):

$$\int_{S_1^{d-1}} d^{d-1}\omega_1 e^{i\lambda k_F \xi(\omega_1-\omega_2)} g(\omega_1) \approx \left(\frac{2\pi}{\lambda k_F}\right)^{d-1} g(\omega_2)\delta^{(d-1)}(\boldsymbol{\xi}_\perp[\omega_2]) , \tag{3.9}$$

where g is a test function on S_1^{d-1}, and $\boldsymbol{\xi}_\perp[\omega_2] := \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \omega_2)\omega_2$.

This yields

$$S^{\text{eff}}_{(\lambda)}(A^{(\lambda)}) \approx \lambda^{d-1} 2 \sum_{l=2}^{\infty} \frac{1}{l} \left(\frac{k_F}{2\pi}\right)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega \int d^{d-1}\xi_\perp \prod_{j=1}^{l} \int d\xi^0_j \int d\xi^{\|}_j \cdot \bar{f}_\omega(a;\xi^0_j,\xi^{\|}_j;\boldsymbol{\xi}_\perp)\, G^{\circ}_\omega(\xi^0_j - \xi^0_{j+1}, \xi^{\|}_j - \xi^{\|}_{j+1}) \tag{3.10}$$

with

$$\bar{f}_\omega(a;\xi^0,\xi^{\|}_j;\boldsymbol{\xi}_\perp) = -iea_0(\xi^0,\xi^{\|};\boldsymbol{\xi}_\perp) + ev_F\boldsymbol{\omega}\mathbf{a}(\xi^0,\xi^{\|};\boldsymbol{\xi}_\perp) .$$

- For $d = 1$, one has to replace the integrals $\int_{S_1^{d-1}} d^{d-1}\omega_j$ in (3.6) by sums $\sum_{\omega_j=\pm 1}$. For $\omega_j \neq \omega_l$, the factor $e^{-ik_F\lambda(\omega_j-\omega_l)(\xi_j^1-\xi_{j+1}^1)}$ oscillates so rapidly, that, after integration over the space variable ξ_j^1, this contribution is of lower order in λ than the one for $\omega_j = \omega_l$. Consequently,

$$S_{(\lambda)}^{\text{eff}}(A^{(\lambda)}) \approx 2\sum_{l=2}^{\infty}\frac{1}{l}\sum_{\omega=\pm 1}\prod_{j=1}^{l}\int d\xi_j^0 d\xi_j^1 \bar{f}_\omega(a;\xi_j^0,\xi_j^1)G_\omega^\circ(\xi_j^0-\xi_{j+1}^0,\xi_j^1-\xi_{j+1}^1)\,, \tag{3.11}$$

with

$$\bar{f}_\omega(a;\xi^0,\xi^1) = -iea_0(\xi^0,\xi^1) + ev_F\omega a_1(\xi^0,\xi^1)$$

and

$$G_\omega^\circ(\xi^\circ,\xi^1) = \int\frac{dk_0}{2\pi}\int\frac{dk_1}{2\pi}\frac{e^{-i(k_0\xi_0+k_1\xi_1)}}{ik_0-\omega v_F k_1}\,.$$

If one compares (3.10) to (3.11), one recognizes that, in each direction $[\boldsymbol{\omega}] := \{\boldsymbol{\omega}\}\cup\{-\boldsymbol{\omega}\}$ in (3.10), there is a one dimensional system equivalent to (3.11). The leading order in (3.11) is identical to the perturbation expansion of the effective gauge field action of the Schwinger model:

$$S^{\text{eff}}(a) = -2\ln\left\{\frac{\int\mathscr{D}(\psi^*,\psi)e^{-\int d^2\xi\psi^*(\xi)D(a)\psi(\xi)}}{\int\mathscr{D}(\psi^*,\psi)e^{-\int d^2\xi\psi^*(\xi)D(0)\psi(\xi)}}\right\}, \tag{3.12}$$

where $D(a) := (\partial_0 - iea_0(\xi))\mathbf{1}_2 + iv_F(\partial_1 - iea_1(\xi))\boldsymbol{\sigma}_3$, and the two-component Grassmann field $\psi(\xi) = \begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}(\xi)$ describes chiral, relativistic electrons with velocity $\mp v_F$. The Green functions $G_{\omega=\pm 1}^\circ$ in (3.11) are forming the two non-zero components of the corresponding propagator

$$G^\circ(\xi-\eta) = \begin{pmatrix}-\langle\psi_R(\xi)\psi_R^*(\eta)\rangle & 0\\ 0 & -\langle\psi_L(\xi)\psi_L^*(\eta)\rangle\end{pmatrix},$$

defined by $D(0)G^\circ(\xi-\eta) = -\mathbf{1}_2\delta^{(2)}(\xi-\eta)$.

As has been known for a long time [12], the effective gauge field action (3.12) of the Schwinger model is proportional to $(a^T)^2$. We give a selfcontained discussion in Appendix D, emphasizing some points which are important in our context. The explicit result is

$$S^{\text{eff}}(a) = \frac{e^2}{2\pi}\frac{2}{v_F}\int d^2\xi\sum_{\rho=0,1}a_\rho^T(\xi)a_\rho^T(\xi) \tag{3.13}$$

with

$$a_\rho^T(\xi) = \sum_{\sigma=0,1}\left(\delta_{\rho\sigma} - \frac{\partial_\rho\partial_\sigma}{\partial_0^2+\partial_1^2}\right)a_\sigma(\xi) \quad\text{and}\quad \partial_0 := \frac{\partial}{\partial\xi_0} \quad \partial_1 := v_F\frac{\partial}{\partial\xi_1}\,.$$

This shows that the leading order of (3.11), and consequently the leading order of (3.10), is given by the term quadratic in the gauge field.

Explicitly, for $d > 1$,

$$S^*(a) = \lambda^{d-1}\left(\frac{k_F}{2\pi}\right)^{d-1} \int_{S_1^{d-1}} \frac{d^{d-1}\omega}{2} \int d^{d-1}\xi_\perp \left\{\frac{2e^2}{2\pi}\frac{1}{v_F}\int d\xi_0 d\xi_1^{\|} \sum_{\rho=0,1} a_\rho^{\|T}(\xi) a_\rho^{\|T}(\xi)\right\}, \tag{3.14}$$

where

$$a_\rho^{\|T}(\xi) = \sum_{\sigma=0,1}\left(\delta_{\rho\sigma} - \frac{\partial_\rho^{\|}\partial_\sigma^{\|}}{\partial_0^2 + \partial_1^{\|2}}\right) a_\sigma^{\|}(\xi)$$

and

$$\partial_0^{\|} := \frac{\partial}{\partial\xi_0} \quad \partial_1^{\|} := v_F\,\boldsymbol{\omega}\frac{\partial}{\partial\boldsymbol{\xi}}$$

$$a_0^{\|}(\xi) := a_0(\xi) \quad a_1^{\|}(\xi) := v_F\,\boldsymbol{\omega}\,\mathbf{a}(\xi)\,.$$

To carry out the integration over the sphere S_1^{d-1}, we pass to momentum space:

$$S^*(a) = \lambda^{d-1}\left(\frac{k_F}{2\pi}\right)^{d-1} \frac{e^2}{v_F\pi} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_{S_1^{d-1}} \frac{d^{d-1}\omega}{2}$$

$$\cdot \sum_{\rho=0,1} \hat{a}_\rho^{\|}(-k)\left(\delta_{\rho,\sigma} - \frac{k_\rho^{\|}k_\sigma^{\|}}{k_0^2 + k_1^{\|2}}\right)\hat{a}_\sigma^{\|}(k)\,. \tag{3.15}$$

The integration then yields the final result

$$S^*(a) = \frac{\lambda^{d-1}}{2}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \sum_{\mu,\nu=0}^{d} \hat{a}_\mu(-k)\,\boldsymbol{\Pi}_*^{\mu,\nu}(k)\,\hat{a}_\nu(k)\,,$$

where the asymptotic form of the euclidean polarization tensor is given by

$$\boldsymbol{\Pi}_*^{00}(k) = \frac{\mathbf{k}^2}{k_0^2}\boldsymbol{\Pi}_l(k), \quad \boldsymbol{\Pi}_*^{0i}(k) = -\frac{k_i}{k_0}\,\boldsymbol{\Pi}_l(k)\,,$$

$$\boldsymbol{\Pi}_*^{ij}(k) = \boldsymbol{\Pi}_t(k)\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) + \boldsymbol{\Pi}_l(k)\frac{k_i k_j}{\mathbf{k}^2} \quad i,j = 1,\ldots,d\,, \tag{3.16}$$

with

$$\boldsymbol{\Pi}_l(k) = \begin{cases} \frac{1}{2\pi}e^2 v_F \frac{k_0^2}{k_0^2+(v_F k)^2} & d=1 \\ e^2\frac{k_F}{\pi}v_F\left[\left(1+\sqrt{1+\left(\frac{v_F k}{k_0}\right)^2}\right)\sqrt{1+\left(\frac{v_F k}{k_0}\right)^2}\right]^{-1} & d=2 \\ e^2\frac{k_F m}{\pi^2}\frac{k_0^2}{k^2}\left(1-\left|\frac{k_0}{v_F k}\right| \operatorname{arctg}\left|\frac{v_F k}{k_0}\right|\right) & d=3 \end{cases},$$

$$\boldsymbol{\Pi}_t(k) = \begin{cases} \text{———} & d=1 \\ \frac{k_F^2}{m}\frac{e^2}{\pi}\frac{|k_0|}{\sqrt{k_0^2+(v_F k)^2}} - \boldsymbol{\Pi}_l(k) & d=2 \\ \frac{1}{2}\left(k_F^2\frac{e^2}{\pi^2}\left|\frac{k_0}{k}\right| \operatorname{arctg}\left|\frac{v_F k}{k_0}\right| - \boldsymbol{\Pi}_l(k)\right) & d=3\,. \end{cases}$$

4. Perturbation of a System of Free Electrons by Pair Interactions

In this section, we consider a system of interacting electrons with pair interactions of the form

$$V(\psi^*,\psi) := \frac{1}{2}\int d^{d+1}x \int d^{d+1}y : \psi^*(x)\psi(x)V(|\mathbf{x}-\mathbf{y}|)\delta(x_0-y_0)\psi^*(y)\psi(y) : . \quad (4.1)$$

The extension of our analysis to more general current-current interactions is immediate. The euclidean action $S_V(\psi^*,\psi,A)$ of the interacting system coupled to the external gauge field A is given by

$$S_V(\psi^*,\psi,A) = S^\circ(\psi^*,\psi,A) + V(\psi^*,\psi)\,, \quad (4.2)$$

where S° denotes the euclidean version of (1.2). We attempt to determine the scaling limit $S_V^*(A)$ of the effective action S_V^{eff} of the gauge field A:

$$\mathrm{e}^{-S_V^{\text{eff}}(A)} = \int \frac{\mathscr{D}(\psi^*,\psi)}{Z_V(0)}\, \mathrm{e}^{-[S^\circ(\psi^*,\psi,A)+V(\psi^*,\psi)]}\,. \quad (4.3)$$

If the following perturbative assumption holds, knowledge of the scaling limit of the non-interacting system $S^*(A)$ (cf. (3.16)) enables us to determine the explicit form of $S_V^*(A)$:

Assumption P: The scaling limit of the perturbed theory coincides with the perturbation of the scaling limit of the non-interacting theory.

First, we work out the consequences of this assumption. Afterwards, we will discuss for what potentials $V(|\mathbf{x}-\mathbf{y}|)$ Assumption P is likely to hold.

With $j^\circ(\psi^*,\psi) = -ie : \psi^*(x)\psi(x) :$, we can rewrite (4.3) in momentum space as

$$\mathrm{e}^{-S_V^{\text{eff}}(A)} = \int \frac{\mathscr{D}(\psi^*,\psi)}{Z_V(0)}\, e^{-\left[S^\circ(\psi^*,\psi)+\int \frac{d^{d+1}p}{(2\pi)^{d+1}} A_\mu(-p)j^\mu(\psi^*,\psi,A;p)\right]} \cdot e^{\frac{1}{2}\frac{1}{e^2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}} j^\circ(\psi^*,\psi,-p)\hat{V}(|\mathbf{p}|)j^\circ(\psi^*,\psi,p)}\,.$$

Introducing the real ("Hubbard–Stratonovich") field $B_0(k)$, we get

$$\begin{aligned}\mathrm{e}^{-S_V^{\text{eff}}(A)} &= \int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)}\, e^{-\left[S^\circ(\psi^*,\psi)+\int \frac{d^{d+1}p}{(2\pi)^{d+1}} A_\mu(p)j^\mu(\psi^*,\psi,A,p)\right]} \\ &\quad \cdot \int \frac{\mathscr{D}B_0}{W_V(0)}\, e^{\int \frac{d^{d+1}p}{(2\pi)^{d+1}} B_0(-p)j^\circ(\psi^*,\psi,p)}\, e^{-\frac{e^2}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}} B_0(-p)\hat{V}^{-1}(|\mathbf{p}|)B_0(p)} \\ &= \int \frac{\mathscr{D}(\psi^*,\psi)}{Z(0)} \int \frac{\mathscr{D}B_0}{W_V(0)}\, e^{-\left[S^\circ(\psi^*,\psi)+\int \frac{d^{d+1}p}{(2\pi)^{d+1}} (A_\mu(p)-\delta_{\mu,0}B_0(p))j^\mu(\psi^*,\psi,A,-p)\right]} \\ &\quad \cdot e^{-\frac{e^2}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}} B_0(-p)\hat{V}^{-1}(|\mathbf{p}|)B_0(p)}\end{aligned} \quad (4.4)$$

We rescale the A- and B-gauge fields as discussed in Sect. 2, i.e.,

$$A_\mu\left(\frac{k}{\lambda}\right) \equiv A_\mu^{(\lambda)}\left(\frac{k}{\lambda}\right) = \lambda^d a_\mu(k) \quad \text{and} \quad B_0\left(\frac{k}{\lambda}\right) = \lambda^d b_0(k)$$

with $p = \frac{k}{\lambda}$, and we assume that, in the limit $\lambda \to \infty$, the leading term of the interaction potential has the scaling behavior

$$\begin{aligned} V^{(\lambda)}(\lambda|\boldsymbol{\xi} - \boldsymbol{\eta}|) &= \lambda^{-\gamma} v(|\boldsymbol{\xi} - \boldsymbol{\eta}|), \quad \text{in } x\text{-space}, \\ \hat{V}^{(\lambda)}\left(\frac{|\mathbf{k}|}{\lambda}\right) &= \lambda^{d-\gamma}\hat{v}(|\mathbf{k}|), \quad \text{in momentum space}, \end{aligned} \tag{4.5}$$

for some positive constant γ.

Under Assumption P, we can use the results of Sect. 3 to integrate over the fermionic fields ψ^*, ψ in expression (4.4), and we get, to leading order,

$$\begin{aligned} e^{-S^{\text{eff}}_{V,(\lambda)}(a)} = \int \frac{\mathscr{D}b_0}{W_V(0)} \exp\left(-\frac{1}{2}\lambda^{d-1}\left\{\int \frac{d^{d-1}k}{(2\pi)^{d+1}} \left[(a_\mu(-k) - \delta_{\mu,0}b_0(-k))\boldsymbol{\Pi}^{\mu\nu}_*(k)\right.\right.\right. \\ \cdot \left.\left.\left.(a_\nu(k) - \delta_{\nu,0}b_0(k))\right] + 0\left(\frac{1}{\lambda}\right)\right\}\right) \\ \cdot \exp\left(-\frac{1}{2}\lambda^{d-1}\left\{\lambda^{\gamma-d}e^2 \int \frac{d^{d+1}k}{(2\pi)^{d+1}} b_0(-k)\hat{v}^{-1}(|\mathbf{k}|)b_0(k)\right\}\right), \end{aligned} \tag{4.6}$$

with $\boldsymbol{\Pi}^{\mu\nu}_*$ defined in Eq. (3.16).

It would seem that, for the consistency of our argument, we have to suppose that $\gamma - d > -1$, so that the terms neglected are of lower order than the contribution of the interaction. Then we can integrate over the b_0 field to obtain

$$S^*_V(a) = \frac{1}{2}\lambda^{d-1} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} a_\mu(-k)\widetilde{\boldsymbol{\Pi}}^{\mu\nu}_*(k)a_\nu(k), \tag{4.7}$$

where

$$\widetilde{\boldsymbol{\Pi}}^{\mu\nu}_*(k) = \boldsymbol{\Pi}^{\mu\nu}_*(k) - \boldsymbol{\Pi}^{\mu 0}_*(-k)\frac{\lambda^{d-\gamma}\hat{v}(|\mathbf{k}|)}{1 + \lambda^{d-\gamma}\hat{v}(|\mathbf{k}|)\boldsymbol{\Pi}^{00}_*(k)}\boldsymbol{\Pi}^{0\nu}_*(k).$$

Remarks.

1. In order for the integration over the b_0 field to be well defined, we must assume that $\text{Re}\left\{\Pi^{00}(k) + \frac{1}{\lambda^{d-\gamma}\hat{v}(|\mathbf{k}|)}\right\}$ is strictly positive which holds for potentials V of positive type (i.e., $\hat{V} > 0$).

2. The result (4.7) is equivalent to the RPA approximation, i.e., under the given assumptions, the RPA approximation becomes exact in the scaling limit. This leads to an effective action which is quadratic in the gauge field a, as for the non-interacting system.

3. Note that if $\gamma > d$, i.e., if the potential is integrable, the effect of the interaction disappears, as $\lambda \to \infty$.

4. The condition $\gamma - d > -1$ is rather misleading. It suggests that the RPA approximation is accurate for potentials which are integrable ($\gamma > d$) or "weakly non-integrable" ($d \geqq \gamma > d - 1$) (for example, the Coulomb-potential in $d = 3$ with $\gamma = 1$ does not belong to this class). This is contrary to what one would expect!

To analyze Assumption P, in other words, to describe the circumstances under which the RPA contributions are the most relevant ones, we have to go beyond the arguments advanced so far. We must study the effects of the interaction (4.1) in many-body perturbation theory. In principle, this is a formidable task. But recently, there has been much progress in identifying the essential weak coupling mechanisms that govern the scaling limit [8, 11, 4]. In the following, we sketch only the central ideas and results; details can be found, for example, in [3, 4].

We perturb the system by the two-body interaction, $g V$, with V as in (4.1). For a given, large value of a scale parameter λ, we assume that the Wilson effective action of the system at momentum scales $\mathbf{p}$, with $\left|\mathbf{p} - k_F \frac{\mathbf{p}}{|\mathbf{p}|}\right| \leqq \frac{1}{2}\left(\frac{\Lambda}{\lambda}\right)$ has the form

$$S_{\text{eff}}^{(\lambda)} \approx S^{\circ}_{(\mu^{(\lambda)})} + g^{(\lambda)} V \,, \tag{4.8}$$

for a renormalized chemical potential, $\mu^{(\lambda)}$, and a renormalized coupling constant, $g^{(\lambda)}$, of order unity, up to perturbatively small irrelevant terms. Λ is an arbitrary momentum reference scale of order unity.

We divide the neighbourhood

$$\Omega_{\left(\frac{\Lambda}{\lambda}\right)} = \left\{\mathbf{p} : \left|\mathbf{p} - k_F \frac{\mathbf{p}}{|\mathbf{p}|}\right| \leqq \frac{1}{2}\left(\frac{\Lambda}{\lambda}\right)\right\}$$

of the Fermi surface into

$$N = \left(\frac{\lambda}{\Lambda}\right)^{d-1} \mathrm{Vol}[S_{k_F}^{(d-1)}]$$

disjoint, congruent blocks of diameter $\frac{\Lambda}{\lambda}$. Up to an error of order $\frac{1}{\lambda}$, we can choose these blocks to be d-dimensional cubes $Q^d\left(\frac{\Lambda}{\lambda}\right)$ with sides of length $\frac{\Lambda}{\lambda}$, centered at points $k_F \boldsymbol{\omega}_i$ on the Fermi surface.

In order to explore the properties of the theory on momentum scales $\mathbf{p} \in \Omega_{\left(\frac{\Lambda}{\lambda}\right)}$, we attempt to expand its Green functions in powers of $\frac{1}{\lambda} \sim \left(\frac{1}{N}\right)^{\frac{1}{d-1}}$, as proposed in [9]. For this, we rescale the momentum variables $\mathbf{p} \in \Omega_{\left(\frac{\Lambda}{\lambda}\right)}$ by $p \to k = \lambda p$, $\mathbf{k} \in \Omega_\Lambda$, and the x-variables by $x \to \xi = \frac{x}{\lambda}$.

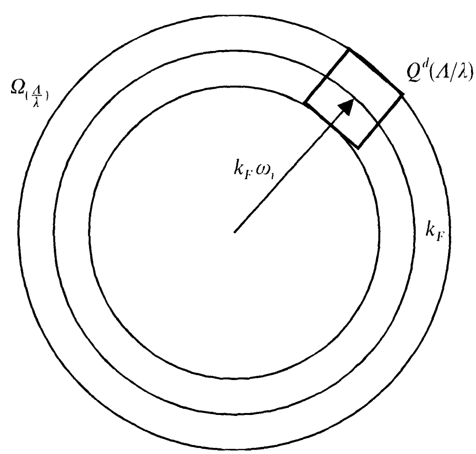


Fig. 1.

By arguments similar to those in Appendix B, one deduces that – after rescaling – the propagator (1.5) of the effective theory can be written as:

$$G^0(\lambda(\xi-\eta)) = -\langle\psi(\lambda\xi)\psi^*(\lambda\eta)\rangle$$
$$\approx \sum_{\omega_i}\int_{\mathbf{R}}\frac{dk_0}{2\pi}\int_{Q^d(\Lambda)}\frac{d^dk}{(2\pi)^d}e^{-i\lambda k_F\boldsymbol{\omega}_i(\boldsymbol{\xi}-\boldsymbol{\eta})}$$
$$\cdot\frac{1}{\lambda^d}\frac{e^{-i(k_0(\xi_0-\eta_0)+\mathbf{k}(\boldsymbol{\xi}-\boldsymbol{\eta}))}}{ik_0+v_F\mathbf{k}\boldsymbol{\omega}_i}\,. \tag{4.9}$$

"$\approx$ " stands for "equal in leading order in $\frac{1}{\lambda}$". The fermionic field ψ decomposes into N independent components

$$\psi(\lambda\xi)\approx\sum_{\omega_i}e^{-i\lambda k_F\boldsymbol{\omega}_i\boldsymbol{\xi}}\frac{1}{\lambda^{d/2}}\psi_{\omega_i}(\xi) \tag{4.10}$$

with propagators

$$G_{\omega_i,\omega_j}(\xi-\eta) = -\left\langle\psi_{\omega_i}(\xi)\psi^*_{\omega_j}(\eta)\right\rangle$$
$$=\delta_{\omega_i,\omega_j}\int_{\mathbf{R}}\frac{dk_0}{2\pi}\int_{Q^d(\Lambda)}\frac{d^dk}{(2\pi)^d}e^{-i(k_0(\xi_0-\eta_0)+\mathbf{k}(\boldsymbol{\xi}-\boldsymbol{\eta}))}\frac{1}{ik_0-v_F\mathbf{k}\boldsymbol{\omega}_i}\,. \tag{4.11}$$

This decomposition implies, that – in the rescaled system – the interaction (4.1) can be written as

$$V_{(\lambda)}(\psi^*,\psi)\approx\frac{1}{2}\frac{g^{(\lambda)}}{\lambda^{d-1}}\sum_{\substack{\omega_1,\omega_2\\ \omega_3,\omega_4}}\delta_{\omega_1+\omega_2,\,\omega_3+\omega_4}\int_{\mathbf{R}\times Q^d(\Lambda)}\frac{d^{d+1}k_1}{(2\pi)^{d+1}}\;\cdots$$
$$\cdots\int_{\mathbf{R}\times Q^d(\Lambda)}\frac{d^{d+1}k_4}{(2\pi)^{d+1}}\delta\left(k_0^1+k_0^2-k_0^3-k_0^4\right)$$
$$\cdot\,\delta^{(d)}\left(\mathbf{k}^1+\mathbf{k}^2-\mathbf{k}^3-\mathbf{k}^4\right)\hat{v}\left(k_F(\boldsymbol{\omega}_4-\boldsymbol{\omega}_1)+\frac{1}{\lambda}(\mathbf{k}_4-\mathbf{k}_1)\right)$$
$$\cdot\,\psi^*_{\omega_4}(k^4)\psi^*_{\omega_3}(k^3)\psi_{\omega_2}(k^2)\psi_{\omega_1}(k^1)\,, \tag{4.12}$$

where

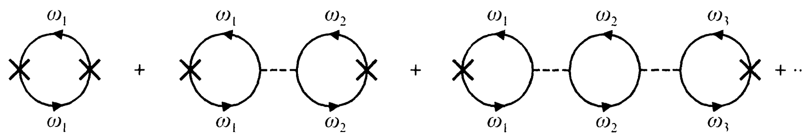
$$\hat{v}(\mathbf{p})=\int d^d\xi e^{i\lambda\boldsymbol{\xi}\mathbf{p}}v(\lambda\boldsymbol{\xi})\qquad\text{and}\qquad\psi_\omega(k)=\int d^{d+1}\xi e^{ik\xi}\psi_\omega(\xi)\,.$$

Hence, in the rescaled reference system, Eq. (4.8) is transformed to

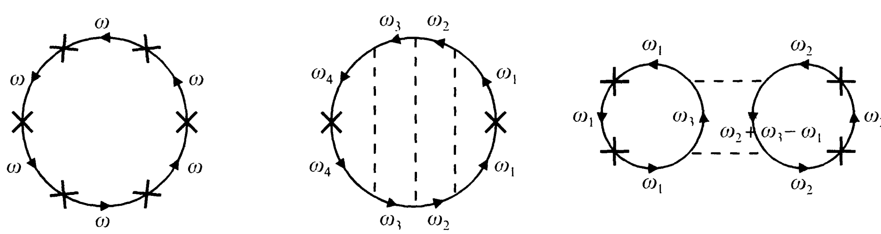
$$S^{(\lambda)}_{\text{eff, resc}}\approx S^{\circ}_{\text{resc}}+\frac{g^{(\lambda)}}{N}V\,.$$

If the effective coupling constant $g^{(\lambda)}$ is of order unity, an expansion in the coupling constant $\frac{g^{(\lambda)}}{N}$ of the reference system yields the desired expansion in powers of $\frac{1}{\lambda}$.

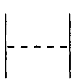
The claim that (4.5) is the leading term in $S_V^*(a)$ reduces, in diagrammatic language, to showing that the RPA contributions

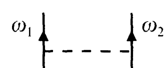
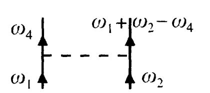


are more relevant than all remaining diagrams, for example,



(i) (ii) (iii)

Here, $\xrightarrow{\omega_i}$ denotes the electron propagator G_{ω_i}, —×— a gauge field insertion, and  an interaction vertex. The components ω_i are indicated to show how many independent summations, $\sum_{\omega_i}(\,\cdot\,)$, over the N blocks in $\Omega_{(\frac{A}{\lambda})}$ are executed. Each interaction vertex carries a factor $\frac{1}{N}$, but each independent summation, $\sum_{\omega_i}(\,\cdot\,)$, contributes a factor N. This leads to an expansion in powers of $\frac{1}{N}$.

- The RPA diagrams, for example, are all of the same (leading) order in $\frac{1}{N}$.
- Diagrams (i) are naïvely of the same order, but their contribution is down by $N^{-\frac{1}{d-1}} \sim \lambda^{-1}$. This is the result of Sect. 3 and Appendix D: each fermion loop of more than two propagator lines, all carrying the same index ω, is $0\left(\frac{1}{\lambda}\right)$.
- Diagrams (ii) are of the same order, too, and one can neglect these diagrams only, if, in our model, the special vertex  is more relevant than the general vertex  with $\omega_4 \neq \omega_1$. **This holds for long-range potentials which become singular at zero momentum-transfer.**
- Diagrams (iii) are of higher order in the $\frac{1}{N}$ expansion.

So far, we have taken into account only the large-N phenomenon and the suppression of loops with many gauge field vertices. But instabilities in the system are typically driven by IR-divergent diagrams like the ones in Fig. 2 where integration over the loop momentum q generates logarithmic divergencies, as $p \to 0$.

For a system with a spherical Fermi surface, corrections to the linearization of the spectrum in the asymptotic form of the propagator G_ω° destroy the

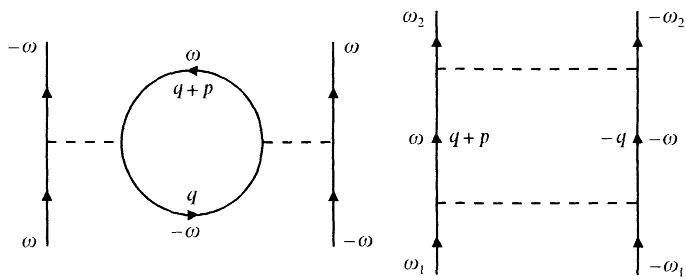


Fig. 2.

divergence of the bubble diagram, the diagram to the left in Fig. 2. Such diagrams become divergent only in systems with a Fermi surface that contains flat pieces.

In contrast, the divergence of the ladder graphs (diagram to the right in Fig. 2) is stable against corrections to the linearization of the spectrum near the Fermi surface. They become relevant for attractive short range potentials and lead to the superconducting instability. We must therefore assume that the superconducting instability is suppressed in our system. A more systematic analysis of the perturbation theory – using RG-techniques – is developed in [4] (see also [8]).

That analysis suggests that, for systems with a spherical Fermi surface ($d > 1$) and strictly positive, long range interaction potential, the leading contribution of $S_V^*(a)$ is given by the RPA diagrams, i.e. (4.5), as long as the Cooper channel (BCS instability) is suppressed.

5. Bosonization

In the discussion of the $1+1$ dimensional Schwinger model (cf. Appendix D) we establish the connection between gauge invariance, i.e., local conservation of the electric current, and the possibility of expressing this current by a bose field. As shown in [2], this can be generalized to arbitrary dimensions, where the conservation of the current density j^μ, $\mu = 0,\ldots,d$, implies, that it can be written in terms of an antisymmetric, bosonic tensor field $b_{\nu_2,\ldots,\nu_d}$ of rank $d-1$,

$$j^\mu = \frac{\varepsilon^{\mu\nu_1\nu_2\cdots\nu_d}}{(d-1)!}\partial_{\nu_1} b_{\nu_2\cdots\nu_d} \quad , \quad \nu_i \in \{0,\ldots,d\} \; , \tag{5.1}$$

where summation over repeated indices is understood. We will see that this general formalism reduces in the special context of Sect. 3 to Luther–Haldane bosonization [13]. In a forthcoming paper [3], we will use this method to calculate the fermion propagator of the $d+1$ dimensional Luttinger-model.

One can retrace the results of Sect. 3 by noting that the fermion field ψ decomposes in the scaling limit into independent radial components

$$\psi(\lambda\xi) \approx \int \frac{d^{d-1}\omega}{\left(\frac{2\pi}{k_F}\right)^{d-1}} e^{-ik_F\lambda\omega\xi}\psi_\omega(\lambda\xi_0, \lambda\xi_{\|}) \tag{5.2}$$

with propagators

$$-\left\langle \psi_\omega(\lambda\xi_0, \lambda\xi_\parallel)\psi^*_{\omega'}(\lambda\eta_0, \lambda\eta_\parallel)\right\rangle = \left(\frac{2\pi}{k_F}\right)^{d-1} \delta^{(d-1)}(\boldsymbol{\omega}-\boldsymbol{\omega}')\frac{1}{\lambda}G_\omega(\xi_0-\eta_0, \xi_\parallel-\eta_\parallel)\,.$$

This is the continuum version of the Eqs. (4.10) and (4.11) which reproduces Eq. (3.5). Again, the symbol "$\approx$" means "equal up to leading order in $\frac{1}{\lambda}$".

The fermionic action takes the form

$$S^{\circ}_{(\lambda)}(\psi^*,\psi) \approx \int\limits_{S_1} \frac{d^{d-1}\omega}{(\frac{2\pi}{k_F})^{d-1}} \int d^{d+1}\xi\, \delta^{(d-1)}(\boldsymbol{\xi}_\perp)\psi^*_\omega(\xi_0,\xi_\parallel)\left(\partial_0 + i\boldsymbol{\omega}\frac{\partial}{\partial\boldsymbol{\xi}}\right)\psi_\omega(\xi_0,\xi_\parallel)$$

$$= \int\limits_{[S_1]} \frac{d^{d-1}[\omega]}{\left(\frac{2\pi}{k_F}\right)^{d-1}} \int d\xi_0 \int d\xi_\parallel \overline{\psi}_{[\omega]}(\xi_0,\xi_\parallel)(\gamma^0\partial_0 + \gamma^1\partial_\parallel)\psi_{[\omega]}(\xi_0,\xi_\parallel)\,. \quad (5.3)$$

For simplicity, we have set $v_F \equiv e \equiv 1$. Further, we combined the vector $\boldsymbol{\omega}$ with its antipode $-\boldsymbol{\omega}$ to the direction $[\omega] := \{\boldsymbol{\omega}, -\boldsymbol{\omega}\}$, in order to introduce the "relativistic notation"

$$\psi_{[\omega]} := \begin{pmatrix} \psi_\omega \\ \psi_{-\omega} \end{pmatrix} \qquad \overline{\psi}_{[\omega]} := \psi^*_{[\omega]}\gamma^0 = (\psi^*_{-\omega}, \psi^*_\omega) \quad (5.4)$$

with

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = \sigma_2\,.$$

The coupling of the gauge field $A^{(\lambda)}_\mu(\lambda\xi) = \frac{1}{\lambda}a_\mu(\xi)$ to the fermionic part reduces to

$$S^a_{(\lambda)} \approx \lambda^{d-1} \int d^{d+1}\xi \int d\omega' \int d\omega\, [-ia_0(\xi) + \boldsymbol{\omega}\mathbf{a}(\xi)] \cdot e^{-i\lambda k_F(\omega-\omega')\xi}$$

$$\cdot\, \psi^*_{\omega'}(\xi_0,\xi'_\parallel)\psi_\omega(\xi_0,\xi_\parallel)$$

$$= \lambda^d \int d^{d+1}\xi \int d[\omega]\, a^\omega_\mu(\xi)\, j^\mu_{(\lambda),[\omega]}(\lambda\xi)\,, \quad (5.5)$$

where we defined

$$\int d\omega(\ldots) := \int\limits_{S} \frac{d^{d-1}\omega}{\left(\frac{2\pi}{k_F}\right)^{d-1}}(\ldots), \qquad \int d[\omega](\ldots) := \int\limits_{[S]} \frac{d^{d-1}[\omega]}{\left(\frac{2\pi}{k_F}\right)^{d-1}}(\ldots)\,,$$

$$\partial^\omega_0 = \partial_0\,, \qquad \partial^\omega_1 \equiv \partial^\parallel_1 = \boldsymbol{\omega}\frac{\partial}{\partial\boldsymbol{\xi}},$$
$$a^\omega_0(\xi) = a_0(\xi), \qquad a^\omega_1(\xi) = \boldsymbol{\omega}\mathbf{a}(\xi)\,,$$

and

$$j^\mu_{(\lambda),[\omega]}(\lambda\xi) = \frac{-1}{\lambda}\int d[\omega']\, e^{-i\lambda k_F(\boldsymbol{\omega}-\boldsymbol{\omega}')\xi}\overline{\psi}_{[\omega']}(\xi_0,\xi'_\parallel)\, i\gamma^\mu\, \psi_{[\omega]}(\xi_0,\xi_\parallel) \quad (5.6)$$

for $\mu = 0, 1$.

The integration over the fermionic modes to calculate the effective action of the gauge field

$$S^*(a) = \text{``}\lim_{\lambda\to\infty}\text{''}\left(-\ln \int \frac{\mathscr{D}(\bar{\psi}_{[\omega]}, \psi_{[\omega]})}{Z(0)} \exp -[S^{\circ}_{(\lambda)} + S^{a}_{(\lambda)}]\right) \tag{5.7}$$

leads to the expansion (3.8). "$\lim_{\lambda\to\infty}$" stands for taking the scaling limit as described in Sect. 2.

One would have been tempted to take the limit $\lambda \to \infty$ of expression (5.6) by using Lemma (3.9), i.e.,

$$\lim_{\lambda\to\infty} j^{\mu}_{(\lambda),[\omega]}(\lambda\xi) \approx \frac{-1}{\lambda^d}\delta^{(d-1)}(\xi_\perp)\bar{\psi}_{[\omega]}(\xi_0, \xi_\parallel)\, i\gamma^{\mu}\, \psi_{[\omega]}(\xi_0, \xi_\parallel)$$

before integrating out the fermionic modes, but the two procedures *do not commute.*

The reduction into independent $1+1$ dimensional Schwinger models along the directions $[\omega]$ and the discussion of Appendix D suggest that one can express the currents $j^{\mu}_{[\omega]}$ by real, massless bose fields $\varphi_{[\omega]}$ (cf. D.25):

$$j^{\mu}_{(\lambda),[\omega]}(\lambda\xi) \approx \frac{i}{\sqrt{\pi}}\varepsilon^{\mu\nu}\frac{\partial^{\parallel}_{\nu}}{\lambda}\varphi_{(\lambda),[\omega]}(\lambda\xi) = \frac{i}{\sqrt{\pi}}\frac{1}{\lambda^d}\varepsilon^{\mu\nu}\partial^{\parallel}_{\nu}\varphi_{[\omega]}(\lambda\xi)\,, \qquad \mu, \nu = 0, 1\,, \tag{5.8}$$

where

$$\varphi_{(\lambda),[\omega]}(\lambda\xi) = \frac{1}{\lambda^{d-1}}\varphi_{[\omega]}(\xi)$$

and

$$\langle(\cdots)\rangle_B = \int \frac{\mathscr{D}\varphi_{[\omega]}}{Z_B} \exp\left(\frac{1}{2\lambda^{d-1}} \int d[\omega] \int d^{d+1}\xi\; \varphi_{[\omega]}(\xi)\partial^{\parallel}_{\mu}\partial^{\parallel}_{\mu}\varphi_{[\omega]}(\xi)\right)(\cdots)\,.$$

In fact, one can easily relate this representation to the local conservation of the electric current – i.e. gauge invariance – in each direction $[\omega]$ separately.

One obtains

$$\begin{aligned}
e^{-S^*(a)} &= \text{``}\lim_{\lambda\to\infty}\text{''} \int \frac{\mathscr{D}(\bar{\psi}_{[\omega]}, \psi_{[\omega]})}{Z(0)} \exp -\left[S^{\circ}_{(\lambda)}(\bar{\psi}_{[\omega]}, \psi_{[\omega]}) + S^{a}_{(\lambda)}(\bar{\psi}_{[\omega]}, \psi_{[\omega]}; a)\right] \\
&= \text{``}\lim_{\lambda\to\infty}\text{''} \int \frac{\mathscr{D}(j^{\mu}_{(\lambda),[\omega]})}{\tilde{Z}(0)} \exp \widetilde{S^{\circ}_{(\lambda)}}(j^{\mu}_{(\lambda),[\omega]}) \\
&\quad \cdot \exp\left[-\lambda^d \int d[\omega] \int d^{d+1}\xi\; j^{\mu}_{(\lambda),[\omega]}(\lambda\xi) a^{\omega}_{\mu}(\xi)\right] \\
&= \int \frac{\mathscr{D}\varphi_{[\omega]}}{Z_B(0)} \exp\left(\frac{1}{2\lambda^{d-1}} \int d[\omega] \int d^{d+1}\xi\; \varphi_{[\omega]}(\xi)\partial^{\parallel}_{\mu}\partial^{\parallel}_{\mu}\varphi_{[\omega]}(\xi)\right) \\
&\quad \cdot \exp\left(-\int d[\omega] \int d^{d+1}\xi\; a^{\omega}_{\mu}(\xi)\varepsilon^{\mu\nu}\frac{i}{\sqrt{\pi}}\partial^{\omega}_{\nu}\varphi_{[\omega]}(\xi)\right) \\
&= \exp\left(-\frac{1}{2\pi}(2)\lambda^{d-1} \int d[\omega] \int d^{d+1}\xi\; a^{\omega}_{\mu}(\xi)\right. \\
&\quad \left. \cdot \left[\delta_{\mu\nu} - \frac{\partial^{\omega}_{\mu}\partial^{\omega}_{\nu}}{(\partial^{\omega}_0)^2 + (\partial^{\omega}_1)^2}\right] a^{\omega}_{\nu}(\xi)\right)\,.
\end{aligned} \tag{5.9}$$

Remarks.

- The factor (2) in the last equation of (5.9) comes from the two spin degrees of freedom which are completely separated. In fact, we always understand $j^{\mu}_{[\omega]} = (j^{\mu}_{[\omega]\uparrow}, j^{\mu}_{[\omega],\downarrow})$ and $\varphi_{[\omega]} = (\varphi_{[\omega]\uparrow}, \varphi_{[\omega]\downarrow})$.
- $\widetilde{S^{\circ}_{(\lambda)}}(j^{\mu}_{(\lambda),[\omega]})$ is the Fourier transform of $S^*(a)$, i.e.,

$$\begin{aligned}\exp \widetilde{S^{\circ}_{(\lambda)}}(j^{\mu}_{(\lambda),[\omega]}) &= \int \mathscr{D}[a^{\omega}_{\mu}(\xi)] \exp[-S^*(a)] \\ &\quad \cdot \exp[\lambda^d \int d[\omega] \int d^{d+1}\xi\, j^{\mu}_{(\lambda),[\omega]}(\lambda\xi) a^{\omega}_{\mu}(\xi)] \\ &= \delta\left(\partial^{\omega}_{\mu}\, j^{\mu}_{(\lambda),[\omega]}(\lambda\xi)\right) \\ &\quad \cdot \exp\left[\frac{\pi}{2}\lambda^{d+1} \int d[\omega] \int d^{d+1}\xi\, j^{\mu}_{(\lambda),[\omega]}(\lambda\xi) j^{\mu}_{(\lambda),[\omega]}(\lambda\xi)\right] .\end{aligned}$$

The bosonization formulas (5.8) are equivalent to (a continuum version of) Luther–Haldane bosonization [13].

To make contact with the general bosonization formalism proposed in [2], one has to note that the euclidean version of the current density $j^{\mu}(\psi^*, \psi, A; x = \lambda\xi)|_{A=0}$, $\mu = 0, \dots, d$, defined in Eq. (1.3), can be written – using Eq. (5.6) – as

$$\begin{aligned} j^0(\lambda\xi) &= \int d[\omega]\, j^0_{(\lambda),[\omega]}(\lambda\xi) , \\ j^k(\lambda\xi) &= \int d[\omega]\; \mathbf{e}^k \cdot \boldsymbol{\omega}\; j^1_{(\lambda)[\omega]}(\lambda\xi), \qquad k = 1, \dots d . \end{aligned} \tag{5.10}$$

This allows us to establish the connection to the bosonic tensor field b, Eq. (5.1):

$$\lambda^d j^0(\lambda\xi) = \int d[\omega] \frac{i}{\sqrt{\pi}} \boldsymbol{\omega} \frac{\partial}{\partial \boldsymbol{\xi}} \varphi_{[\omega]}(\xi) = \frac{\varepsilon^{0\nu\rho_1\dots\rho_{d-1}}}{(d-1)!} \partial_{\nu} b_{\rho_1\dots\rho_{d-1}}(\xi) , \tag{5.11}$$

$$\lambda^d j^k(\lambda\xi) = \int d[\omega] \frac{-i}{\sqrt{\pi}}\; \mathbf{e}^k \cdot \boldsymbol{\omega}\; \partial_0 \varphi_{[\omega]}(\xi) = \frac{\varepsilon^{k\nu\rho_1\dots\rho_{d-1}}}{(d-1)!} \partial_{\nu} b_{\rho_1\dots\rho_{d-1}}(\xi) , \tag{5.12}$$

The current density j^{μ} specifies the tensor-field $b_{\rho_1\dots\rho_{d-1}}$ only up to the exterior derivative of an arbitrary, antisymmetric, rank $d-2$ tensor-field, $\chi_{\rho_2\dots\rho_{d-1}}(\xi)$, $\rho_\iota \in \{0, \dots, d\}$:

$$j^{\mu} = \frac{\varepsilon^{0\nu\rho_1\dots\rho_{d-1}}}{(d-1)!} \partial_{\nu} b_{\rho_1\dots\rho_{d-1}} = \frac{\varepsilon^{0\nu\rho_1\dots\rho_{d-1}}}{(d-1)!} \partial_{\nu} b^{\chi}_{\rho_1\dots\rho_{d-1}} , \tag{5.13}$$

where $b^{\chi}_{\rho_1\dots\rho_{d-1}} := b_{\rho_1\dots\rho_{d-1}} + \partial_{\rho_1}\chi_{\rho_2\dots\rho_{d-1}}$. I.e., $b_{\rho_1\dots\rho_{d-1}}$ is a rank $d-1$ gauge field and the transformation $b \to b^{\chi}$ is a gauge transformation which does not change physics (for details see [2], where the more convenient language of differential forms is used).

Let us introduce the following, "geometric" notation. Once a basis $\{e^1, \dots, e^{d+1}\}$ of $\mathbf{R}^{d+1}$ is chosen, the (oriented) array $(\rho_1, \dots, \rho_{d-1})$ of $d-1$ distinct indices $\rho_i \in \{0, \dots, d+1\}$ specifies the (oriented) $d-1$ dimensional plane $\Pi^{(d-1)}_{\rho_1\dots\rho_{d-1}}$ generated by the vectors $\{e^{\rho_1}, \dots, e^{\rho_{d-1}}\}$. So, each component of the tensorfield $b_{\rho_1\dots\rho_{d-1}}$ is associated to a $d-1$ dimensional plane $\Pi^{(d-1)}_{\rho_1\dots\rho_{d-1}}$ in $\mathbf{R}^{d+1}$. We denote by

$\Pi_{\perp}^{d-1}(\rho_1,\rho_2)$ the $d-1$ dimensional plane orthogonal in $\mathbf{R}^{d+1}$ to the plane generated by $\{e^{\rho_1},e^{\rho_2}\}$. Equation (4.11) can then be written as $j^\mu = \varepsilon^{\mu\nu\Pi_\perp(\mu\nu)}\partial_\nu b_{\Pi_\perp(\mu\nu)}$.

Let us now choose the "temporal" gauge

$$b_{\Pi_\perp(k,l)} \equiv 0, \quad \text{for all} \quad k,l \in \{1,\dots,d\}\,, \tag{5.14}$$

i.e., components of b associated to planes which contain the time-direction vanish. In this gauge, it follows from Eq. (5.12) that

$$\varepsilon^{k0\Pi_\perp(k,0)}\partial_0 b_{\Pi_\perp(k,0)}(\xi) = \int d[\omega]\frac{-i}{\sqrt{\pi}}\ \mathbf{e}^k\cdot\boldsymbol{\omega}\ \partial_0\varphi_{[\omega]}(\xi)\,, \quad k=1,\dots,d$$

or

$$\varepsilon^{0k\Pi_\perp(0,k)}b_{\Pi_\perp(0,k)}(\xi) + f_k(\xi) = \int d[\omega]\frac{i}{\sqrt{\pi}}\ \mathbf{e}^k\cdot\boldsymbol{\omega}\ \varphi_{[\omega]}(\xi)\,, \tag{5.15}$$

where $f_k(\boldsymbol{\xi})$, $k=1,\dots,d$, are arbitrary functions which do not depend on the time variable ξ_0. Inserting (5.15) in Eq. (5.11), one obtains the consistency condition $\partial_k f_k(\boldsymbol{\xi}) = 0$, which can be solved by $f_k(\boldsymbol{\xi}) = \varepsilon^{0kl\Pi_\perp^{(d-2)}(0,k,l)}\partial_l\tilde{\chi}_{\Pi_\perp^{(d-2)}(0,k,l)}(\boldsymbol{\xi})$. I.e., the functions $f_k(\boldsymbol{\xi})$ can be absorbed into the b field by a further gauge transformation, consistent with the temporal gauge (5.14).

From a complementary point of view, we observe that Eq. (5.8) determines the bose field $\varphi_{[\omega]}(\xi)$ only up to functions $\chi_{[\omega]}(\boldsymbol{\xi}_\perp(\boldsymbol{\omega}))$ which do not depend on ξ_0 and $\xi_\parallel$. As a consequence, the functions $f_k(\boldsymbol{\xi})$ can be absorbed into a redefinition, $\varphi^\chi_{[\omega]}$, of the Bose field, given by $\varphi^\chi_{[\omega]}(\xi) = \varphi_{[\omega]}(\xi) + \chi_{[\omega]}(\boldsymbol{\xi}_\perp(\boldsymbol{\omega}))$, where $\int d[\omega]\mathbf{e}^k\boldsymbol{\omega}\frac{i}{\sqrt{\pi}}\chi_{[\omega]}(\boldsymbol{\xi}_\perp(\boldsymbol{\omega})) = f_k(\boldsymbol{\xi})$.

In the "b-formalism," the calculation of $S^*(a)$ for a free electron system takes the form (cf. [2]):

$$\begin{aligned}
e^{-S^*(a)} = \int\frac{[\mathscr{D}b]}{Z(0)}\ \exp\Bigg[-\frac{1}{2\lambda}\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\ &\varepsilon^{\mu\nu\rho_1\cdots\rho_{d-1}}(-ik_\nu)b_{\rho_1\cdots\rho_{d-1}}(-k)\\
&\cdot(\boldsymbol{\Pi}_*^{-1}(k))^{\mu\kappa}\varepsilon^{\kappa\lambda\sigma_1\cdots\sigma_{d-1}}(ik_\lambda)b_{\sigma_1\cdots\sigma_{d-1}}(k)\Bigg]\\
&\cdot\exp\left[\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\ a_\mu(-k)\varepsilon^{\mu\nu\rho_1\cdots\rho_{d-1}}(ik_\nu)b_{\rho_1\cdots\rho_{d-1}}(k)\right]
\end{aligned} \tag{5.16}$$

with $\boldsymbol{\Pi}_*^{\mu\nu}(k)$ given by Eq. (3.16).

The "b-formalism" is a general formalism which is based only on the local conservation law for the $U(1)$-current j^μ. In contrast, the introduction of the $\varphi_{[\omega]}$ fields is based on the decomposition of the $d+1$ dimensional electron system into independent, $1+1$ dimensional systems along the directions $[\omega]$, the currents $j^\mu_{[\omega]}$ along the directions $[\omega]$ being conserved separately. As a consequence of this supplementary symmetry, each of these currents $j^\mu_{[\omega]}$ can be bosonized separately. This is a special property of the free, non-relativistic electron system with a Fermi surface.

This remains true if one adds to the free system (5.1) an interaction of the form

$$S^{TL}_{(\lambda)} = -\lambda^{d+1} \int d[\omega_1] \int d[\omega_2] g_{[\omega_1][\omega_2]} \int d^{d+1}\xi \sum_{\mu=0}^{1} j^{\mu}_{(\lambda),[\omega_1]}(\lambda\xi) j^{\mu}_{(\lambda),[\omega_2]}(\lambda\xi), \quad (5.17)$$

which is the higher-dimensional analogue of a $1+1$ dimensional Luttinger Liquid [14] (for $g_{[\omega_1][\omega_2]} \sim \delta_{[\omega_1],[\omega_2]}$, it reduces to the so-called Tomographic Luttinger Liquid). The electron propagator of this system can be calculated exactly by the bosonization techniques presented in [2]. The calculation and the presentation of the results requires additional discussion deferred to the forthcoming paper [3]. There, we also analyse under which conditions the general two-body interaction (4.12) reduces, in the scaling limit, to an interaction of the form (5.17). This is done by using renormalization-group techniques.

Appendix A: Derivation of (3.4)

We rewrite

$$e^{-S^{\text{eff}}(A)} = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \prod_{i=1}^{n} \int d^{d+1}x_i$$

$$\cdot \sum_{s_i=\pm} \sum_{\substack{\sigma\in S_n \\ \sigma(i)\neq i}} \text{sgn}(\sigma) f(A; x_i) G^{\circ}_{s_i, s_{\sigma(i)}}(x_i - x_{\sigma(i)}) \quad (A.1)$$

in a more convenient form by using the following elementary facts about the symmetric group S_n:

- Each permutation σ of S_n can be written as a product of cycles c_l of length l: $\sigma = \prod_i c_{l_i}$ with $\sum_i l_i = n$.
- The permutations of S_n, which can be written as a product of ν_1 cycles of length 1, ν_2 cycles of length $2, \ldots, \nu_n$ cycles of length n, are forming the conjugate class $(1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n})$ of S_n.
- The conjugate class $(1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n})$ consists of exactly $n_{(\nu)} = \dfrac{n!}{\nu_1! 2^{\nu_2} \nu_2! 3^{\nu_3} \nu_3! \ldots\ldots n^{\nu_n} \nu_n!}$ different permutations of S_n.
- For $\sigma \in (1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n})$, $\quad \text{sgn}(\sigma) = \sum_{i=1}^{n} (-1)^{\nu_i(i-1)} = (-1)^n (-1)^{\nu_1 + \cdots + \nu_n}$.

The expression for $e^{-S^{\text{eff}}(A)}$ takes now the form :

$$e^{-S^{\text{eff}}(A)} = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \sum_{\substack{(\nu_2,\ldots,\nu_2) \\ 2\nu_2+\cdots+n\nu_n=n}} \sum_{\substack{\sigma\in S_n \\ \sigma\in(2^{\nu_2},\ldots,n^{\nu_n})}} (-1)^n (-1)^{\nu_2+\cdots+\nu_n}$$

$$\cdot \prod_{i=1}^{n} \sum_{s_i=\pm} \int d^{d+1}x_i f(x_i) G^{\circ}_{s_i, s_{\sigma(i)}}(x_i - x_{\sigma(i)})$$

$$= 1 + \sum_{n=2}^{\infty} \sum_{\substack{(\nu_2,\ldots,\nu_n) \\ 2\nu_2+\cdots+n\nu_n=n}} \frac{1}{n!} \frac{n!}{\nu_2!\ldots\nu_n!}$$

$$\cdot \underbrace{\prod_{\alpha=1}^{n} \left(-\frac{1}{\alpha} \prod_{\substack{\iota_\alpha=1 \\ (\alpha+1=1)}}^{\alpha} \sum_{s_\iota} \int d^{d+1} x_{i_\alpha} f(x_{\iota_\alpha}) G^{\circ}_{s_\iota, s_{\sigma(i)}}(x_{i_\alpha} - x_{\iota_\alpha+1}) \right)^{\nu_\alpha}}_{=:(-\frac{2}{\alpha}(fG)^\alpha)^{\nu_\alpha}}$$

$$= 1 + \sum_{N=1}^{\infty} \sum_{\substack{(\nu_{l_1},\ldots,\nu_{l_N}) \\ \sum_{\alpha=1}^{N} \nu_{l_\alpha}=N \\ l_\alpha \in \mathbb{N}\setminus\{0,1\} \\ \nu_{l_\alpha} \in \mathbb{N}\cup\{0\}}} \prod_{\alpha=1}^{N} \frac{1}{\nu_{l_\alpha}!} \left(\frac{-2}{l_\alpha} (fG)^{l_\alpha} \right)^{\nu_{l_\alpha}}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \left\{ \sum_{\substack{(\nu_{l_1},\ldots,\nu_{l_N}) \\ \ldots\ldots}} N! \prod_{\alpha=1}^{N} \frac{1}{\nu_{l_\alpha}!} \left(\frac{-2}{l_\alpha} (fG)^{l_\alpha} \right)^{\nu_{l_\alpha}} \right\}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \left(-1 \sum_{l=2}^{\infty} \frac{2}{l} (fG)^l \right)^N$$

$$= \exp\left(-\sum_{l=2}^{\infty} \frac{2}{l} (fG)^l \right) .$$

Appendix B : Asymptotic Form of the Electron Propagator in $d + 1$ Dimensions

In euclidean notation, the free, electronic propagator is given by

$$G(x_0, \mathbf{x}; k_F) = \frac{1}{(2\pi)^{d+1}} \int dk_0 \int d^d k \; \frac{e^{-\iota(k_0 x_0 + \mathbf{k}\mathbf{x})}}{ik_0 - \varepsilon_{\mathbf{k}}} , \tag{B.1}$$

where $\varepsilon_{\mathbf{k}} = \frac{1}{2m}(\mathbf{k}^2 - k_F^2)$ and $k_F = \sqrt{2m\mu}$. We want to determine the asymptotic form of this propagator which emerges from the scaling procedure ($x = \lambda\xi \rightarrow \xi$), described in Sect. 2. We show that, for $x_0 \neq 0$, it takes the form:

$$G^{(\lambda)}(\lambda\xi_0, \lambda\boldsymbol{\xi}; k_F) = \frac{1}{\lambda} \left\{ \left(\frac{k_F}{2\pi} \right)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega \; e^{-ik_F \lambda \boldsymbol{\omega}\boldsymbol{\xi}} \cdot G_{\boldsymbol{\omega}}(\xi_0, \xi_\|; k_F) \left[1 + 0\left(\frac{1}{\lambda} \right) \right] \right\}$$

where $\quad \xi_\| := \boldsymbol{\omega}\boldsymbol{\xi} \quad$ and $\quad G_{\boldsymbol{\omega}}(\xi_0, \xi_\|) = \int_{\mathbf{R}} \frac{dk_0}{2\pi} \int_{\mathbf{R}} \frac{dk_\|}{2\pi} \frac{e^{-\iota(k_0\xi_0 + k_\| \xi_\|)}}{ik_0 - v_F k_\|}. \quad$ (B.2)

The Fermi velocity v_F is defined by $v_F := \frac{k_F}{m}$. For $d = 1$, the integral over the $d - 1$ dimensional unit sphere, $\int_{S_1^{d-1}} d^{d-1}\omega$, must be replaced by the sum $\sum_{\omega=\pm 1}$.

Proof. For $x_0 \neq 0$, Jordan's lemma allows us to transform (B.1) in the following way:

$$G(x_0, \mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} \left\{ \theta(x_0) \oint \frac{dk_0}{2\pi i} + \theta(-x_0) \oint \frac{dk_0}{2\pi i} \right\} \frac{e^{-i(k_0 x_0 + \mathbf{kx})}}{k_0 + i\varepsilon_{\mathbf{k}}}$$

$$= \int \frac{d^d k}{(2\pi)^d} \{ -\theta(x_0)\theta(\varepsilon_{\mathbf{k}}) + \theta(-x_0)\theta(-\varepsilon_{\mathbf{k}}) \} \exp\{ -\varepsilon_{\mathbf{k}} x_0 - i\mathbf{kx} \} .$$

We introduce the variables $\xi = \frac{1}{\lambda} x$:

$$G^{(\lambda)}(\lambda\xi_0, \lambda\boldsymbol{\xi}) = \int \frac{d^d k}{(2\pi)^d} \left\{ -\theta(\lambda\xi_0)\theta\left(\frac{1}{2m}(\mathbf{k}^2 - k_F^2)\right) + \theta(-\lambda\xi_0)\theta\left(\frac{-1}{2m}(\mathbf{k}^2 - k_F^2)\right) \right\}$$

$$\cdot \exp\left\{ -\frac{1}{2m}(\mathbf{k}^2 - k_F^2)\lambda\xi_0 - i\lambda\mathbf{k}\boldsymbol{\xi} \right\} .$$

With $\mathbf{k} := (k_F + k)\omega, |\omega| = 1$, it follows that

$$G^{(\lambda)}(\lambda\xi_0, \lambda\boldsymbol{\xi}) = \frac{1}{(2\pi)^d} \int_{-k_F}^{\infty} dk (k + k_F)^{d-1} \int_{S_1^{d-1}} d^{d-1}\omega \{ -\theta(\lambda\xi_0)\theta(v_F k)$$

$$+ \theta(-\lambda\xi_0)\theta(-v_F k) \} \cdot \exp\left\{ -\frac{1}{2m}(k^2 + 2k_F k)\lambda\xi_0 \right\} e^{-i\lambda(k_F + k)\omega\xi}$$

$$\overset{p:=\lambda k}{=} \frac{1}{(2\pi)^d} \int_{-\lambda k_F}^{\infty} \frac{dp}{\lambda} \left(\frac{p}{\lambda} + k_F \right)^{d-1} \int d^{d-1}\omega \{ -\theta(\xi_0)\theta(v_F p)$$

$$+ \theta(-\xi_0)\theta(-v_F p) \} \cdot \exp\left\{ -\frac{1}{2m}\left(\frac{p^2}{\lambda} + 2k_F p \right)\xi_0 \right\} e^{-i\lambda k_F \omega\xi} e^{-ip\omega\xi} .$$

For $\lambda \to \infty, \xi_0 \neq 0$, one obtains:

$$G^{(\lambda)}(\lambda\xi_0, \lambda\boldsymbol{\xi}) = \left(\frac{k_F}{2\pi} \right)^{d-1} \frac{1}{\lambda} \int d^{d-1}\omega e^{-i\lambda k_F \omega\xi} \int_{\mathbf{R}} \frac{dp}{2\pi} e^{-ip\omega\xi}$$

$$\cdot \{ -\theta(\xi_0)\theta(v_F p) + \theta(-\xi_0)\theta(-v_F p) \} e^{-v_F p \xi_0} \left[1 + 0\left(\frac{1}{\lambda} \right) \right]$$

$$= \left(\frac{k_F}{2\pi} \right)^{d-1} \frac{1}{\lambda} \int_{S_1^{d-1}} d^{d-1}\omega e^{-i\lambda k_F \omega\xi} \int \frac{dp}{2\pi}$$

$$\cdot \int \frac{dp_0}{2\pi} \frac{e^{-i(p_0\xi_0 + p\omega\xi)}}{ip_0 - v_F p} \left[1 + 0\left(\frac{1}{\lambda} \right) \right] ,$$

where, again, Jordan's lemma has been used.

Similarly, one shows that for $x_0 = \lambda\xi_0 \mp 0$,

$$\frac{\partial}{\partial(\lambda\xi)} G^{(\lambda)}(\lambda\xi_0, \lambda\xi) = \left(\frac{k_F}{2\pi}\right)^{d-1} \int\limits_{S_1^{d-1}} d^{d-1}\omega e^{-ik_F\lambda\omega\xi} \left(-\frac{ik_F\omega}{\lambda}\right) \cdot G_\omega\left(\xi_0, \xi_\|\right)\left[1 + 0\left(\frac{1}{\lambda}\right)\right].$$

Appendix C : Proof of Lemma (3.21)

Lemma (3.21) states:

For $d > 1, \lambda \to \infty$

$$\int\limits_{\mathbf{R}^d} h(\xi) \left\{ \int\limits_{S_1^{d-1}} d^{d-1}\omega e^{i\lambda k_F \xi(\omega - \sigma)} g(\omega) \right\} = \int\limits_{\mathbf{R}^d} d^d\xi h(\xi) \left\{ \left(\frac{2\pi}{\lambda k_F}\right)^{d-1} g(\sigma) \left[\delta^{(d-1)}(\xi_\perp[\sigma]) + 0\left(\frac{1}{\lambda}\right)\right] \right\},$$

where h, g are test functions, $\omega, \sigma \in S_1^{d-1} = \{\omega \in \mathbf{R}^1; |\omega| = 1\}$ and $\xi_\perp[\sigma] := \xi - \sigma(\xi \cdot \sigma)$.

Proof. One verifies easily that, for an arbitrary small neighbourhood $U_\varepsilon(\sigma) := \{\omega \in S_1^{d-1} : |\omega - \sigma| < \varepsilon^{d-1}\}$, the value of the integral

$$\int\limits_{\mathbf{R}^d} d^d\xi\, h(\xi) \int\limits_{S_1^{d-1}\setminus U_\varepsilon(\sigma)} d^{d-1}\omega\, e^{i\lambda k_F \xi(\omega-\sigma)} g(\omega)$$

is of order $\frac{c(\varepsilon)}{\lambda^d}$. We will show that this contribution is negligible in comparison to

$$I_{U_\varepsilon(\sigma)} := \int d^d\xi\, h(\xi) \int\limits_{U_\varepsilon(\sigma)} d^{d-1}\omega\, e^{i\lambda k_F \xi(\omega-\sigma)} g(\omega).$$

We choose the coordinate system so that $\sigma = (1, 0, \ldots, 0)$. Parametrising the vectors $\omega \in U_\varepsilon(\sigma)$ by

$$\omega = \left(\sqrt{1 - \sum_{i=2}^{d} \eta_i^2}, \eta_2, \ldots, \eta_d\right), \qquad \eta_i \in (-\varepsilon, \varepsilon) \quad i = 2, .., d$$

we can write

$$I_{u_\varepsilon(\sigma)} = \int d^d\xi \, h(\boldsymbol{\xi}) \int_{-\varepsilon}^{\varepsilon} d\eta_2 .... \int_{-\varepsilon}^{\varepsilon} d\eta_d \, \exp\left[i\lambda k_F \left(\sum_{i=2}^{d} \eta_i \xi_i + \xi_1 \left(\sqrt{1 - \sum_{i=2}^{d} \eta_i^2} - 1 \right) \right) \right]$$

$$\cdot \, g\left(\sqrt{1 - \sum_{i=2}^{d} \eta_i^2}, \eta_2, ..., \eta_d \right)$$

$$= \int d^d\xi' \int_{-\varepsilon}^{\varepsilon} d\eta_2 .... \int_{-\varepsilon}^{\varepsilon} d\eta_d \left| \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\xi}'} \right| \exp\left[i\lambda k_F \sum_{i=2}^{d} \eta_i \xi_i' \right] h'(\boldsymbol{\xi}')$$

$$\cdot \, g\left(\sqrt{1 - \sum_{i=2}^{d} \eta_i^2}, \eta_2, ..., \eta_d \right),$$

where we introduced the coordinate transformation

$$\xi_1' = \xi_1, \qquad \xi_i' = \xi_i - \eta_i u_i(\eta) \xi_1, \qquad i = 2, .., d \, .$$

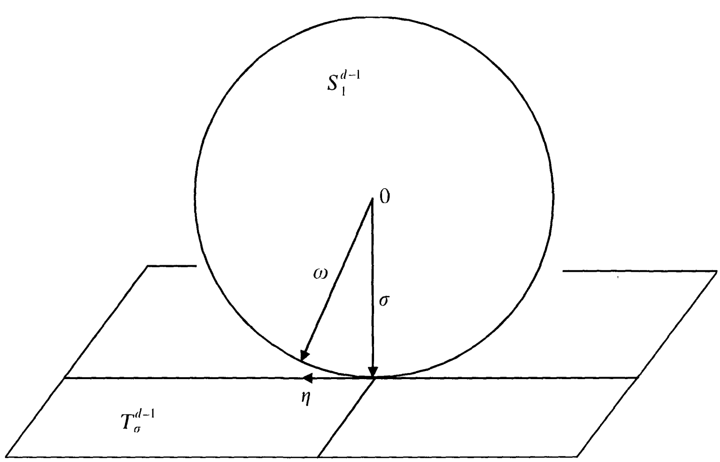
The functions $u_i(\eta)$ are smooth functions, and $h'(\boldsymbol{\xi}') := h(\boldsymbol{\xi}(\boldsymbol{\xi}'))$.

Lemma (3.21) follows by using that

$$\int_{\mathbf{R}^2}\int dx dy \, e^{i\lambda xy} f_1(x) f_2(y) = \frac{2\pi}{\lambda} \left(f_1(0) f_2(0) + 0\left(\frac{1}{\lambda}\right) \right),$$

for test functions f_1, f_2.

The lemma states that, on the LHS of Eq. (3.9), the ω-integral over the sphere S_1^{d-1} can be replaced – to leading order – by an integral over the tangent plane $T_{\boldsymbol{\sigma}}^{d-1}$ of the sphere at the point $\boldsymbol{\sigma}$.



Appendix D : Calculation of the Effective Gauge Field Action of the Schwinger Model

The Schwinger model [10] describes relativistic electrons in $1+1$ dimensions coupled to an abelian gauge field A. In euclidean space, the fermionic part of its action is given by

$$S(\bar{\psi},\psi;A) = \int d^2x\, \bar{\psi}(x)D(A)\psi(x)$$

with $D(A) := \gamma^\mu(\partial_\mu + iqA_\mu)$ and $x := (x_0,x_1)$ .We interpret A as a classical, external gauge field, but quantize the fermions. Quantization is accomplished by using functional integrals. The field ψ denotes a two-component Grassmann field, and $\bar{\psi} := \psi^*\gamma^0$, where ψ^* is an independent Grassmann field. Choosing the chiral representation for the γ-matrices,

$$\gamma^0 = \sigma_1, \qquad \gamma^1 = \sigma_2, \qquad \gamma^5 = -i\gamma^0\gamma^1 = \sigma_3\ ,$$

the two components $\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}$ of ψ are antiholomorphic and holomorphic modes, resp., which are the euclidean analogues of right- and left-movers, resp. We are interested in the effective gauge field action $S^{\text{eff}}(A)$:

$$\mathrm{e}^{-S^{\text{eff}}(A)} = \frac{Z(A)}{Z(0)} = \frac{\int \mathscr{D}(\bar{\psi},\psi)e^{-S(\bar{\psi},\psi;A)}}{\int \mathscr{D}(\bar{\psi},\psi)e^{-S(\bar{\psi},\psi;A=0)}} = \int \frac{\mathscr{D}(\bar{\psi},\psi)}{Z(0)} e^{-\int d^2x(\bar{\psi}\gamma^\mu\partial_\mu\psi + A_\mu j^\mu)}\ , \quad \text{(D.1)}$$

where the currents are defined by:

$$j^\mu(\bar{\psi},\psi;x) := q\bar{\psi}(x)i\gamma^\mu\psi(x)\ .$$

Imposing gauge invariance, $S^{\text{eff}}(A_\mu) = S^{\text{eff}}(A_\mu + \partial_\mu\chi)$, the following Ward identities must be satisfied:

$$\frac{\partial}{\partial x_i^{\mu_i}} \langle j^{\mu_1}(x_1)\cdot\dots\cdot j^{\mu_i}(x_i)\cdot\dots\cdot j^{\mu_n}(x_n)\rangle^{\text{con}} = 0, \qquad i = 1,\dots,n\ . \quad \text{(D.2)}$$

The connected expectation values of products of currents are given by:

$$\left\langle \prod_{i=1}^n j^{\mu_i}(x_i)\right\rangle^{\text{con}} = (-1)^n \prod_{i=1}^n \frac{\delta}{\delta A_{\mu_i}(x_i)} \log\left\{ \int \frac{\mathscr{D}(\bar{\psi},\psi)}{Z(0)} e^{-\int d^2x(\bar{\psi}\gamma^\mu\partial_\mu\psi + A_\mu j^\mu)}\right\}\Bigg|_{A=0}\ . \quad \text{(D.3)}$$

To calculate expression (D.1), we expand it in powers of the gauge field A:

$$\mathrm{e}^{-S^{\text{eff}}(A)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left[-\int d^2x A_\mu(x) j^\mu(\bar{\psi},\psi;x)\right]^n \right\rangle \quad \text{(D.4)}$$

with

$$< \cdots > := \int \frac{\mathscr{D}(\bar{\psi},\psi)}{Z(0)} e^{-\int d^2x \bar{\psi}\gamma^\mu\partial_\mu\psi}(\dots)\ .$$

The problem reduces to calculating expectation values of products of $\bar{\psi}$ and ψ. They can be obtained from the generating functional

$$Z(\bar{u},u) = \left\langle e^{\int d^2x(\bar{\psi}(x)u(x) + \bar{u}(x)\psi(x))}\right\rangle \quad \text{(D.5)}$$

by differentiating with respect to the anticommuting sources $\bar{u}, u$:

$$\left\langle \prod_{i=1}^{n} \psi(x_i)\bar{\psi}(y_i) \right\rangle = (-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta \bar{u}(x_i)} \frac{\delta}{\delta u(y_i)} Z(\bar{u}, u) \Big|_{\bar{u}, u=0} \tag{D.6}$$

The generating functional $Z(\bar{u}, u)$ is obtained by calculating a gaussian Berezin integral and is given by

$$Z(\bar{u}, u) = \exp\left[-\int d^2x d^2y \bar{u}(x) \mathbf{G}(x-y) u(x)\right] , \tag{D.7}$$

where the propagator $\mathbf{G}(x-y) = -\langle \psi(x)\bar{\psi}(y) \rangle$ is the solution of the differential equation $\gamma^\mu \partial_\mu \mathbf{G}(x-y) = -\delta^{(2)}(x-y)\mathbf{1}_2$. The result is

$$\begin{aligned} \mathbf{G}(x-y) &= \gamma^\mu \partial_\mu \left[-\frac{1}{2\pi} \ln |x-y| \right] \\ &= \begin{pmatrix} 0 & \frac{-i}{2\pi} \frac{1}{-i(x_0-y_0)+(x_1-y_1)} \\ \frac{i}{2\pi} \frac{1}{i(x_0-y_0)+(x_1-y_1)} & 0 \end{pmatrix} . \end{aligned} \tag{D.8}$$

To evaluate the expansion coefficients in (D.4), one uses Wick's theorem. In order to get only well defined quantities, we omit selfcontractions (which, formally, vanish anyway). This is equivalent to replacing the currents j^μ by their normal ordered expressions.

$$\begin{aligned} &\left\langle \prod_{i=1}^{n} : \bar{\psi}(x_i) \gamma^{\mu_i} \psi(x_i) : \right\rangle \\ &= -1 \operatorname{tr} \left\langle \gamma^{\mu_1} \psi(x_1) \bar{\psi}(x_2) \gamma^{\mu_2} \psi(x_2) \cdots \bar{\psi}(x_n) \gamma^{\mu_n} \psi(x_n) \bar{\psi}(x_1) \right\rangle \Big|_{\substack{\text{no self-} \\ \text{contractions}}} \\ &= (-1)^{n+1} \sum_{\substack{\sigma \in S_n \\ \sigma(i+1) \neq i}} \operatorname{sgn}(\sigma) \operatorname{tr} \left(\prod_{i=1}^{n} \gamma^{\mu_i} \mathbf{G}(x_i - x_{\sigma(i+1)}) \right) \\ &= \sum_{\substack{\tau \in S_n \\ \tau(i) \neq i}} \operatorname{sgn}(\tau) \operatorname{tr} \left(\prod_{i=1}^{n} \gamma^{\mu_i} \mathbf{G}(x_i - x_{\tau(i)}) \right) . \end{aligned} \tag{D.9}$$

After some elementary transformations (similar to those in Appendix A), expansion (D.4) becomes

$$\begin{aligned} e^{-S^{\text{eff}}(A)} = \exp \Bigg[&- \sum_{l=2}^{\infty} \frac{q^l}{l} \int d^2x_1 \cdots \int d^2x_l \operatorname{tr} \left(\prod_{\substack{j=1 \\ (l+1=1)}}^{l} i\gamma^{\mu_j} \mathbf{G}(x_j - x_{j+1}) \right) \\ &\cdot A_{\mu_1}(x_1) \cdots A_{\mu_l}(x_l) \Bigg] . \end{aligned} \tag{D.10}$$

In formula (D.10) we observe that $S^{\text{eff}}(A)$ is the sum of loops of length $l \geqq 2$, composed of l propagator lines $\mathbf{G}$ and l insertions of the external gauge potential A. Because of the special form of the propagator $\mathbf{G}$ (cf. (D.8)), loops of length $l > 2$ vanish (we postpone a direct verification of this fact to the end of this appendix). It remains to calculate the loop of length 2. To avoid UV as well as IR divergencies, we use, for example, dimensional regularization which preserves gauge invariance and give temporarily a mass to the fermion field. The result of this straightforward calculation (cf. 16) is

$$\langle j^\mu(x) j^\nu(y)\rangle^{\text{con}} = \langle j^\mu(x) j^\nu(y)\rangle = -\frac{q^2}{\pi}\left(\delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial_0^2+\partial_1^2}\right)\delta^{(2)}(x-y)$$

so that

$$S^{\text{eff}}(A) = \frac{1}{2}\frac{q^2}{\pi}\int d^2x\; A^T_\mu(x) A^T_\mu(x)\,. \tag{D.11}$$

Note that only the transversal part of the gauge field enters, as required by gauge invariance:

$$A^T_\mu(x) = \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial_0^2+\partial_1^2}\right)A_\nu(x)\,.$$

For pedagogical reasons, we analyse the expansion (D.4) from a second point of view (cf. 15) .

As discussed in Sect. 2, in the construction of the scaling limit of (1.1), we lost local terms which are vital for gauge invariance. If one calculates the expansion coefficients of (D.4) or its logarithm directly in x-space using (D.8) and (D.9), one encounters non-integrable singularities at coinciding arguments. A priori, the current distributions are only defined on spaces of test functions that vanish at coinciding arguments. We wish to extend them to tempered distributions, i.e., define them on arbitrary Schwartz space test functions, in such a way that the Ward identities of $U(1)$-gauge invariance are satisfied. It turns out that the requirement of gauge invariance uniquely determines their extension to coinciding arguments. In the process of our calculations, we will understand, in a natural way, why only loops of length $l = 2$ contribute to $S^{\text{eff}}(A)$. This is the key fact that enables us to express the current j^μ in terms of a free bose field.

We require the following notations:

$$\begin{aligned} z &:= ix_0 + x_1, \qquad & \partial &:= \tfrac{\partial}{\partial z} = \tfrac{1}{2}(-i\partial_0 + \partial_1)\,, \\ \bar{z} &:= -ix_0 + x_1, \qquad & \bar{\partial} &:= \tfrac{\partial}{\partial \bar{z}} = \tfrac{1}{2}(i\partial_0 + \partial_1)\,, \end{aligned} \tag{D.12}$$

and

$$\psi(x) := \begin{pmatrix} \bar{b}(\bar{z}) \\ c(z) \end{pmatrix}, \qquad \bar{\psi}(x) := (b(z), \bar{c}(\bar{z}))\,.$$

Expressing the currents j_μ by the b- and c- fields,

$$\begin{aligned} j^0(x) &= iq :\bar{\psi}_R(x)\psi_L(x) + \bar{\psi}_L(x)\psi_R(x) := iq : b(z)c(z) - \bar{b}(\bar{z})\bar{c}(\bar{z}) :\,, \\ j^1(x) &= q :\bar{\psi}_R(x)\psi_L(x) - \bar{\psi}_L(x)\psi_R(x) := q : b(z)c(z) + \bar{b}(\bar{z})\bar{c}(\bar{z}) :\,, \end{aligned} \tag{D.13}$$

the explicit calculation of the expansion coefficients in (D.4) reduces to evaluating expressions of the form

$$\left\langle \prod_{i=1}^{n} : b(z_i)c(z_i) : \right\rangle\,, \tag{D.14}$$

$$\left\langle \prod_{i=1}^{n} : \bar{b}(\bar{z}_i)\bar{c}(\bar{z}_i) : \right\rangle\,, \tag{D.15}$$

$$\left\langle \prod_{i=1}^{n} : b(z_i)c(z_i) : \prod_{j=1}^{m} : \bar{b}(\bar{z}_j)\bar{c}(\bar{z}_j) : \right\rangle . \tag{D.16}$$

With (D.8) and Wicks theorem, it follows e.g. for (D.14) that

$$\begin{aligned} \left\langle \prod_{i=1}^{n} : b(z_i)c(z_i) : \right\rangle &= \sum_{\substack{\sigma\in S_n \\ \sigma(\imath)\neq\imath}} \operatorname{sgn}(\sigma) \prod_{\imath=1}^{n} \langle b(z_i)c(z_{\sigma(i)})\rangle \\ &= \left(\frac{i}{2\pi}\right)^n \sum_{\substack{\sigma\in S_n \\ \sigma(\imath)\neq\imath}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \frac{1}{z_i - z_{\sigma(i)}} \\ &= \det a_{ij} \end{aligned} \tag{D.17}$$

with $a_{ij} = \frac{i}{2\pi}\frac{1}{z_\imath - z_j}$ for $i \neq j$ and $a_{ii} = 0$ $i,j = 1,\ldots,n$.

By antisymmetry of (a_{ij}) this determinant vanishes, if n is odd. Moreover, in the sum over the permutations σ, only those permutations contribute which factorize completely in a product of (disjoint) 2-cycles. To keep the present argument short and coherent, we postpone the proof of this assertion to the end of the appendix. (By transformations analogous to those in Appendix A, one can see that this result implies that, in (D.10), only loops of length 2 contribute).

Then (D.17) becomes

$$\left\langle \prod_{i=1}^{n=2m} : b(z_i)c(z_i) : \right\rangle = \left(\frac{i}{2\pi}\right)^{2m} \sum_{\substack{\text{pairings } p \\ p=[(p_1,p_2),\ldots,(p_{2m-1},p_{2m})]}} \prod_{i=1}^{m} \left(\frac{1}{z_{p_{2\jmath-1}} - z_{p_{2j}}}\right)^2 . \tag{D.18}$$

Similarly, (D.15) is seen to be given by

$$\left\langle \prod_{i=1}^{n=2m} : \bar{b}(\bar{z}_i)\bar{c}(\bar{z}_i) : \right\rangle = \left(\frac{i}{2\pi}\right)^{2m} \sum_{p} \prod_{i=1}^{m} \left(\frac{1}{\bar{z}_{p_{2j-1}} - \bar{z}_{p_{2j}}}\right)^2 . \tag{D.19}$$

Here, one encounters an important fact. One obtains the same result by substituting

$$: b(z)c(z) : \longrightarrow \pm\frac{1}{\sqrt{\pi}}\partial\varphi(x) , \tag{D.20}$$

$$: \bar{b}(\bar{z})\bar{c}(\bar{z}) : \longrightarrow \pm\frac{1}{\sqrt{\pi}}\bar{\partial}\varphi(x) , \tag{D.21}$$

where φ is a real, free zero-mass bose field with a propagator

$$\left\langle \partial_\imath^{(1)}\varphi(x_1)\partial_j^{(2)}\varphi(x_2)\right\rangle = \frac{1}{2\pi}\frac{\delta_{ij}}{|x_1 - x_2|^2} - \frac{1}{\pi}\frac{(x_1 - x_2)^i(x_1 - x_2)^j}{|x_1 - x_2|^4} . \tag{D.22}$$

The fact that, in the fermionic system, only pairs of currents contract implies, after bosonization, that φ is a free field. The bosonic field theory is given by the Gaussian functional integral,

$$\langle \ldots \rangle_B = \int \mathscr{D}\varphi e^{\frac{1}{2}\int d^2x\varphi(x)\partial_\mu\partial_\mu\varphi(x)}(\ldots) . \tag{D.23}$$

The possibility of expressing the fermionic current j^μ by a bose field is intimately related to gauge invariance (for a more detailed discussion see [2]). Classically, gauge invariance implies the continuity equation for the current $\partial_\mu j^\mu(x) = 0$. This equation can be solved (locally) by

$$j^\mu(x) \sim \varepsilon^{\mu\nu}\partial_\nu \varphi(x)\,, \tag{D.24}$$

where φ is a real, scalar field. To reproduce this formula, one has to choose a relative minus sign between (D.20) and (D.21). With (D.13) one then obtains

$$\begin{aligned} j^0(x) &= \pm i\frac{q}{\sqrt{\pi}}(\partial + \underline{\partial})\varphi(x) = \pm i\frac{q}{\sqrt{\pi}}\partial_1\varphi(x)\,, \\ j^1(x) &= \pm\frac{2}{\sqrt{\pi}}(\partial - \underline{\partial})\varphi(x) = \pm(-i)\frac{q}{\sqrt{\pi}}\partial_0\varphi(x)\,, \\ \text{i.e.}\quad j^\mu(x) &= \pm i\frac{q}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\varphi(x)\,. \end{aligned} \tag{D.25}$$

Moreover, the (global) chiral invariance of the fermionic system (massless electrons!) implies that $\partial_\mu j_5^\mu(x) = 0$, where $j_5^\mu(x) \sim \varepsilon^{\mu\nu} j_\nu(x)$, and hence

$$(\partial_0^2 + \partial_1^2)\varphi(x) = 0\,. \tag{D.26}$$

Thus, chiral invariance implies that φ is a free, massless bose field.

We now return to (D.16). By (D.8) and Wick's theorem, one might expect that after extension to coinciding arguments, one still has holomorphic factorization!

$$\begin{aligned} &\left\langle \prod_{i=1}^{n} :\bar{b}(\bar{z}_i)\bar{c}(\bar{z}_i): \prod_{j=1}^{m} :b(z_j)c(z_j): \right\rangle \\ &\quad = \left\langle \prod_{i=1}^{n} :\bar{b}(\bar{z}_i)\bar{c}(\bar{z}_i): \right\rangle \left\langle \prod_{j=1}^{m} :b(z_j)c(z_j): \right\rangle, \end{aligned}$$

in particular, for $n = m = 1$,

$$\langle :\bar{b}(\bar{z}_1)\bar{c}(\bar{z}_1)::b(z_2)c(z_2): \rangle = 0\,.$$

Actually, this is in conflict with gauge invariance. On the basis of the bosonization formulas (D.20) and (D.21), one expects that

$$\langle :\bar{b}(\bar{z}_1)\bar{c}(\bar{z}_1)::b(z_2)c(z_2): \rangle = -\frac{1}{4\pi}\delta^{(2)}(x_1 - x_2)\,. \tag{D.27}$$

This turns out to be the correct extension of current Green functions to coinciding arguments. By Eq. (D.24), this local term – which couples the holomorphic and the antiholomorphic modes of the systems – guarantees the gauge invariance of the theory.

With the bosonization identities (D.20) and (D.21), it is straightforward to calculate (D.4). The result is identical to (D.11).

Finally, we prove the following two assertions announced above:

(I) In Eq. (D.10), only loops of length 2 contribute.

(II) For the calculation of the determinant (D.17), only permutations contribute, which factorize completely in a product of (disjoint) 2-cycles.

Clearly, (II) implies (I), but, for fun, we present two independent proofs.

(I) Denoting the two non-zero components of the propagator $\mathbf{G}$ (cf. Eq. (D.8)) by:

$$\mathbf{G}(x) = \begin{pmatrix} 0 & G_R(x) \\ G_L(x) & 0 \end{pmatrix} ,$$

Eq. (D.10) can be rewritten as

$$e^{-S^{\mathrm{eff}}(A)} = \exp\left\{ -\sum_{l=2}^{\infty} \frac{q^l}{l} \int d^2x_1 \cdots \int d^2x_l \left[\prod_{j=1}^{l} (iA_0 - A_1)(x_j)\, G_R(x_j - x_{j+1}) \right.\right.$$
$$\left.\left. +(iA_0 + A_1)(x_j)\, G_L(x_j - x_{j+1}) \right] \right\} .$$

In momentum space, each summand takes the form

$$\int d^2x_1 \cdots \int d^2x_l \left[\prod_{j=1}^{l} (iA_0 + A_1)(x_j)\, G_R(x_j - x_{j+1}) + (iA_0 - A_1)(x_j)\, G_L(x_j - x_{j+1}) \right]$$

$$= \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2p_1}{(2\pi)^2} \cdots \int \frac{d^2p_{l-1}}{(2\pi)^2}$$
$$\times \left\{ [(i\hat{A}_0 - \hat{A}_1](p_1)\, \hat{G}_R(q + p_1) \cdot \left[i\hat{A}_0 - \hat{A}_1\right](p_2)\, \hat{G}_R(q + p_1 + p_2) \right.$$
$$\cdots \left[i\hat{A}_0 - \hat{A}_1\right](p_{l-1})\, \hat{G}_R\left(q + \sum_{j=1}^{l-1} p_j\right) \cdot \left[i\hat{A}_0 - \hat{A}_1\right]\left(-\sum_{j=1}^{l-1} p_j\right) \hat{G}_R(q)$$
$$+ \left[(i\hat{A}_0 + \hat{A}_1\right](p_1)\, \hat{G}_L(q + p_1) \cdot \left[i\hat{A}_0 + \hat{A}_1\right](p_2)\, \hat{G}_L(q + p_1 + p_2)$$
$$\left. \cdots \left[i\hat{A}_0 + \hat{A}_1\right](p_{l-1})\, \hat{G}_L\left(q + \sum_{j=1}^{l-1} p_j\right) \cdot \left[i\hat{A}_0 + \hat{A}_1\right]\left(-\sum_{j=1}^{l-1} p_j\right) \hat{G}_L(q) \right\} ,$$

where $\hat{G}_{R/L}(k) = \frac{1}{ik_0 \mp k_1}$.

The gauge field insertions do not depend on the loop momentum q, so that one can integrate over this variable. Assertion (I) reduces then to the following lemma:

The function

$$f(x^1, \ldots, x^{l-i}) := \int_{\mathbf{R}} dq_0 \int_{\mathbf{R}} dq_1 \frac{1}{iq_0 + q_1}$$
$$\cdot \frac{1}{i(q_0 + x_0^1) + (q_1 + x_1^1)} \cdots \frac{1}{i(q_0 + x_0^{l-1}) + (q_1 + x_1^{l-1})} \tag{D.28}$$

with $x^i := (x_0^i, x_1^i) \in \mathbf{R}^2$**, is identically zero for** $l > 2$.

We define

$$h(q_1;z^1,\dots,z^{l-1}) := \frac{1}{i^l}\int dq_0 \frac{1}{q_0-iq_1}\,\frac{1}{q_0-iq_1-z^1}\cdots\frac{1}{q_0-iq_1-z^{l-1}}$$

with $z^i := -x_0^i + ix_1^i \in \mathbf{C}$, so that

$$f(x^1,\dots,x^{l-1}) = \int_{\mathbf{R}} dq_1\, h(q_1;z^1,\dots,z^{l-1})\,.$$

The following steps of the proof are only sketched :

- One chooses $i_0 \in \{1,\dots,l-1\}$ and considers $h(q_1;z^1,\dots,z^{l-1})$ as function of z^{i_0}, where the other variables q_1 and $(z^1,\dots,\widehat{z^{i_0}},\dots,z^{l-1})$ are kept fixed.
- One shows that, for an arbitrary configuration $(z^1,\dots,\widehat{z^{i_0}},\dots,z^{l-1})$, this is a holomorphic function in z^{i_0}, which is bounded on the entire complex plane, $\forall q_1 \in \mathbf{R}\setminus\mathscr{D}[(z^1,\dots,\widehat{z^{i_0}},\dots,z^{l-1})]$, where $\mathscr{D}$ is a finite set of points depending on the configuration $(z^1,\dots,\widehat{z^{i_0}},\dots,z^{l-1})$. So, it must be a constant.
- Further, for $|z^{i_0}|\to\infty$, this function tends to 0, which proves Lemma (D.28).

(II) For assertion (II), we present a purely algebraic proof.

- Each permutation $\sigma\in S_{2m}$ can be written as a product of *disjoint* cycles $\sigma=\prod_{j\in J} c_j$. The permutations in S_{2m} with fixed numbers $v_1,\dots,v_n$ of length $l=1,\dots,n$ (where $\sum_{l=1}^n v_l\cdot l \overset{!}{=} 2m$) form a conjugacy class. The sign of the permutation is given by the product of the signs of the cycles. But, the sign of a cycle of length l is $(-1)^{l-1}$.
- All permutations which contain 1-cycles do not contribute to (D.17).
- Let $\sigma=\prod_{j\in\mathscr{J}} c_j$ contain at least one cycle c_{j_0} of odd length : $\sigma=\prod_{j\in\mathscr{J}\setminus\{j_0\}}(c_j)\circ c_{j_0}$. To the sum (D.17), both, the permutation σ and the permutation $\bar\sigma=\prod_{j\in\mathscr{J}\setminus\{j_0\}}(c_j)\circ c_{j_0}^{-1}$ contribute. $\bar\sigma$ has the same sign as σ. But $\prod_{i=1}^{2m}\frac{1}{z_i-z_{\bar\sigma(i)}} = -1\prod_{i=1}^{2m}\frac{1}{z_i-z_{\sigma(i)}}$, so that they cancel each other. Consequently, only permutations which factorize into *even* cycles contribute to (D.17).
- Finally, we show that the permutations which contain at least one cycle of length $l\geqq 4$ do not contribute.

Let $\sigma=\prod_{j\in\mathscr{J}\setminus\{j_0\}}(c_j)\circ c_{j_0}$ denote such a permutation, where $c_{j_0}=(i_1\cdots i_{l=2k})$, $k\geqq 2$, is a cycle of length $l\geqq 4$. Given the set of pairwise distinct numbers, $I_{j_0}:=\{i_1,\dots,i_{2k}\,|\,i_\alpha\in\{1,\dots,2m\},\,i_\alpha\neq i_\beta\ \alpha,\beta=1,\dots,2k\}$, there exist $(2k-1)!$ different $2k$-cycles. One can generate all these different cycles, $\mathscr{C}^{(2k)}_{I_{j_0}}$, by fixing one number in the cycle – for example i_1 – and permuting all the others $\mathscr{C}^{(2k)}_{I_{j_0}} = \{(i_1 i_{\rho(2)}\cdots i_{\rho(2k)})|\rho\in S_{2k-1}^{\{2,3,\dots,2k\}}\}$.

For our purposes it is useful to define the product of transpositions

$$\begin{array}{cccc} T_{\alpha,\beta} & : & \mathscr{C}^{(2k)} & \longrightarrow & \mathscr{C}^{(2k)} \\ & & (i_1\cdots i_\alpha\cdots i_\beta\cdots i_{2k}) & \mapsto & (i_1\cdots i_\beta\cdots i_\alpha\cdots i_{2k}) \end{array},$$

which enables us to associate a "string" of cycles to a given cycle c_{j_0}

$$\mathrm{Str}^{(2k)}[c_{j_0}] := \{c_{j_0}\} \cup \left\{ \prod_{\alpha=2}^{A} T_{\alpha+1,\alpha}\, c_{j_0} \,\middle|\, A = 2,3,\ldots,2k-1 \right\} ,$$

where $\prod_{\alpha=2}^{A} T_{\alpha+1,\alpha} := T_{A+1,A} \circ \cdots \circ T_{3,2}$ and $\prod_{\alpha=2}^{2k} T_{\alpha+1,\alpha}\, c_{j_0} = c_{j_0}$.

So, one can generate $\mathscr{C}_{I_{j_0}}^{(2k)}$ by fixing 2 numbers – for example i_1 and i_2 –, permuting the others and collecting all the associated strings :

$$\mathscr{C}_{I_{j_0}}^{(2k)} = \bigcup_{\rho \in S_{2k-2}^{\{3,\ldots,2k\}}} \mathrm{Str}^{(2k)}[(i_1 i_2 i_{\rho(3)} \cdots i_{\rho(2k)})] .$$

It turns out that

$$\sum_{c \in \mathrm{Str}^{(2k)}[c_{j_0}]} \prod_{i=1}^{2k} \frac{1}{z_i - z_{c(i)}} = 0 , \tag{D.29}$$

so that

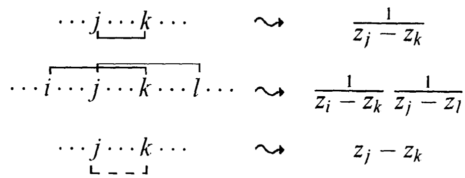
$$\sum_{\substack{\tau \in S_{2m} \\ \tau = \prod_{j \in \mathscr{J}\setminus\{j_0\}} (c_j) \circ c \\ c \in \mathrm{Str}^{(2k)}[c_{j_0}]}} \frac{1}{z_l - z_{\tau(l)}} = 0 .$$

Then assertion (II) follows, because the set of permutations $\{\prod_{j\in\mathscr{J}\setminus\{j_0\}}(c_j) \circ c \,|\, c \in \mathscr{C}_{I_{j_0}}^{(2k)}\} \subset S_{2m}$ decomposes completely into disjoint strings.

- One proves (D.29) by explicit calculation. As an example, we consider the string $\mathrm{Str}^{(2k)}[e]$ associated to the "standard" cycle $e := (1,2,\ldots,2k)$. The generalization is only a matter of notation. We define

$$e^{(A)} := \prod_{\alpha=2}^{A} (T_{\alpha+1,\alpha})\, e \qquad (2 \leqq A \leqq 2k-1) .$$

To show the effect of summing over the string, we use a graphical representation of the algebraic expressions, defined by the following dictionary:



Different such pairings in a cycle imply the multiplication of the algebraic expressions, for example:

$$(1\ 2\ 3\ \cdots\ 2k) \rightsquigarrow (-1)\prod_{i=1}^{2k} \frac{1}{z_i - z_{e(i)}} .$$

We will now sum, consecutively, over the elements of the string. To represent the corresponding result graphically, it is advantageous to always display the last cycle , $e^{(k)}$, which has been summed over :

i)

$$\prod_i \frac{1}{z_i - z_{e(i)}} + \prod_i \frac{1}{z_i - z_{e^{(2)}(i)}}$$

$$= (-1)\left(1 \quad 3 \quad 2 \quad 4 \quad 5 \quad \cdots \quad 2k \right)$$

ii)

$$\prod_i \frac{1}{z_i - z_{e(i)}} + \prod_i \frac{1}{z_i - z_{e^{(2)}(i)}} + \prod_i \frac{1}{z_i - z_{e^{(3)}(i)}}$$

$$= (-1)\left(1 \quad 3 \quad 4 \quad 2 \quad 5 \quad 6 \quad \cdots \quad 2k \right)$$

iii)

$$\prod_i \frac{1}{z_i - z_{e(i)}} + \sum_{A=2}^{L<2k-2} \prod_i \frac{1}{z_i - z_{e^{(A)}(i)}}$$

$$= (-1)\left(1 \quad 3 \quad 4 \quad \cdots \quad L \quad (L+1) \quad 2 \quad (L+2) \quad \cdots \quad 2k \right)$$

iv)

$$\prod_i \frac{1}{z_i - z_{e(i)}} + \sum_{A=2}^{2k-2} \prod_i \frac{1}{z_i - z_{e^{(A)}(i)}}$$

$$= (-1)\left(1 \quad 3 \quad 4 \quad \cdots \quad (2k-2) \quad (2k-1) \quad 2 \quad 2k \right)$$

$$= \frac{1}{z_1 - z_3} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{2k-2} - z_{2k-1}} \frac{1}{z_{2k-1} - z_{2k}} \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_{2k}} \frac{z_1 - z_{2k}}{z_{2k} - z_1}$$

$$= (-1)\left(1 \quad 3 \quad 4 \quad \cdots \quad (2k-1) \quad 2k \quad 2 \right)$$

$$= (-1)\prod_i \frac{1}{z_i - z_{e^{(2k-1)}(i)}}$$

$$\Longrightarrow \sum_{c \in \mathrm{Str}^{(2k)}[e]} \prod_{i=1}^{2k} \frac{1}{z_i - z_{e(i)}} = 0 . \qquad \square$$

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