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Analytic Bethe Ansatz for Fundamental Representations of Yangians

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Abstract: We study the analytic Bethe ansatz in solvable vertex models associated with the Yangian $Y(X_r)$ or its quantum affine analogue $U_q(X_r^{(1)})$ for $X_r = B_r$, C_r and D_r. Eigenvalue formulas are proposed for the transfer matrices related to all the fundamental representations of $Y(X_r)$. Under the Bethe ansatz equation, we explicitly prove that they are pole-free, a crucial property in the ansatz. Conjectures are also given on higher representation cases by applying the T-system, the transfer matrix functional relations proposed recently. The eigenvalues are neatly described in terms of Yangian analogues of the semi-standard Young tableaux.

1. Introduction

1.1. General Remarks. Among many studies on solvable lattice models, the Bethe ansatz is one of the most successful and widely applied machineries. It was invented at the very dawn of the field [1] and is still providing rich insights. Meanwhile, Bethe's original idea has evolved into several versions of the Bethe ansätze called by the adjectives "thermodynamic" [2], "algebraic" [3], "analytic" [4, 5], "functional" [6] and so forth. These are all powerful techniques involving some profound aspects. We have yet to understand their full contents, a challenge raised on Feynman's "last blackboard" [7].

In this paper we step towards it by developing our recent works [8–11] further. We shall propose eigenvalue formulas for several transfer matrices in the models with the Yangian symmetry [12] or its quantum affine analogue [13–15]. An interesting interplay will thereby be exposed between the representation theory of these algebras and the analytic Bethe ansatz. Let us explain our basic setting of the problem.

1.2. Yang–Baxter Equation and Transfer Matrices. Consider the quantum affine algebra $U_q(X_r^{(1)})$ [13, 14] associated with any classical simple Lie algebra X_r of

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rank r. Throughout the paper we assume that q is generic. Let $W_m^{(a)}$ $(1 \leqq a \leqq r, m \in \mathbf{Z}_{\geqq 1})$ be the irreducible finite dimensional $U_q(X_r^{(1)})$-module as sketched in Sect. 2.1. See also [16] and [8]. For $W, W' \in \{W_m^{(a)} | 1 \leqq a \leqq r, m \in \mathbf{Z}_{\geqq 1}\}$, let $R_{W,W'}(u) \in \mathrm{End}(W \otimes W')$ denote the quantum R-matrix satisfying the Yang–Baxter equation [17]

$$R_{W,W'}(u) R_{W,W''}(u+v) R_{W',W''}(v) = R_{W',W''}(v) R_{W,W''}(u+v) R_{W,W'}(u)\,. \qquad (1.1)$$

Here, $u, v \in \mathbf{C}$ denote the spectral parameters and $R_{W,W'}(u)$ is supposed to act as identity on W'', etc. As is well known, one has a solvable vertex model on the planar square lattice by regarding the matrix elements of the R-matrix as local Boltzmann weights. For $R_{W,W'}(u)$, the vertices take dim W-states (resp. dim W'-states) on, say, horizontal (resp. vertical) edges. The row-to-row transfer matrix under the periodic boundary condition is defined by

$$T_m^{(a)}(u) = \mathrm{Tr}_{W_m^{(a)}}(R_{W_m^{(a)}, W_s^{(p)}}(u - w_1) \cdots R_{W_m^{(a)}, W_s^{(p)}}(u - w_N)) \qquad (1.2)$$

up to an overall scalar multiple. Here N is the system size, $w_1, \ldots, w_N$ are complex parameters representing the inhomogeneity, $1 \leqq a, p \leqq r$ and $m, s \in \mathbf{Z}_{\geqq 1}$. Following the QISM terminology [3], we say that (1.2) is the row-to-row transfer matrix with the *auxiliary space* $W_m^{(a)}$ that acts on the *quantum space* $(W_s^{(p)})^{\otimes N}$. (More precisely, $W_m^{(a)}(u)$ and $\otimes_{j=1}^N W_s^{(p)}(w_j)$, respectively. See Sect. 2.1.) Note that in (1.2) we have suppressed the quantum space dependence on the lhs. Thanks to the Yang–Baxter equation (1.1), the transfer matrices form a commuting family

$$[T_m^{(a)}(u), T_{m'}^{(a')}(u')] = 0\,. \qquad (1.3)$$

They can be simultaneously diagonalized and we shall write their eigenvalues as $\Lambda_m^{(a)}(u)$, which is also dependent on p and s. Our aim is to find an explicit formula for them. So far, the full answer is known only for $X_r = A_r$ [18, 19] and $X_r = C_2$ [10]. In this paper we extend the results in [20, 21] for $X_r = B_r$, C_r and D_r further by combining the two basic ingredients, the analytic Bethe ansatz [5] and the transfer matrix functional relations (T-system) [8, 9]. Our approach renders a new insight into the base structure of the module $W_m^{(a)}$ and leads to several conjectures on $\Lambda_m^{(a)}(u)$. Below we shall illustrate our idea along an exposition of the analytic Bethe ansatz (Sect. 1.3) and the T-system (Sect. 1.4) for the simplest example $X_r = sl(2)$.

1.3. Analytic Bethe Ansatz. We write $T_m(u)$ for $T_m^{(1)}(u)$, etc. since the rank of $sl(2)$ is 1. Then W_m denotes the $(m+1)$-dimensional irreducible representation of $U_q(\widehat{sl}(2))$. For simplicity, we assume that $s = 1$ in (1.2). Then $T_1(u)$ is just the 6-vertex model transfer matrix acting on the vectors labeled by length N sequences of $+$ or $-$ states. We take the local vertex Boltzmann weights as $R_u(\pm, \pm, \pm, \pm) = [2 + u]$, $R_u(\pm, \mp, \pm, \mp) = [u]$ and $R_u(\pm, \mp, \mp, \pm) = [2]$, where the local states $+$ or $-$ are ordered anti-clockwise from the left edge of the vertex. The function $[u]$ is defined by

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}\,.$$

The eigenvalue $\Lambda_1(u)$ is well known and given by

$$\Lambda_1(u) = \frac{Q(u-1)}{Q(u+1)}\phi(u+2) + \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.4a}$$

$$Q(u) = \prod_{j=1}^{n} \left[u - iu_j\right], \quad \phi(u) = \prod_{j=1}^{N} \left[u - w_j\right] . \tag{1.4b}$$

Here, $0 \leqq n \leqq N/2$ is the number of the $-$states in the eigenvector, which is preserved under the action of $T_1(u). u_j \in \mathbf{C}$ are any solution of the Bethe ansatz equation (BAE)

$$-\frac{\phi(iu_k+1)}{\phi(iu_k-1)} = \frac{Q(iu_k+2)}{Q(iu_k-2)} . \tag{1.5}$$

On the result (1.4–5), one makes a few observations.

(i) The eigenvalue has the "dressed vacuum form (DVF)," which means the following. The "vacuum vector" $+, +, \ldots, +$ is the obvious eigenvector with the vacuum eigenvalue

$$\prod_{j=1}^{N} R_{u-w_j}(+,+,+,+) + \prod_{j=1}^{N} R_{u-w_j}(-,+,-,+) = \phi(u+2) + \phi(u) . \tag{1.6}$$

Equation (1.4) tells that general eigenvalues have a modified form of this with the "dress" factors Q/Q which are certainly 1 when $n = 0$. In particular, the number of the terms in $\Lambda_1(u)$ is the dimension of the auxiliary space $\dim W_1 = 2$.

(ii) The BAE (1.5) ensures that the eigenvalues are free of poles for finite u. The apparent pole at $u = iu_k - 1$ in (1.4a) is spurious as the residues from the two terms cancel due to (1.5). The eigenvalues must actually be pole-free because the local Boltzmann weight, hence the matrix elements of $T_1(u)$ are so.

(iii) Properties inherited from the $|u| \to \infty$ behavior and the first/second inversion relations of the R-matrix (vertex Boltzmann weights). For example, one has $\Lambda_1(u) = (-)^N \Lambda_1(-2-u)|_{w_j \to -w_j,\, u_j \to -u_j}$ from the last property. See also the remark after (2.12).

The analytic Bethe ansatz is the hypothesis that the postulates (i)–(iii) essentially determine a function of u uniquely and that the so obtained is the actual transfer matrix eigenvalue. As the input data, it only uses the BAE and the R-matrix (or the vacuum eigenvalue (1.6)) which should be normalized to be an entire function of u. It was formulated in [5] by extracting the idea from Baxter's solution of the 8-vertex model [4]. See [10, 11, 20, 21] for other applications. In Sect. 2.4, we will introduce a few more conditions than (i)–(iii) above.

1.4. Transfer Matrix Functional Relations. The transfer matrix (1.2) obeys various functional relations. For $X_r = sl(2)$ and $s = 1$ in (1.2), it is known that [18, 22]

$$T_m(u+1)T_m(u-1) = T_{m+1}(u)T_{m-1}(u) + g_m(u)\mathrm{Id} ,$$

$$g_m(u) = \prod_{k=0}^{m-1} \phi(u+2k-m)\phi(u+4+2k-m) , \tag{1.7}$$

where $m \geqq 0$ and $T_0(u) = \mathrm{Id}$. Since $T_m(u)$'s can be simultaneously diagonalized, (1.7) may be regarded as an equation for the eigenvalues $\Lambda_m(u)$. By using (1.4a) and $\Lambda_0(u) = 1$ as the initial condition, it is easy to solve the recursion (1.7) to find

$$\Lambda_m(u) = \left(\prod_{k=1}^{m-1} \phi(u+m+1-2k)\right) \sum_{j=0}^{m} \frac{Q(u-m)Q(u+m+2)\phi(u+m+1-2j)}{Q(u+m-2j)Q(u+m+2-2j)}, \tag{1.8}$$

in agreement with [18]. To observe a representation theoretical content, we now set

$$\boxed{1} = \frac{Q(u-1)}{Q(u+1)}\phi(u+2), \quad \boxed{2} = \frac{Q(u+3)}{Q(u+1)}\phi(u), \tag{1.9}$$

where we assume on the lhs that the spectral parameter u is implicitly attached to the single box as well. In this notation (1.4a) reads as $\Lambda_1(u) = \boxed{1} + \boxed{2}$. Moreover, the result (1.8) for general m can be expressed as follows.

$$\Lambda_m(u) = \sum_{j=0}^{m} \overbrace{\boxed{1}\ \cdots\ \boxed{1}}^{m-j}\overbrace{\boxed{2}\ \cdots\ \boxed{2}}^{j}. \tag{1.10}$$

Here we interpret the tableau as the product of the m functions (1.9) with the spectral parameter u shifted to $u-m+1, u-m+3, \ldots, u+m-1$ from the left to the right. Notice that the tableaux appearing in (1.10) are exactly the semi-standard ones that label the weight vectors in the $(m+1)$-dimensional irreducible representation W_m of $U_q(\widehat{sl}(2))$ (plainly, the spin $\frac{m}{2}$ representation of $sl(2)$). In this sense the eigenvalues $\Lambda_m(u)$ are analogues ("Yang–Baxterizations") of the characters of the auxiliary space W_m, which may be natural from (1.2). The functional relation (1.7) for $\Lambda_m(u)$ thereby plays the role of a character identity.

1.5. General X_r Case. Having seen the $sl(2)$ example, an immediate question then would be, how the "tableau construction" of the eigenvalues as (1.10) can be generalized to the other algebra cases. For $X_r = A_r$, the $U_q(A_r^{(1)})$-module $W_m^{(a)}$ (the auxiliary space) is a q-analogue of the $sl(r+1)$-module corresponding to the $a \times m$ rectangular Young diagram representation. The eigenvalue $\Lambda_m^{(a)}(u)$ for the corresponding RSOS model [23] has been constructed [19] as in (1.10) from the set of the usual semi-standard tableaux labeling the weight vectors.

An interesting feature emerges for $X_r \neq A_r$, where $U_q(X_r^{(1)})$-module $W_m^{(a)}$ is a q-analogue of a *reducible* X_r-module in general. Evaluation of $\Lambda_m^{(a)}(u)$ amounts to finding the tableau-like objects that label the base of such $W_m^{(a)}$. This can actually be done by postulating the *T-system*, the transfer matrix functional relations, proposed in [8]. It is a generalization of (1.7) into the arbitrary X_r case and can be solved for $\Lambda_m^{(a)}(u)$ in terms of $\Lambda_1^{(a)}(u + \text{shift})(1 \leqq a \leqq r)$(and $\Lambda_0^{(a)}(u) = 1$). Thus one can play the following game.

Step 1. Find $\Lambda_1^{(1)}(u), \ldots, \Lambda_1^{(r)}(u)$ by the analytic Bethe ansatz.

Step 2. Find such "tableaux" that the *Step* 1 result is expressed in an analogous manner to (1.10).

Step 3. Solve the T-system for $\Lambda_m^{(a)}(u)$ recursively by taking the *Step* 1,2 results as the initial condition.

We shall completely execute *Step* 1 and 2 for $X_r = B_r$, C_r and D_r and achieve *Step* 3 in several cases. The resulting tableau label for the base of $W_m^{(a)}$ exhibits an interesting contrast with those for the crystal base [24,25] concerning the irreducible X_r-modules.

1.6. Summary of Main Results. Let us briefly sketch our main results concerning *Step* 1 and 2. We introduce the boxes containing a letter from the set

$$J = \begin{cases} \{1,2,\ldots,r,\bar{r},\ldots,\bar{2},\bar{1}\} & \text{for } C_r \text{ and } D_r\,, \\ \{1,2,\ldots,r,0,\bar{r},\ldots,\bar{1}\} & \text{for } B_r\,. \end{cases}$$

J is equipped with an order $\prec$ as specified in (3.5), (4.5) and (5.7) for C_r, B_r and D_r, respectively. As in (1.9), the boxes represent the dressed vacuum in (3.4), (4.4) and (5.6) and depend on the spectral parameter u. We find that $\Lambda_1^{(a)}(u)$ for non-spin representations ($1 \leqq a \leqq r$ for $C_r, 1 \leqq a \leqq r-1$ for $B_r, 1 \leqq a \leqq r-2$ for D_r) is expressed as

$$\Lambda_1^{(a)}(u) = (\text{scalar}) \sum \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_a \\ \hline \end{array}\;,$$

where the tableau means the product of the boxes with the spectral parameter shifts from the top to the bottom; $u + \frac{a-1}{2}, u + \frac{a-3}{2}, \ldots, u - \frac{a-1}{2}$ for C_r and $u + a - 1, u + a - 3, \ldots, u - a + 1$ for B_r, D_r. The sum extends over the tableaux obeying the rules:

$$\begin{aligned} &C_r : 1 \preceq i_1 \prec i_2 \prec \cdots \prec i_a \preceq \bar{1}, \quad \text{if } i_k = c \text{ and } i_l = \bar{c}, \text{ then } r + k - l \geqq c\,, \\ &B_r : i_k \prec i_{k+1} \text{ or } i_k = i_{k+1} = 0 \quad \text{for any } 1 \leqq k \leqq a-1\,, \\ &D_r : i_k \prec i_{k+1} \text{ or } (i_k, i_{k+1}) = (r, \bar{r}) \text{ or } (i_k, i_{k+1}) = (\bar{r}, r) \quad \text{for any } 1 \leqq k \leqq a-1\,. \end{aligned}$$

For spin representations ($a = r$ for B_r and $a = r-1, r$ for D_r), we find it convenient to introduce another kind of boxes containing a $\pm$ sequence of length r:

$$\Lambda_1^{(a)}(u) = \sum_{\{\mu_j = \pm\}} \boxed{\mu_1, \mu_2, \ldots, \mu_r}\,.$$

They are defined by recursion relations with respect to r as in (4.25) and (5.13). Under the BAE (2.7) the pole-freeness of these DVFs is explicitly proved in the main text. Furthermore, several conjectures on $\Lambda_m^{(a)}(u)$ are given in Sects. 3.4, 4.4 and 5.3 in terms of the semi-standard-like tableaux as above.

These results extend earlier ones in [5,10,11,19–21]. The DVFs for $\Lambda_m^{(a)}(u)$ are Yang–Baxterizations of the characters of the auxiliary spaces $W_m^{(a)}$. Our tableaux are natural objects that label the base of the irreducible finite dimensional modules $W_m^{(a)}$ over the Yangians or the quantum affine algebras.

1.7. Plan of the Paper. In the next section, we begin by fixing our notations and recall the family of the modules $W_m^{(a)}$, the T-system [8] and the BAE [26,21] for models with $U_q(X_r^{(1)})$ symmetry. The Yangian case $Y(X_r)$ corresponds to a smooth rational limit $q \to 1$ of them. Then we discuss the analytic Bethe ansatz and propose a few more hypotheses, "dress universality," "top term" and "coupling rule." They supplement (i)–(iii) in Sect. 1.3 and work efficiently for models with general $U_q(X_r^{(1)})$ symmetry. Sections 3,4 and 5 are devoted to the cases $X_r = C_r, B_r$ and

D_r, respectively. A peculiarity for the latter two algebras is the presence of the spin representations, whose $U_q(X_r^{(1)})$-analogues are certainly in question. ($W_1^{(r)}$ for B_r and $W_1^{(r-1)}, W_1^{(r)}$ for D_r.) For these algebras, we introduce two kinds of elementary boxes corresponding to the bases of the vector and the spin representations. Appendices A and B describe their relation, which reflects the fact that the former representation is contained in a tensor product of the latter. Section 6 is devoted to discussions.

2. *T*-System, BAE and Analytic Bethe Ansatz

2.1. Modules $W_m^{(a)}$. Let us fix our notations for the data from the simple Lie algebras X_r. Let $\alpha_a, \omega_a (1 \leqq a \leqq r)$ and $(\,\cdot\,|\,\cdot\,)$ denote the simple roots, the fundamental weights and the invariant bilinear form on X_r. We identify the Cartan subalgebra and its dual via $(\,\cdot\,|\,\cdot\,)$ and normalize it as $(\alpha|\alpha) = 2$ for $\alpha =$ long root. Put

$$t_a = \frac{2}{(\alpha_a|\alpha_a)} \quad 1 \leqq a \leqq r, \quad g = \text{dual Coxeter number of } X_r\,. \tag{2.1}$$

By the definition $t_a = 1, 2$ or 3 and $(\omega_a|\alpha_b) = \delta_{ab}/t_a$. Enumeration of the nodes $1 \leqq a \leqq r$ on the Dynkin diagram is the same as Table I in [8]. For $X_r = B_r (r \geqq 2), C_r (r \geqq 2)$ and $D_r (r \geqq 4)$, (2.1) reads explicitly as

$$\begin{aligned} &g = 2r-1,\ t_1 = \cdots = t_{r-1} = 1, t_r = 2 \quad \text{for } B_r\,,\\ &g = r+1,\ t_1 = \cdots = t_{r-1} = 2, t_r = 1 \quad \text{for } C_r\,,\\ &g = 2r-2,\ \forall t_a = 1 \quad \text{for } D_r\,, \end{aligned} \tag{2.2}$$

Now we recall the family of modules $\{W_m^{(a)} | 1 \leqq a \leqq r, m \in \mathbf{Z}_{\geqq 1}\}$ first introduced in [16] for the Yangian $Y(X_r)$ extending the earlier examples [26]. Precisely speaking, Yangian modules carry a spectral parameter hence the auxiliary and the quantum spaces in (1.2) are to be understood as $W_m^{(a)}(u)$ and $\otimes_{j=1}^N W_s^{(p)}(w_j)$, respectively. See [27, 28] and Sect. 3.2 in [8]. Then $W_m^{(a)}(u)$ has a characterization by the Drinfel'd polynomials [27, 28] $\{P_a(v) | 1 \leqq a \leqq r\}$ as

$$P_b(v) = \begin{cases} \left(v - u + \frac{m-2}{t_a}\right)\left(v - u + \frac{m-4}{t_a}\right) \cdots \left(v - u - \frac{m}{t_a}\right) & \text{for } b = a \\ 1 & \text{otherwise} \end{cases}. \tag{2.3}$$

In [28], $W_1^{(a)}(u) (1 \leqq a \leqq r)$ is called the *fundamental representation* of $Y(X_r)$. Viewed as a module over $X_r \subset Y(X_r)$, $W_m^{(a)}(u)$ is reducible in general but the contained irreducible components are independent of u. Thus we let simply $W_m^{(a)}$ denote the X_r-module so obtained. Then it is known that [16]

C_r;

$$W_m^{(a)} \simeq \begin{cases} \oplus V(k_1\omega_1 + \cdots + k_a\omega_a) & 1 \leqq a \leqq r-1 \\ V(m\omega_r) & a = r \end{cases}, \tag{2.4a}$$

B_r and D_r;

$$W_m^{(a)} \simeq \oplus V\left(k_{a_0}\omega_{a_0} + k_{a_0+2}\omega_{a_0+2} + \cdots + k_a\omega_a\right) \quad 1 \leqq a \leqq r'\,, \tag{2.4b}$$

$$r' = \begin{cases} r & \text{for } B_r \\ r-2 & \text{for } D_r \end{cases}, \qquad a_0 \equiv a \bmod 2, \quad a_0 = 0 \text{ or } 1, \tag{2.4c}$$

$$W_m^{(a)} \simeq V(m\omega_a) \quad a = r-1, r \quad \text{only for } D_r\,. \tag{2.4d}$$

Here $\omega_0 = 0$ and $V(\lambda)$ denotes the irreducible X_r-module with highest weight λ. The sum in (2.4a) is taken over non-negative integers $k_1, \dots, k_a$ that satisfy $k_1 + \cdots + k_a \leqq m, k_j \equiv m\delta_{ja}$ mod 2 for all $1 \leqq j \leqq a$. The sum in (2.4b) extends over non-negative integers $k_{a_0}, k_{a_0+2}, \dots, k_a$ obeying the constraint $t_a(k_{a_0} + k_{a_0+2} + \cdots + k_{a-2}) + k_a = m$. If one depicts the highest weights in the sum (2.4a) and (resp. (2.4b)) by Young diagrams as usual, they correspond to those obtained from the $a \times m$ rectangular one by successively removing 1×2 and (resp. 2×1) pieces.

As mentioned in Sect. 3.2 of [8], we assume in this paper that there exists a natural q-analogue of these modules over the quantum affine algebra $U_q(X_r^{(1)})$, which will also be denoted by $W_m^{(a)}$. When referring it as an X_r-module, it means that the corresponding $Y(X_r)$-module in the $q \longrightarrow 1$ limit has been regarded so.

2.2. T-system. Consider the transfer matrix (1.2) acting on the quantum space $\otimes_{j=1}^N W_s^{(p)}(w_j)$. We shall reserve the letters p and s for this meaning throughout the paper. (See also the end of Sect. 2.4.) In [8], a set of functional relations, the T-system, was conjectured for $U_q(X_r^{(1)})$ symmetry models for any X_r. For $X_r = B_r, C_r$ and D_r they read as follows:

B_r :

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u)\,,$$
$$1 \leqq a \leqq r-2\,,$$

$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) + g_m^{(r-1)}(u)T_m^{(r-2)}(u)T_{2m}^{(r)}(u)\,,$$

$$T_{2m}^{(r)}\left(u-\frac{1}{2}\right)T_{2m}^{(r)}\left(u+\frac{1}{2}\right) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) + g_{2m}^{(r)}(u)T_m^{(r-1)}\left(u-\frac{1}{2}\right)T_m^{(r-1)}\left(u+\frac{1}{2}\right),$$

$$T_{2m+1}^{(r)}\left(u-\frac{1}{2}\right)T_{2m+1}^{(r)}\left(u+\frac{1}{2}\right) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) + g_{2m+1}^{(r)}(u)T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u)\,. \tag{2.5a}$$

C_r :

$$T_m^{(a)}\left(u-\frac{1}{2}\right)T_m^{(a)}\left(u+\frac{1}{2}\right) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u)T_m^{(a-1)}(u)T_m^{(a+1)}(u)\,,$$
$$1 \leqq a \leqq r-2\,,$$

$$T_{2m}^{(r-1)}\left(u-\frac{1}{2}\right)T_{2m}^{(r-1)}\left(u+\frac{1}{2}\right) = T_{2m+1}^{(r-1)}(u)T_{2m-1}^{(r-1)}(u) + g_{2m}^{(r-1)}(u)T_{2m}^{(r-2)}(u)T_m^{(r)}\left(u-\frac{1}{2}\right)T_m^{(r)}\left(u+\frac{1}{2}\right),$$

$$T^{(r-1)}_{2m+1}\left(u-\frac{1}{2}\right)T^{(r-1)}_{2m+1}\left(u+\frac{1}{2}\right) = T^{(r-1)}_{2m+2}(u)T^{(r-1)}_{2m}(u)$$

$$+ g^{(r-1)}_{2m+1}(u)T^{(r-2)}_{2m+1}(u)T^{(r)}_{m}(u)T^{(r)}_{m+1}(u)\,,$$

$$T^{(r)}_m(u-1)T^{(r)}_m(u+1) = T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u) + g^{(r)}_m(u)T^{(r-1)}_{2m}(u)\,. \qquad (2.5b)$$

D_r :

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)T^{(a-1)}_m(u)T^{(a+1)}_m(u)\,,$$

$$1 \leqq a \leqq r-3\,,$$

$$T^{(r-2)}_m(u-1)T^{(r-2)}_m(u+1) = T^{(r-2)}_{m+1}(u)T^{(r-2)}_{m-1}(u)$$

$$+ g^{(r-2)}_m(u)T^{(r-3)}_m(u)T^{(r-1)}_m(u)T^{(r)}_m(u)\,,$$

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)T^{(r-2)}_m(u) \quad a = r-1, r\,. \qquad (2.5c)$$

Here the subscripts of the transfer matrices are non-negative and $T^{(0)}_m(u) = T^{(a)}_0(u) \equiv \mathrm{Id}$. $g^{(a)}_m(u)$ is a scalar function that depends on $W^{(p)}_s$ and satisfies

$$g^{(a)}_m\left(u-\frac{1}{t_a}\right)g^{(a)}_m\left(u+\frac{1}{t_a}\right) = g^{(a)}_{m+1}(u)g^{(a)}_{m-1}(u)\,. \qquad (2.6)$$

See Eq. (3.18) in [8]. We have slightly changed the convention from [8] so that $T^{(a)}_m(u+j)$ there corresponds to $T^{(a)}_m(u+2j)$ here, etc. A wealth of consistency for the T-system have been observed in [8, 9, 10, 11] for any X_r and we shall assume (2.5) henceforth. Owing to the commutativity (1.3), one can regard (2.5) as the functional relations on the eigenvalues $\Lambda^{(a)}_m(u)$. ($\Lambda^{(0)}_m(u) = \Lambda^{(a)}_0(u) = 1$.) Then it can be recursively solved for $\Lambda^{(a)}_m(u)$ in terms of $\Lambda^{(1)}_1(u+\text{shift}), \dots, \Lambda^{(r)}_1(u+\text{shift})$. In fact, $\Lambda^{(a)}_m(u)$ will be obtainable within a *polynomial* in these functions as argued in [8]. This process corresponds to *Step* 3 mentioned in Sect. 1.5.

2.3. Bethe Ansatz Equation. As in (1.4), the eigenvalues $\Lambda^{(a)}_m(u)$ will be expressed by the solutions to the BAE [26, 21]:

$$-\frac{\phi(iu^{(a)}_k + \frac{s}{t_p}\delta_{ap})}{\phi(iu^{(a)}_k - \frac{s}{t_p}\delta_{ap})} = \prod_{b=1}^{r}\frac{Q_b(iu^{(a)}_k + (\alpha_a|\alpha_b))}{Q_b(iu^{(a)}_k - (\alpha_a|\alpha_b))}\,, \qquad (2.7)$$

where s and p are the labels of the quantum space $\otimes_{j=1}^{N} W^{(p)}_s(w_j)$, $\phi(u)$ is given in (1.4b) and $Q_a(u)$ is defined by

$$Q_a(u) = \prod_{j=1}^{N_a}[u - iu^{(a)}_j] \;\; 1 \leqq a \leqq r\,. \qquad (2.8)$$

Here N_a is a non-negative integer analogous to n in (1.4b). The system size N in $\phi(u)$ and N_a are to be taken so that $\omega^{(p)}_s \overset{\mathrm{def}}{=} Ns\omega_p - \sum_{a=1}^{r} N_a\alpha_a \in \sum_{a=1}^{r}\mathbf{R}_{\geqq 0}\omega_a$. In

Sect. 5, we will consider a slightly modified version of (2.7) that suits the diagram automorphism symmetry in $X_r = D_r$.

2.4. Empirical Rules in Analytic Bethe Ansatz. As in (1.4a), the functions $Q_a(u)$ and $\phi(u)$ are the constituents of the dress and the vacuum parts in the analytic Bethe ansatz, respectively. In handling the formulas as (1.9), we find it convenient to specify these parts as dr(tableau) and vac(tableau). For example,

$$\boxed{1} = dr\boxed{1}\,vac\boxed{1}\,, \quad dr\boxed{1} = \frac{Q(u-1)}{Q(u+1)}, \quad vac\boxed{1} = \phi(u+2)\,. \tag{2.9a}$$

In general the DVF reads

$$\Lambda_m^{(a)}(u) = \sum \frac{Q_{a_1}(u+x_1)\cdots Q_{a_n}(u+x_n)}{Q_{a_1}(u+y_1)\cdots Q_{a_n}(u+y_n)}\phi(u+z_1)\cdots\phi(u+z_k)\,, \tag{2.9b}$$

in which ratios of Q_a's are the dress parts and products of ϕ's are the vacuum parts. Using these notations we now introduce three hypotheses, "dress universality," "top term" and "coupling rule" in the analytic Bethe ansatz. They are the properties of mathematical interest rendering valuable insights into the auxiliary space $W_m^{(a)}$ as the $U_q(X_r^{(1)})$ or the Yangian modules. Roughly speaking, the latter two are the information on the "highest weight vector" and the "action" of the Chevalley-like generators. The hypotheses have been confirmed in several examples and we believe they should rightly be added to the postulates (i)–(iii) explained in Sect. 1.3.

Dress universality. Let $T_m^{(a)}(u)$ and $T_m'^{(a)}(u)$ be the transfer matrices with the same auxiliary space $W_m^{(a)}(u)$ but acting on the different quantum spaces $\otimes_{k=1}^{N} W_s^{(p)}(w_k)$ and $\otimes_{k=1}^{N'} W_{s'}^{(p')}(w_k')$, respectively. Denote by $Q_a(u)$ and $Q_a'(u)$ the functions (2.8) specified from the solutions to the BAE (2.7) for these quantum space choices. Suppose one got their eigenvalues in the DVFs,

$$\Lambda_m^{(a)}(u) = \sum_{j=1}^{\dim W_m^{(a)}} \mathrm{tab}_j, \quad \Lambda_m'^{(a)}(u) = \sum_{j=1}^{\dim W_m^{(a)}} \mathrm{tab}_j'\,, \tag{2.10}$$

where tab_j and tab_j' denote the terms whose vacuum parts correspond to the same (i.e., "j-th") vector from $W_m^{(a)}$ in the trace (1.2). Then the dress universality is stated as

$$dr(\mathrm{tab}_j) = dr(\mathrm{tab}_j')|_{Q_a'(u)\to Q_a(u)} \quad \text{for all } j\,. \tag{2.11}$$

Namely, the dress part is independent of the quantum space choice if it is expressed in terms of $Q_a(u)$. On the contrary, one has $vac(\mathrm{tab}_j) \neq vac(\mathrm{tab}_j')|_{N'\to N, w_k'\to w_k}$ in general if $(p', s') \neq (p, s)$.

Top term. Among the $\dim W_m^{(a)}$ terms in (2.10), let tab_1 denote the one corresponding to the "highest weight vector" in $W_m^{(a)}$. By this we mean more precisely the unique vector of weight $m\omega_a$ when $W_m^{(a)}$ is regarded as an X_r-module in the sense of Sect. 2.1. Plainly, tab_1 is the analogue of the first term on the rhs of (1.4a). Then the top term hypothesis reads

$$dr(\mathrm{tab}_1) = \frac{Q_a\left(u - \frac{m}{t_a}\right)}{Q_a\left(u + \frac{m}{t_a}\right)} \tag{2.12}$$

in (2.10), which is certainly consistent to the dress universality. It follows from (2.12) that

$$\Lambda_m^{(a)}(u)\,(\Lambda_m^{(a)}(-u)|_{w_j\to -w_j, u_k^{(b)}\to -u_k^{(b)}}) = \Phi(u)\,(\Phi(-u)|_{w_j\to -w_j}) + \cdots ,$$

where $\Phi(u) = vac(\mathrm{tab}_1)$ is a product of ϕ's. This is essentially Eq. (5) in [21], which is a consequence of the first inversion relation of the relevant R-matrix.

Coupling rule. Regard the auxiliary space $W_m^{(a)}$ as an X_r-module in the sense of Sect. 2.1 and let λ be a weight without multiplicity

$$\mathrm{mult}_\lambda\, W_m^{(a)} = 1\,. \tag{2.13}$$

Then it makes sense to denote by $\boxed{\lambda}$ the term in (2.10) corresponding to the λ-weight vector from $W_m^{(a)}$. Thus $\Lambda_m^{(a)}(u) = \cdots + \boxed{\lambda} + \cdots$. Now the coupling rule is stated as follows.

If λ and μ are multiplicity-free weights such that $\lambda - \mu = \alpha_c$, then

(a) $\boxed{\lambda}$ and $\boxed{\mu}$ share common poles of the form $1/Q_c(u+\xi)$ for a certain ξ depending on λ and c. (2.14a)

(b) The BAE (2.7) guarantees $Res_{u=-\xi+\imath u_k^{(c)}}\left(\boxed{\lambda}+\boxed{\mu}\right) = 0$ in such a way that

$$\frac{dr\boxed{\mu}}{dr\boxed{\lambda}} = \prod_{b=1}^{r} \frac{Q_b(u+\xi+(\alpha_c|\alpha_b))}{Q_b(u+\xi-(\alpha_c|\alpha_b))}\,. \tag{2.14b}$$

The hypothesis tells that for $\lambda - \mu = \alpha_c$, spurious "poles of color c" in $\boxed{\lambda}$ and $\boxed{\mu}$ couple into a pair yielding zero residue in total. To determine ξ is a non-trivial task in general. From (2.14b), (2.7) and $\boxed{\lambda} = dr\boxed{\lambda}\,vac\boxed{\lambda}$ etc, one deduces

$$\frac{vac\boxed{\lambda}}{vac\boxed{\mu}} = \frac{\phi\left(u+\xi+\frac{s}{t_p}\delta_{cp}\right)}{\phi\left(u+\xi-\frac{s}{t_p}\delta_{cp}\right)} \tag{2.15}$$

for the vacuum parts. The last equation in (2.14b) excludes the possibility to exchange λ and μ in (2.14b) and (2.15) simultaneously, in which case the BAE could also have ensured the pole-freeness. The coupling rule is certainly valid in (1.8) and (1.9) for $sl(2)$. We will visualize (2.14) and (2.15) as

$$\boxed{\lambda}\xrightarrow{c}\boxed{\nu}\,,$$

where c signifies the color of the pole shared by the two boxes.

There are two more postulates that embody the asymptotics and the second inversion properties mentioned in (iii) in Sect. 1.3. The first one is stated as

Character limit. As said in the end of Sect. 1.4, the eigenvalue $\Lambda_m^{(a)}(u)$ is a Yang–Baxterization of the character of the auxiliary space $W_m^{(a)}(u)$ viewed as an X_r-module. Indeed, the latter can be recovered from the former as

$$\lim_{u\to\sigma_1\infty,(|q|^{\sigma_2}>1)} q^{\tau(\sigma_1,\sigma_2)}\Lambda_m^{(a)}(u) = \sum_\lambda (\mathrm{mult}_\lambda\, W_m^{(a)})\, q^{2\sigma_1\sigma_2(\omega_s^{(p)}|\lambda)} \quad \sigma_1,\sigma_2 = \pm 1\,, \tag{2.16}$$

where the sum extends over all the weights in $W_m^{(a)}$, $q^{\tau(\sigma_1,\sigma_2)}$ is some convergence factor and $\omega_s^{(p)}$ has been specified after (2.8). One readily sees that (2.16) is consistent with (2.14b) and (2.15) by computing the asymptotics of $\boxed{\lambda}/\boxed{\mu}$. Equation (2.16) is also asserting that DVFs always contain Q_a via the combination $Q_a(u+\cdots)/Q_a(u+\cdots)$ as in (2.9b) and that they are homogeneous polynomials w.r.t $\phi(u+\cdots)$. Thus k is common in all the terms in (2.9b). In [8,9,29], the rhs of (2.16) was denoted by $Q_m^{(a)}(\omega_s^{(p)})$. It obeys the Q-system, the character identity in [16], which was extensively used to formulate the conjectures on dilogarithm identity [29,30,8,9], q-series formula for an $X_r^{(1)}$ string function [31] and to find the T-system [8]. The limit (2.16) is essentially Eq. (12) in [21]. Now we state the second postulate.

Crossing symmetry. Most R-matrices enjoy the so-called crossing symmetry, Eq. (4) in [21], from which the second inversion relation follows. The eigenvalue $\Lambda_m^{(a)}(u)$ inherits the following property from it:

$$\Lambda_m^{(a)}(u) = (-)^{kN}\, \Lambda_m^{(a)}(-g-u)\big|_{w_j \to -w_j, u_i^{(b)} \to -u_i^{(b)}} \,. \tag{2.17}$$

Here g is defined in (2.1), k is the order of the DVF w.r.t. ϕ as in (2.9b) and N is the number of lattice sites entering ϕ via (1.4b). This is essentially Eq. (6) in [21], which we call the crossing symmetry as well. Note that the BAE (2.7) remains unchanged under the simultaneous replacement $w_j \to -w_j$ and $u_k^{(b)} \to -u_k^{(b)}$. In particular, if $\pm\lambda$ are multiplicity-free weights of $W_m^{(a)}$, the combination $\boxed{\lambda}+\boxed{-\lambda}$ in $\Lambda_m^{(a)}(u)$ becomes the same on both sides of (2.17) as

$$\boxed{-\lambda} = (-)^{kN}\boxed{\lambda}\big|_{u \to -g-u, w_j \to -w_j, u_k^{(b)} \to -u_k^{(b)}} \,. \tag{2.18}$$

From the definitions of $\phi(u)$ (1.4b) and $Q_a(u)$ (2.8), the rhs of (2.17) is then obtained from (2.9b) by the simultaneous replacements

$$x_l \to g - x_l,\ y_l \to g - y_i,\ z_i \to g - z_i \,. \tag{2.19}$$

The dress universality, top term, coupling rule, character limit and crossing symmetry severely limit the possible form of the DVF in the analytic Bethe ansatz. In particular if all the weights in $W_m^{(a)}$ are multiplicity-free, (2.12), (2.14) and (2.15) determine the DVF for $\Lambda_m^{(a)}(u)$ completely up to an overall scalar multiple. In such cases, one even does not need the vacuum parts a priori hence can avoid a tedious computation of the R-matrices. The DVFs given in the subsequent sections have actually been derived in that manner for such cases. Except for a few cases, it is yet to be verified if those DVFs with $\forall Q_a(u) = 1$ yield the actual vacuum eigenvalues obtainable from the relevant R-matrix as in (1.6). In a sense we have partially absorbed the postulate (i) of Sect. 1.3 into (2.11)–(2.15) here, which may be viewed as a modification of the analytic Bethe ansatz itself.

Let us include a remark before closing this section. Suppose one has found the DVF when the quantum space is $\otimes_{j=1}^N W_1^{(p)}(w_j)$. Then, the one for $\otimes_{j=1}^N W_s^{(p)}(w_j)$ can be deduced from it by the replacement

$$\phi(u) \to \phi_s(u) \overset{\mathrm{def}}{=} \prod_{k=1}^{s} \phi\left(u + \frac{s+1-2k}{t_p}\right) . \tag{2.20}$$

To see this one just notes that the lhs of (2.7) is equal to $-\frac{\phi_s(iu_k^{(a)}+\delta_{ap}/t_p)}{\phi_s(iu_k^{(a)}-\delta_{ap}/t_p)}$. See also (2.15). Thus we shall exclusively consider the $s = 1$ case with no loss of generality.

3. Eigenvalues for C_r

3.1. Eigenvalue $\Lambda_1^{(1)}(u)$. The family of $U_q(C_r^{(1)})$-modules $\{W_m^{(a)} | 1 \leqq a \leqq r, m \in \mathbf{Z}_{\geqq 1}\}$ is generated by decomposing tensor products of $W_1^{(1)}$ as suggested in [8]. Thus we first do the analytic Bethe ansatz for the fundamental eigenvalue $\Lambda_1^{(1)}(u)$. The relevant auxiliary space is $W_1^{(1)} \simeq V(\omega_1)$ as an C_r-module from (2.4a), which is the vector representation. Then all the weights are multiplicity-free and one can apply the coupling rule (2.14). To be concrete, we introduce the orthogonal vectors $\varepsilon_a, 1 \leqq a \leqq r$ normalized as $(\varepsilon_a|\varepsilon_b) = \delta_{ab}/2$ and realize the root system as follows:

$$\alpha_a = \begin{cases} \varepsilon_a - \varepsilon_{a+1} & \text{for } 1 \leqq a \leqq r-1 \\ 2\varepsilon_r & \text{for } a = r \end{cases},$$
$$\omega_a = \varepsilon_1 + \cdots + \varepsilon_a\,. \tag{3.1}$$

Then the weights in $V(\omega_1)$ are ε_a and $-\varepsilon_a (1 \leqq a \leqq r)$, which we will abbreviate to a and $\bar{a}$, respectively. In this notation the set of weights reads

$$J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{2}, \bar{1}\}\,. \tag{3.2}$$

Starting from the top term (2.12), one successively applies the coupling rule (2.14) to find the DVF

$$\Lambda_1^{(1)}(u) = \sum_{a \in J} \boxed{a}\,, \tag{3.3}$$

with the elementary boxes defined by

$$\boxed{a} = \psi_a(u) \frac{Q_{a-1}\left(u + \frac{a+1}{2}\right) Q_a\left(u + \frac{a-2}{2}\right)}{Q_{a-1}\left(u + \frac{a-1}{2}\right) Q_a\left(u + \frac{a}{2}\right)} \quad 1 \leqq a \leqq r-1\,,$$

$$\boxed{r} = \psi_r(u) \frac{Q_{r-1}\left(u + \frac{r+1}{2}\right) Q_r\left(u + \frac{r-3}{2}\right)}{Q_{r-1}\left(u + \frac{r-1}{2}\right) Q_r\left(u + \frac{r+1}{2}\right)}\,,$$

$$\boxed{\bar{r}} = \psi_{\bar{r}}(u) \frac{Q_{r-1}\left(u + \frac{r+1}{2}\right) Q_r\left(u + \frac{r+5}{2}\right)}{Q_{r-1}\left(u + \frac{r+3}{2}\right) Q_r\left(u + \frac{r+1}{2}\right)}\,,$$

$$\boxed{\bar{a}} = \psi_{\bar{a}}(u) \frac{Q_{a-1}\left(u + \frac{2r-a+1}{2}\right) Q_a\left(u + \frac{2r-a+4}{2}\right)}{Q_{a-1}\left(u + \frac{2r-a+3}{2}\right) Q_a\left(u + \frac{2r-a+2}{2}\right)} \quad 1 \leqq a \leqq r-1\,, \tag{3.4a}$$

where we have set $Q_0(u) = 1$. The vacuum part $\psi_a(u) = vac\boxed{a}$ is given by

$$\psi_a(u) = \begin{cases} \phi\left(u + \frac{p+1}{2}\right) \phi\left(u + \frac{2r-p+3}{2}\right) & 1 \preceq a \preceq p \\ \phi\left(u + \frac{p-1}{2}\right) \phi\left(u + \frac{2r-p+3}{2}\right) & p+1 \preceq a \preceq \overline{p+1} \\ \phi\left(u + \frac{p-1}{2}\right) \phi\left(u + \frac{2r-p+1}{2}\right) & \bar{p} \preceq a \preceq \bar{1} \end{cases} \tag{3.4b}$$

depending on the quantum space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$. The symbol $\prec$ here stands for a total order in the set J specified as

$$1 \prec 2 \prec \cdots \prec r \prec \bar{r} \prec \cdots \prec \bar{2} \prec \bar{1} \,. \tag{3.5}$$

When $p = r$, the second possibility in (3.4b) is absent. The case $p = 1$ was obtained in [21]. Note that $\boxed{1}$ is the top term (2.12). By the construction, p enters only the vacuum parts (3.4b) hence the dress universality (2.11) is valid. The crossing symmetry (2.18) holds between $\boxed{\bar{a}}$ and $\boxed{a}$. Under the BAE (2.7), (3.3) is pole-free due to the coupling rule (2.14) and (2.15) as follows:

$$Res_{u=-\frac{b}{2}+iu_k^{(b)}} \left(\boxed{b} + \boxed{b+1} \right) = 0 \quad 1 \leqq b \leqq r-1 \,, \tag{3.6a}$$

$$Res_{u=-\frac{r+1}{2}+iu_k^{(r)}} \left(\boxed{r} + \boxed{\bar{r}} \right) = 0 \,, \tag{3.6b}$$

$$Res_{u=-\frac{2r-b+2}{2}+iu_k^{(b)}} \left(\boxed{\overline{b+1}} + \boxed{\bar{b}} \right) = 0 \quad 1 \leqq b \leqq r-1 \,. \tag{3.6c}$$

Following Sect. 2.4, this can be summarized in the diagram

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}} \,.$$

This turns out to be identical with the crystal graph [24, 25].

3.2. Eigenvalue $\Lambda_1^{(a)}(u)$. Let us proceed to $\Lambda_1^{(a)}(u)$, which can be constructed from the elementary boxes (3.4). For $1 \leqq a \leqq r$, let $\mathcal{T}_1^{(a)}$ be the set of the tableaux of the form

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_a \\ \hline \end{array} \tag{3.7a}$$

with entries $i_k \in J$ obeying the following conditions:

$$1 \preceq i_1 \prec i_2 \prec \cdots \prec i_a \preceq \bar{1} \,, \tag{3.7b}$$

$$\text{If } i_k = c \text{ and } i_l = \bar{c}, \text{ then } r + k - l \geqq c \,. \tag{3.7c}$$

We remark that these constraints are very similar but different from the crystal base [24, 25], where (3.7c) is replaced by $a + 1 + k - l \leqq c$. We identify each element (3.7a) of $\mathcal{T}_1^{(a)}$ with the product of (3.4) with the following spectral parameters,

$$\prod_{k=1}^{a} \boxed{i_k}\Big|_{u \to u + \frac{a+1-2k}{2}} \,. \tag{3.8}$$

Then the analytic Bethe ansatz yields the following DVF:

$$\Lambda_1^{(a)}(u) = \sum_{T \in \mathcal{T}_1^{(a)}} T \quad 1 \leqq a \leqq r \,, \tag{3.9}$$

which reduces to (3.3) when $a = 1$. Let us observe consistency of this result before proving that it is pole-free in Sect. 3.3. Firstly, the dress part of

$$\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline a \\ \hline \end{array}$$

is $Q_a(u - \frac{1}{t_a})/Q_a(u + \frac{1}{t_a})$, telling that the above tableau indeed gives the top term (2.12). Secondly, the set $\mathscr{T}_1^{(a)}$ is invariant under the interchange of the two tableaux

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_a \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \overline{i_a} \\ \hline \vdots \\ \hline \overline{i_1} \\ \hline \end{array}$$

and the crossing symmetry (2.18) is valid among them. Thirdly, the character limit (2.16) can be proved. This is essentially done by showing

$$\#\mathscr{T}_1^{(a)} = \dim\, V(\omega_a) = \binom{2r}{a} - \binom{2r}{a-2}\,, \qquad (3.10)$$

which corresponds to the $q \to 1$ limit of (2.16) since $W_1^{(a)} \simeq V(\omega_a)$ as a C_r-module by (2.4a). We have verified (3.10) by building injections in both directions between the sets of depth a tableaux (3.7a) breaking (3.7c) and the depth $a - 2$ ones only obeying the constraint (3.7b). Once (3.10) is established, the weight counting in (2.16) for $q \neq 1$ is shown consistent with Eq. (2.2.2) of [25] by noting that the injections are weight preserving and $\lim_{u \to \infty, |q| > 1} q^* \boxed{a} = q^{2(\omega_1^{(p)}|\varepsilon_a)}$ for some $*$.

3.3. Pole-freeness of $\Lambda_1^{(a)}(u)$. The DVF (3.9) passes the crucial condition in the analytic Bethe ansatz, namely,

Theorem 3.3.1. $\Lambda_1^{(a)}(u)(1 \leqq a \leqq r)$ (3.9) *is free of poles provided that the BAE* (2.7) (*for* $s = 1$) *is valid.*

For the proof we prepare a few lemmas, which follow directly from (3.4).

Lemma 3.3.2. *For* $1 \leqq b \leqq r - 1$, *the products*

$$\begin{array}{|c|}\hline b \\ \hline b+1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \overline{b+1} \\ \hline \bar{b} \\ \hline \end{array} \qquad (3.11)$$

with the spectral parameter v $(v - 1)$ *for the upper* (*lower*) *box do not involve* Q_b *function.*

Lemma 3.3.3. *For* $1 \leqq b \leqq r - 1$, *put*

$$\boxed{b}_{v}\boxed{\overline{b+1}}_{v-r+b} = \frac{Q_b\left(v + \frac{b}{2} - 1\right)}{Q_b\left(v + \frac{b}{2} + 1\right)} X_1\,, \qquad (3.12a)$$

$$\boxed{b}_{v}\boxed{\bar{b}}_{v-r+b} = \frac{Q_b\left(v+\frac{b}{2}-1\right)Q_b\left(v+\frac{b}{2}+2\right)}{Q_b\left(v+\frac{b}{2}\right)Q_b\left(v+\frac{b}{2}+1\right)}X_2\,, \tag{3.12b}$$

$$\boxed{b+1}_{v}\boxed{\bar{b}}_{v-r+b} = \frac{Q_b\left(v+\frac{b}{2}+2\right)}{Q_b\left(v+\frac{b}{2}\right)}X_3\,, \tag{3.12c}$$

where the indices specify the spectral parameters attached to the boxes (3.4). *Then X_i's do not involve Q_b function.*

The point is that (3.12a) and (3.12c) have only one Q_b function in their denominators after some cancellations owing to the spectral parameter choice $v, v-r+b$.

Lemma 3.3.4. *For* $1 \leqq b \leqq r-1$, *let the tableaux*

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \zeta \\ \hline \end{array} \quad \textit{or} \quad \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \eta \\ \hline \bar{b} \\ \hline \zeta \\ \hline \end{array} \tag{3.13}$$

be the elements in $\mathcal{T}_1^{(a)}$ such that the columns $\boxed{\xi}$, $\boxed{\eta}$ and $\boxed{\zeta}$ do not contain the boxes with entries $b, b+1, \overline{b+1}$ and $\bar{b}$. Then the length of $\boxed{\eta}$ is less than $r-b$.

One can easily derive a contradiction supposing the length $\geqq r-b$.

Proof of Theorem 3.3.1. We shall show that color b singularity is spurious, i.e., $Res_{u=u_k^{(b)}+\cdots}\Lambda_1^{(a)}(u) = 0$ for each $2 \leqq b \leqq r-1$. The remaining cases $b=1$ and r can be verified similarly and more easily. Among the elementary boxes (3.4a), the factor $1/Q_b(u+\cdots)$ enters only $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\bar{b}}$. Thus one has to keep track of only these four boxes appearing in (3.7a). Accordingly, let us write (3.9) as $\Lambda_1^{(a)}(u) = S_0 + S_1 + \cdots + S_4$, where S_k denotes the partial sum over the tableaux (3.7a) containing precisely k boxes among the above four. Obviously S_0 is free of $1/Q_b(u+\cdots)$. So is S_4 because the relevant tableaux involve both of the 2×1 patterns in (3.11) and therefore do not contain Q_b by Lemma 3.3.2. Next consider S_1 which is the sum over the tableaux of the form

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \eta \\ \hline \end{array} \quad \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \eta \\ \hline \end{array} \quad \begin{array}{|c|}\hline \xi \\ \hline \overline{b+1} \\ \hline \eta \\ \hline \end{array} \quad \begin{array}{|c|}\hline \xi \\ \hline \bar{b} \\ \hline \eta \\ \hline \end{array}$$

Here $\boxed{\xi}$ and $\boxed{\eta}$ stand for columns with total length $a-1$ and they do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\bar{b}}$. From (3.6), color b residues in the first and second (third and fourth) tableaux sum up to zero. By the same reason S_3 is free of color b singularities since the relevant tableaux must contain one of (3.11). Thus we are left with S_2, whose summands are classified into the following four types:

$$\begin{array}{|c|}\hline \xi\\\hline b\\\hline \eta\\\hline \overline{b+1}\\\hline \zeta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline b\\\hline \eta\\\hline \bar{b}\\\hline \zeta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline b+1\\\hline \eta\\\hline \overline{b+1}\\\hline \zeta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline b+1\\\hline \eta\\\hline \bar{b}\\\hline \zeta\\\hline\end{array} \tag{3.14}$$

Here, $\boxed{\xi}$, $\boxed{\eta}$ and $\boxed{\zeta}$ are columns without $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\bar{b}}$. Denoting their lengths by $k-1, l-k-1$ and $a-l$, respectively, we consider the cases $r+k-l \geqq b+1, r+k-l=b$ and $r+k-l \leqq b-1$ separately. If $r+k-l \geqq b+1$, all the four tableaux (3.14) actually belong to $\mathscr{T}_1^{(a)}$ and the pole-freeness of their sum follows straightforwardly from (3.6). If $r+k-l=b$, the third tableau in (3.14) is absent since it breaks (3.7c). Up to an overall factor not containing Q_b, the remaining three terms are proportional to those in (3.12) for some v. From Lemma 3.3.3 their sum has zero residue both at $v=-\frac{b}{2}+iu_k^{(b)}$ by (3.6a) and at $v=-\frac{b}{2}-1+iu_k^{(b)}$ by (3.6c). Finally, we consider the case $r+k-l \leqq b-1$, when the second and third tableaux in (3.14) do not exist because they both break (3.7c). In fact, the first and the fourth ones are also absent. This is because $r+k-l \leqq b-1$ is equivalent to saying that the length of $\boxed{\eta}$ is not less than $r-b$ against Lemma 3.3.4. Thus S_2 is free of color b poles, which completes the proof of the theorem.

3.4 Eigenvalue $\Lambda_m^{(1)}(u)$. The result (3.9) accomplishes *Step* 2 in Sect. 1.5. The remaining task is *Step* 3, i.e., to find the eigenvalues $\Lambda_m^{(a)}(u)$ for higher m by solving the T-system (2.5b) with

$$\begin{aligned} g_m^{(a)}(u) &= 1 \quad \text{for } 1 \leqq a \leqq r-1\,, \\ g_m^{(r)}(u) &= \prod_{k=1}^{m} g_1^{(r)}(u+m+1-2k)\,, \\ g_1^{(r)}(u) &= \psi_1\left(u+\frac{r+1}{2}\right)\psi_{\bar{1}}\left(u-\frac{r+1}{2}\right)\,, \end{aligned}$$

under the initial conditions $\Lambda_0^{(a)}(u)=1$ and (3.9). So far we have done this only partially to get a conjecture on $\Lambda_m^{(1)}(u)$. To present it we introduce a set $\mathscr{T}_m^{(1)}(m \in \mathbf{Z}_{\geqq 1})$ of the tableaux having the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline i_1 & \cdots & i_k & \bar{r} & r & \cdots & \bar{r} & r & \bar{j_l} & \cdots & \bar{j_1} \\\hline\end{array} \tag{3.15a}$$

where the entries $\bar{r}\ r \cdots \bar{r}\ r$ are grouped by $\overbrace{\qquad\qquad}^{2n}$

with the conditions

$$k,n,l \geqq 0, \quad k+2n+l=m\,, \tag{3.15b}$$

$$1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_k \leqq r, \quad 1 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_l \leqq r\,. \tag{3.15c}$$

Writing (3.15a) simply as $\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_m \\ \hline\end{array}$ with $i_k \in J$ (3.2), we identify it with the product of (3.4) with the following spectral parameters,

$$\prod_{k=1}^{m} \boxed{i_k}\Big|_{u\to u-\frac{m+1-2k}{2}} \,. \tag{3.16}$$

Then we conjecture that the T-system (2.5b) with the initial condition (3.9) leads to

$$\Lambda_m^{(1)}(u) = \sum_{T\in\mathcal{T}_m^{(1)}} T \qquad m \in \mathbf{Z}_{\geqq 1} \,. \tag{3.17}$$

This is just (3.3) when $m = 1$. Equation (3.17) consists of the correct number of terms,

$$\#\mathcal{T}_m^{(1)} = \dim W_m^{(1)} \,. \tag{3.18}$$

To see this, note from (2.4a) that the rhs is equal to

$$\dim V(m\omega_1) + \dim V((m-2)\omega_1) + \cdots + \begin{cases} \dim V(0) & m \text{ even} \\ \dim V(\omega_1) & m \text{ odd} \end{cases} \,. \tag{3.19}$$

On the other hand, the set $\mathcal{T}_m^{(1)}$ is the disjoint union of those tableaux (3.15a) with $n = 0, 1, 2, \ldots$. Thus it suffices to check

$$\dim V(m\omega_1) = \#\{(3.15\text{a}) \in \mathcal{T}_m^{(1)} | n = 0\} \,. \tag{3.20}$$

Obviously the rhs is $\binom{m+2r-1}{m}$, which agrees with the lhs calculated from Weyl's dimension formula.

3.5. C_2 case. For C_2 it is possible to provide the full solution $\Lambda_m^{(1)}(u), \Lambda_m^{(2)}(u)$ to the T-system [10]. In terms of the tableaux, $\Lambda_m^{(1)}(u)$ in [10] is certainly given by (3.17) up to an inessential overall scalar reflecting a different convention on $\Lambda_0^{(a)}(u)$. To present the other eigenvalue $\Lambda_m^{(2)}(u)$ there, we introduce a set $\mathcal{T}_m^{(2)}$ of $2 \times m$ tableaux

$$\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_m \\ \hline j_1 & \cdots & j_m \\ \hline\end{array} \tag{3.21a}$$

obeying the conditions

$$\text{Every column belongs to } \mathcal{T}_1^{(2)}(3.7) \text{ for } C_2 \,, \tag{3.21b}$$

$$i_1 \preceq \cdots \preceq i_m, \quad \text{and} \quad j_1 \preceq \cdots \preceq j_m \,, \tag{3.21c}$$

$$\text{The column } \begin{array}{|c|}\hline 1 \\ \hline \bar{1} \\ \hline\end{array} \text{ is contained at most once.} \tag{3.21d}$$

We identify each element (3.21a) in $\mathcal{T}_m^{(2)}$ with the product of (3.4) with the spectral parameters as follows:

$$\prod_{k=1}^{m} \boxed{i_k}\Big|_{u\to u-m-\frac{1}{2}+2k} \prod_{k=1}^{m} \boxed{j_k}\Big|_{u\to u-m-\frac{3}{2}+2k} \,. \tag{3.22}$$

Namely, the shifts increase by 2 from the left to the right, decrease by 1 from the top to the bottom and their average is 0. Then the result in [10] reads

$$\Lambda^{(2)}_m(u) = \sum_{T\in\mathcal{T}^{(2)}_m} T\,. \tag{3.23}$$

4. Eigenvalues for B_r

As in the C_r case we first introduce elementary boxes attached to the vector representation. Using them as the building blocks, we will construct the DVF for $\Lambda^{(a)}_1(u)(1 \leqq a \leqq r-1)$ and prove its pole-freeness under the BAE. We also conjecture $\Lambda^{(a)}_m(u)(1 \leqq a \leqq r-1)$ in terms of tableaux made of these boxes.

Compared with the C_r case, a distinct feature in B_r (and D_r) is the existence of the spin representation. Any finite dimensional irreducible B_r-module is generated by decomposing a tensor product of the spin representations. Thus we introduce another kind of elementary boxes attached to the spin representation. It enables a unified description of the DVFs for all the fundamental eigenvalues $\Lambda^{(a)}_1(u)(1 \leqq a \leqq r)$. An explicit relation between the two kinds of elementary boxes is given in Appendix A.

4.1. Eigenvalue $\Lambda^{(1)}_1(u)$. Let $\varepsilon_a, 1 \leqq a \leqq r$ be the orthonormal vectors $(\varepsilon_a|\varepsilon_b) = \delta_{ab}$ realizing the root system as follows:

$$\alpha_a = \begin{cases} \varepsilon_a - \varepsilon_{a+1} & \text{for } 1 \leqq a \leqq r-1 \\ \varepsilon_r & \text{for } a = r \end{cases},$$

$$\omega_a = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_a & \text{for } 1 \leqq a \leqq r-1 \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r) & \text{for } a = r \end{cases}. \tag{4.1}$$

The auxiliary space relevant to $\Lambda^{(1)}_1(u)$ is $W^{(1)}_1 \simeq V(\omega_1)$ as an B_r-module. This is the vector representation, whose weights are $\varepsilon_a, -\varepsilon_a(1 \leqq a \leqq r)$ and 0. By abbreviating them to $a, \bar{a}$ and 0, the set of weights is given by

$$J = \{1, 2, \ldots, r, 0, \bar{r}, \ldots, \bar{1}\}\,. \tag{4.2}$$

All the weights are multiplicity-free, therefore one can determine the DVF from (2.12) and (2.14). The result reads

$$\Lambda^{(1)}_1(u) = \sum_{a\in J} \boxed{a}\,, \tag{4.3}$$

which is formally the same with (3.3). The elementary boxes here are defined by

$$\begin{aligned} \boxed{a} &= \psi_a(u)\frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \quad 1 \leqq a \leqq r\,, \\ \boxed{0} &= \psi_0(u)\frac{Q_r(u+r-2)Q_r(u+r+1)}{Q_r(u+r)Q_r(u+r-1)}\,, \\ \boxed{\bar{a}} &= \psi_{\bar{a}}(u)\frac{Q_{a-1}(u+2r-a-2)Q_a(u+2r-a+1)}{Q_{a-1}(u+2r-a)Q_a(u+2r-a-1)} \quad 1 \leqq a \leqq r\,, \end{aligned} \tag{4.4a}$$

where we have set $Q_0(u) = 1$. The vacuum parts $\psi_a(u)$ depend on the quantum space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$ and are given by

$$\psi_a(u) = \begin{cases} \phi\left(u + p + \frac{1}{t_p}\right)\phi\left(u + 2r - p - 1 + \frac{1}{t_p}\right)\Phi^r_p(u) \text{ for } 1 \preceq a \preceq p \\ \phi\left(u + p - \frac{1}{t_p}\right)\phi\left(u + 2r - p - 1 + \frac{1}{t_p}\right)\Phi^r_p(u) \text{ for } p+1 \preceq a \preceq \overline{p+1} \\ \phi\left(u + p - \frac{1}{t_p}\right)\phi\left(u + 2r - p - 1 - \frac{1}{t_p}\right)\Phi^r_p(u) \text{ for } \bar{p} \preceq a \preceq \bar{1} \end{cases}, \tag{4.4b}$$

where

$$\begin{aligned}\Phi^r_p(u) &= \prod_{j=1}^{p-1} \phi\left(u + p - 2j - \frac{1}{t_p}\right)\phi\left(u + 2r - p + 2j - 1 + \frac{1}{t_p}\right) \\ &= \Phi^r_p(-2r + 1 - u)|_{w_k \to -w_k} \,.\end{aligned} \tag{4.4c}$$

The common factor $\Phi^r_p(u)$ here will play a role in Appendix A, where the boxes here are related to those in Sect. 4.5. The order $\prec$ in the set J (4.2) is defined by

$$1 \prec 2 \prec \cdots \prec r \prec 0 \prec \bar{r} \prec \cdots \prec \bar{2} \prec \bar{1} \,. \tag{4.5}$$

Note the top term $\boxed{1}$ (2.12), the dress universality (2.11) and the crossing symmetry (2.18) for the pairs $\boxed{a} \leftrightarrow \boxed{\bar{a}}$ and $\boxed{0} \leftrightarrow \boxed{0}$. Under the BAE (2.7), (4.3) is pole-free because the coupling rule (2.14) and (2.15) have been embodied as

$$Res_{u=-b+iu_k^{(b)}}\left(\boxed{b} + \boxed{b+1}\right) = 0 \quad 1 \leqq b \leqq r-1\,, \tag{4.6a}$$

$$Res_{u=-r+iu_k^{(r)}}\left(\boxed{r} + \boxed{0}\right) = 0\,, \tag{4.6b}$$

$$Res_{u=-r+1+iu_k^{(r)}}\left(\boxed{0} + \boxed{\bar{r}}\right) = 0\,, \tag{4.6c}$$

$$Res_{u=-2r+b+1+iu_k^{(b)}}\left(\boxed{\overline{b+1}} + \boxed{\bar{b}}\right) = 0 \quad 1 \leqq b \leqq r-1\,. \tag{4.6d}$$

Thus we have a diagram

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{0} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}\,.$$

This is again identical with the crystal graph [24, 25]. For $p = 1$, (4.3) has been known earlier in [21].

4.2. Eigenvalue $\Lambda_1^{(a)}(u)$ for $1 \leqq a \leqq r-1$. For $1 \leqq a \leqq r-1$, let $\mathscr{T}_1^{(a)}$ be the set of the tableaux of the form (3.7a) with $i_k \in J$ (4.2) obeying the condition

$$i_k \prec i_{k+1} \text{ or } i_k = i_{k+1} = 0 \text{ for any } 1 \leqq k \leqq a-1\,. \tag{4.7}$$

We identify each element (3.7a) of $\mathscr{T}_1^{(a)}$ with the product of (4.4a) with the following spectral parameters,

$$\prod_{k=1}^{a} \boxed{i_k}|_{u \to u+a+1-2k} \,. \tag{4.8}$$

Then the analytic Bethe ansatz yields the following DVF:

$$\Lambda_1^{(a)}(u) = \frac{1}{F_a^{(p,r)}(u)} \sum_{T \in \mathscr{T}_1^{(a)}} T \quad 1 \leqq a \leqq r-1\,, \tag{4.9a}$$

where the scalar $F_a^{(p,r)}(u)$ is defined by

$$\begin{aligned} F_a^{(p,r)}(u) &= \prod_{j=1}^{a-1} \prod_{k=0}^{p-1} \phi\left(u+p+a-1-\frac{1}{t_p}-2j-2k\right) \\ &\quad \times \phi\left(u+2r-p-a+\frac{1}{t_p}+2j+2k\right) \\ &= F_a^{(p,r)}(-2r+1-u)|_{w_k \to -w_k}\,. \end{aligned} \tag{4.9b}$$

Notice that $F_1^{(p,r)}(u) = 1$ hence (4.9a) reduces to (4.3) when $a = 1$. From (4.4c) and (4.27c) in Sect. 4.5, (4.9b) can also be written as

$$\begin{aligned} F_a^{(p,r)}(u) &= \prod_{j=1}^{a-1} \phi\left(u+p+a-1-\frac{1}{t_p}-2j\right) \phi\left(u+2r+p+a-2+\frac{1}{t_p}-2j\right) \\ &\quad \times \prod_{j=1}^{a-1} \Phi_p^r(u+a-1-2j) \qquad (4.10a) \\ &= \prod_{j=1}^{a-1} \psi_0^{(p,r)}\left(u+r-a-\frac{1}{2}+2j\right) \psi_p^{(p,r)}\left(u-r+a+\frac{1}{2}-2j\right). \end{aligned}$$

$$(4.10b)$$

By using (4.10a), it can be checked that each summand T in (4.9a) contains the factor $F_a^{(p,r)}(u)$ and $\Lambda_1^{(a)}(u)$ is homogeneous of order $2p$ w.r.t $\phi(u+\cdots)$. This will be seen more manifestly in (A.4). One can observe the top term and the crossing symmetry in (4.9a) as done after (3.9). The character limit (2.16) is also valid. To see this, we introduce a map χ from $\mathscr{T}_1^{(a)}$ to the Laurent polynomials $\mathbf{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}]$ by

$$\chi\left(\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_a \\ \hline \end{array}\right) = y_{i_1} \cdots y_{i_a}\,, \tag{4.11a}$$

$$y_0 = 1, y_a = z_a, y_{\bar{a}} = z_a^{-1}, 1 \leqq a \leqq r\,. \tag{4.11b}$$

In view of $\lim_{u\to\infty, |q|>1} q^* \boxed{a} = q^{2(\omega_1^{(p)}|\varepsilon_a)}$ for some $*$, this corresponds to taking the limit (2.16) of the element (3.7a). Since $W_1^{(a)} \simeq V(\omega_a) \oplus V(\omega_{a-2}) \oplus \cdots$ from (2.4b), we are to show

$$\sum_{T \in \mathscr{T}_1^{(a)}} \chi(T) = chV(\omega_a) + chV(\omega_{a-2}) + \cdots\,, \tag{4.11c}$$

for $1 \leqq a \leqq r-1$. Here chV denotes the classical character of the B_r-module V

on letters $z_1,\dots,z_r$. This can be easily proved from (4.7) and the known formula

$$chV(\omega_a) = \sum_{\substack{i_1,\dots,i_a\in J\\ i_1\prec\cdots\prec i_a}} y_{i_1}\cdots y_{i_a}\,, \tag{4.11d}$$

for $1 \leqq a \leqq r-1$. Equation (4.11d) originates in $so(2r+1) \hookrightarrow gl(2r+1)$.

4.3. Pole-Freeness of $\Lambda_1^{(a)}(u)$ for $1 \leqq a \leqq r-1$. Here we show

Theorem 4.3.1. $\Lambda_1^{(a)}(u)(1 \leqq a \leqq r-1)$ (4.9) *is free of poles provided that the BAE* (2.7) (*for* $s=1$) *is valid.*

Lemma 4.3.2. *For* $n \in \mathbf{Z}_{\geqq 0}$, *put*

$$\prod_{j=0}^{n} \boxed{0}_{v-2j} = \frac{Q_r(v+r+1)Q_r(v+r-2n-2)}{Q_r(v+r)Q_r(v+r-2n-1)}X_1\,, \tag{4.12a}$$

$$\boxed{r}_v \prod_{j=1}^{n} \boxed{0}_{v-2j} = \frac{Q_r(v+r-1)Q_r(v+r-2n-2)}{Q_r(v+r)Q_r(v+r-2n-1)}X_2\,, \tag{4.12b}$$

$$\boxed{\bar{r}}_{v-2n} \prod_{j=0}^{n-1} \boxed{0}_{v-2j} = \frac{Q_r(v+r+1)Q_r(v+r-2n)}{Q_r(v+r)Q_r(v+r-2n-1)}X_3\,, \tag{4.12c}$$

$$\boxed{r}_v\boxed{\bar{r}}_{v-2n} \prod_{j=1}^{n-1} \boxed{0}_{v-2j} = \frac{Q_r(v+r-1)Q_r(v+r-2n)}{Q_r(v+r)Q_r(v+r-2n-1)}X_4\,, \tag{4.12d}$$

where the indices specify the spectral parameters attached to the boxes (4.4). *Then*

$$X_i\text{'s do not involve } Q_r \text{ function}\ , \tag{4.13a}$$

$$\frac{Q_r(v+r\pm 1)}{Q_r(v+r)} \text{ comes from the box } \boxed{*}_v\,, \tag{4.13b}$$

$$\frac{Q_r(v+r-2n-1\pm 1)}{Q_r(v+r-2n-1)} \text{ comes from the box } \boxed{*}_{v-2n}\,, \tag{4.13c}$$

where $* = r, \bar{r}$ *or* 0.

This can be verified by a direct calculation.

Lemma 4.3.3. *If the BAE* (2.7) ($s=1$) *is valid, then*

$$\begin{aligned} &Res_{v=-r+iu_k^{(r)}}((4.12\text{a})+(4.12\text{b})) = Res_{v=-r+iu_k^{(r)}}((4.12\text{c})+(4.12\text{d})) = 0\,,\\ &Res_{v=-r+2n+1+iu_k^{(r)}}((4.12\text{a})+(4.12\text{c})) = Res_{v=-r+2n+1+iu_k^{(r)}}((4.12\text{b})+(4.12\text{d}))\\ &\qquad = 0\,. \end{aligned} \tag{4.14}$$

This follows from (4.13b, c) and (4.6b, c). Now we proceed to

Proof of Theorem 4.3.1. As remarked after (4.10), there is no pole originated from the overall scalar $1/F_a^{(p,r)}(u)$ in (4.9a). Thus one has only to show that the apparent color b poles $1/Q_b(u+\cdots)$ in $\sum_{T\in\mathcal{T}_1^{(a)}} T$ are spurious for all $1 \leqq b \leqq r$ under the BAE. For $1 \leqq b \leqq r-1$, this can be done similarly to the proof of Theorem 3.3.1. In fact the present case is much easier since (4.7) is so compared with (3.7b, c). Henceforth we focus on the $b=r$ case which needs a separate consideration. From (4.4a), we have only to keep track of the boxes $\boxed{r}, \boxed{0}$ and $\boxed{\bar{r}}$ containing Q_r. Let us classify the tableaux (3.7a) in $\mathcal{T}_1^{(a)}$ (4.7) into the sectors labeled by the number n of $\boxed{0}$'s contained in them. In each sector, we further divide the tableaux into four types according to the entries (u,d) in the boxes just above and below the consecutive $\boxed{0}$'s.

$$\begin{aligned} &\text{type } 1_n : u \neq r \text{ and } d \neq \bar{r}\,, &&\text{type } 2_n : u = r \text{ and } d \neq \bar{r}\,,\\ &\text{type } 3_n : u \neq r \text{ and } d = \bar{r}\,, &&\text{type } 4_n : u = r \text{ and } d = \bar{r}\,. \end{aligned}$$

Thus we have

$$\sum_{T\in\mathcal{T}_1^{(a)}} T = \sum_{n=0}^{a}\sum_{i=1}^{4} S_{n,i}\,, \tag{4.15a}$$

$$S_{n,i} = \sum_{T\in \text{type } i_n} T\,, \tag{4.15b}$$

$$S_{a,2} = S_{a,3} = S_{a,4} = S_{a-1,4} = 0\,. \tag{4.15c}$$

Consider the following quartet of the tableaux of types $1_{n+1}, 2_n, 3_n$ and 4_{n-1}, respectively:

$$\begin{array}{|c|}\hline \xi\\\hline 0\\\hline 0\\\hline \vdots\\\hline 0\\\hline 0\\\hline \eta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline r\\\hline 0\\\hline \vdots\\\hline 0\\\hline 0\\\hline \eta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline 0\\\hline 0\\\hline \vdots\\\hline 0\\\hline \bar{r}\\\hline \eta\\\hline\end{array}\qquad \begin{array}{|c|}\hline \xi\\\hline r\\\hline 0\\\hline \vdots\\\hline 0\\\hline \bar{r}\\\hline \eta\\\hline\end{array} \tag{4.16}$$

Here, $\boxed{\xi}$ and $\boxed{\eta}$ are the columns with total length $a-n-1$ and they do not contain $\boxed{r}, \boxed{0}$ and $\boxed{\bar{r}}$. In view of (4.8) and (4.13a), the tableaux (4.16) are proportional to the four terms (4.12) with some v up to an overall factor not containing Q_r. Thus from Lemma 4.3.3, their sum is free of color r singularity. That is true for any fixed $\boxed{\xi}$ and $\boxed{\eta}$ such that the tableaux (4.16) belong to $\mathcal{T}_1^{(a)}$. Therefore $S_{n+1,1}+S_{n,2}+S_{n,3}+S_{n-1,4}$ is free of color r singularity for each $1 \leqq n \leqq a-1$. Due to (4.15c), the remaining terms in (4.15a) are $S_{1,1}, S_{0,1}, S_{0,2}$ and $S_{0,3}$. By the definition $S_{0,1}$ is independent of Q_r and it is straightforward to check that $S_{1,1}+S_{0,2}+S_{0,3}$ is free of color r singularity by using (4.6b, c). This establishes the theorem.

4.4. Eigenvalue $\Lambda^{(a)}_m(u)$ for $1 \leqq a \leqq r-1$. Starting from (4.9) and $\Lambda^{(r)}_1(u)$ that will be described in Sect. 4.5, we are to solve the T-system (2.5a) with

$$g^{(a)}_m(u) = 1 \quad \text{for } 2 \leqq a \leqq r\,,$$

$$g^{(1)}_m(u) = \prod_{k=1}^{m} F^{(p,r)}_2(u+m+1-2k)\,, \tag{4.17}$$

where $F^{(p,r)}_2(u)$ has been given in (4.9b) and (4.10). The solution will yield a DVF for the general eigenvalue $\Lambda^{(a)}_m(u)$. Here we shall present the so derived conjecture for $1 \leqq a \leqq r-1$.

Let $\mathcal{T}^{(a)}_m (1 \leqq a \leqq r-1)$ be the set of the $a \times m$ rectangular tableaux containing $\boxed{i_{jk}}, i_{jk} \in J$ at the (j,k) position.

$$\begin{array}{|c|c|c|}
\hline
i_{11} & \cdots & i_{1m} \\
\hline
\vdots & \ddots & \vdots \\
\hline
i_{a1} & \cdots & i_{am} \\
\hline
\end{array}$$

The entries are to obey the conditions

$$i_{jk} \prec i_{j+1k} \text{ or } i_{jk} = i_{j+1k} = 0 \text{ for any } 1 \leqq j \leqq a-1, 1 \leqq k \leqq m\,, \tag{4.18a}$$

$$i_{jk} \prec i_{jk+1} \text{ or } i_{jk} = i_{jk+1} \in J\backslash\{0\} \text{ for any } 1 \leqq j \leqq a, 1 \leqq k \leqq m-1\,. \tag{4.18b}$$

We identify each element of $\mathcal{T}^{(a)}_m$ as above with the following product of (4.4a):

$$\prod_{j=1}^{a}\prod_{k=1}^{m} \boxed{i_{jk}}_{\,u \to u+a-m-2j+2k}\,. \tag{4.19}$$

Then we conjecture that the T-system (2.5a) with (4.17) and the initial condition (4.9) leads to

$$\Lambda^{(a)}_m(u) = \frac{1}{\prod_{k=1}^{m} F^{(p,r)}_a(u-m-1+2k)} \sum_{T \in \mathcal{T}^{(a)}_m} T \quad 1 \leqq a \leqq r-1, m \in \mathbf{Z}_{\geqq 1}\,. \tag{4.20}$$

From (4.18a) and the remark after (4.10), the rhs is homogeneous of degree $2pm$ w.r.t. ϕ. The conjecture (4.20) reduces to (4.9a) when $m = 1$. For B_2, (4.20) is certainly true because $\Lambda^{(1)}_m(u)$ of B_2 equals $\Lambda^{(2)}_m(u)$ of C_2 given in (3.23) under the exchange $Q_1(u) \leftrightarrow Q_2(u)$. The cases $m = 2, a = 1, 2$ have also been checked directly for B_3 and B_4. As a further support, we have verified $\sharp\mathcal{T}^{(a)}_m = \dim W^{(a)}_m$ by computer for several values of a and m. For example, both sides yields 247500 for $B_5, a = m = 3$. We emphasize that the set $\mathcal{T}^{(a)}_m$ is specified by a remarkably simple rule (4.18).

4.5. Eigenvalue $\Lambda_1^{(r)}(u)$. From (2.4b) the relevant auxiliary space is $W_1^{(r)} \simeq V(\omega_r)$ as a B_r-module. This is the spin representation, whose weights are all multiplicity-free

$$\frac{1}{2}(\mu_1\varepsilon_1 + \cdots + \mu_r\varepsilon_r), \quad \mu_1, \ldots, \mu_r = \pm\,. \tag{4.21}$$

Thus we shall introduce another kind of elementary boxes $\boxed{\mu_1, \mu_2, \ldots, \mu_r}$ by which the DVF can be written as

$$\Lambda_1^{(r)}(u) = \sum_{\{\mu_j = \pm\}} \underset{p}{\overbrace{\boxed{\mu_1, \mu_2, \ldots, \mu_r}}^{r}}\,. \tag{4.22}$$

We let $\mathscr{T}_1^{(r)}$ denote the set of $\dim W_1^{(r)} = 2^r$ boxes as above. The indices r and p here signify the rank of B_r and the quantum space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$, respectively. Each box is identified with a product of dress and vacuum parts that are defined via certain recursion relations w.r.t. these indices. To describe them we introduce the operators τ_γ^u, τ^Q and τ_γ^C acting on the DVF (2.9b) as follows:

$$\tau_\gamma^u : u \to u + \gamma\,, \tag{4.23a}$$

$$\tau^Q : Q_a(u) \to Q_{a+1}(u)\,, \tag{4.23b}$$

$$\begin{aligned} \tau_\gamma^C :\ & Q_a(u+x) \to Q_a(u+\gamma-x), \\ & \phi(u+x) \to \phi(u+\gamma-x) \text{ for any } x\,. \end{aligned} \tag{4.23c}$$

By the definition they obey the relations

$$\tau^Q\tau_\gamma^u = \tau_\gamma^u\tau^Q, \quad \tau^Q\tau_\gamma^C = \tau_\gamma^C\tau^Q\,, \tag{4.24a}$$

$$\tau_\gamma^C\tau_{\gamma'}^C = \tau_{\gamma-\gamma'}^u, \quad \tau_\gamma^u\tau_{\gamma'}^u = \tau_{\gamma+\gamma'}^u\,. \tag{4.24b}$$

In view of (2.8), τ^Q is equivalent to $N_a \to N_{a+1}$ and $u_j^{(a)} \to u_j^{(a+1)}$. It is to be understood as replacing $Q_a(u)$ with $1 \leqq a \leqq r-1$ for B_{r-1} by $Q_{a+1}(u)$ for B_r. The operator τ_γ^C will be used to describe the transformation (2.19) concerning the crossing symmetry. Now the recursion relations read,

$$\underset{p}{\overbrace{\boxed{+,+,\xi}}^{r}} = \phi\left(u + r + p - \frac{3}{2} + \frac{1}{t_p}\right)\tau^Q \underset{p-1}{\overbrace{\boxed{+,\xi}}^{r-1}}, \tag{4.25a}$$

$$\underset{p}{\overbrace{\boxed{+,-,\xi}}^{r}} = \phi\left(u + r + p - \frac{3}{2} + \frac{1}{t_p}\right)\frac{Q_1\left(u+r-\frac{5}{2}\right)}{Q_1\left(u+r-\frac{1}{2}\right)}\tau^Q \underset{p-1}{\overbrace{\boxed{-,\xi}}^{r-1}}, \tag{4.25b}$$

$$\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}} = \phi\left(u + r - p + \frac{1}{2} - \frac{1}{t_p}\right)\frac{Q_1\left(u+r+\frac{3}{2}\right)}{Q_1\left(u+r-\frac{1}{2}\right)}\tau_2^u\tau^Q \underset{p-1}{\overbrace{\boxed{+,\xi}}^{r-1}}, \tag{4.25c}$$

$$\underset{p}{\overbrace{\boxed{-,-,\xi}}^{r}} = \phi\left(u + r - p + \frac{1}{2} - \frac{1}{t_p}\right)\tau_2^u\tau^Q \underset{p-1}{\overbrace{\boxed{-,\xi}}^{r-1}}, \tag{4.25d}$$

where ξ denotes arbitrary sequence of $\pm$ symbols with length $r-2$. The recursions (4.25) are valid for $1 \leqq p \leqq r$ and $r \geqq 3$. The initial condition is given by

$$dr\underset{p}{\overbrace{\boxed{+,+}}^{2}} = \frac{Q_2\left(u-\frac{1}{2}\right)}{Q_2\left(u+\frac{1}{2}\right)}, \qquad vac\underset{p}{\overbrace{\boxed{+,+}}^{2}} = \begin{cases} \phi\left(u+\frac{5}{2}\right) & p=1 \\ \phi(u+1)\phi(u+3) & p=2 \end{cases},$$

$$dr\underset{p}{\overbrace{\boxed{+,-}}^{2}} = \frac{Q_1\left(u-\frac{1}{2}\right)Q_2\left(u+\frac{3}{2}\right)}{Q_1\left(u+\frac{3}{2}\right)Q_2\left(u+\frac{1}{2}\right)}, \qquad vac\underset{p}{\overbrace{\boxed{+,-}}^{2}} = \begin{cases} \phi\left(u+\frac{5}{2}\right) & p=1 \\ \phi(u)\phi(u+3) & p=2 \end{cases},$$

$$dr\underset{p}{\overbrace{\boxed{-,+}}^{2}} = \frac{Q_1\left(u+\frac{7}{2}\right)Q_2\left(u+\frac{3}{2}\right)}{Q_1\left(u+\frac{3}{2}\right)Q_2\left(u+\frac{5}{2}\right)}, \qquad vac\underset{p}{\overbrace{\boxed{-,+}}^{2}} = \begin{cases} \phi\left(u+\frac{1}{2}\right) & p=1 \\ \phi(u)\phi(u+3) & p=2 \end{cases},$$

$$dr\underset{p}{\overbrace{\boxed{-,-}}^{2}} = \frac{Q_2\left(u+\frac{7}{2}\right)}{Q_2\left(u+\frac{5}{2}\right)}, \qquad vac\underset{p}{\overbrace{\boxed{-,-}}^{2}} = \begin{cases} \phi\left(u+\frac{1}{2}\right) & p=1 \\ \phi(u)\phi(u+2) & p=2 \end{cases}. \tag{4.26a}$$

Note that one formally needs the dress and the vacuum parts for $p=0$ when applying (4.25) with $p=1$. As for the vacuum parts we fix this by putting

$$vac\underset{0}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = 1 \quad \text{for any } r \text{ and } \{\mu_j\}\,. \tag{4.26b}$$

As for the dress parts we simply let $dr\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}$ be the same for any $0 \leqq p \leqq r$. This is consistent with (4.26a) and the dress universality (2.11). Under these setting the recursions (4.25) and the initial condition (4.26) provide a complete characterization of our $\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}$ for any $0 \leqq p \leqq r, r \geqq 2$ and $\{\mu_j\}$. Thus we have presented the DVF (4.22) for the eigenvalue $\Lambda_1^{(r)}(u)$. In the rational case ($q \to 1$) with $p=1$, a similar recursive description is available in [5].

Let us observe various features of our DVF (4.22) before proving that it is polefree in Sect. 4.6. Firstly, it is easy to calculate the vacuum parts explicitly.

$$vac\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = \psi_n^{(p,r)}(u)\,, \tag{4.27a}$$

$$n = \sharp\{j|\mu_j = -, 1 \leqq j \leqq p\}\,, \tag{4.27b}$$

$$\begin{aligned} \psi_n^{(p,r)}(u) = &\prod_{j=0}^{n-1}\phi\left(u+r-p+2j+\frac{1}{2}-\frac{1}{t_p}\right) \\ &\times \prod_{j=n}^{p-1}\phi\left(u+r-p+2j+\frac{1}{2}+\frac{1}{t_p}\right) \quad 0 \leqq n \leqq p\,. \end{aligned} \tag{4.27c}$$

This is order p w.r.t. ϕ. Secondly, the top term is given by

$$\underset{p}{\overbrace{\boxed{+,+,\ldots,+}}^{r}} = \psi_0^{(p,r)}(u)\frac{Q_r\left(u-\frac{1}{2}\right)}{Q_r\left(u+\frac{1}{2}\right)}\,. \tag{4.28}$$

This is consistent with (2.12) since the above box is associated with the highest weight $(\varepsilon_1 + \cdots + \varepsilon_r)/2 = \omega_r$ from (4.1) and (4.21). Thirdly, the crossing symmetry $\Lambda_1^{(r)}(u) = (-)^{pN}\Lambda_1^{(r)}(-2r+1-u)|_{w_j \to -w_j, u_i^{(a)} \to -u_i^{(a)}}$ is valid, which is precisely (2.17) with the order $k = p$ as remarked above. At the level of the boxes, this is due to

$$\tau_{2r-1}^{C}\,\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = \underset{p}{\overbrace{\boxed{-\mu_1,\ldots,-\mu_r}}^{r}}\,, \tag{4.29}$$

where the effect of $(-)^{pN}$ has been absorbed into τ_{2r-1}^{C} as explained in (2.19). In the sequel, we will write such ± sequences as above simply as μ and $\overline{\mu}$, etc.

4.6. Pole-freeness of $\Lambda_1^{(r)}(u)$. In Sect. 4.5, we have formally allowed $p = 0$ in the boxes that consist of the DVF (4.22). Correspondingly, we find it convenient to consider the BAE with $p = 0$ as the one obtained from (2.7) by setting its lhs always -1. We shall quote (2.7) as BAE_p^r. Our aim here is to establish

Theorem 4.6.1. *For $r \geqq 2$ and $0 \leqq p \leqq r, \Lambda_1^{(r)}(u)$ (4.22) is free of poles provided that the BAE_p^r (2.7) (for $s = 1$) is valid.*

We are to show that color a poles $1/Q_a$ are spurious for each $1 \leqq a \leqq r$. The poles are located by

Lemma 4.6.2. *For $1 \leqq a \leqq r-1$ the factor $1/Q_a$ is contained in the box $\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}$ if and only if $(\mu_a, \mu_{a+1}) = (+,-)$ or $(-,+)$. Any two such boxes $\underset{p}{\overbrace{\boxed{\eta,+,-,\xi}}^{r}}$ and $\underset{p}{\overbrace{\boxed{\eta,-,+,\xi}}^{r}}$ share a common color a pole $1/Q_a(u+y)$ for some y.*

The factor $1/Q_r$ is contained in all the boxes. For $\varepsilon = \pm$, any two boxes $\underset{p}{\overbrace{\boxed{\zeta,\varepsilon,\varepsilon}}^{r}}$ and $\underset{p}{\overbrace{\boxed{\zeta,\varepsilon,-\varepsilon}}^{r}}$ share a common color r pole $1/Q_r(u+z)$ for some z.

The assertions are immediate consequences of (4.25) and (4.26). If one puts $\lambda = (\eta,+,-,\xi), \mu = (\eta,-,+,\xi)$ and identifies them with the weights via (4.21), one has $\lambda - \mu = \varepsilon_a - \varepsilon_{a+1} = \alpha_a$ for $1 \leqq a \leqq r-1$ by (4.21). A similar relation holds for $a = r$ as well. Thus the above lemma is another example of the coupling rule (2.14a). In this view Theorem 4.6.1 is a corollary of

Theorem 4.6.3. *For $1 \leqq a \leqq r-1$, let η, ξ and ζ be any ± sequences with lengths $a-1, r-a-1$ and $r-2$, respectively. If the BAE_p^r (2.7) (for $s = 1$) is valid,* *then*

$$\mathrm{Res}_{u=-y+iu_k^{(a)}}\left(\underset{p}{\overbrace{\boxed{\eta,+,-,\xi}}^{r}}+\underset{p}{\overbrace{\boxed{\eta,-,+,\xi}}^{r}}\right)=0\,, \tag{4.30a}$$

$$\mathrm{Res}_{u=-z+iu_k^{(r)}}\left(\underset{p}{\overbrace{\boxed{\zeta,\pm,\pm}}^{r}}+\underset{p}{\overbrace{\boxed{\zeta,\pm,\mp}}^{r}}\right)=0\,, \tag{4.30b}$$

where y and z are those in Lemma 4.6.2.

The rest of the present subsection is devoted to a proof of this theorem. In fact the proof is done essentially by establishing (2.14b) and (2.15). It follows that the character limit (2.16) is also valid for $\Lambda_1^{(r)}(u)$ (4.22). It is easy to show

Lemma 4.6.4. *Let ξ be any sequence of $\pm$ with length $r-1$. Then*

$$\tau^C_{2r+1}\underset{p}{\overbrace{\boxed{-,\bar{\xi}}}^{r}}=\left(\frac{\phi\left(u+r+p+\frac{1}{2}+\frac{1}{t_p}\right)}{\phi\left(u+r-p+\frac{1}{2}-\frac{1}{t_p}\right)}\right)^{1-\delta_{p0}}\frac{Q_1\left(u+r-\frac{1}{2}\right)}{Q_1\left(u+r+\frac{3}{2}\right)}\underset{p}{\overbrace{\boxed{-,\bar{\xi}}}^{r}} \tag{4.31}$$

for $0\leqq p\leqq r$.

Proof of Theorem 4.6.3. It is straightforward to check (4.30) for $r=2$ by (4.26a). We assume that the theorem is true for B_{r-1} and use induction on r. We shall only verify $a=1$ case of (4.30a), for $a=2$ case is more easy and $a\geqq 3$ case follows immediately from the induction assumption. When $a=1$, the two boxes in (4.30a) are $\underset{p}{\overbrace{\boxed{+,-,\xi}}^{r}}$ and $\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}}$. From (4.25b,c) they share a color $a=1$ pole at $u=-r+\frac{1}{2}+iu_k^{(1)}$. Let us rewrite the latter as follows:

$$\begin{aligned}\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}}&=\tau^C_{2r-1}\underset{p}{\overbrace{\boxed{+,-,\bar{\xi}}}^{r}}\\&=\phi\left(u+r-p+\frac{1}{2}-\frac{1}{t_p}\right)\frac{Q_1\left(u+r+\frac{3}{2}\right)}{Q_1\left(u+r-\frac{1}{2}\right)}\tau^Q\tau^C_{2r-1}\underset{p-1}{\overbrace{\boxed{-,\bar{\xi}}}^{r-1}}\,,\end{aligned}$$

where we have used (4.29) and (4.25b). In the last line, $\tau^C_{2r-1}\underset{p-1}{\overbrace{\boxed{-,\bar{\xi}}}^{r-1}}$ can be further rewritten by applying Lemma 4.6.4 with $r\to r-1, p\to p-1$. Dividing the resulting expression by the rhs of (4.25b) we obtain

$$\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}} / \underset{p}{\overbrace{\boxed{+,-,\xi}}^{r}} = \frac{Q_1\left(u+r+\frac{3}{2}\right)Q_2\left(u+r-\frac{3}{2}\right)}{Q_1\left(u+r-\frac{5}{2}\right)Q_2\left(u+r+\frac{1}{2}\right)}$$

$$\times\left(\frac{\phi\left(u+r-p+\frac{1}{2}-\frac{1}{t_p}\right)}{\phi\left(u+r+p-\frac{3}{2}+\frac{1}{t_p}\right)}\right)^{\delta_{1p}} .$$

At the pole location $u=-r+\frac{1}{2}+iu_k^{(1)}$, this is just -1 owing to BAE_p^r (2.7) with $a=1$. Therefore (4.30a) is free of color 1 poles. Equation (4.30b) can be shown similarly. This completes the proof of Theorem 4.6.3 hence Theorem 4.6.1.

The two kinds of boxes introduced here and Sect. 4.1 are related reflecting the fact that the vector representation is contained in a tensor product of the spin ones. The precise relation has been described in Appendix A.

5. Eigenvalues for D_r

Our results for $D_r = so(2r)$ are quite parallel with those for $B_r = so(2r+1)$ in many respects. In fact many formulas here become those in Sect. 4 through a formal replacement $r \to r+\frac{1}{2}$. Thus we shall state them without a proof, which can be done in a similar manner to the B_r case. We will introduce two kinds of boxes associated with the vector and the spin representations. Their relation is clarified in Appendix B. A distinct feature in D_r is that there are two spin representations, $V(\omega_{r-1})$ and $V(\omega_r)$, each having the quantum affine analogue $W_1^{(r-1)}$ and $W_1^{(r)}$, respectively. They are interchanged under the Dynkin diagram automorphism. In order to respect the symmetry under it, we modify the quantum spaces $\otimes_{j=1}^N W_1^{(p)}(w_j)$ for $p=r-1$ and r into $\otimes_{j=1}^N W_1^{(\pm)}(w_j)$, where $W_1^{(\pm)}(w) = W_1^{(r)}(w\mp 2)\otimes W_1^{(r-1)}(w\pm 2)$. Pictorially, one may view this as arranging the vertical lines on the square lattice endowed with the modules $V(\omega_r), V(\omega_{r-1})$ alternately and with the inhomogeneity as $w_1 \mp 2, w_1 \pm 2, w_2 \mp 2, w_2 \pm 2, \ldots$. This pattern has been introduced to utilize the degeneracy of the spin-conjugate spin R-matrix [32] $\mathrm{Im} R_{W_1^{(r)}, W_1^{(r-1)}}(u=4) \simeq V(\omega_r+\omega_{r-1})$, where the image becomes manifestly symmetric under the automorphism. The BAE (2.7) (with $s=1$) is thereby unchanged as long as $p=1,2,\ldots,r-2$. Instead of $p=r-1$ and r, we now take $p=\pm$, for which the BAE reads

$$-\frac{\phi_p^+(iu_k^{(a)}+\delta_{ar})\phi_p^-(iu_k^{(a)}+\delta_{ar-1})}{\phi_p^+(iu_k^{(a)}-\delta_{ar})\phi_p^-(iu_k^{(a)}-\delta_{ar-1})} = \prod_{b=1}^{r}\frac{Q_b(iu_k^{(a)}+(\alpha_a|\alpha_b))}{Q_b(iu_k^{(a)}-(\alpha_a|\alpha_b))} . \tag{5.1}$$

Here the functions in the lhs are defined via $\phi(u)$ (1.4b) by

$$\phi_{\pm}^{\pm}(u) = \phi(u+2), \quad \phi_{\mp}^{\pm}(u) = \phi(u-2) . \tag{5.2}$$

5.1. Eigenvalue $\Lambda_1^{(1)}(u)$. Let $\varepsilon_a, 1 \leqq a \leqq r$ be the orthonormal vectors $(\varepsilon_a|\varepsilon_b) = \delta_{ab}$ realizing the root system as follows:

$$\alpha_a = \begin{cases} \varepsilon_a - \varepsilon_{a+1} & \text{for } 1 \leqq a \leqq r-1 \\ \varepsilon_{r-1} + \varepsilon_r & \text{for } a = r \end{cases} ,$$

$$\omega_a = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_a & \text{for } 1 \leqq a \leqq r-2 \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r) & \text{for } a = r-1 \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r) & \text{for } a = r \end{cases} \tag{5.3}$$

The auxiliary space relevant to $\Lambda_1^{(1)}(u)$ is $W_1^{(1)} \simeq V(\omega_1)$ as an D_r-module by (2.4b). This is the vector representation, whose weights are all multiplicity-free and given by ε_a and $-\varepsilon_a (1 \leqq a \leqq r)$. By abbreviating them to a and $\bar{a}$, the set of weights and the DVF are given follows:

$$J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{1}\}, \tag{5.4}$$

$$\Lambda_1^{(1)}(u) = \sum_{a \in J} \boxed{a}\,. \tag{5.5}$$

This is formally the same with (3.2–3). The elementary boxes are defined by

$$\boxed{a} = \psi_a(u) \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \quad 1 \leqq a \leqq r-2,$$

$$\boxed{r-1} = \psi_{r-1}(u) \frac{Q_{r-2}(u+r)Q_{r-1}(u+r-3)Q_r(u+r-3)}{Q_{r-2}(u+r-2)Q_{r-1}(u+r-1)Q_r(u+r-1)},$$

$$\boxed{r} = \psi_r(u) \frac{Q_{r-1}(u+r+1)Q_r(u+r-3)}{Q_{r-1}(u+r-1)Q_r(u+r-1)},$$

$$\boxed{\bar{r}} = \psi_{\bar{r}}(u) \frac{Q_{r-1}(u+r-3)Q_r(u+r+1)}{Q_{r-1}(u+r-1)Q_r(u+r-1)},$$

$$\boxed{\overline{r-1}} = \psi_{\overline{r-1}}(u) \frac{Q_{r-2}(u+r-2)Q_{r-1}(u+r+1)Q_r(u+r+1)}{Q_{r-2}(u+r)Q_{r-1}(u+r-1)Q_r(u+r-1)},$$

$$\boxed{\bar{a}} = \psi_{\bar{a}}(u) \frac{Q_{a-1}(u+2r-a-3)Q_a(u+2r-a)}{Q_{a-1}(u+2r-a-1)Q_a(u+2r-a-2)} \quad 1 \leqq a \leqq r-2, \tag{5.6a}$$

where we have set $Q_0(u) = 1$. The vacuum part $\psi_a(u)$ depends on the quantum

space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$ and is given by
if $1 \leqq p \leqq r-2$

$$\psi_a(u) = \begin{cases} \phi(u+p+1)\phi(u+2r-p-1)\Phi^r_p(u) & \text{for } 1 \preceq a \preceq p \\ \phi(u+p-1)\phi(u+2r-p-1)\Phi^r_p(u) & \text{for } p+1 \preceq a \preceq \overline{p+1} \\ \phi(u+p-1)\phi(u+2r-p-3)\Phi^r_p(u) & \text{for } \overline{p} \preceq a \preceq \overline{1} \end{cases} ,$$

if $p = \pm$

$$\psi_a(u) = \begin{cases} \phi^+_p(u+r)\phi^-_p(u+r)\Phi^r_{r+1}(u) & \text{for } 1 \preceq a \preceq r-1 \\ \phi^+_p(u+r)\phi^-_p(u+r-2)\Phi^r_{r+1}(u) & \text{for } a = r \\ \phi^-_p(u+r)\phi^+_p(u+r-2)\Phi^r_{r+1}(u) & \text{for } a = \bar{r} \\ \phi^-_p(u+r-2)\phi^+_p(u+r-2)\Phi^r_{r+1}(u) & \text{for } \overline{r-1} \preceq a \preceq \overline{1} \end{cases} , \tag{5.6b}$$

$$\begin{aligned} \Phi^r_p(u) &= \prod_{j=1}^{p-1} \phi(u+p-2j-1)\phi(u+2r-p+2j-1) \\ &= \Phi^r_p(-2r+2-u)|_{w_k \to -w_k} . \end{aligned} \tag{5.6c}$$

The order $\prec$ in the set J (5.4) is specified by

$$1 \prec 2 \prec \cdots \prec r-1 \prec \begin{matrix} r \\ \bar{r} \end{matrix} \prec \overline{r-1} \prec \cdots \bar{2} \prec \bar{1} . \tag{5.7}$$

We impose no order between r and $\bar{r}$. The DVF (5.5) possesses all the features explained in Sect. 2.4. In particular it is pole-free under the BAE (2.7) and (5.1) thanks to the coupling rule (2.14). It can be summarized in the diagram

$$\begin{tikzcd}
 & & & & \boxed{r} \arrow[dr, "r"] & & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "r-2"] & \boxed{r-1} \arrow[ur, "r-1"] \arrow[dr, "r"'] & & \boxed{\overline{r-1}} \arrow[r, "r-2"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \\
 & & & & \boxed{\bar{r}} \arrow[ur, "r-1"'] & & & &
\end{tikzcd}$$

in the same sense with those in Sect. 3.1 and 4.1. This is again identical with the crystal graph [24,25]. For $p = 1$, the DVF (5.5–6) has been known earlier in [21].

5.2. Eigenvalue $\Lambda_1^{(a)}(u)$ for $1 \leqq a \leqq r-2$. For $1 \leqq a \leqq r-2$, let $\mathscr{T}_1^{(a)}$ be the set of the tableaux of the form (3.7a) with $i_k \in J$ (5.4) obeying the condition

$$i_k \prec i_{k+1} \text{ or } (i_k, i_{k+1}) = (r, \bar{r}) \text{ or } (i_k, i_{k+1}) = (\bar{r}, r) \quad \text{for any } 1 \leqq k \leqq a-1 . \tag{5.8}$$

We identify each element (3.7a) of $\mathscr{T}_1^{(a)}$ with the product of (5.6a) by the same rule as (4.8). Then the analytic Bethe ansatz yields the following DVF:

$$\Lambda_1^{(a)}(u) = \frac{1}{F_a^{(p,r)}(u)} \sum_{T \in \mathscr{T}_1^{(a)}} T, \quad 1 \leqq a \leqq r-2 , \tag{5.9a}$$

$$
\begin{aligned}
&F_a^{(p,r)}(u)\\
&=\begin{cases}\prod_{j=1}^{a-1}\psi_0^{(p,r)}(u+r-a-1+2j)\psi_p^{(p,r)}(u-r+a+1-2j) & \text{for } 1\leqq p\leqq r-2\\ \prod_{j=1}^{a-1}\psi_{0,+}^{(p,r)}(u+r-a-1+2j)\psi_{r-1,-}^{(p,r)}(u-r+a+1-2j) & \text{for } p=\pm\end{cases}\\
&=F_a^{(p,r)}(-2r+2-u)|_{w_k\to -w_k}\,,
\end{aligned}
\tag{5.9b}
$$

where $\psi_n^{(p,r)}(u)$ and $\psi_{n,\pm}^{(p,r)}(u)$ are specified in (5.15) in Sect. 5.4. Notice that $F_1^{(p,r)}(u)=1$, hence (5.9a) reduces to (5.5) when $a=1$. It can be shown that each summand T in (5.9a) contains the factor $F_a^{(p,r)}$. This will be seen manifestly in Theorem B.1. The DVF (5.9) for $\Lambda_1^{(a)}(u)$ is homogeneous w.r.t. ϕ of order $2p$ if $1\leqq p\leqq r-2$ and order $2r+2$ if $p=\pm$.

One can observe the top term and the crossing symmetry in the DVF (5.9) as done after (3.9). To check the character limit (2.16) is also similar to (4.11). From (2.4b) we must show (4.11c) again for $1\leqq a\leqq r-2$ under the absence of y_0 in (4.11b). But this is straightforward from (5.8) and by noting that the character formula (4.11d) is still valid for D_r if J is taken as (5.4).

By a similar method to Theorem 4.3.1 one can prove

Theorem 5.2.1. $\Lambda_1^{(a)}(u)(1\leqq a\leqq r-2)$ (5.9) *is free of poles provided that the BAE* (2.7) (*with* $s=1$) *for* $1\leqq p\leqq r-2$ *and* (5.1) *for* $p=\pm$ *are valid.*

5.3. Eigenvalue $\Lambda_m^{(1)}(u)$. Starting from (5.9) and the DVFs of $\Lambda_1^{(r)}(u),\Lambda_1^{(r-1)}(u)$ that will be given in Sect. 5.4, we are to solve the T-system (2.5c). The scalar $g_m^{(a)}(u)$ there is to be taken as (4.17) with $F_2^{(p,r)}(u)$ determined from (5.9b). This program is yet to be executed completely. Here we shall only present a conjecture on $\Lambda_m^{(1)}(u)$.

For $m\in\mathbf{Z}_{\geqq 1}$, let $\mathcal{T}_m^{(1)}$ denote the set of tableaux of the form

$$\boxed{i_1}\boxed{\;\cdots\;}\boxed{i_m}$$

with $i_k\in J$ (5.4) obeying the condition

$$
\begin{aligned}
&i_k\preceq i_{k+1}\quad\text{for any } 1\leqq k\leqq m-1\,,\\
&r \text{ and } \bar{r} \text{ do not appear simultaneously}\,.
\end{aligned}
\tag{5.10}
$$

We identify each element of $\mathcal{T}_m^{(1)}$ with the product $\prod_{k=1}^{m}\boxed{i_k}|_{u\to u-m-1+2k}$ of (5.6a). Then we have the conjecture

$$\Lambda_m^{(1)}(u)=\sum_{T\in\mathcal{T}_m^{(1)}}T\,,\tag{5.11}$$

which reduces to (5.5) when $m=1$. It is easy to prove $\sharp\mathcal{T}_m^{(1)}=\dim W_m^{(1)}$. We have checked (5.11) up to $m=4$ for D_4 and $m=3$ for D_5.

5.4. Eigenvalues $\Lambda_1^{(r-1)}(u)$ *and* $\Lambda_1^{(r)}(u)$. Now the relevant auxiliary spaces are $W_1^{(r-1)}\simeq V(\omega_{r-1})$ and $W_1^{(r)}\simeq V(\omega_r)$ as D_r-modules. They are the two spin representations, whose weights are all multiplicity-free and given by (4.21) for $V(\omega_{r-1})$

if $\mu_1\mu_2\cdots\mu_r = -$ and for $V(\omega_r)$ if $\mu_1\mu_2\cdots\mu_r = +$. As in the B_r case we shall build the boxes $\boxed{\mu_1,\mu_2,\ldots,\mu_r}$ by which the DVF can be written as

$$\Lambda_1^{(r-1)}(u) = \sum_{\left\{\mu_j=\pm;\prod_{j=1}^{r}\mu_j=-\right\}} \underset{p}{\overbrace{\boxed{\mu_1,\mu_2,\ldots,\mu_r}}^{r}}\,, \tag{5.12a}$$

$$\Lambda_1^{(r)}(u) = \sum_{\left\{\mu_j=\pm;\prod_{j=1}^{r}\mu_j=+\right\}} \underset{p}{\overbrace{\boxed{\mu_1,\mu_2,\ldots,\mu_r}}^{r}}\,. \tag{5.12b}$$

We let $\mathscr{T}_1^{(r-1)}$ and $\mathscr{T}_1^{(r)}$ denote the sets of $\dim W_1^{(r-1)} = \dim W_1^{(r)} = 2^{r-1}$ boxes in (5.12a) and (5.12b), respectively. The indices r and $p \in \{1,2,\ldots,r-2,+,-\}$ signify the rank of D_r and the quantum space $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$, respectively. The boxes are again defined by the recursion relations w.r.t. these indices. By using the operators (4.24), they read,

for $1 \leqq p \leqq r-2$,

$$\underset{p}{\overbrace{\boxed{+,+,\xi}}^{r}} = \phi(u+r+p-1)\tau^{Q}\underset{p-1}{\overbrace{\boxed{+,\xi}}^{r-1}}\,, \tag{5.13a}$$

$$\underset{p}{\overbrace{\boxed{+,-,\xi}}^{r}} = \phi(u+r+p-1)\frac{Q_1(u+r-3)}{Q_1(u+r-1)}\tau^{Q}\underset{p-1}{\overbrace{\boxed{-,\xi}}^{r-1}}\,, \tag{5.13b}$$

$$\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}} = \phi(u+r-p-1)\frac{Q_1(u+r+1)}{Q_1(u+r-1)}\tau_2^{u}\tau^{Q}\underset{p-1}{\overbrace{\boxed{+,\xi}}^{r-1}}\,, \tag{5.13c}$$

$$\underset{p}{\overbrace{\boxed{-,-,\xi}}^{r}} = \phi(u+r-p-1)\tau_2^{u}\tau^{Q}\underset{p-1}{\overbrace{\boxed{-,\xi}}^{r-1}}\,, \tag{5.13d}$$

for $p = \pm$,

$$\underset{p}{\overbrace{\boxed{+,+,\xi}}^{r}} = \phi(u+2r)\tau^{Q}\underset{p}{\overbrace{\boxed{+,\xi}}^{r-1}}\,, \tag{5.13e}$$

$$\underset{p}{\overbrace{\boxed{+,-,\xi}}^{r}} = \phi(u+2r)\frac{Q_1(u+r-3)}{Q_1(u+r-1)}\tau^{Q}\underset{p}{\overbrace{\boxed{-,\xi}}^{r-1}}\,, \tag{5.13f}$$

$$\underset{p}{\overbrace{\boxed{-,+,\xi}}^{r}} = \phi(u-2)\frac{Q_1(u+r+1)}{Q_1(u+r-1)}\tau_2^{u}\tau^{Q}\underset{p}{\overbrace{\boxed{+,\xi}}^{r-1}}\,, \tag{5.13g}$$

$$\underset{p}{\overbrace{\boxed{-,-,\xi}}^{r}} = \phi(u-2)\tau_2^{u}\tau^{Q}\underset{p}{\overbrace{\boxed{-,\xi}}^{r-1}}\,. \tag{5.13h}$$

Here ξ denotes arbitrary sequence of ± symbols with length $r-2$. The recursions (5.13) involve both boxes in $\mathscr{T}_1^{(r-1)}$ and $\mathscr{T}_1^{(r)}$ and hold for $r \geqq 5$. As in B_r case, we formally consider boxes with $p=0$ and fix them by (4.26b) and the convention explained after it. We are yet to specify the initial condition, i.e., data for D_4 case. As for the dress parts, they are given by

$$
\begin{aligned}
dr\,\boxed{+,+,+,+} &= \frac{Q_4(u-1)}{Q_4(u+1)}\,,\\
dr\,\boxed{+,+,-,-} &= \frac{Q_2(u)Q_4(u+3)}{Q_2(u+2)Q_4(u+1)}\,,\\
dr\,\boxed{+,-,+,-} &= \frac{Q_1(u+1)Q_2(u+4)Q_3(u+1)}{Q_1(u+3)Q_2(u+2)Q_3(u+3)}\,,\\
dr\,\boxed{+,-,-,+} &= \frac{Q_1(u+1)Q_3(u+5)}{Q_1(u+3)Q_3(u+3)}\,,\\
dr\,\boxed{-,+,+,-} &= \frac{Q_1(u+5)Q_3(u+1)}{Q_1(u+3)Q_3(u+3)}\,,\\
dr\,\boxed{-,+,-,+} &= \frac{Q_1(u+5)Q_2(u+2)Q_3(u+5)}{Q_1(u+3)Q_2(u+4)Q_3(u+3)}\,,\\
dr\,\boxed{-,-,+,+} &= \frac{Q_2(u+6)Q_4(u+3)}{Q_2(u+4)Q_4(u+5)}\,,\\
dr\,\boxed{-,-,-,-} &= \frac{Q_4(u+7)}{Q_4(u+5)}\,.
\end{aligned}
\tag{5.14a}
$$

The other 8 are deduced from the $r=4$ case of

$$
dr\,\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_{r-1},\mu_r}}^{r}} = dr\,\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_{r-1},-\mu_r}}^{r}}\,\Big|_{Q_{r-1}(u)\leftrightarrow Q_r(u)}\,. \tag{5.14b}
$$

This is consistent with the diagram symmetry and (5.13). As for the vacuum parts, we shall give their general form that includes the initial condition ($r=4$) and fulfills the recursions (5.13),

$$
vac\,\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = \begin{cases} \psi_n^{(p,r)}(u) & \text{for } 1 \leqq p \leqq r-2 \\ \psi_{n,\mu_r}^{(p,r)}(u) & \text{for } p = \pm \end{cases}\,, \tag{5.15a}
$$

$$
n = \begin{cases} \sharp\{j \mid \mu_j = -, 1 \leqq j \leqq p\} & \text{for } 1 \leqq p \leqq r-2 \\ \sharp\{j \mid \mu_j = -, 1 \leqq j \leqq r-1\} & \text{for } p = \pm \end{cases}\,, \tag{5.15b}
$$

$$\psi_n^{(p,r)}(u) = \prod_{\substack{j=0\\ j\neq n}}^{p} \phi(u+r-p+2j-1)\,, \tag{5.15c}$$

$$\psi_{n,+}^{(+,r)}(u) = \psi_{n,-}^{(-,r)}(u) = \prod_{\substack{j=0\\ j\neq n+1}}^{r+1} \phi(u+2j-2)\,, \tag{5.15d}$$

$$\psi_{n,-}^{(+,r)}(u) = \psi_{n,+}^{(-,r)}(u) = \phi(u+2n) \prod_{\substack{j=0\\ j\neq n,n+2}}^{r+1} \phi(u+2j-2)\,. \tag{5.15e}$$

By the definition, n ranges over $0 \leqq n \leqq p$ in (5.15c) and $0 \leqq n \leqq r-1$ in (5.15d,e). This completes the characterization of all the 2^r boxes hence the DVF (5.12) for any $r \geqq 4, p \in \{0,1,\ldots,r-2,+,-\}$. In the rational case ($q \to 1$) with $p = 1$, a similar recursive description is available in [5].

Let us list a few features explained in Sect. 2.4. Firstly, the top term (2.12) corresponds to

$$dr\, \underset{p}{\overbrace{\boxed{+,\ldots,+,\pm}}^{r}} = \frac{Q_{r-1+\frac{1\pm1}{2}}(u-1)}{Q_{r-1+\frac{1\pm1}{2}}(u+1)}\,, \tag{5.16}$$

where the lhs' are indeed associated with the highest weights ω_{r-1} and ω_r in view of (5.3) and (4.21). Secondly, the crossing symmetry (2.18, 19) holds.

$$\tau_{2r-2}^{C}\, \underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = \underset{p}{\overbrace{\boxed{-\mu_1,\ldots,-\mu_r}}^{r}} \quad \text{for } 1 \leqq p \leqq r-2\,,$$

$$\tau_{2r-2}^{C}\, \underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}} = \underset{-p}{\overbrace{\boxed{-\mu_1,\ldots,-\mu_r}}^{r}} \quad \text{for } p = \pm\,. \tag{5.17}$$

Thirdly, the coupling rule (2.14a) is valid due to

Lemma 5.4.1. *For* $1 \leqq a \leqq r-1$ *the factor* $1/Q_a$ *is contained in the box* $\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}$ *if and only if* $(\mu_a, \mu_{a+1}) = (+,-)$ *or* $(-,+)$*. Any two such boxes* $\underset{p}{\overbrace{\boxed{\eta,+,-,\xi}}^{r}}$ *and* $\underset{p}{\overbrace{\boxed{\eta,-,+,\xi}}^{r}}$ *share a common color a pole* $1/Q_a(u+y)$ *for some y. The factor* $1/Q_r$ *is contained in the box* $\underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}$ *if and only if* $\mu_{r-1} = \mu_r$*. Any two such boxes* $\underset{p}{\overbrace{\boxed{\zeta,+,+}}^{r}}$ *and* $\underset{p}{\overbrace{\boxed{\zeta,-,-}}^{r}}$ *share a common color r pole* $1/Q_r(u+z)$ *for some z.*

As introduced in the beginning of Sect. 4.6, let $\mathrm{BAE}^r_{p=0}$ be (2.7) with the lhs being always -1. Under the BAE, the pair of the coupled boxes yield zero residue in total. We claim this in

Theorem 5.4.2. *For* $1 \leqq a \leqq r-1$, *let* η, ξ *and* ζ *be any* $\pm$ *sequences with lengths* $a-1, r-a-1$ *and* $r-2$, *respectively. If the* BAE_p^r (2.7) (*with* $s=1$) *for* $0 \leqq p \leqq r-2$ *and* (5.1) *for* $p = \pm$ *are valid, then*

$$Res_{u=-y+iu_k^{(a)}} \left(\underset{p}{\overbrace{\boxed{\eta, +, -, \xi}}^{r}} + \underset{p}{\overbrace{\boxed{\eta, -, +, \xi}}^{r}} \right) = 0\,, \tag{5.18a}$$

$$Res_{u=-z+iu_k^{(r)}} \left(\underset{p}{\overbrace{\boxed{\zeta, +, +}}^{r}} + \underset{p}{\overbrace{\boxed{\zeta, -, -}}^{r}} \right) = 0\,, \tag{5.18b}$$

where y *and* z *are those in Lemma* 5.4.1.

The proof is similar to that for Theorem 4.6.3. In particular (2.14b) and (2.15) can be shown, therefore the character limit (2.16) is valid for the DVFs (5.12). (When $p = \pm$, one modifies the $\omega_{s=1}^{(p)}$ in (2.16) suitably.) Notice that both of the coupled boxes in (5.18) belong to the same set $\mathscr{T}_1^{(r-1)}$ or $\mathscr{T}_1^{(r)}$. Thus Lemma 5.4.1 and Theorem 5.4.2 lead to

Theorem 5.4.3. *For* $r \geqq 4$ *and* $p \in \{0, 1, \ldots, r-2, +, -\}$, $\Lambda_1^{(r-1)}(u)$ *and* $\Lambda_1^{(r)}(u)$ *in* (5.12) *are free of poles provided that the* BAE_p^r (2.7) (*with* $s=1$) *for* $0 \leqq p \leqq r-2$ *and* (5.1) *for* $p = \pm$ *are valid.*

6. Discussions

Let us indicate further applications of our approach. As can be observed through Sects. 2 to 5, the hypotheses called the top term (2.12) and the coupling rule (2.14), (2.15) severely restrict possible DVFs. This is especially significant when as many weight spaces as possible are multiplicity-free (2.13) in the auxiliary space. An interesting example of such a situation is the Yangian analogue of the adjoint representation. Below we exclude the case $X_r = A_r$, where the DVF for general eigenvalues is already available [19]. Then it is known [12, 28] that the Yangian $Y(X_r)$ admits the irreducible representation W_{adj} isomorphic to $V(\theta) \oplus V(0)$ as an X_r-module. Here θ denotes the highest root hence $V(\theta)$ means the adjoint representation of X_r. One can identify W_{adj} in the family $\{W_m^{(a)}\}$ by θ and the data in Appendix A of [33].

$$(\theta, W_{adj}) = \begin{cases} (\omega_1, W_1^{(1)}) & E_7, E_8, F_4, G_2 \\ (2\omega_1, W_2^{(1)}) & C_r \\ (\omega_2, W_1^{(2)}) & B_r, D_r \\ (\omega_6, W_1^{(6)}) & E_6 \end{cases}$$

Thus the cases $X_r = B_r, C_r$ and D_r are already covered in this paper. For G_2, the DVF of $\Lambda_1^{(1)}(u)$ has been obtained recently [11]. Let us turn to the remaining cases, $\Lambda_1^{(1)}(u)$ of $E_{7,8}, F_4$ and $\Lambda_1^{(6)}(u)$ of E_6. By the definition, $\dim W_{adj} = \dim X_r + 1$. All the weights in W_{adj} are multiplicity-free except the null one, $\mathrm{mult}_0 W_{adj} = r + 1$. Thus one may try to apply the top term (2.12), the coupling rule (2.14,15) and the crossing symmetry (2.18) to possibly determine the $\dim X_r - r$ terms in the DVF corresponding to the root vectors. We have checked that this certainly works consistently and fix those terms uniquely. Moreover, we have found that pole-freeness under the BAE requires exactly $r+1$ more terms that make the null weight contribution $(r+1)q^0$ in the character limit (2.16). These features are equally valid in the trigonometric case as well. Thus the resulting DVFs are candidates of the transfer matrix eigenvalues for the trigonometric vertex models associated with $U_q(E_8^{(1)})$, etc. The details will appear elsewhere. It still remains to understand the hypotheses (2.12), (2.14) and (2.18) intrinsically and thus to unveil the full aspects of the analytic Bethe ansatz.

Appendix A. Relations Between Two Kinds of Boxes in B_r

Here we clarify the relation between the two kinds of the boxes $\boxed{a}$ and $\boxed{\mu_1,\ldots,\mu_r}$ introduced in Sect. 4.1 and 4.5, respectively. In terms of the relevant auxiliary spaces, they are associated with the vector and the spin representations. To infer their relation, recall the classical tensor product decomposition

$$V(\omega_r)\otimes V(\omega_r) = V(2\omega_r)\oplus V(\omega_{r-1})\oplus\cdots\oplus V(\omega_1)\oplus V(0)\,. \tag{A.1}$$

Correspondingly, there exists an $U_q(B_r^{(1)})$ quantum R-matrix $R_{W_1^{(r)},W_1^{(r)}}(u)$ [32] acting on the q-analogue of the above. On each component $V(\omega)$ of the rhs, it acts as a constant $\rho_\omega(u)$ that depends on the spectral parameter u. A little investigation of the spectrum $\rho_\omega(u)$ in [32] tells that only $\rho_{\omega_a}(u), \rho_{\omega_{a-2}}(u),\ldots$ are non-zero at $u = -2(r-a)+1$ for $1\leqq a\leqq r-1$. From this and (2.4b) we see that the specialized R-matrix $R_{W_1^{(r)},W_1^{(r)}}(-2(r-a)+1)$ yields the embedding

$$W_1^{(a)}(u)\hookrightarrow W_1^{(r)}\left(u+r-a-\frac{1}{2}\right)\otimes W_1^{(r)}\left(u-r+a+\frac{1}{2}\right) \tag{A.2}$$

in the notation of [8]. According to the arguments there, (A.2) imposes the following functional relation among the transfer matrices having the relevant auxiliary spaces:

$$\begin{aligned} &T_1^{(r)}\left(u+r-a-\frac{1}{2}\right)T_1^{(r)}\left(u-r+a+\frac{1}{2}\right)\\ &\quad = T_1^{(a)}(u)+T'(u)\quad \text{for } 1\leqq a\leqq r-1\,. \end{aligned} \tag{A.3}$$

Here $T'(u)$ denotes some matrix commuting with all $T_m^{(b)}(v)$'s. When $a=r-1$, (A.3) is just the last equation in (2.5a) with $m=0$, hence $T'(u)=T_2^{(r)}(u)$. Viewed as a relation among the eigenvalues, (A.3) implies that each term in the DVF (4.9a) can be expressed as a product of certain two boxes in Sect. 4.5 with the spectral parameters $u+r-a-\frac{1}{2}$ and $u-r+a+\frac{1}{2}$. Actually we have

Theorem A.1. *For* $1\leqq a\leqq r-1$, $k,n,l\in \mathbf{Z}_{\geqq 0}$ *such that* $k+n+l=a$, *take any integers* $1\leqq i_1<\cdots<i_k\leqq r$ *and* $1\leqq j_1<\cdots<j_l\leqq r$. *Then the following equality holds between the elements of* $\mathscr{T}_1^{(a)}$ *and* $\mathscr{T}_1^{(r)}$ *defined in* (4.7, 8) *and* (4.25, 26), *respectively,*

$$\frac{1}{F_a^{(p,r)}(u)} \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline 0 \\ \hline \vdots \\ \hline 0 \\ \hline \bar{j}_l \\ \hline \vdots \\ \hline \bar{j}_1 \\ \hline \end{array} = \left(\tau^u_{-r+a+\frac{1}{2}} \underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}\right)\left(\tau^u_{r-a-\frac{1}{2}} \underset{p}{\overbrace{\boxed{\nu_1,\ldots,\nu_r}}^{r}}\right), \tag{A.4a}$$

where there are $n\,\boxed{0}$*'s in the lhs and* $F_a^{(p,r)}(u)$ *is defined in* (4.9b) *and* (4.10). *The* $\pm$ *sequences in the rhs are specified by*

$$\mu_b = \begin{cases} + & \text{if } b \in \{i_1,\ldots,i_k\} \\ - & \text{otherwise} \end{cases}, \quad \nu_b = \begin{cases} - & \text{if } b \in \{j_1,\ldots,j_l\} \\ + & \text{otherwise} \end{cases}. \tag{A.4b}$$

Note that both sides of (A.4a) are of order $2p$ w.r.t. ϕ and carry the same weight $\varepsilon_{i_1} + \cdots + \varepsilon_{i_k} - \varepsilon_{j_1} - \cdots - \varepsilon_{j_l}$. The theorem is proved by induction on the rank r.

Appendix B. Relations Between Two Kinds of Boxes in D_r

The elementary boxes $\boxed{a}$ and $\boxed{\mu_1,\ldots,\mu_r}$ introduced in Sects. 5.1 and 5.4 are related by

Theorem B.1. *For* $1 \leqq a \leqq r-2, k, n, l \in \mathbf{Z}_{\geqq 0}$ *such that* $k + 2n + l = a$, *take any integers* $1 \leqq i_1 < \cdots < i_k \leqq r$ *and* $1 \leqq j_1 < \cdots < j_l \leqq r$. *Then the following equality holds between the elements of* $\mathscr{T}_1^{(a)}$ *and* $\mathscr{T}_1^{(r-1)} \cup \mathscr{T}_1^{(r)}$ *defined in Sect.* 5.2 *and* (5.13–15), *respectively.*

$$\frac{1}{F_a^{(p,r)}(u)} \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \bar{r} \\ \hline \vdots \\ \hline r \\ \hline \bar{j}_l \\ \hline \vdots \\ \hline \bar{j}_1 \\ \hline \end{array} = \left(\tau^u_{-r+a+1} \underset{p}{\overbrace{\boxed{\mu_1,\ldots,\mu_r}}^{r}}\right)\left(\tau^u_{r-a-1} \underset{p}{\overbrace{\boxed{\nu_1,\ldots,\nu_r}}^{r}}\right), \tag{B.1}$$

where there are n $\boxed{\begin{array}{c}\bar{r}\\ \hline r\end{array}}$ *'s in the lhs and* $F_a^{(p,r)}$ *is defined in* (5.9b). *The* ± *sequences* μ *and* ν *in the rhs are determined by* (A.4b).

Put $a \equiv r + \sigma \bmod 2$ where $\sigma = 0$ or 1. Then the tableaux in the rhs of (B.1) belong to the following sets:

$$\underset{p}{\overbrace{\boxed{\quad\mu\quad}}^{r}} \in \begin{cases} \mathscr{T}_1^{(r-\sigma)} & l \text{ even} \\ \mathscr{T}_1^{(r-1+\sigma)} & l \text{ odd} \end{cases}, \quad \underset{p}{\overbrace{\boxed{\quad\nu\quad}}^{r}} \in \begin{cases} \mathscr{T}_1^{(r)} & l \text{ even} \\ \mathscr{T}_1^{(r-1)} & l \text{ odd} \end{cases}. \tag{B.2}$$

One can rewrite the rhs of (B.1) so as to interchange the parity of l in (B.2). Given any ± sequences $\mu = (\mu_1, \dots, \mu_r)$ and $\nu = (\nu_1, \dots, \nu_r)$, we set

$$e_k(\mu, \nu) = \sharp\{j \mid 1 \leqq j \leqq k, \mu_j = -\} - \sharp\{j \mid 1 \leqq j \leqq k, \nu_j = -\}, \tag{B.3a}$$

$$d_y(\mu, \nu) = \min(\{\infty\} \cup \{k \mid 1 \leqq k \leqq r-1, e_k(\mu, \nu) = y\}). \tag{B.3b}$$

Then we have

Lemma B.2. *For any* $1 \leqq a \leqq r-2$ *and any* ± *sequences* $\mu = (\mu_1, \dots, \mu_r), \nu = (\nu_1, \dots, \nu_r)$, *one has*

$$\begin{aligned} &\Big(\tau^u_{-r+a+1} \underset{p}{\overbrace{\boxed{\mu_1, \dots, \mu_r}}^{r}}\Big)\Big(\tau^u_{r-a-1} \underset{p}{\overbrace{\boxed{\nu_1, \dots, \nu_r}}^{r}}\Big) \\ &\quad = \Big(\tau^u_{-r+a+1} \underset{p}{\overbrace{\boxed{\mu'_1, \dots, \mu'_r}}^{r}}\Big)\Big(\tau^u_{r-a-1} \underset{p}{\overbrace{\boxed{\nu'_1, \dots, \nu'_r}}^{r}}\Big), \end{aligned} \tag{B.4a}$$

where μ'_j *and* ν'_j *are determined by*

$$(\mu'_j, \nu'_j) = \begin{cases} (\mu_j, \nu_j) & \text{if } 1 \leqq j \leqq d_{r-a-1}(\mu, \nu) \\ (\nu_j, \mu_j) & \text{otherwise} \end{cases}. \tag{B.4b}$$

The lemma enables the interchange of those μ_j and ν_j with $j > d_{r-a-1}(\mu, \nu)$ in the products (B.4a). In case $d_{r-a-1}(\mu, \nu) = \infty$, the assertion is trivial. One may apply Lemma B.2 to rewrite the rhs of (B.1). A little inspection tells that $1 \leqq d_{r-a-1}(\mu, \nu) \leqq r-1$ for any of those μ and ν appearing there. Moreover, for such $d = d_{r-a-1}(\mu, \nu)$ one can evaluate the difference

$$\sharp\{j \mid d < j \leqq r, \mu_j = -\} - \sharp\{j \mid d < j \leqq r, \nu_j = -\} = 2n + 1 \in 2\mathbf{Z} + 1,$$

in terms of the n in Theorem B.1. Thus Lemma B.2 expresses the rhs of (B.1) by the tableaux such that

$$\underset{p}{\overbrace{\boxed{\quad\mu'\quad}}^{r}} \in \begin{cases} \mathscr{T}_1^{(r-1+\sigma)} & l \text{ even} \\ \mathscr{T}_1^{(r-\sigma)} & l \text{ odd} \end{cases}, \quad \underset{p}{\overbrace{\boxed{\quad\nu'\quad}}^{r}} \in \begin{cases} \mathscr{T}_1^{(r-1)} & l \text{ even} \\ \mathscr{T}_1^{(r)} & l \text{ odd} \end{cases}, \tag{B.5}$$

which is opposite to (B.2). Based on these observations, we can give a similar argument to Appendix A that backgrounds Theorem B.1. There is a degeneracy point $u = -2(r-a-1)$ of the $U_q(D_r^{(1)})$ quantum R-matrix [32] where it yields embedding

$$W_1^{(a)}(u) \hookrightarrow W_1^{(r-1)}(u+r-a-1) \otimes W_1^{(r-1+\sigma)}(u-r+a+1)\,,$$

$$W_1^{(a)}(u) \hookrightarrow W_1^{(r)}(u+r-a-1) \otimes W_1^{(r-\sigma)}(u-r+a+1)\,. \qquad \text{(B.6)}$$

According to [8], (B.6) implies the functional relations

$$T_1^{(r-1)}(u+r-a-1)T_1^{(r-1+\sigma)}(u-r+a+1) = T_1^{(a)}(u) + T'(u)\,, \qquad \text{(B.7a)}$$

$$T_1^{(r)}(u+r-a-1)T_1^{(r-\sigma)}(u-r+a+1) = T_1^{(a)}(u) + T''(u)\,, \qquad \text{(B.7b)}$$

where $T'(u)$ and $T''(u)$ are some matrices commuting with all $T_m^{(b)}(v)$'s. In particular if $a = r-2 (\sigma = 0)$, (B.7) is the last equation in (2.5c) with $m = 1$, hence $T'(u) = T_2^{(r-1)}(u)$ and $T''(u) = T_2^{(r)}(u)$. One may regard (B.7a,b) as equations on the eigenvalues and substitute (5.9a) and (5.12). Then Theorem B.1 tells how one can pick up the DVF for $\Lambda_1^{(a)}(u)$ from the lhs. For example in (B.7a), one depicts the terms in $\Lambda_1^{(a)}(u)$ as the lhs of (B.1). Then the l odd terms are indeed contained in $\Lambda_1^{(r-1)}(u+r-a-1)\Lambda_1^{(r-1+\sigma)}(u-r+a+1)$ due to (B.1) and (B.2). The l even terms can also be found by expressing the above product in terms of the tableaux in (B.5).

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Note added in proof. The conjecture (4.20) has now been proved.

References

1. Bethe, H.A.: Zur Theorie der Metalle, I. Eigenwerte und Eigenfunktionen der linearen Atomkette. Z. Physik **71**, 205–226 (1931)
2. Yang, C.N., Yang, C.P.: Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction. J. Math. Phys. **10**, 1115–1122 (1969)
3. Sklyanin, E.K., Takhtajan, L.A., Faddeev, L.D.: Quantum inverse problem method I. Theor. Math. Phys. **40**, 688–706 (1980); Kulish, P.P., Sklyanin, E.K.: Quantum spectral transform method. Recent developments. Lect. Note. Phys. **151**, 61–119 Springer, 1982
4. Baxter, R.J.: Partition function of the eight-vertex lattice model. Ann. Phys. **70**, 193–228 (1972)
5. Reshetikhin, N.Yu.: The functional equation method in the theory of exactly soluble quantum systems. Sov. Phys. JETP **57**, 691–696 (1983)
6. Sklyanin, E.K.: Quantum inverse scattering method. Selected topics. In: Quantum Group and Quantum Integrable Systems. ed. Mo-Lin Ge, Singapore: World Scientific, 1992
7. Paz, R.: Feynman's office: The last blackboards. Phys. Today **42**, 88 (1989)
8. Kuniba, A., Nakanishi, T., Suzuki, J.: Functional relations in solvable lattice models I: Functional relations and representation theory. Int. J. Mod. Phys. **A9** (1994) 5215–5266

9. Kuniba, A., Nakanishi, T., Suzuki, J.: Functional relations in solvable lattice models II: Applications. Int. J. Mod. Phys. **A9** (1994) 5267–5312
10. Kuniba, A.: Analytic Bethe ansatz and T-system in $C_2^{(1)}$ vertex models. J. Phys. A: Math. Gen. **27**, L113-L118 (1994)
11. Suzuki, J.: Fusion $U_q(G_2^{(1)})$vertex models and analytic Bethe ansätze. Phys. Lett. **A195** (1994) 190–197
12. Drinfel'd, V.G.: Hopf algebras and quantum Yang–Baxter equation. Sov. Math. Dokl. **32**, 254–258 (1985)
13. Drinfel'd, V.G.: Quantum groups. ICM Proceedings, Berkeley, pp. 798–820 (1986)
14. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang–Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
15. Kac, V.G.: Infinite Dimensional Lie Algebras. Cambridge: Cambridge University Press, 1990
16. Kirillov, A.N., Reshetikhin, N.Yu.: Representations of Yangians and multiplicity of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras. J. Sov. Math. **52**, 3156–3164 (1990)
17. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. London: Academic Press, 1982
18. Kirillov, A.N., Reshetikhin, N.Yu.: Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I. The ground state and the excitation spectrum. J. Phys. A: Math. Gen. **20**, 1565–1585 (1987)
19. Bazhanov, V.V., Reshetikhin, N.Yu.: Restricted solid on solid models connected with simply laced algebras and conformal field theory. J. Phys. A: Math. Gen. **23**, 1477–1492 (1990)
20. Reshetikhin, N.Yu.: Integrable models of quantum one-dimensional magnets with $O(n)$ and $Sp(2k)$ symmetry. Theor. Math. Phys. **63**, 555–569 (1985)
21. Reshetikhin, N.Yu.: The spectrum of the transfer matrices connected with Kac–Moody algebras. Lett. Math. Phys. **14**, 235–246 (1987)
22. Klümper, A., Pearce, P.A.: Conformal weights of RSOS lattice models and their fusion hierarchies. Physica **A183**, 304–350 (1992)
23. Jimbo, M., Kuniba, A., Miwa, T., Okado, M.: The $A_n^{(1)}$ face models. Commun. Math. Phys. **119**, 543–565 (1988)
24. Kashiwara, M., Nakashima, T.: Crystal graphs for representations of the q-analogue of classical Lie algebras. RIMS preprint **767** (1991)
25. Nakashima, T.: Crystal base and a generalization of the Littlewood–Richardson rule for the classical Lie algebras. Commun. Math. Phys. **154**, 215–243 (1993)
26. Ogievetsky, E.I., Wiegmann, P.B.: Factorized S-matrix and the Bethe ansatz for simple Lie groups. Phys. Lett. **B168**, 360–366 (1986)
27. Drinfel'd, V.G.: A new realization of Yangians and quantum affine algebras. Sov. Math. Dokl. **36**, 212–216 (1988)
28. Chari, V., Pressley, A.: Fundamental representations of Yangians and singularities of R-matrices. J. reine angew. Math. **417**, 87–128 (1991)
29. Kuniba, A., Nakanishi, T.: Spectra in conformal field theories from the Rodgers dilogarithm. Mod. Phys. Lett. **A7**, 3487–3494 (1992)
30. Kirillov, A.N.: Identities for the Rodgers dilogarithm function connected with simple Lie algebras. J. Sov. Math. **47**, 2450–2459 (1989)
31. Kuniba, A., Nakanishi, T., Suzuki, J.: Characters in conformal field theories from thermodynamic Bethe ansatz. Mod. Phys. Lett. **A8**, 1649–1659 (1993)
32. Reshetikhin, N.Yu.: Algebraic Bethe ansatz for $SO(n)$ invariant transfer-matrices. Zapiski nauch. LOMI **169**, 122–140 (1984) (in Russian); Okado, M.: Quantum R matrices related to the spin representation of B_r and D_r. Commun. Math. Phys. **134**, 467–486 (1990)
33. Kuniba, A.: Thermodynamics of the $U_q(X_r^{(1)})$ Bethe ansatz system with q a root of unity. Nucl. Phys. **B389**, 209–244 (1993)

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