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The Set of Maps $F_{a,b}: x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any Given Rotation Interval is Contractible

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Abstract: Consider the two-parameter family of real analytic maps $F_{a,b}: x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ which are lifts of degree one endomorphisms of the circle. The purpose of this paper is to provide a proof that for any closed interval I, the set of maps $F_{a,b}$ whose rotation interval is I, form a contractible set.

1. Introduction

Orientation preserving homeomorphisms and diffeomorphisms of the circle have attracted the attention of mathematicians and physicists for a long time because they arise as Poincaré maps induced by non-singular flows on the two-dimensional torus [2, 7, 14, 29]. More recently, families of circle endomorphisms which are deformations of rotations have appeared as approximate models for some scenarios of transition to "chaos," or more technically, transitions from zero to positive topological entropy. One can observe these transitions by varying parameters of flows in 3-space so that tori supporting non-singular flows get wrinkled and then get destroyed. More generally, these scenarios are typical for a huge variety of systems of coupled oscillators so that one sees them everywhere. As a matter of fact, in many cases when one is lead to study an endomorphism of the interval as a model for a natural science experiment, some circle endomorphism is a more adequate model.

While the simplest endomorphisms of the interval depend on a single parameter, say the non-linearity, the simplest reasonably complete family of circle endomorphisms containing the rotations has to depend on two parameters: the non-linearity and some form of mean rotation speed. In the coupled oscillators picture, these parameters correspond respectively to the strength of the forcing and the frequency ratio of the coupled oscillators. The paradigm for interval endomorphisms is the

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quadratic family. For the circle, it is the so-called *Arnold* or *standard* two-parameter family:

$$f_{A,b}\colon \theta \mapsto \left(\theta + A + \frac{b}{2\pi}\sin(2\pi\theta)\right)_1 ,$$

with $(A, b) \in [0, 1[\times \mathbb{R}^+$, and $(\theta)_n \stackrel{\text{def}}{=} \theta \bmod n$.

Under an orientation preserving homeomorphism of the circle, the orbits of all points wrap around the circle at the same average speed [29]. For non-invertible maps this is no longer necessarily the case, but the set of average speeds form a closed interval. With the concepts roughly recalled so far, we can give an example of a result that is a corollary of our main theorem: for any average speed ω, the set $\{(A, b)\}$ of pairs such that all orbits under the map $f_{A,b}$ wrap around the circle at speed ω, is connected. Our main result is in fact a similar statement in the more general setting where average speeds vary in an interval.

Precise definitions and statements are contained in Sect. 2. In Sect. 3, we reduce our main theorem to a rigidity result: this reduction is merely well known material, but some proofs are sketched for completeness. In Sect. 3, we have also included some material not strictly needed for the proof of Theorem A, but intended to help some readers to build an intuitive picture of what the main result is all about. The rigidity property, formalized in Theorem D, is proved in Sect. 4. Our proof of Theorem D is one more example of the efficiency of complex analytic methods in dealing with questions arising naturally in a real analytic framework.

2. Definitions and Statement of the Results

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle and $\Pi\colon \mathbb{R} \to \mathbb{T}$ the canonical projection. The real continuous map F is a lift of the continuous circle map $f\colon \mathbb{T} \to \mathbb{T}$ if and only if

$$f \circ \Pi = \Pi \circ F .$$

The integer d such that

$$F(x+1) = F(x) + d ,$$

for all real numbers x is called the *degree* of f (or of F). The identity map, and more generally the rotations, have degree one. Since the degree varies continuously for continuous deformations of circle maps, and since we are interested in a parametrized continuous family containing rotations, we shall only consider degree one maps in the rest of the paper. Hence, *circle map* will always mean "degree-one continuous circle map," and a real map will be called a *lift* if and only if it is the lift of a degree one circle map.

Let f be a circle map, and let F be a lift of f (each time both symbols f and F appear conjoined in the paper, they are related in the same way). We define

$$\underline{\rho}_F(x) = \liminf_{n\to\infty} \frac{F^n(x)}{n} ,$$

and

$$\overline{\rho}_F(x) = \limsup_{n\to\infty} \frac{F^n(x)}{n} .$$

The *rotation interval* of F [28] is then

$$I(F) = [\alpha, \beta],$$

where

$$\alpha = \inf_{x \in \mathbb{R}} \underline{\rho}_F(x), \qquad \beta = \sup_{x \in \mathbb{R}} \overline{\rho}_F(x).$$

When $I(F)$ is a singleton $\{\omega\}$, we sometimes use the classical language and say that $\rho(F) \stackrel{\text{def}}{=} \{\omega\}$ is the *rotation number* of F.

We will focus on the standard family $f_{A,b}$ with parameter space $[0, 1[\times \mathbb{R}^+$, and the corresponding degree one lifts $F_{a,b}$ with parameter space $\mathbb{R} \times \mathbb{R}^+$, where the correspondence is given by $A = a \bmod 1$. To state our main result we need the following

Definition. *An arc or curve $a = \phi(b)$ in parameter space $\mathbb{R} \times \mathbb{R}^+$ is called an L-curve if ϕ is uniformly Lipschitz with bound $\frac{1}{2\pi}$.*

Theorem A. *For each closed interval I, the set R_I of standard lifts with rotation interval I corresponds to a contractible region, also denoted R_I, in the parameter space $\mathbb{R} \times \mathbb{R}^+$. More precisely,*

- *For any irrational number ω, $R_{\{\omega\}}$ is an L-curve.*
- *If one bound of I is irrational while the second bound is a rational number, R_I is an L-curve.*
- *If the bounds of I are distinct irrational numbers, R_I is an L-curve.*
- *When both bounds of I are rational, R_I is a lens shaped domain bounded by two L-curves that meet at their endpoints.*

Convention. To simplify the language and the notation, we shall continue to identify sets of standard lifts with the corresponding regions in parameter space, as we did in the statement of Theorem A; when the distinction is relevant the context should tell which space we mean.

Conjecture B. *If the bounds of I are distinct irrational numbers, R_I is a point.*

The content of Theorem A is illustrated in Figs. 1 to 3:

- Figure 1 is a schematic picture of part of the parameter space for the lifts of the standard family. The small inserts represent the graph of $F^{q'}$ for F in various regions, lines and points.
- Figure 2 shows a situation proved to not exist in the standard family by Theorem A.
- Figure 3 illustrates the situation conjectured not to occur in the standard family (in Conjecture B).

All labeling appearing in these figures is defined in Sects. 2 and 3.

As an important particular case, relevant for example in the description of the boundary of chaos [20], Theorem A contains the following result which was conjectured in [4] (p. 378 and Fig. 13) and [20] (p. 213):

Corollary C. *For each real ω, the set $R_{\{\omega\}}$ is connected.*

Remark. Corollary C (which was known for a long time to hold in the case when ω is irrational) was recently proved for some families of piecewise affine lifts; for these families, the proof only uses elementary real variable methods [32]. Later, and in parallel to the work presented here, more sophisticated real variable methods [11] were used to get the counterpart to Theorem A as well as a proof of Conjecture B for these piecewise lifts [32].

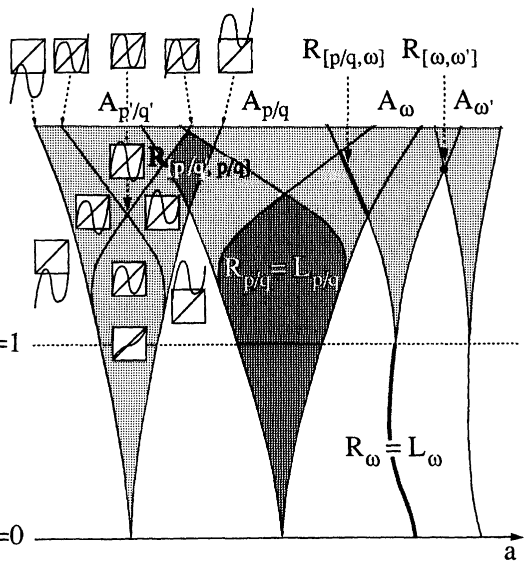

Fig. 1.

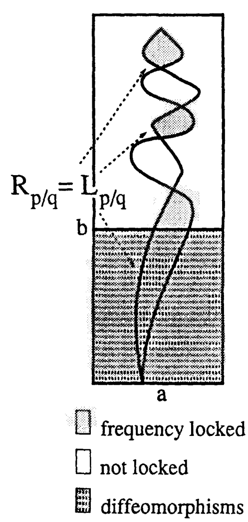

Fig. 2.

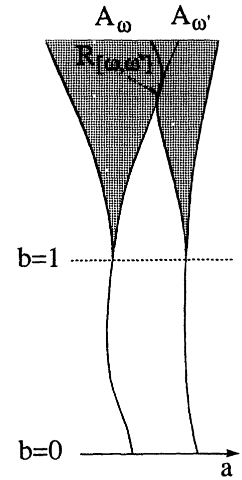

Fig. 3.

3. Proof of Theorem A, Part I: Real Analytic Part

3.1. Non-Decreasing Lifts. We shall denote by $\mathscr{F}^k(\mathbb{R})$ the space of C^k non-decreasing lifts, equipped with the C^k norm. The next theorem recalls some classical results [2, 14], usually formulated for lifts of degree one homeomorphisms, but generalizable *verbatim* to $\mathscr{F}^0(\mathbb{R})$.

Theorem 3.1.1.

1. *The rotaion number, as a function* $\rho: \mathscr{F}^0(\mathbb{R}) \to \mathbb{R}$ *is continuous.*
2. *For* F *and* G *in* $\mathscr{F}^0(\mathbb{R})$,

$$F \geqq G \Rightarrow \rho(F) \geqq \rho(G) ,$$

$$F > G \;\&\; \rho(F) \text{ or } \rho(G) \text{ irrational } \Rightarrow \rho(F) > \rho(G) .$$

3. *If* $\rho(F)$ *is irrational and* f *has a dense orbit, then,*

$$F \geqq G \quad \text{and} \quad F \neq G \Rightarrow \rho(F) > \rho(G) .$$

The next result which can also be found in the above references, is more specific to our problem:

Theorem 3.1.2. *For* $b \neq 0$, *no iterate of a standard lift is affine.*

Set

$$\mathbf{A}'_\omega = \{F \in \mathscr{F}^0(\mathbb{R}) \mid \omega \in I(F)\} ,$$

Theorems 3.1.1 and 3.1.2 yield a partition of the subset $\mathbb{R} \times [0, 1]$ of the parameter space of the standard family in the $\mathbf{A}'_\omega$'s, with the following properties [2]:

P1. For ω irrational, $\mathbf{A}'_\omega$ is an arc crossing each line $b = constant \leqq 1$ at a single point,

P2. For a rational number $\frac{p}{q}$, $\mathbf{A}'_{\frac{p}{q}}$, is often called an *Arnold tongue*; it crosses each line $0 < b = \text{constant} \leqq 1$ on an interval of positive length.

Remark. Property P2 describes an aspect of the phenomenon of "frequency locking," first described by Huyggens, in the context of clocks hanging from the same wall, and described in modern terms, in the simplest cases, as the structural stability of generic degree one circle diffeomorphisms with rational rotation numbers.

3.2. Some Special Sets. Following [4] and [20] (both of whom extended the above mentioned work in [2] from homeomorphisms to endomorphisms), for each real number ω, we define

$$\mathbf{A}_\omega = \{F \in \{F_{a,b}\}_{(a,b)\in\mathbb{R}\times\mathbb{R}^+} \mid \omega \in I(F)\} ,$$

and

$$\mathbf{L}_\omega = \{F \in \{F_{a,b}\}_{(a,b)\in\mathbb{R}\times\mathbb{R}^+} \mid \{\omega\} = I(F)\} .$$

According to our notational convention, $\mathbf{A}_\omega$ and $\mathbf{L}_\omega$ can also be understood as subspaces of the parameter space. The following theorem enables us to use the results of Sect. 3.1 to analyze these subspaces.

Theorem 3.2.1 ([5, 27]).

1. *For any lift* $F \in C_1^0(\mathbb{R})$,

$$I(F) = [\rho(F^-), \rho(F^+)] ,$$

where F^+ *is the monotonic upper-bound of* F, *and* F^- *is the monotonic lower-bound* (*see Fig.* 4). *In formulas we have*:

$$F^+(x) = \sup_{y \leqq x} (F(y)) ,$$

$$F^-(x) = \inf_{y \geqq x} (F(y)) .$$

2. *For each* $\omega \in I(F)$, *there is a non-decreasing lift* F_ω *with* $\rho(F_\omega) = \omega$, *and such that* F_ω *coincides with* F *where it is not locally constant.*

Theorem 3.2.1. is quite easy to prove for the maps in the standard family.

3.3. Simple Properties of the $\mathbf{A}_\omega$*'s and* $\mathbf{L}_\omega$*'s.* Theorem 3.2.1. allows us to use Theorem 3.1.1. in the study of non-invertible maps. In particular, one gets easily that the $\mathbf{A}_\omega$'s are connected and, for ω rational, they intersect each line $b =$ constant > 1 on a segment of non-zero length. We discuss the case when ω is irrational in Sect. 3.4. A fundamental role will be played by the boundaries of the $\mathbf{A}_\omega$'s and some of their accumulation sets defined as follows:

$$\mathbf{B}^l_\omega = \lim_{\theta\to\omega^+} \mathbf{A}^l_\theta ,$$

$$\mathbf{B}^r_\omega = \lim_{\theta\to\omega^-} \mathbf{A}^r_\theta .$$

These boundaries, pieces of which form all the boundaries of the sets R_I of Theorem A, are described in the following result due to Boyland [4]. We include a proof for the sake of completeness.

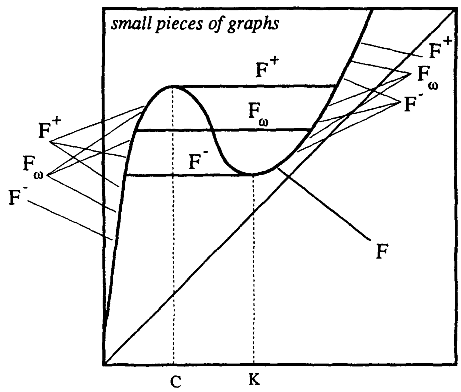

Fig. 4.

Theorem 3.3.1 ([4]).

1. *For any real number* ω, *the left and right bounds* $\mathbf{A}^l_\omega$ *and* $\mathbf{A}^r_\omega$ *of* $\mathbf{A}_\omega$ *in* $\mathbb{R} \times \mathbb{R}^+$, *are* L-*curves.*
2. *When* ω *is a rational number these curves intersect at a single point which has* $b = 0$ *as second coordinate.*
3. $$\mathbf{B}^l_{\frac{p}{q}} = \lim_{\frac{p'}{q'} \to \frac{p}{q}+} \mathbf{A}^l_{\frac{p'}{q'}} \quad \textit{and} \quad \mathbf{B}^r_{\frac{p}{q}} = \lim_{\frac{p'}{q'} \to \frac{p}{q}-} \mathbf{A}^r_{\frac{p'}{q'}} .$$

   Moreover, the sets $\mathbf{B}^l_{\frac{p}{q}}$ *and* $\mathbf{B}^r_{\frac{p}{q}}$ *are* L-*curves.*
4. *For* ω *irrational,*
$$\mathbf{A}^l_\omega = \mathbf{B}^l_\omega = \lim_{\frac{p}{q} \to \omega-} \mathbf{A}^l_{\frac{p}{q}} ,$$
$$\mathbf{A}^r_\omega = \mathbf{B}^r_\omega = \lim_{\frac{p}{q} \to \omega+} \mathbf{A}^r_{\frac{p}{q}} .$$

Proof. With no loss of generality, we prove statement 1 for $\mathbf{A}^l_{\frac{p}{q}}$. To do this we consider the vertical cone in $\mathbb{R} \times \mathbb{R}^+$ with vertex at (a, b) and boundaries made by lines with slopes 2π and -2π, and show that it contains $\mathbf{A}^l_{\frac{p}{q}}$. A point to the right of the cone has coordinates $a' = a + \delta + \varepsilon$, $b' = b \pm 2\pi\delta$, while a point to the left of the cone has coordinates $a'' = a - \delta - \varepsilon$, $b'' = b \mp 2\pi\delta$, for some non-negative δ and ε. Consequently, using the continuity of the rotation number of $F^+_{a,b}$ as a function of the parameters, and its monotonicity as a function of a, the inclusion of $\mathbf{A}^l_{\frac{p}{q}}$ in the cone follows from

$$\forall \delta \geqq 0, \quad \forall \varepsilon \geqq 0: \quad \delta + \varepsilon \pm \delta \sin 2\pi x \geqq 0 .$$

Statement 2 follows from Theorem 3.1.2. Statement 3 follows from statement 1 by continuity of the rotation number applied to the monotonic bounds. Statement 4 is a consequence of the same continuity property. Q.E.D.

3.4. Theorem A in the Simplest Case. We recall here the proof of Theorem A in the case when I is the singleton $\{\omega\}$ for some irrational number ω. We begin with a weak form of a theorem by Denjoy [7]

Theorem 3.4.1 ([7]). *For F in $\mathscr{F}^2(\mathbb{R})$, if $\rho(F) = \omega$ for some irrational number ω, then the circle map f with lift F has a dense orbit.*

The next result is a particular case of a theorem obtained by Block and Franke (see also [5]) as a consequence of the Denjoy theory:

Theorem 3.4.2 ([3]). *If $b > 1$ and $\rho = \rho(F^-_{a,b}) = \rho(F^+_{a,b})$, then $\rho \in \mathbf{Q}$.*

Proof. We first remark that there exist distinct C^2 smooth lifts F_0 and F_1 such that, $\forall x \in \mathbb{R}$, $F^-(x) \leqq F_0(x) \leqq F_1(x) \leqq F^+(x)$. If the claim were false, by Theorem 3.1.1–2, $\rho = \rho(F_0) = \rho(F_1) = \rho(F^+)$ for some $\rho \notin \mathbf{Q}$. But if $\rho \notin \mathbf{Q}$, Theorem 3.4.1 implies F_0 (and F_1) has a dense orbit. Hence the claim follows from Theorem 3.1.1–3. Q.E.D.

To finish the analysis of the case when $\rho(F) = \omega \notin \mathbf{Q}$, we just have to check that, as a consequence of Theorem 3.4.2, all $\mathbf{L}_\omega$ are contained in the region $b \leqq 1$ described in Sects. 3.1 and 3.3. In summary, we have:

Lemma 3.4.3. *For $\omega \notin \mathbf{Q}$, $R_{\{\omega\}}$ is an* L*-curve contained in the region $b \leqq 1$.*

3.5. Intersections of the Boundaries of the A_ω's: Existence. For any A, the narrowest diagonal strip with sides parallel to and centered on the main diagonal that contains the graph of $F_{a,b}$, can be made arbitrarily wide by choosing b large enough. Hence

Lemma 3.5.1. *For any $\omega \in \mathbb{R}$, and any $a \in \mathbb{R}$, ω is contained in the interior of $I(F_{a,b})$ as soon as b is large enough.*

Corollary 3.5.2. *If $\omega < 0$, $\mathbf{A}^r_\omega$ and $\mathbf{B}^r_\omega$ intersect $\mathbf{A}^l_\omega$ and $\mathbf{B}^r_\omega$. Also, $\boldsymbol{B}^r_{\frac{p}{q}}$ intersects $\boldsymbol{B}^l_{\frac{p}{q}}$ for any rational $\frac{p}{q}$.*

3.6. Intersections of the Boundaries of the A_ω's: Combinatorics. In order to prove Theorem A using Theorem 3.3.1–4, we may restrict our attention to the intersection points of the boundaries of $A_{\frac{p}{q}}$ and $A_{\frac{p'}{q'}}$. Before we begin the analysis of these intersection points, let us recall that the Schwarzian derivative of a map g is defined as

$$Sg = \frac{g''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

A direct computation then gives

Lemma 3.6.1. *For $b > 1$ and $F'_{a,b}(x) \neq 0$, $SF_{a,b}(x) < 0$.*

This will be used to prove Lemma 3.6.2.

Using a lift O of any periodic orbit o of $f_{A,b}$, we can analyze the local behavior of $f_{A,b}$ near o in terms of the derivatives of $F_{a,b}$ at q successive points of O, x_0, $x_1 = F_{a,b}(x_0), \ldots, x_{q-1} = F_{a,b}(x_{q-2})$. Define the *multiplier* of o as $m_o = F'_{a,b}(x_0) \cdot F'_{a,b}(x_1) \ldots F'_{a,b}(x_{q-1})$. We call o

- *attracting* if $|m_o| < 1$,
- *neutral* if $|m_o| = 1$, and in particular *parabolic* if m_o is a root of unity,
- *hyperbolic* if $|m_o| > 1$.

Clearly m_o only depends on O.

The periodic orbits of the circle map $f_{A,b}$, which are also orbits of some homeomorphism of the circle, and lifts of these orbits will play an important role in our discussion. If a point of such a periodic orbit o of $f_{A,b}$ has a lift with rotation number $\frac{p}{q}$ under $F_{a,b}$, o has period q and lifts to p distinct orbits of $F_{a,b}$.

Lemma 3.6.2. *Suppose $\frac{p}{q} \in I(F_{a,b})$. Then $F_{a,b}$ has an orbit O such that*

1. *O projects to a periodic orbit o of $f_{A,b}$.*
2. *There is a monotone $G \in \mathscr{F}^0(\mathbb{R})$ such that O is an orbit of G.*
3. *No point of O is in an interval where $F_{a,b}$ is decreasing.*
4. *If $m_o \geqq 1$, O is uniquely determined up to integer translation. If we relax the multiplier condition, there are, up to integer translation, at most two distinct orbits. When there are two orbits distinct under integer translation, denote them by O and O'. Then*
   (a) *O and O' bound intervals that are lifts of q pairwise disjoint arcs on which $f_{A,b}$ is orientation preserving,*
   (b) *the interiors of these intervals are in the immediate basin of the attracting periodic orbit which lifts to O',*
   (c) *at least one critical point is in the immediate basin of the attracting orbit which lifts to O'.*

Proof. For properties 1 to 3, the existence follows from Theorem 3.2.1.

Uniqueness under the condition $m_o \geqq 1$ follows from the well known fact (proved by a direct computation) that the absolute value of the (usual) derivative of a function on the real line, whose Schwarzian derivative is negative off the critical set, has no local non-zero minimum. That there are at most two orbits distinct under integer translation when the multiplier condition is relaxed follows in a similar fashion from the negative Schwarzian derivative property.

Property 4 (a) comes from the fact that we only use the restriction of $F_{a,b}$ to the intervals where it is increasing; property 4 (b) is immediate (draw a graph); and property 4 (c) is a classical result in holomorphic dynamics (see the second remark in Sect. 4.1). Q.E.D.

Let

$$\mathbf{o}_{\frac{P}{Q}} = \{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{Q-1}\} \quad \text{and} \quad \mathbf{o}'_{\frac{P}{Q}} = \{\mathbf{p}'_0, \mathbf{p}'_1, \ldots, \mathbf{p}'_{Q-1}\}$$

be the projections of $\mathbf{O}_{\frac{p}{q}}$, and $\mathbf{O}'_{\frac{p}{q}}$ respectively, where $f_{A,b}(\mathbf{p}_j) = \mathbf{p}_{(j+1)_Q}$, $f_{A,b}(\mathbf{p}'_j) = \mathbf{p}'_{(j+1)_Q}$, and $\frac{P}{Q} = (\frac{p}{q})_1$.

Assume that $b > 1$. It follows from the properties of $\mathbf{o}_{\frac{P}{Q}}$ that the two critical points $\mathbf{c}$ and $\mathbf{k}$ of $f_{A,b}$ are in an arc Γ bounded by two successive points $\mathbf{p}_j$ and

$\mathbf{p}_k$ of $\mathbf{o}_{\frac{P}{Q}}$. When $Q=1$, $\mathbf{p}_j$ and $\mathbf{p}_k$ coincide. When $\mathbf{o}_{\frac{P}{Q}}$ exists, they lie in an arc Γ' bounded by two successive points $\mathbf{p}'_j$ and $\mathbf{p}'_k$ of $\mathbf{o}'_{\frac{P}{Q}}$. Let P_j be a lift of $\mathbf{p}_j$, P_k be the lift of $\mathbf{p}_k$ immediately to the right of P_j, and let C and K be the lifts of $\mathbf{c}$ and $\mathbf{k}$ in $[P_j, P_k]$. Let P'_j be the lift of $\mathbf{p}'_j$ immediately to the left of C, and P'_k be the lift of $\mathbf{p}'_k$ immediately to the right of P'_j (and of K). Let $F_{a,b}$ be the lift of $f_{A,b}$ such that P_j and P'_j have rotation number $\frac{p}{q}$ under $F_{a,b}$, i.e., $\underline{\rho}_F(P'_j)=\overline{\rho}_F(P'_j)=\frac{p}{q}$. We then have the following result whose first part follows easily from Theorem 3.2.1, and whose second part is a standard bifurcation theory result.

Lemma 3.6.3 ([4, 20]).
(i) *With the above notation,*

$$(a,\,b)\in \mathbf{B}^l_{\frac{p}{q}}\setminus \mathbf{A}^l_{\frac{p}{q}} \iff F_{a,b}(K)=F_{a,b}(P_j)\,,$$

and

$$(a,\,b)\in \mathbf{B}^r_{\frac{p}{q}}\setminus \mathbf{A}^r_{\frac{p}{q}} \iff F_{a,b}(C)=F_{a,b}(P_k)\,.$$

(ii) *Furthermore, for* $b\neq 0$,

$$(a,\,b)\in \mathbf{A}^l_{\frac{p}{q}}\cap \mathbf{A}^r_{\frac{p}{q}} \iff \mathbf{O}_{\frac{p}{q}} \textit{ is parabolic and } \mathbf{O}'_{\frac{p}{q}} \textit{ does not exist.}$$

From the picture which emerges from the discussion so far, Theorem A would follow from the uniqueness of the intersections described in Corollary 3.5.2. The general case then follows by Theorem 3.3.1–3. We shall prove this uniqueness property in Sect. 4 using the fact that the labels of the curves that intersect determine the topological conjugacy classes of the maps at the intersections.

Lemma 3.6.4. *At any crossing of two boundary curves* $\mathbf{C}^l_{\frac{p}{q}}$ *and* $\mathbf{D}^r_{\frac{p'}{q'}}$, *(where* $\mathbf{C}$ *and* $\mathbf{D}$ *stand for either* $\mathbf{A}$ *or* $\mathbf{B}$*), the way the orbits* $\mathbf{O}_{\frac{p}{q}}$, $\mathbf{O}_{\frac{p'}{q'}}$ *and the critical points intertwine is determined by the pair* $(\frac{p}{q},\frac{p'}{q'})$. *Furthermore, the itineraries of* $\mathbf{O}_{\frac{p}{q}}$ *and* $\mathbf{O}_{\frac{p'}{q'}}$ *are determined by the pair* $(\frac{p}{q},\frac{p'}{q'})$.

Proof. Take any two standard lifts F_{a_0,b_0} and F_{a_1,b_1}, which both possess the pair of orbits $(\mathbf{O}_{\frac{p}{q}},\mathbf{O}_{\frac{p'}{q'}})$. One can find a piecewise linear lift G as in Fig. 5 that contains all periodic itineraries of both F_{a_0,b_0} and F_{a_1,b_1}. Choosing G to have all its slopes greater than one in absolute value, it is easy to check that for $\omega\in\{\frac{p}{q},\frac{p'}{q'}\}$ it possesses only one orbit which

- Is invariant by a non-decreasing lift with rotation number ω,
- Has no point in the segments where G is decreasing, and
- Is a lift of a periodic orbit of the circle map g.

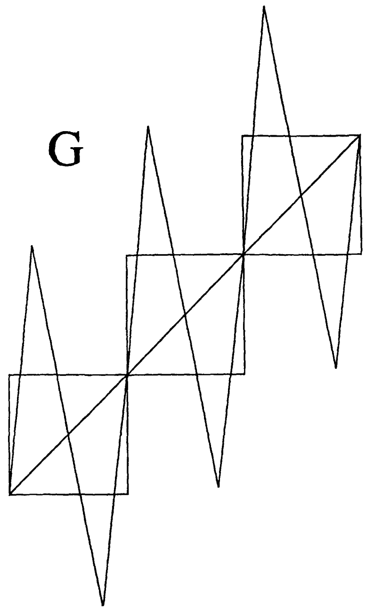

Fig. 5.

By standard kneading theory arguments we get ([1, 26]):

- The two orbits (for $\omega = \frac{p}{q}$ and $\omega = \frac{p'}{q'}$) obtained this way and the turning points of G are intertwined in the same way as the corresponding orbits and critical points of F_{a_0,b_0} and F_{a_1,b_1}
- The kneading information about these orbits can be read from G as well. Q.E.D.

Lemma 3.6.5. *The maps corresponding to all intersections of the two boundary curves* $\mathbf{C}^l_{\frac{p}{q}}$ *and* $\mathbf{D}^r_{\frac{p'}{q'}}$, (*where again* $\mathbf{C}$ *and* $\mathbf{D}$ *stand for either* $\mathbf{A}$ *or* $\mathbf{B}$) *are topologically conjugate.*

Proof. This statement is a standard result of the topological classification of maps with negative Schwarzian derivative, and we refer to ([24] Chap. 2.3) for a more general discussion; we give only a sketch of the arguments.

Using Lemmas 3.6.3 and 3.6.4, we know that all such maps have the same kneading data. Because these are smooth maps with isolated critical points, it follows that they have the same sets of itineraries. The fact that maps with negative Schwarzian derivative off the critical set have no homterval (intervals of positive length, not in the basin of a stable periodic orbit, but where all iterates of the map are homeomorphisms [19, 21]) yields the conjugacy. Points with similar itineraries are paired by the conjugacy, except for points belonging to the basins of stable or semi-stable periodic orbits. For these points, the connected components of the basins, on each side of the periodic points and their preimages, can be paired in any way that respects the orbit structure. Hence the conjugacy is not necessarily unique. Q.E.D.

4. Proof of Theorem A, Part II: Complex Analytic Part

To complete the proof of Theorem A we must show that the boundary curves described in Sect. 3.5 have unique intersection points; that is, that the conjugacy classes in Lemma 3.6.5 correspond to a single map. This is the content of Theorem D. Before we can prove this lemma, however, we need to introduce some techniques from complex analysis. References for the basic theory of complex dynamics are [16, 25]. References for Teichmüller theory are [12, 18] and references for its application to dynamical systems are [8, 15, 23, 31].

4.1. Basic Theory of Complex Dynamics. We define a point to be *normal* for a family of holomorphic functions if the functions in the family are locally uniformly bounded in a neighborhood of the point. The set of normal points is open by definition. We are interested in the normal sets of families generated by iterating a single holomorphic self-map of the punctured plane $\mathbf{C}^*$.

A *singular value* for a holomorphic map is either a *critical value* (the image of a critical point) or an *asymptotic value* (a limiting value of the image of a path tending to infinity). A map with only finitely many singular values is called a *finite type map.* The points 0 and ∞ are asymptotic values for holomorphic self-maps of $\mathbf{C}^*$ but it will be more convenient to exclude them from the singular value set.

The non-normal set divides the normal set into connected components. The normal set is forward and backward invariant and its components are mapped to one another. If a finite type map has a periodic cycle $o = \{z_0, z_1 = f(z_0), \ldots, z_{q-1} = f(z_{q-2})\}$ we define the multiplier in the same way we do for real maps, that is, $m_o = f'(z_{q-2}) \cdot f'(z_{q-2}) \cdots f'(z_0)$.

The following theorem classifies the behavior of the components of the normal set.

Classification Theorem. *Given a finite type holomorphic self map of* $\mathbf{C}^*$, *the orbits of the components of the normal set are characterized as follows*:

- *they fall onto a periodic cycle of components containing a periodic cycle with multiplier* $|m_o| < 1$ (*attracting domain if* $|m_o| \neq 0$ *or super-attracting domain if* $|m_o| = 0$);
- *they fall onto a periodic cycle with multiplier a root of unity* (*parabolic domain*);
- *orbits eventually fall into a domain on which an iterate of the map is holomorphically conjugate to an irrational rotation* (*rotation domain*).

Remark. The classification of periodic normal behavior was done by Fatou [9, 10], Siegel [30] and Herman [14]. The eventual periodicity of all normal components (often called the Non-Wandering Theorem) was proved for rational maps by Sullivan [31]. For finite type holomorphic self-maps of $\mathbf{C}^*$ the Non-Wandering Theorem was proved in [15].

Although arbitrary holomorphic self-maps of $\mathbf{C}^*$ may have normal components whose orbits fall onto a periodic cycle of domains in which points are attracted to zero or infinity (essentially parabolic domains), it was proved in [17] that finite type holomorphic self-maps of $\mathbf{C}^*$ have no essentially parabolic domains.

Remark. Each cycle of periodic components uses a singular value in the following sense: cycles of super-attracting periodic normal domains contain singular values by definition, cycles of attracting and parabolic domains each contain the infinite forward orbit of a singular value, and in fact one of the domains in the cycle contains the singular point; finally, the boundary of any rotation domain is contained in the closure of the forward orbit of some singular value. Proofs of these facts go back to Fatou. Among these facts is the statement in Lemma 3.6.2–4(c).

Definition. *The closure of the forward orbits of the singular values is called the post-singular set and is denoted by* $PS(f)$.

We shall be interested in a special subclass of finite type maps.

Definition. *A finite type map is geometrically finite if every infinite forward orbit of a singular value tends to a periodic cycle.*

It may happen that no singular value has an infinite forward orbit; such orbits are periodic or pre-periodic. These maps are trivially geometrically finite.

Standard arguments (see e.g. [8, 16, 25]) show that for geometrically finite maps

- There are no rotation domains, and
- Every infinite forward singular orbit lies in the normal set and is attracted to a (necessarily attracting or parabolic) periodic cycle.

4.2. Combinatorial Equivalence.

Definition. *A combinatorial equivalence of finite type maps is a pair of homeomorphisms* (ϕ, ψ) *such that*

$$\phi \circ f_0 = f_1 \circ \psi$$

and such that ϕ *and* ψ *are isotopic rel* $(PS(f_0))$.

Definition. *A homeomorphism* $\phi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ *is called K-quasiconformal, or K-*QC *for short, if there exists a* $K \geqq 1$ *such that the field of infinitesimally small circles is mapped almost everywhere onto a field of infinitesimally small ellipses of eccentricity bounded by* $k = \frac{K-1}{K+1}$. *A map that is K-*QC *for some K is called quasiconformal.*

Definition. *A combinatorial equivalence* (ϕ, ψ) *is K-*QC (*or just* QC *if we do not care about the constant*) *if* ϕ *and* ψ *are K-quasiconformal; it is strong if* ϕ *and* ψ *agree in a neighborhood of each super-attracting, attracting and parabolic cycle* (*and hence define a conjugacy in these neighborhoods*).

Next we prove,

Lemma 4.2.1. *A strong combinatorial equivalence of holomorphic geometrically finite maps can be isotoped* (*rel the post singular set*) *through strong combinatorial equivalences to a strong* QC *combinatorial equivalence.*

Proof. Let (ϕ, ψ) be the given strong combinatorial equivalence. Let N be the union of the neighborhoods of the super-attracting, attracting and parabolic periodic cycles of f_0 on which ϕ and ψ agree.

The first step is to isotop $\phi|_N = \psi|_N$ in N rel $(PS(f_0) \cap N) \cup \partial N$ to a quasi-conformal homeomorphism that we again call ϕ. To do this, we use the canonical local picture associated to each periodic cycle and determined by the multiplier of the cycle. For a more complete description of the local behavior see e.g. [25].

Case 1. Suppose first that p is an attracting periodic point of f_0 with multiplier λ and N_p is the component of N containing p. Then there is an integer k such that f_0^k is the first return map for N_p and a conformal homeomorphism $h: N_p \to \Delta$, where Δ is the unit disk, such that $h(0) = 0$ and $h \circ f_0^k(z) = \lambda h(z)$.

We can use the first return map f_0^k to identify points in $N_p - \{p\}$ and obtain a torus of modulus λ. The projection $N_p - \{p\} \to N_p - \{p\}/f_0^k$ is a branched covering map of the torus and the conjugation ϕ projects to this torus. Isotopy classes rel the finitely many marked points for a torus are known to contain K-QC maps for some $K > 1$, so the projection of ϕ may be isotoped rel the branch points to a K-QC map. Since the homotopy lifting property holds, (see e.g. [13]), and the projection is holomorphic, the K-QC map lifts to a K-QC map on $N_p - \{p\}$ and may be extended to $N_p \cup \partial N_p$ so that the lift is isotopic to ϕ rel $(PS(f_0) \cap N_p) \cup \partial N_p$. (If, as in our application to the standard family, there is a single branch point, and the tori have the same modulus, the isotopy class contains a conformal map but we do not use this fact.)

Case 2. If p is super-attracting the first return map is holomorphically conjugate in N_p to a map of the form $z \mapsto z^k$ on Δ, for some $k \geqq 2$. If p attracts no other singular points, we may push ϕ to Δ, isotop the map on Δ to a conformal map keeping the boundary values fixed, and pull the isotoped map back to a conformal map on N_p rel ∂N_p. If p does attract singular values the argument has to be modified somewhat to take these orbits into account and the isotopy will be only quasiconformal. In our application we have two singular values but we assume each is attracted to a *distinct* periodic orbit. Hence we omit the details for the case where a superattractive p attracts a second singular value and refer the interested reader to [22].

Case 3. It remains to describe the local behavior when p belongs to a parabolic cycle. The picture in this case is known as the Leau–Fatou flower. We make two simplifying assumptions: first that p is a non-degenerate parabolic fixed point, that is, $f_0'(p) = 1$ and $f_0''(p) \neq 0$, and second that p attracts only one singular value. Full details of the Leau–Fatou flower in the context of rational maps may be found in [25], Sect. 7. The details for geometrically finite maps may be found in [8]. A small neighborhood N_p of p is covered by a pair of overlapping attracting and repelling *petals*, U and U', such that $f_0(U) \subset (U)$ and $f_0(U') \supset U'$. If we conjugate f_0 by $w = -1/(z - p)$, p is mapped to infinity and the petals U and U' are transformed into the two overlapping regions D_R and D_L shown in Fig. 6. The conjugated map in a neighborhood of infinity takes the form

$$F: w \mapsto w + 1 + o(1)\,.$$

We see therefore that D_L contains a left half plane $\{\Re w < -M\}$ for some $M > 0$, in which F is holomorphically conjugate to right translation by 1; similarly, D_R contains a right half plane $\{\Re w > M'\}$ for some $M' > 0$ in which F^{-1} is holomorphically conjugate to left translation by 1.

The orbit of the singular value in U is transformed into the attracting region D_R and since the map acts almost as translation the imaginary parts of points in

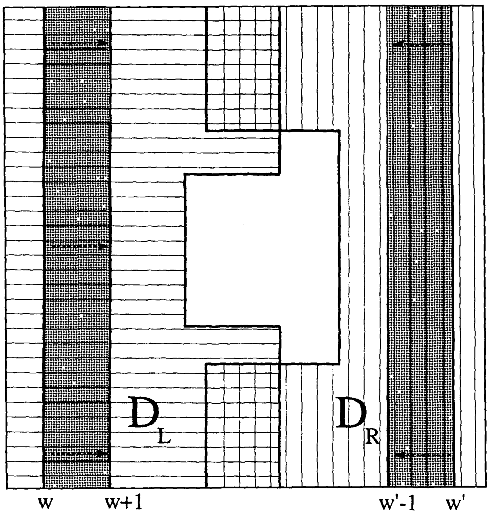

Fig. 6.

the orbit are bounded. The repelling petal U' is transformed into the domain D_L. Hence D_L contains a piece of the non-normal set and so is not invariant under the conjugated map.

For each map f_i, $i = 1, 2$ we form the *Ecalle cylinder* E_R by identifying orbits in D_R under the map F. The singular orbit projects to a single marked point. Similarly we form E_L by identifying orbits in D_L under the map F^{-1}. The conjugacy ϕ projects to the cylinders and we can find, for some $K > 1$, a K-QC map in the isotopy class of this projected ϕ. Now we lift this isotoped ϕ to the exterior of a large rectangle in the w-plane.

We thus have a quasiconformal conjugacy in a neighborhood of the parabolic point (perhaps smaller than N_p). To obtain the strong K-QC equivalence we must extend the conjugacy to a full neighborhood of the full post-singular set. Since ϕ is K-QC on N, where by definition $\phi = f_1 \circ \phi \circ f_0^{-1}$, we may lift it as a K-QC map to $f_0^{-1}(N)$. To extend this lift quasiconformally to the closures of these neighborhoods we need to know that any intersections of ∂N and $\partial f_0^{-1}(N)$ are transverse. We can assure this by modifying our original choice of N if necessary, and using the normal form for parabolic points again. Since f_0 is geometrically finite, we may lift a finite number of times to obtain a K-QC conjugacy on a neighborhood N' of the full post-singular set which is isotopic (rel $PS(f_0)$) to the original combinatorial equivalence on N' and agrees with it in the complement of N'.

To complete the proof, we isotop ϕ in the complement of N' to any globally K-QC map and set $\psi = f_1^{-1} \circ \phi \circ f_0$, where we choose the branch of the inverse to preserve the isotopy. Note that these branches are well defined since there are no singular values in this region. Q.E.D.

4.3. Application to the Standard Family. The circle maps $f_{A,b}$ have a natural extension to $\mathbf{C}^*$. To see this note that the family of lifts $F_{a,b}$ extends to $\mathbf{C}$, by the formula

$$F_{a,b}: z \mapsto z + a + (b/2\pi)\sin(2\pi z) .$$

Using the projection of $\mathbf{C}$ to $\mathbf{C}^*$ given by the exponential map, we obtain holomorphic self-maps of $\mathbf{C}^*$ that are holomorphic extensions of the family $f_{A,b}$. For readability, we keep the same notation. These maps have exactly two critical values and no asymptotic values so are of finite type.

4.4. Extending Real Conjugacies. The following lemma appears in various guises in the literature. To prove a version suited to our needs we require

Definition. *Let I be an open interval in $\mathbb{R}$ or $\mathbb{T}$. A homeomorphism $\phi: I \to I$ is called K-quasisymmetric, or K-QS, if there exists a $K > 1$ such that for every triple (a, b, c) of points in I, where $a < b < c$, ϕ satisfies*

$$\frac{1}{K} < \frac{\phi(c) - \phi(b)}{\phi(b) - \phi(a)} < K .$$

The homeomorphism $\psi: \mathbb{T} \to \mathbb{T}$ is K-quasisymmetric if its restriction to every subinterval is K-quasisymmetric.

Remark. The restriction of a K-QC homeomorphism of $\mathbf{C}^*$ is K-QS on $\mathbb{T}$ and any K-QS homeomorphism of $\mathbb{T}$ has a (not necessarily unique) K-QC extension to $\mathbf{C}^*$ (see [12, 18]).

Lemma 4.3.1. *Let g_0, g_1 be topologically conjugate maps of $\mathbb{T}$ in the family $f_{A,b}$ whose extensions f_0, f_1 to $\mathbf{C}^*$ have the property that their post-singular orbits remain in $\mathbb{T}$. Then there is a strong K-QC combinatorial equivalence (ϕ, ψ) for f_0, f_1.*

Proof. We need only show how to use the given real conjugacy Φ for g_0, g_1 to obtain a strong combinatorial equivalence for f_0, f_1 because we may then apply Lemma 4.2.1 to complete the proof.

The first step is to replace Φ by a K-QS homeomorphism which agrees with Φ on the closed post-singular set $PS(g_0) = PS(f_0)$. We can do this since $PS(g_0)$ consists of isolated points plus points accumulating at attracting or parabolic cycles. The attracting and parabolic cycles are distinguishable by their local topological behavior. Near each cycle we use the local normal form to replace Φ by a K-QS homeomorphism for some K; we then use the circle map g_0 to pull this K-QS homeomorphism back to the closures of the basins of the cycles in $\mathbb{T}$; finally, we extend by continuity to $\mathbb{T}$. Since g_0 is the restriction of a holomorphic map, the new map, which we again call Φ, is K-quasisymmetric.

The second step is to extend the K-QS map Φ to a K-QC self-map ϕ of $\mathbf{C}^*$. For each attracting or parabolic cycle P, let N_P be a neighborhood of P in $\mathbf{C}^*$ with smooth boundary. Using the local normal form again, we extend the K-QS map Φ to N_P so that it is K-QC. Extending this way for all the cycles defines a germ ϕ for a K-QC conjugacy between f_0 and f_1. Now we extend ϕ arbitrarily as a K-QC homeomorphism of $\mathbf{C}^*$.

The final step is to define a lift ψ of ϕ so that the pair (ϕ, ψ) are isotopic rel the post-singular sets and are the desired strong combinatorial equivalence. Denote the critical value set of the map f_i by S_i, $i = 0, 1$; each set consists of two points. The maps f_i are covering maps of $\mathbf{C}^* - f_i^{-1}(S_i)$ onto $\mathbf{C}^* - S_i$. Extend these covering maps to fix the "ends" zero and infinity of $\mathbf{C}^*$. Because the maps f_0, f_1 are in the same family, that is, given by a formula of the form $\alpha\zeta \exp \beta(\zeta - 1/\zeta)$ for constants α and β, and variable $\zeta \in \mathbf{C}^*$, they are built up from a sequence of elementary maps whose lifting properties are known. The lift $\psi = f_0^{-1} \circ \phi \circ f_0$ may therefore be defined uniquely so that it agrees with ϕ on any and hence all the points $PS(f_0)$. Q.E.D.

Remark. If the post-singular set is actually finite, the situation is much simpler. Every singular point is superattracting or else its orbit eventually lands on a repelling periodic cycle. We can choose an arbitrary topological extension to $\mathbf{C}^*$ as the homeomorphism ϕ and define ψ by the formula $\psi = f_1^{-1} \circ \phi \circ f_0$, where again the branch of the inverse is chosen so that (ϕ, ψ) are isotopic rel the post-singular sets. By the easy parts of Lemma 4.2.1 there are automatically quasiconformal homeomorphisms in this isotopy class.

4.5. Statement of the Rigidity Theorem

Theorem D (Rigidity). *Suppose that the functions f_0 and f_1 are both intersections of boundary curves $C^1_{\frac{p}{q}}$ and $D^r_{\frac{p'}{q'}}$, where C and D stand for either A or B as in Lemma* 3.6.5. *Then $f_0 = f_1$.*

From Lemma 3.6.3 we see that at the intersections of the boundary curves one of the following holds:

α – Both singular orbits are attracted by distinct parabolic cycles,
β – Both singular orbits are preperiodic, or
γ – One singular orbit is preperiodic and the other is attracted by a parabolic cycle.

It follows that the extensions of standard maps corresponding to these intersection points are geometrically finite.

4.6. Basic Teichmüller Theory. To prove the Rigidity Theorem D we follow the version of the proof of a rigidity result for rational maps carried out by McMullen in [23]. In particular, we shall use some standard Teichmüller theory. Below we state those facts we require in a form suited to our needs. A good basic reference for this material is [12], Chap. 6. Thurston and Sullivan were the first to apply these techniques in the context of rigidity in holomorphic dynamics.

Let X be a compact Riemann surface and let C be a closed subset of X containing at most countably many points. Then the *Teichmüller space* of X with boundary C is the set of isotopy classes of quasiconformal homeomorphisms of X rel C. We denote it by $\mathcal{T}(X, C)$. We shall be interested in $\mathcal{T}(X, C)$, where $X = \mathbf{C}^*$ and $C = PS(f)$ for f in the standard family.

The Teichmüller space is finite dimensional if X has finite genus and C is a finite point set: in our case, if $PS(f)$ is finite.

If ϕ is a quasiconformal homeomorphism of X, its *Beltrami differential* is $\mu(z) = \phi_{\bar{z}}/\phi_z$, where the derivatives are taken in the generalized sense. The infinitesimal ellipse field is determined by $\mu(z)$: the eccentricity of the ellipse at the point z is $|\mu(z)|$ and the major axis has argument arg $\mu(z)$.

The *maximal dilatation* of ϕ is

$$K_\phi(X) = \max_z(1 + ||\mu||_\infty)/(1 - ||\mu||_\infty) < \infty .$$

Given an isotopy class of quasiconformal homeomorphisms X rel C one can ask if there is a map that is *extremal*; that is, its maximal dilatation is minimal over all maps in its class.

A quasiconformal map is called a *Teichmüller map* if it is locally an affine stretch: that is, its Beltrami differential has the form $\mu = t\bar{q}/|q|$, where q is a holomorphic quadratic differential such that $||q|| = \int_X |q| < \infty$ and $|t| < 1$.

Teichmüller's Theorem. *Let $\mathcal{T}(X, C)$ be a finite dimensional Teichmüller space. Then every isotopy class contains an extremal map. Moreover, this extremal is unique and is a Teichmüller map. If $\mathcal{T}(X, C)$ is not finite dimensional, the extremal map exists but it is not necessarily unique nor is any such extremal a Teichmüller map.*

Since the post-singular set is not always finite we need to consider infinite dimensional Teichmüller spaces. To this end, we introduce the concept of boundary dilatation. Let $S = X - C$ and let R be any compact subset of S. Set $K^0_\phi(S - R) = \inf_{\psi\sim\phi}(K_\psi(S - R))$. Define the *boundary dilatation* $H(\phi)$ as the direct limit of the numbers $K^0_\phi(S - R)$ as R increases to S.

Strebel's Frame Mapping Condition. *Let ϕ be a quasiconformal homeomorphism of S to another surface and suppose $H(\phi) < K^0_\phi(S)$. Then the isotopy class of ϕ* (rel C) *contains a unique extremal map and this map is a Teichmüller map.*

4.7. Proof of Theorem D. It suffices to prove that if f_0 and f_1 are topologically conjugate maps in the standard family whose singular orbits satisfy one of the conditions $\alpha - \gamma$ of Sect. 4.5 then they are equal.

Since their extensions to $\mathbf{C}^*$ are geometrically finite, by Lemma 4.3.1 there is a strong K-QC combinatorial equivalence (ϕ, ψ) between them.

Suppose first that both singular orbits are preperiodic. Then the post-singular set is finite and any K-QC combinatorial equivalence is trivially strong. Moreover, $\mathcal{T}(\mathbf{C}^*, PS(f_0))$ is finite dimensional and by Teichmüller's theorem, there is a unique extremal map in every isotopy class; denote the extremal map in the isotopy class of ϕ and ψ by $\hat{\phi}$. Now we replace ϕ by $\hat{\phi}$ as we did in the last step of the proof of Lemma 4.2.1 and set $\hat{\phi} = f_1^{-1} \circ \hat{\phi} \circ f_0$, choosing the branch that preserves the isotopy. Since f_0 and f_1 are holomorphic the infinitesimal ellipse fields determined by $\hat{\phi}$ and $\hat{\psi}$ are the same and $\hat{\psi}$ is extremal. By uniqueness $\hat{\phi} = \hat{\psi}$; denote the extremal quasiconformal conjugacy $\hat{\phi}$ by ϕ again.

In the other two cases, there is at least one singular orbit attracted by a parabolic cycle. It is important to note that no parabolic cycle attracts more that one singular orbit. The quasiconformal homeomorphisms (ϕ, ψ) we obtained in the proof of Lemma 4.2.1 agree in a neighborhood N of the post-singular set. We need to modify this ϕ in a neighborhood of a parabolic point p containing the forward orbit of one singular value so that it satisfies the Frame Mapping Condition.

As above we conjugate f_0 to $F(w) = w + 1 + o(1)$ by sending p to infinity. We follow the argument in [8], Sect. 4.2, Lemma 78. An application of the Schwarz lemma shows that $|F'(w)|$ is uniformly close to 1 in a neighborhood of infinity. This means that for η large, the image of $F(t \pm i\eta), t \in \mathbb{R}$ is a curve that stays very close to horizontal. Hence, given any $\varepsilon > 0$, we can find M such that for $|\eta| > M, \phi$ is isotopic to a map (again called ϕ) with dilatation less than $1 + \varepsilon$. Next, using the images of the endpoints of vertical lines inside the closed large rectangle to control the images of these lines, and noting that we have arranged it so that there are no points of $PS(f)$ inside the large rectangle, we can isotop ϕ in the part of $D_L \cup D_R$ inside the rectangle so that it is quasiconformal.

This new map together with its lift in the same isotopy class gives us a combinatorial equivalence (ϕ, ψ) which is no longer strong but is still K-QC. This new ϕ satisfies Strebel's Frame Mapping Condition for ε small enough. Therefore, just as in the preperiodic case, we may replace both maps in the equivalence with the unique extremal Teichmüller map in their isotopy class and obtain a quasiconformal conjugation, denoted again by ϕ.

Finally, we complete the proof of the lemma by showing that ϕ is conformal and hence a homothety.

If ϕ is not conformal, its Beltrami differential determines a quadratic differential q on S. Since ϕ is a conjugacy, and the maps f_0 and f_1 are holomorphic, the infinitesimal ellipse fields determined by the Beltrami differential μ and the Beltrami differential $f_0^*\mu$ of $f_1^{-1} \circ \phi \circ f_0 = \phi$ are the same; that is, $f_0^*\mu = \mu$. Since $f_0^*\mu$ is again the Beltrami differential of a Teichmüller map, it has the form $f_0^*\mu = t\overline{f_0^*q}/|f_0^*q|$, where f_0^*q is the pull-back quadratic differential. Now on the one hand, the norm of the pullback differential $||f_0^*q||$ is given by $||q||$ times the degree of f_0 so since f_0 has infinite degree, $||f_0^*q||$ is unbounded. On the other hand however, $f_0^*\mu = t\overline{f_0^*q}/|f_0^*q|$, so that

$$\bar{f}_0^*q/|f_0^*q| = \bar{q}/|q| \, .$$

If $h = f_0^*q/q$, then $\bar{h} = |h|$ and h is real valued. But h is meromorphic on S and any meromorphic function taking only real values must be constant. Thus $f_0^*q = cq$ for some $c > 0$. Since $||q||$ is bounded, we have a contradiction and ϕ is conformal.

If we conjugate $f_{A,b}$ by a homothety, we obtain an equivalent dynamical system. Since the homothety preserves the unit circle, the factor must have modulus 1; its argument only appears in the sine term and does not change any of the dynamical properties. Q.E.D.

5. Concluding Remarks

In our study of the standard family we used real analytic techniques to get good control of the boundary curves in the parameter plane of regions with a given lower or upper bound on the rotation number. In order to control the intersections of these curves we needed to apply rigidity properties found in families of complex analytic maps. Previously, complex analytic techniques were used to obtain rigidity in one parameter families of maps with a single critical point. Our description of the parameter space of the standard family is still incomplete and we pose some open problems here. They do not seem amenable to the methods used so far and new ideas are needed.

Conjectures.

- $R_{\frac{p_0}{q_0}}$ is homeomorphic to $R_{\frac{p_1}{q_1}}$ by a homeomorphism $H_{\frac{p_0}{q_0},\frac{p_1}{q_1}}$ having the following property:
  If $H_{\frac{p_0}{q_0},\frac{p_1}{q_1}}(f_0) = f_1$, then f_0 partitions the circle into q_0 intervals, $I_1,\dots,I_0$, and f_1 partitions the circle into q_1 intervals, $J_1,\dots,J_{q_1}$ so that $f_0^{q_0}|_{I_j}$ is topologically conjugate to $f_1^{q_1}|_{I_k}, j \in \{1,\dots,q_0\}, k \in \{1,\dots,q_1\}$.
- The set of maps with a given topological entropy is connected.

Both conjectures are proved in [32] for some two parameter families of piecewise affine maps. A similar entropy conjecture for cubic maps is discussed in [6].

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