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Invariants of 3-Manifolds and Projective Representations of Mapping Class Groups via Quantum Groups at Roots of Unity

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Abstract: An example of a finite dimensional factorizable ribbon Hopf $\mathbb{C}$-algebra is given by a quotient $H = u_q(\mathfrak{g})$ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ at a root of unity q of odd degree. The mapping class group $M_{g,1}$ of a surface of genus g with one hole projectively acts by automorphisms in the H-module $H^{*\otimes g}$, if H^* is endowed with the coadjoint H-module structure. There exists a projective representation of the mapping class group $M_{g,n}$ of a surface of genus g with n holes labeled by finite dimensional H-modules $X_1, \dots, X_n$ in the vector space $\mathrm{Hom}_H(X_1 \otimes \cdots \otimes X_n, H^{*\otimes g})$. An invariant of closed oriented 3-manifolds is constructed. Modifications of these constructions for a class of ribbon Hopf algebras satisfying weaker conditions than factorizability (including most of $u_q(\mathfrak{g})$ at roots of unity q of even degree) are described.

After works of Moore and Seiberg [44], Witten [63], Reshetikhin and Turaev [51], Walker [62], Kohno [22, 23] and Turaev [60] it became clear that any semisimple abelian ribbon category with a finite number of simple objects satisfying some non-degeneracy condition gives rise to projective representations of mapping class groups of surfaces as well as to invariants of closed 3-manifolds. It was proposed in [38] to get rid of semisimplicity and to extend so the class of categories which serve as the set of labels for a modular functor.

In this article we describe (eventually non-semisimple) ribbon Hopf algebras H, whose modules form a category with the required properties, thereby giving representations of mapping class groups. These algebras are called 2-modular. All finite dimensional factorizable ribbon Hopf algebras have those properties.

As a byproduct we obtain a projective representation of the mapping class group $M_{g,1}$ of a surface of genus g with one hole in the vector space $H^{*\otimes g}$. If H^* is endowed with the coadjoint H-module structure, $M_{g,1}$ acts by automorphisms of the H-module. For genus 1 and factorizable Hopf algebras this representation was obtained by Majid and the author [40]. In the case of Drinfeld's doubles another proof of modular relations for genus 1 was given by Kerler [16]. The projective

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representations of $M_{1,1}$ thus obtained for finite dimensional $H = u_q(\mathfrak{sl}(2))$ are very close to those of Crivelli, Felder and Wieczerkowski [3], which come from conformal field theories on the torus based on $SU(2)$.

We describe also an intermediate class of categories and Hopf algebras between factorizable and 2-modular ones. These categories and Hopf algebras are called 3-modular and they give rise to invariants of closed oriented 3-manifolds. In the case of semisimple factorizable categories this is the Reshetikhin–Turaev invariant [51]. In the case of Hopf algebras it turns out to be the Hennings' invariant [10] (in the form of Kauffman and Radford [14]).

The results of the paper apply to both main known classes of ribbon categories: semisimple ones and the categories of all modules over a ribbon Hopf algebra. For the former we obtain already known results, the latter gives new representations. Although the language of abelian tensor categories is the most suitable for our purposes, the reader is advised to restrict consideration to the categories of modules.

Finite dimensional quotients $u_q(\mathfrak{g})$ of quantized universal enveloping algebras $U_q(\mathfrak{g})$ at roots q of unity are studied as an example. We show that if the degree l of the root q of unity is odd, the algebra is factorizable, if it is even, then $u_q(\mathfrak{g})$ will be 2-modular or not, depending on arithmetical properties of q.

Conformal field theories is a powerful source of ribbon categories. Kazhdan and Lusztig constructed non-trivial braided tensor subcategories in the category of modules over an affine Lie algebra [15], motivated by CFT. These categories can be semisimple as well as not. Furthermore, Gaberdiel [9] associates with each CFT a braided tensor category, namely, the category of modules over the chiral symmetry algebra with a non-standard tensor product. By the very nature of CFT one expects [44] appearance of representations of mapping class groups (this is obvious for TQFT). It turns out [38] that such representations can be constructed from ribbon categories even if they are not describing the fusion in some CFT.

Turaev proved that in the semisimple case the modular functor extends to a TQFT [60]. Moreover, he constructs the modular functor (that is, representations of mapping class groups) as a part of a bigger functor (TQFT), assigning linear maps to 3-cobordisms. In the non-semisimple case the word-by-word repetition of his approach is impossible, which forces one to seek for a direct construction of the modular functor as was done in [38]. Besides, some remnants of TQFT-structure survive in non-semisimple case; this is under consideration now.

We recall basic facts about ribbon abelian categories in Sect. 1. The quantum Fourier transform is discussed in Sect. 2. Ordinary ribbon Hopf algebras produce braided Hopf algebras in Sect. 3. Representations of mapping class groups are described in Sect. 4. New invariants of closed 3-manifolds are discussed in Sect. 5. We construct finite dimensional ribbon Hopf algebras $u_q(\mathfrak{g})$ in Appendix A and single out factorizable and 3-modular ones in Appendix B.

Acknowledgement. I am grateful to R. R. Hall, T. H. Jackson and A. Sudbery for useful discussions of number theory questions, which arose in this paper. I thank C. De Concini for discussions of 3-manifold invariants and the structure of quantum groups at roots of unity. I am indebted to the referee for a useful remark which simplified the proof of one of the theorems and for suggestions which improved the paper.

1. Introduction

1.1. Notations and conventions. k denotes a field. In this paper a *Hopf algebra* means a k-bialgebra with an invertible antipode. Associative comultiplication is denoted $\Delta x = x_{(1)} \otimes x_{(2)}$, counit is denoted by ε, antipode in Hopf algebras is denoted γ. If H is a Hopf algebra, H^{op} denotes the same coalgebra H with opposite multiplication, H_{op} denotes the same algebra H with the opposite comultiplication. The category of H-modules is denoted H-Mod, and its subcategory of finite dimensional H-modules is denoted H-mod. In the particular case $H = k$ we use k-Vect and k-vect respectively. The category of H-comodules is denoted H-Comod, and its subcategory of finite dimensional H-comodules is denoted H-comod. The left adjoint action of $x \in H$ in a Hopf algebra H means

$$\operatorname{ad} x.y = x_{(1)} y \gamma(x_{(2)}),$$

where $y \in H$.

Let X be an H-module, denote X^* the space of linear functionals on X. Denote by $X^\vee$ and ${}^\vee X$ the two different structures of the H-module on X^*, the former being $(h.\xi)(x) = \xi(\gamma^{-1}(h).x)$, the latter being $(h.\xi)(x) = \xi(\gamma(h).x)$ for $h \in H$, $\xi \in X^*$, $x \in X$. Iterating this definition we get $X^{\vee\vee}$, ${}^{\vee\vee}X$. Notice that $({}^\vee X)^\vee$ and ${}^\vee(X^\vee)$ are naturally identified with X, so we can use the general notation $X^{(m\vee)}$, $m \in \mathbb{Z}$, such that $X^{(-2\vee)} = {}^{\vee\vee}X$, $X^{(-1\vee)} = {}^\vee X$, $X^{(0\vee)} = X$, $X^{(1\vee)} = X^\vee$, $X^{(2\vee)} = X^{\vee\vee}$, etc.

$\mathfrak{g}$ will denote a complex semi-simple Lie algebra of rank n with Borel and Cartan subalgebras $\mathfrak{b}_+$, $\mathfrak{b}_-$, $\mathfrak{h}$. The root lattice, generated by the simple roots $\alpha_1, \dots, \alpha_n$, will be denoted Q. The weight lattice, generated by the fundamental weights $\omega_1, \dots, \omega_n$ is denoted P. We write the group Q also in multiplicative notations $K_\alpha = \alpha \in Q$, using $K_i = \alpha_i$ as generators. There is a perfect bilinear pairing

$$\langle , \rangle : Q \times P \to \mathbb{Z}, \qquad \langle \alpha_i, \omega_j \rangle = \delta_{ij}.$$

The Cartan matrix a_{ij} determines an inclusion

$$Q \hookrightarrow P, \qquad \alpha_j = \sum_{i=1}^{r} a_{ij} \omega_i,$$

and the inner product

$$(|) : Q \times Q \to \mathbb{Z}, \qquad (\alpha_i | \alpha_j) = d_i a_{ij},$$

where $d_i = 1, 2, 3$.

q will denote an indeterminate or a primitive l^{th} root of unity $q = \varepsilon \in \mathbb{C}$. This root of unity is assumed to satisfy $\varepsilon^{2m} \neq 1$ for all $1 \leqslant m \leqslant \max_i d_i$. Let l_i be the smallest positive integers such that $\varepsilon^{2d_i l_i} = 1$. We use the following notations for q-numbers:

$$(n)_q = \frac{q^n - 1}{q - 1}, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$(n)_q! = \prod_{m=1}^{n} (m)_q, \qquad [n]_q! = \prod_{m=1}^{n} [m]_q.$$

$U_h(\mathfrak{g})$ (resp. $U_q(\mathfrak{g})$) is a topological $\mathbb{C}[[h]]$-algebra (resp. $\mathbb{Q}(q)$-algebra), generated by H_i (resp. $K_i^{\pm 1}$), E_i, F_i, the quantum group of Drinfeld [5] and Jimbo [11]. Equipped with the comultiplication

$$\Delta H_i = H_i \otimes 1 + 1 \otimes H_i, \ \ \Delta E_i = E_i \otimes 1 + K_i^{-1} \otimes E_i, \ \ \Delta F_i = F_i \otimes K_i + 1 \otimes F_i$$

(in $U_h(\mathfrak{g})$ K_i denotes $e^{hd_iH_i}$) these algebras become Hopf algebras. Lusztig's divided powers algebra $\Gamma(\mathfrak{g})$ [33] is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_i^{(m)} = E_i^m/[m]_{q_i}!$, $F_i^{(m)} = F_i^m/[m]_{q_i}!$ and some Laurent polynomials of K_i, where $q_i = q^{d_i}$.

Choosing a reduced expression $s_{i_1}s_{i_2}\dots s_{i_N}$ of the longest element w_0 of the Weyl group of $\mathfrak{g}$, we get a total ordering of the positive part Δ^+ of the root system Δ:

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = s_{i_1}\alpha_2, \ \dots, \ \beta_N = s_{i_1}\dots s_{i_{N-1}}\alpha_{i_N}.$$

Following [20, 25, 33] introduce the corresponding root vectors in $\Gamma(\mathfrak{g})$

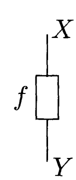
$$E_{\beta_k} = T_{i_1}\dots T_{i_{k-1}}E_{i_k}, \qquad F_{\beta_k} = T_{i_1}\dots T_{i_{k-1}}F_{i_k},$$

where $T_i : \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ are Lusztig's automorphisms [33, 34]. In the products like $\prod_\alpha E_\alpha^{m_\alpha}$ we always assume that α runs over Δ^+ according to the above total order. We use also $q_{\beta_k} = q_{i_k}$ and $l_{\beta_k} = l_{i_k}$.

An R-matrix will be often denoted $R = \sum_i R_i' \otimes R_i''$.

1.2. Ribbon abelian categories. *Ribbon* (also *tortile* [57]) category is the following thing: a braided monoidal category $\mathscr{C}$ [12] with the tensor product $\otimes$, the associativity $a : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, the braiding (commutativity) $c : X \otimes Y \to Y \otimes X$ and a unit object I, such that $\mathscr{C}$ is rigid (for any object $X \in \mathscr{C}$ there are dual objects ${}^\vee X$ and $X^\vee$ with evaluations $\mathrm{ev} : {}^\vee X \otimes X \to I$, $\mathrm{ev} : X \otimes X^\vee \to I$ and coevaluations $\mathrm{coev} : I \to X \otimes {}^\vee X$, $\mathrm{coev} : I \to X^\vee \otimes X$) and possess a ribbon twist ν. A ribbon twist [12, 50, 57] $\nu = \nu_X : X \to X$ is a self-adjoint ($\nu_{X^\vee} = \nu_X^t$) functorial automorphism such that $c^2 = \nu_X^{-1} \otimes \nu_Y^{-1} \circ \nu_{X \otimes Y}$.

Morphisms constructed from braidings and (co)evaluations are often described by tangles. In conventions of [36] we denote

a morphism $f : X \to Y$ by 

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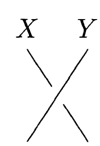
the braiding $c : X \otimes Y \to Y \otimes X$ by 

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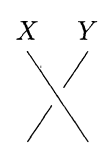
the inverse braiding $c^{-1} : X \otimes Y \to Y \otimes X$ by 

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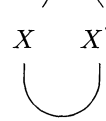
the evaluation $\mathrm{ev} : X \otimes X^\vee \to k$ by 

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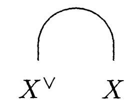
the coevaluation $\mathrm{coev} : k \to X^\vee \otimes X$ by 

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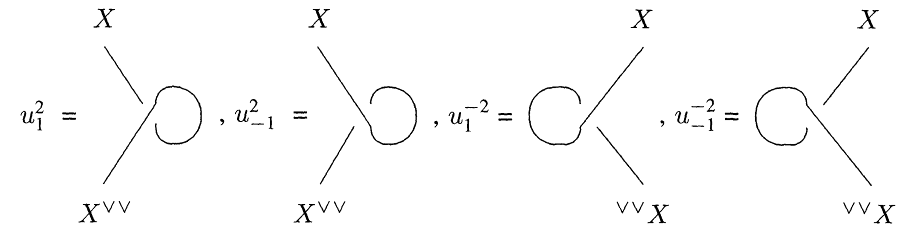
Consistency of these notations is due to the functor Φ from the category of $\mathscr{C}$-colored tangles to the category $\mathscr{C}$ itself [8].

In a ribbon category there are functorial isomorphisms [36]



$$u_0^2 = u_1^2 \circ \nu^{-1} = u_{-1}^2 \circ \nu : X \to X^{\vee\vee}, \quad u_0^{-2} = u_1^{-2} \circ \nu^{-1} = u_{-1}^{-2} \circ \nu : X \to {}^{\vee\vee}X.$$

Changing the category $\mathscr{C}$ by an equivalent one, we can (and we will) assume that ${}^\vee X = X^\vee$, $X^{\vee\vee} = {}^{\vee\vee}X = X$ and $u_0^2 = u_0^{-2} = \mathrm{id}_X$ (see [36]).

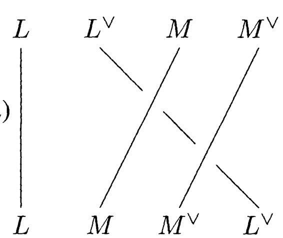
Warning. In the category $\mathscr{C} = H$-mod, where H is a ribbon Hopf algebra, the equation $X^\vee = {}^\vee X$ is not satisfied, nevertheless $X^\vee$ is canonically isomorphic to ${}^\vee X$. We identify these modules via $u_0^2 : {}^\vee X \to X^\vee$ (see Sect. 3.2).

If in addition $\mathscr{C}$ is additive, it is k-linear with $k = \mathrm{End}\, I$. We assume in the following that k is a field, in which each element has a square root. (In fact we need a square root only for one element of k which depends on $\mathscr{C}$.) In this paper $\mathscr{C}$ will be a noetherian abelian category with finite dimensional k-vector spaces $\mathrm{Hom}_{\mathscr{C}}(A, B)$. (One more technical condition: isomorphism classes in $\mathscr{C}$ form a set.) In such a case there exists a coend $F = \int X \otimes X^\vee$ as an object of a cocompletion $\hat{\mathscr{C}}$ [36] of $\mathscr{C}$. Recall that this coend can be defined via an exact sequence

$$(1.2.1) \qquad \bigoplus_{f:A\to B\in\mathscr{C}} A \otimes B^\vee \xrightarrow{f\otimes B^\vee - A\otimes f^t} \bigoplus_{L\in\mathscr{C}} L \otimes L^\vee \xrightarrow{\oplus i_L} F \to 0,$$

where $f^t : B^\vee \to A^\vee$ is the transposed to a morphism $f : A \to B$. For a general definition of a coend see [41].

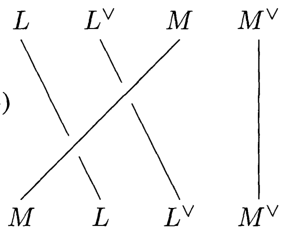
F is a Hopf algebra in the category $\hat{\mathscr{C}}$ (see [37, 40, 42]). The multiplication $m_F : F \otimes F \to F$ is described in [37] by any of the following $\mathscr{C}$-F-tangles

(1.2.2) 

or

$$\begin{array}{ccc} L \otimes L^\vee \otimes (M \otimes M^\vee) & \xrightarrow{i_L \otimes i_M} & F \otimes F \\ {\scriptstyle L\otimes c}\big\downarrow & & \exists \big\downarrow {\scriptstyle m_F} \\ L \otimes M \otimes (L \otimes M)^\vee & \xrightarrow{i_{L\otimes M}} & F \end{array}$$

(1.2.3)

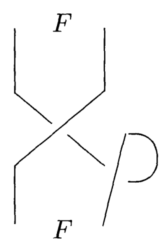


or

$$\begin{array}{ccc} (L\otimes L^\vee)\otimes M\otimes M^\vee & \xrightarrow{i_L\otimes i_M} & F\otimes F \\ {\scriptstyle c\otimes M^\vee}\downarrow & & \exists\downarrow{\scriptstyle m_F} \\ M\otimes L\otimes(M\otimes L)^\vee & \xrightarrow{i_{M\otimes L}} & F \end{array}$$

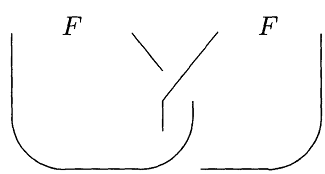
The antipode $\gamma_F : F \to F$ is given by

(1.2.4) $\gamma_F =$



There is a Hopf pairing $\omega : F \otimes F \to I$ [37],

(1.2.5) $\omega =$



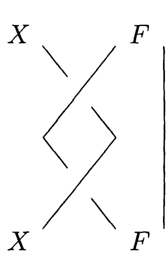
such that

$$\operatorname{Ann}\omega = \operatorname{Ann}^{\text{left}}\omega = \operatorname{Ann}^{\text{right}}\omega \in \hat{\mathscr{C}}.$$

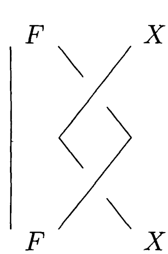
The quotient $\mathbf{f} = F/\operatorname{Ann}\omega \in \hat{\mathscr{C}}$ is also a Hopf algebra.

The morphisms called monodromies $\Omega_l = \Omega^l_{X,F} : X \otimes F \to X \otimes F$, $\Omega_r = \Omega^r_{F,X} : F \otimes X \to F \otimes X$, $\Omega = \Omega_{F,F} : F \otimes F \to F \otimes F$ are defined via tangles

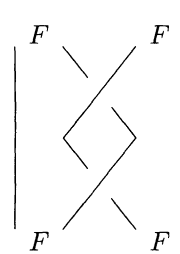
$\Omega_l =$



, $\Omega_r =$



, $\Omega =$



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They project to $\mathbf{f}$ as morphisms

$$\Omega_l = \Omega^l_{X,\mathbf{f}} : X \otimes \mathbf{f} \to X \otimes \mathbf{f},\quad \Omega_r = \Omega^r_{\mathbf{f},X} : \mathbf{f} \otimes X \to \mathbf{f} \otimes X,\quad \Omega = \Omega_{\mathbf{f},\mathbf{f}} : \mathbf{f} \otimes \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$$

also called monodromies.

1.3. 2-modular categories. The first modular axiom is [37]

(M1) $\mathbf{f}$ is an object of $\mathscr{C}$ (and not only of a cocompletion $\hat{\mathscr{C}}$)

(more scrupulously, it means that there exists an exact sequence $0 \to \operatorname{Ann}\omega \to F \to \mathbf{f} \to 0$ in $\hat{\mathscr{C}}$, where $\mathbf{f}$ is an object from $\mathscr{C} \subset \hat{\mathscr{C}}$).

Being the coend $\int X \otimes X^\vee$, the object $F \in \hat{\mathscr{C}}$ has an automorphism $\underline{\nu \otimes 1} \overset{\text{def}}{=} \int \nu \otimes 1$ (notations are from [37], see also Sect. 3.8.1). The second modular axiom says [37]

(M2) $\underline{\nu \otimes 1}(\operatorname{Ann}\omega) \subset \operatorname{Ann}\omega$

(more scrupulously, there exist morphisms $T' : \operatorname{Ann}\omega \to \operatorname{Ann}\omega \in \hat{\mathscr{C}}$, $T : \mathbf{f} \to \mathbf{f} \in \mathscr{C}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ann}\omega & \to & F & \to & \mathbf{f} & \to & 0 \\ & & \downarrow{\scriptstyle T'} & & \downarrow{\scriptstyle \underline{\nu\otimes 1}} & & \downarrow{\scriptstyle T} & & \\ 0 & \to & \operatorname{Ann}\omega & \to & F & \to & \mathbf{f} & \to & 0 \end{array}$$

commutes).

An equivalent form of (M2) is [37]

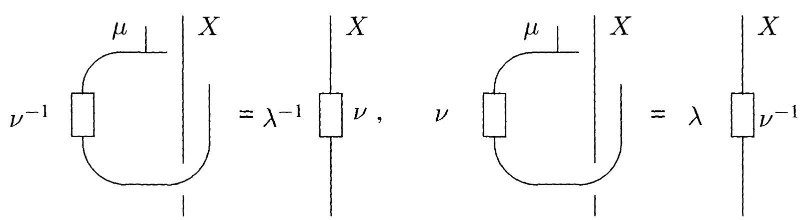
(M2′) There exists a morphism $\theta : I \to \mathbf{f}$ such that for any $X \in \mathscr{C}$ the ribbon twist $\nu : X \to X$ coincides with the composition

$$X \simeq I \otimes X \xrightarrow{\theta \otimes X} \mathbf{f} \otimes X \xrightarrow{\Omega_r} \mathbf{f} \otimes X \xrightarrow{\varepsilon \otimes X} I \otimes X \simeq X.$$

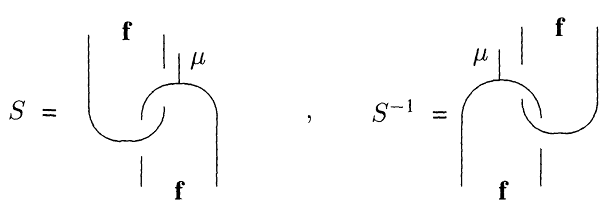
Definition 1.3.1. *A noetherian abelian ribbon category $\mathscr{C}$ with finite dimensional k-vector spaces of morphisms* $\operatorname{Hom}_{\mathscr{C}}(A, B)$ *is called 2-modular, if axioms (M1), (M2) are satisfied.*

Here 2- refers to the dimension of a surface which will be the main application.

It was shown in [37] that in the case of a modular category there exists a morphism $\mu : I \to \mathbf{f}$, which is the integral on the dual Hopf algebra ${}^\vee\mathbf{f} \simeq \mathbf{f}$, and



for some invertible constant $\lambda \in k^\times$. The pair (μ, λ) is unique up to a sign. Morphisms $S, S^{-1} : \mathbf{f} \to \mathbf{f}$



are inverse to each other. Morphisms S and T (defined via (M2)) yield a projective representation of the mapping class group of a torus with one hole:

$$(ST)^3 = \lambda S^2, \qquad S^2 = \gamma_{\mathbf{f}}^{-1}, \qquad T\gamma_{\mathbf{f}} = \gamma_{\mathbf{f}} T, \qquad \gamma_{\mathbf{f}}^2 = \nu.$$

Here $\gamma_{\mathbf{f}} : \mathbf{f} \to \mathbf{f}$ is the antipode of the Hopf algebra $\mathbf{f}$, given by the same tangle as (1.2.4).

2. Modular transformations in F

Here we reproduce results of [37] in special assumptions, which permit to prove more. Let $\mathscr{C}$ be a 2-modular category. Fix a morphism $\alpha : I \to F$ such that

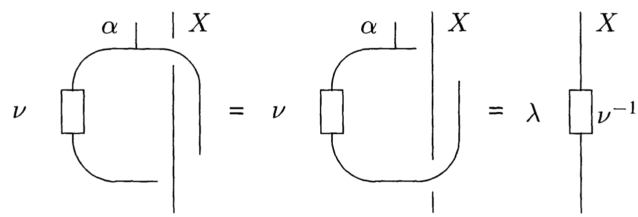
$$\gamma_F \alpha = \alpha : I \to F \qquad \text{and} \qquad \mu = (I \xrightarrow{\alpha} F \xrightarrow{\pi} \mathbf{f})$$

(if there is one). In this section we adopt the convention $AB = A \circ B$ for the composition.

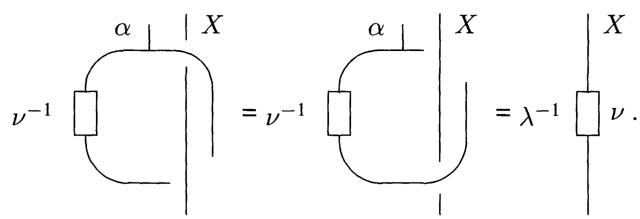
2.1. The quantum Fourier transform

Theorem 2.1.1 (cf. Theorem 6.2 [37]). *For any $X \in \mathscr{C}$ we have*

(2.1.1)



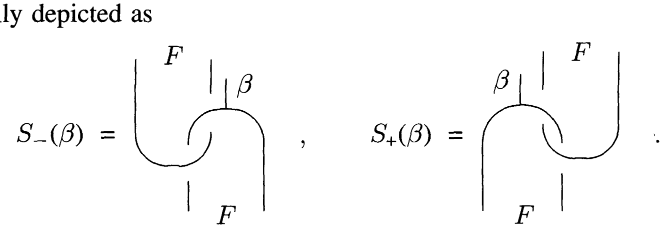
(2.1.2)



Notation. Let $\beta : I \to F$ be an arbitrary morphism. Introduce morphisms $F \to F$

$$S_{\mp}(\beta) = \left(F \simeq F \otimes I \xrightarrow{F\otimes\beta} F \otimes F \xrightarrow{\Omega^{\pm 1}} F \otimes F \xrightarrow{\varepsilon \otimes F} I \otimes F \simeq F\right),$$

graphically depicted as



We shall use the shorthand $S_* = S_{\pm}(\beta) : F \to F$ (a sign is chosen arbitrarily) and

$$S_+ = \lambda S_+(\alpha), \qquad S_- = \lambda^{-1} S_-(\alpha).$$

Proposition 2.1.2 (cf. Proposition 6.3 [37]). *We have*

$$S_-(\alpha)\gamma_F = S_+(\alpha) = \gamma_F S_-(\alpha).$$

In particular, $S_\pm$ *commute with* γ_F.

Theorem 2.1.3 (cf. Theorem 6.5 [37]). *We have*

$$S_*T^{-1}S_+ = TS_*T, \qquad S_*TS_- = T^{-1}S_*T^{-1}.$$

Corollary 2.1.4. $S_-TS_- = T^{-1}S_-T^{-1}$.

Lemma 2.1.5 (cf. Lemma 6.7 [37]). *There are identities*

$$S_*TS_-S_+ = S_*T, \qquad S_*S_+TS_- = TS_*,$$
$$S_*T^{-1}S_+S_- = S_*T^{-1}, \qquad S_*S_-T^{-1}S_+ = T^{-1}S_*.$$

Lemma 2.1.6 (cf. Lemma 6.8 [37]). *The morphism* T *commutes with* S_*S_+ *and* S_*S_-. *We have*

$$S_*S_+S_- = S_* = S_*S_-S_+.$$

Corollary 2.1.7. *The morphisms* $P_1 = S_-S_+$ *and* $P_2 = S_+S_-$ *are projections such that* $P_1P_2 = P_1$ *and* $P_2P_1 = P_2$.

Proposition 2.1.8. *The following kernels and images coincide*

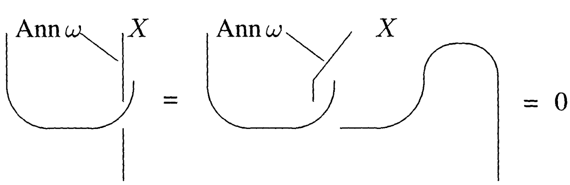
$$\operatorname{Ker} S_+ = \operatorname{Ker} S_- = \operatorname{Ker} P_1 = \operatorname{Ker} P_2 = \operatorname{Ker}\pi = \operatorname{Ann}\omega, \tag{2.1.3}$$

$$\operatorname{Im} S_+ = \operatorname{Im} S_- = \operatorname{Im} P_1 = \operatorname{Im} P_2. \tag{2.1.4}$$

In particular, $P_1 = P_2$.

Proof. We get by Corollary 2.1.7 $\operatorname{Ker} S_+ \subset \operatorname{Ker} P_1$, $\operatorname{Ker} S_- \subset \operatorname{Ker} P_2$, $\operatorname{Ker} P_1 = \operatorname{Ker} P_2$. Lemma 2.1.6 gives $\operatorname{Ker} P_1 \subset \operatorname{Ker} S_\pm$, $\operatorname{Ker} P_2 \subset \operatorname{Ker} S_\pm$. Therefore, $\operatorname{Ker} S_+ = \operatorname{Ker} S_- = \operatorname{Ker} P_1 = \operatorname{Ker} P_2$.

Since



we have $\operatorname{Ann}\omega \subset \operatorname{Ker} S_-$. This identity also implies

$$\left(F \simeq F \otimes I \xrightarrow{F\otimes\alpha} F \otimes F \xrightarrow{\Omega} F \otimes F \xrightarrow{\varepsilon\otimes F} I \otimes F \simeq F \xrightarrow{\pi} \mathbf{f}\right) =$$
$$= \left(F \xrightarrow{\pi} \mathbf{f} \simeq \mathbf{f} \otimes I \xrightarrow{\mathbf{f}\otimes\alpha} \mathbf{f} \otimes F \xrightarrow{\Omega} \mathbf{f} \otimes F \xrightarrow{\varepsilon\otimes F} I \otimes F \simeq F \xrightarrow{\pi} \mathbf{f}\right)$$
$$= \left(F \xrightarrow{\pi} \mathbf{f} \simeq \mathbf{f} \otimes I \xrightarrow{\mathbf{f}\otimes\mu} \mathbf{f} \otimes \mathbf{f} \xrightarrow{\Omega} \mathbf{f} \otimes \mathbf{f} \xrightarrow{\varepsilon\otimes\mathbf{f}} I \otimes \mathbf{f} \simeq \mathbf{f}\right)$$
$$= \left(F \xrightarrow{\pi} \mathbf{f} \xrightarrow{S} \mathbf{f}\right),$$

hence $\pi S_- = S\pi$. Thus, $\operatorname{Ker}(\pi S_-) = \operatorname{Ker}(S\pi) = \operatorname{Ker}\pi$ and $\operatorname{Ker} S_- \subset \operatorname{Ker}\pi = \operatorname{Ann}\omega$, whence Eq. (2.1.3) follows.

Proposition 2.1.2 implies $\operatorname{Im} S_-(\alpha) = \operatorname{Im} S_+(\alpha)$, so $\operatorname{Im} S_- = \operatorname{Im} S_+$. Corollary 2.1.7 gives that $\operatorname{Im} S_- \supset \operatorname{Im} P_1$, $\operatorname{Im} S_+ \supset \operatorname{Im} P_2$. Since $S_-S_+S_- = S_-$ and $S_+S_-S_+ = S_+$ by Lemma 2.1.6, we get $\operatorname{Im} S_- \subset \operatorname{Im} P_1$, $\operatorname{Im} S_+ \subset \operatorname{Im} P_2$. Therefore, Eq. (2.1.4) holds. □

Notation. Denote $P = P_1 = P_2 = S_+S_- = S_-S_+$.

Theorem 2.1.9. *The morphisms* $S_+(\alpha), S_-(\alpha), T, \gamma_F : F \to F$ *commute with the projection* $P : F \to F$. *The restrictions of* $S_\pm(\alpha)$ *to* $\operatorname{Ker} P = \operatorname{Ann}\omega$ *vanish. The restrictions to* $\operatorname{Im} P$ *(which depends on* α*)*

$$S = S_-(\alpha)|_{\operatorname{Im} P}, \qquad S^{-1} = S_+(\alpha)|_{\operatorname{Im} P},$$

inverse to each other, are identified with $S^{\pm 1} : \mathbf{f} \to \mathbf{f}$ *by an isomorphism* $\operatorname{Im} P \xrightarrow{\pi} \mathbf{f}$. *Therefore,*

$$(ST)^3 = \lambda S^2, \qquad S^2 = \gamma_F^{-1} \tag{2.1.5}$$

on $\operatorname{Im} P$.

Proof. Lemma 2.1.6 implies that T commutes with P and $PS_- = S_- = S_-P$, $PS_+ = S_+ = S_+P$. Proposition 2.1.2 implies $P\gamma_F = \gamma_F P$. Other statements follow by Proposition 2.1.8, Corollary 2.1.4 and Proposition 2.1.2. □

A certain similarity of the properties of S with the properties of the ordinary Fourier transform [39] suggests the name of *quantum Fourier transform* for this morphism.

2.2. 3-modular categories. Here 3- refers to applications to 3-manifolds. Recall that $\mu : I \to \mathbf{f}$ is a two-sided integral and $\gamma_F \mu = \mu$. If $F \in \mathscr{C}$ (analogue of being finite dimensional), then it has an invertible object of left integrals Int_l [37]. The canonical projection $\pi : F \to \mathbf{f}$ sends Int_l to $\operatorname{Im}\mu$.

Definition 2.2.1. *A* 3-modular category *is a 2-modular category* $\mathscr{C}$ *with an additional property*

(M3) $F \in \mathscr{C}$ *and it has a two-sided integral* $\sigma : I \to F$ *such that* $\pi\sigma = \mu$.

The assumption $F \in \mathscr{C}$ is not needed in this paper and can be consistently omitted. However, it holds in all known examples.

Proposition 2.2.1. *Suppose (M3) holds. Then* $\gamma_F \sigma = \sigma$.

Proof. Clearly, $\gamma_F\sigma$ is another two-sided integral in F. Therefore, it must be proportional to σ [37]. Projecting the equation $\gamma_F\sigma = C\sigma$, $C \in k^\times$, to $\mathbf{f}$ we get $\mu = \gamma_{\mathbf{f}}\mu = C\mu$, hence, $C = 1$. □

Definition 2.2.2. *A* perfect modular category *is a 2-modular category with a condition*

(PM) $\operatorname{Ann}\omega = 0$ *(equivalently,* $\pi : F \to \mathbf{f}$ *is an isomorphism).*

A perfect modular category is a particular case of a 3-modular category (and the easiest to deal with). The reader is advised to assume $\mathscr{C}$ perfect for the first reading whenever appropriate.

3. Ribbon and Braided Hopf Algebras

In this section we reformulate the results obtained in the abstract setting of ribbon abelian categories in the case of the category of finite dimensional modules $\mathscr{C} = H$-mod over a Hopf algebra H. This job was already partially done by Majid and the author [40], so we shall omit most of the proofs.

3.1. Quasitriangular Hopf algebras. Let H be a Hopf k-algebra with an invertible antipode. There are several (more or less equivalent) ways to make $\mathscr{C} = H$-mod into a braided category, if it permits this structure. We choose the most direct one. Assume that H has an *R-matrix*, which is an element $R \in H \otimes H$ (algebraic tensor product!) satisfying the relations of Drinfeld [5]

$$(\Delta \otimes 1)R = R^{13}R^{23}, \qquad (1 \otimes \Delta)R = R^{13}R^{12}, \qquad \Delta^{\mathrm{op}}a = R\Delta(a)R^{-1}$$

for any $a \in H$, so (H, R) is *quasitriangular.*

For a finite dimensional H-module V as for any vector space there is a canonical linear map $v_0^2 : V \to V^{\vee\vee}$ such that $\langle v, y\rangle = \langle y, v_0^2(v)\rangle$ for $v \in V$, $y \in V^\vee$. Its square gives $v_0^4 = (V \xrightarrow{v_0^2} V^{\vee\vee} \xrightarrow{v_0^2} V^{(4\vee)})$. On the other hand, in $\mathscr{C}$ as in any rigid braided category there are morphisms $u_0^{-4} = (V \xrightarrow{u_1^{-2}} {}^{\vee\vee}V \xrightarrow{u_{-1}^{-2}} V^{(-4\vee)})$. Composing them we get linear bijections

$$g_V : V \xrightarrow{u_0^{-4}} V^{(-4\vee)} \xrightarrow{v_0^4} V.$$

They are decomposable into a product of the two bijections

$$u_1 : V \xrightarrow{v_0^2} V^{\vee\vee} \xrightarrow{u_{-1}^{-2}} V, \qquad u_4 : V \xrightarrow{v_0^2} V^{\vee\vee} \xrightarrow{u_1^{-2}} V.$$

Theorem 3.1.1 (Drinfeld [6]). *The maps u_1, u_4, g are given by the action of the following elements:*

$$u_1 = \gamma(R'')R', \qquad u_4 = \gamma^2(R')R'' = \gamma(u_1)^{-1}, \qquad g = u_1u_4.$$

The element g is grouplike ($\varepsilon(g) = 1$, $\Delta g = g \otimes g$) and for any $a \in H$ we have $gag^{-1} = \gamma^4(a)$.

To find g we can use the following. Let $\mathbf{h}_+$ (resp. $\mathbf{h}_-$) be the minimal subspace of H such that $R \in \mathbf{h}_+ \otimes H$ (resp. $R \in H \otimes \mathbf{h}_-$). Then $\mathbf{h}_+$, $\mathbf{h}_-$ are Hopf subalgebras. The finite dimensional subspace spanned by their products $\mathbf{h} = \mathbf{h}_+\mathbf{h}_-$ coincides with $\mathbf{h}_-\mathbf{h}_+$, therefore, $\mathbf{h}$ is also a Hopf subalgebra. Moreover, it is a minimal quasitriangular Hopf subalgebra of (H, R) [45]. All elements u_1, u_4, g belong to $\mathbf{h}$.

Pick a basis $(a_i) \subset \mathbf{h}_+$ and a basis $(b_i) \subset \mathbf{h}_-$ for which $R = \sum_i a_i \otimes b_i$. Introduce a non-degenerate pairing $\pi : \mathbf{h}_- \otimes \mathbf{h}_+ \to k$, $\pi(b_i, a_j) = \delta_j^i$. Axioms of R-matrix imply that $\pi : \mathbf{h}_{-\mathrm{op}} \otimes \mathbf{h}_+ \to k$ is a Hopf pairing, i.e.

$$\pi(ab, c) = \pi(a, c_{(1)})\pi(b, c_{(2)}), \qquad \pi(a, cd) = \pi(a_{(1)}, d)\pi(a_{(2)}, c).$$

Now construct the double $D(\mathbf{h}_+)$ generated by its Hopf subalgebras $\mathbf{h}_+, \mathbf{h}_-$. The natural projection $j : D(\mathbf{h}_+) \to \mathbf{h}$ is a homomorphism of quasitriangular Hopf algebras. Clearly, $g_H = g_{\mathbf{h}} = j(g_{D(\mathbf{h}_+)})$ is the relationship between the elements g for the three algebras. To find $g_{D(\mathbf{h}_+)}$ use the following

Theorem 3.1.2 (Drinfeld [6], Kauffman and Radford [13]). *Let $\delta_+ \in \mathbf{h}_+$, $\delta_- \in \mathbf{h}_-$ be left integrals, that is, $x\delta_\pm = \varepsilon(x)\delta_\pm$ for any $x \in \mathbf{h}_\pm$. Then $g_{D(\mathbf{h}_+)} = a_+^{-1}a_-$ for grouplike elements (called moduli) $a_+ \in \mathbf{h}_+$, $a_- \in \mathbf{h}_-$ such that*

$$\delta_+ y = \pi(a_-, y)\delta_+ \quad \text{for any } y \in \mathbf{h}_+,$$
$$\delta_- y = \pi(y, a_+)\delta_- \quad \text{for any } y \in \mathbf{h}_-.$$

3.2. Ribbon Hopf algebras. Assume now that $\mathscr{C} = H$-mod has a ribbon structure. Then there is a morphism $u_0^{-2} = u_{-1}^{-2}\nu = u_1^{-2}\nu^{-1} : V \to {}^{\vee\vee}V$ for any finite dimensional H-module V. One can prove that the map

$$\kappa_V : V \xrightarrow{u_0^{-2}} {}^{\vee\vee}V \xrightarrow{v_0^2} V$$

commutes with all morphisms and satisfies $\kappa_{X\otimes Y} = \kappa_X \otimes \kappa_Y$ and $\kappa_X^2 = g_X$. If H is finite dimensional, we deduce that κ_V is the action of a grouplike element κ of H.

Definition 3.2.1 (comp. [50]). *A* ribbon Hopf algebra (H, R, κ) *is a quasitriangular Hopf algebra* (H, R) *and a grouplike element* $\kappa \in H$ *such that*

$$\kappa^2 = g, \qquad \kappa a \kappa^{-1} = \gamma^2(a)$$

for any $a \in H$.

In the category of finite dimensional modules over a ribbon Hopf algebra we have canonical isomorphisms $u_0^2 : V \to V^{\vee\vee}$, $u_0^2(v) = v_0^2(\kappa v) = \kappa v_0^2(v)$, which we use to identify these modules.

The following is essentially proved by Kauffman and Radford.

Theorem 3.2.1 (cf. [13]). *If* (H, R, κ) *is a ribbon Hopf algebra, then the category* H*-mod is a ribbon braided category with the ribbon twist given by multiplication by the central element*

$$\nu = \gamma^2(R')R''\kappa^{-1} = R''\gamma^2(R')\kappa = R''\kappa R' = R'\kappa^{-1}R''.$$

The following holds

$$\nu^{-1} = R'\gamma(R'')\kappa = \gamma(R'')R'\kappa^{-1},$$

$$\varepsilon(\nu) = 1, \qquad \gamma(\nu) = \nu, \qquad \Delta\nu = (R^{21}R^{12}) \cdot \nu \otimes \nu. \tag{3.2.1}$$

Remark 3.2.1. Definition 3.2.1 is equivalent to the definition of a ribbon Hopf algebra of Reshetikhin and Turaev [50]. The element ν^{-1} was denoted v in [50].

3.3. A braided Hopf algebra. Here we describe explicitly the braided Hopf algebra F and its dual algebra U for the case of $\mathscr{C} = H$-mod. Let H be a ribbon Hopf algebra and let H° be its dual [58]. Assume that H has enough finite dimensional modules, so that the pairing $H \otimes H^\circ \to k$ is non-degenerate. Define the Hopf algebra $Fun = (H^\circ)_{\text{op}}$ as H° with the opposite coproduct (note that usually the algebra of functions is a subalgebra of H°, not of $(H^\circ)_{\text{op}}$). We have an equivalence of categories $\mathscr{C} = H$-mod $\simeq Fun$-comod. The pairing $\langle , \rangle : H \otimes Fun \to k$ satisfies

$$\langle x, fg\rangle = \langle x_{(1)}, f\rangle\langle x_{(2)}, g\rangle, \qquad \langle xy, f\rangle = \langle x, f_{(2)}\rangle\langle y, f_{(1)}\rangle$$

for $x, y \in H$, $f, g \in Fun$, where $\Delta f = f_{(1)} \otimes f_{(2)}$ is the coproduct in Fun.

Consider linear maps $i_L : L \otimes L^\vee \to Fun$, $l_a \otimes l_b \mapsto t_{La}{}^b$, where $t_{La}{}^b$ is the matrix element of the H-module L with a basis (l_a), that is, $t_{La}{}^b$ is a linear function on H given by $\langle u, t_{La}{}^b\rangle = \langle u.l_a, l^b\rangle$ for $u \in H$. The maps i_L become homomorphisms of H-modules if Fun is given the coadjoint H-module structure

$$u \triangleright f = \langle u, f_{(1)}\gamma(f_{(3)})\rangle f_{(2)}$$

for $u \in H$, $f \in Fun$. The vector space Fun with this H-module structure will be denoted F.

Theorem 3.3.1 (cf. [4, 54, 55, 64]). *The family* $(i_L : L \otimes L^\vee \to F)_{L\in\mathscr{C}}$ *is a coend of the bifunctor* $\mathscr{C} \times \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$, $(A, B) \mapsto A \otimes B^\vee$, *so we can write* $F = \int^L L \otimes L^\vee$. *In other words, the sequence (1.2.1) is exact.*

Being the coend, F is a Hopf algebra in the category $\widehat{\mathscr{C}} = Fun$-Comod $\subset H$-Mod (braided Hopf algebra), as we have seen in Sect. 1.2. The Hopf structure of F described by tangles in [37] converts to the following. As the coalgebra F coincides with Fun. The multiplication in F is expressed via the multiplication in Fun as

$$\begin{aligned} m_F(f \otimes g) &= \rho(\gamma(f_{(2)}) \otimes g_{(1)}\gamma(g_{(3)}))f_{(1)}g_{(2)} \\ &= \rho(f_{(2)} \otimes g_{(3)}\gamma^{-1}(g_{(1)}))f_{(1)}g_{(2)} \\ &= \rho(f_{(1)}\gamma(f_{(3)}) \otimes g_{(1)})g_{(2)}f_{(2)} \end{aligned}$$

by (1.2.2) and (1.2.3), where

$$\rho(a \otimes b) = \langle R, a \otimes b\rangle = \langle R', a\rangle\langle R'', b\rangle .$$

The unit of F is the same as the unit of Fun. The antipode γ_F of F is expressed via the antipode γ of Fun:

$$\gamma_F(f) = \rho(f_{(1)} \otimes \gamma(f_{(4)}))\langle \nu\kappa^{-1}, f_{(2)}\rangle\gamma(f_{(3)})$$

for $f \in F$. The inverse antipode is

$$\gamma_F^{-1}(f) = \rho(f_{(4)}, f_{(1)})\langle \kappa^{-1}\nu^{-1}, f_{(3)}\rangle\gamma^{-1}(f_{(2)}).$$

All the structure maps $\Delta, \varepsilon, m_F, \gamma_F$ are homomorphisms of H-modules.

3.4. The dual braided Hopf algebra. In order to define a Hopf algebra dual to F we shall not consider the rational part $^\circ F$ of the H-module $^*F$ of all linear functionals on F. Instead we denote by U the H-submodule $H \subset {}^*F$. This amounts to consider the adjoint action

$$\operatorname{ad} a.x = a_{(1)}x\gamma(a_{(2)})$$

for $a \in H$, $x \in U$. Since $^*F$ is an algebra in the category H-Mod (being dual to the coalgebra $F \in H$-Mod), so is U with the usual multiplication m of H. We want to introduce a comultiplication $\nabla : U \to U \otimes U$ which would be dual to m_F in the proper sense:

$$\langle \nabla u, f \otimes g\rangle \equiv \langle u^{(1)} \otimes u^{(2)}, f \otimes g\rangle \equiv \langle u^{(2)}, f\rangle\langle u^{(1)}, g\rangle = \langle u, m_F(f \otimes g)\rangle$$

for $u \in U$, $f, g \in F$. By dualising the formulae for m_F one arrives to the following

$$\nabla u = \operatorname{ad} R''.u_{(2)} \otimes R'u_{(1)} \tag{3.4.1}$$

$$= u_{(1)}\gamma(R'') \otimes \operatorname{ad} R'.u_{(2)}. \tag{3.4.2}$$

The counit of U coincides with the counit of H. The operations m, ∇, ε make U into a braided Hopf algebra in H-Mod, the antipode $\gamma_U : U \to U$ being

$$\gamma_U(u) = \nu\kappa^{-1}\gamma(R'')\gamma(u)R'$$

and the inverse antipode being

$$\gamma_U^{-1}(u) = R'\kappa^{-1}\nu^{-1}\gamma(u)R'' = \gamma^2(R')\gamma^{-1}(u)R''\kappa^{-1}\nu^{-1}.$$

This is the unique Hopf algebra structure dual to F on the H-module H.

3.5. The algebra **f**. The algebra Fun^* acts in every finite dimensional H-module X. Consider special elements $l^-_V(v \otimes w) \in Fun^*$ determined for any $w \in W^\vee$, $v \in V \in H$-mod as the operators in X:

$$(3.5.1) \quad l^-_V(v \otimes w)(x) = (\mathrm{ev} \otimes 1)(1 \otimes c^2)(v \otimes w \otimes x) = \sum_{i,j} \langle v, R''_j R'_i w \rangle R'_j R''_i x.$$

The subspace $\mathbf{u}$ spanned by $l^-_V(v \otimes w)$ is contained in H and even in $\mathbf{h}$, so it is finite dimensional. As shown in [37] $\mathbf{u}$ is a braided Hopf subalgebra of ${}^\circ F$, therefore, it is closed under the operations of U and constitutes a finite dimensional braided Hopf subalgebra of U.

The map

$$l^- : \bigoplus_{V \in \mathscr{C}} V \otimes V^\vee \xrightarrow{\oplus l^-_V} U$$

factors through the coend (1.2.1), therefore, determines a map $l^- : F \to U$, which is a homomorphism of Hopf algebras in H-Mod [37]. The image of l^- is $\mathbf{u}$. The form $\omega : F \otimes F \to k$ (1.2.5) can be presented as

$$\omega(f \otimes t_{La}{}^b) = \langle l^-(f).l_a, l^b \rangle = \langle l^-(f), t_{La}{}^b \rangle,$$

hence, $\mathrm{Ann}^{\mathrm{left}}\, \omega = \mathrm{Ker}\, l^-$. Therefore, the braided Hopf algebras $\mathbf{f} = F/\mathrm{Ann}\, \omega$ and $\mathbf{u}$ are isomorphic [37, Corollary 3.10]. Thus the first modular axiom (M1) is always satisfied for the considered algebras H.

By definition the subspace $\mathbf{u} \subset \mathbf{h}$ is the minimal subspace such that $R^{12}R^{21} \in \mathbf{u} \otimes \mathbf{h}$. Since $(\gamma \otimes \gamma)(R^{12}R^{21}) = R^{21}R^{12}$, the minimal subspace $B \subset \mathbf{h}$ such that $R^{12}R^{21} \in \mathbf{h} \otimes B$ is $\gamma(\mathbf{u})$. It does not coincide necessarily with $\mathbf{u}$, for $\mathbf{u}$ is not an ordinary Hopf subalgebra. Repeating the reasoning we get $\gamma^2(\mathbf{u}) = \mathbf{u}$ and conclude that $R^{12}R^{21} \in \mathbf{u} \otimes \gamma(\mathbf{u})$. Similarly $(R^{12}R^{21})^{-1} \in \mathbf{u} \otimes \gamma(\mathbf{u})$.

3.6. 2-modular Hopf algebras. We already know by Eq. (3.5.1) the image

$$(3.6.1) \quad l^-(f) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f \rangle R'_j R''_i$$

for any $f \in F$. Notice that $\gamma^{-1}(R''_j R'_i) \otimes R'_j R''_i \in \mathbf{u} \otimes \mathbf{u}$. The pairing $\mathbf{u} \otimes F \hookrightarrow U \otimes F \to k$ factorizes through a perfect pairing $\mathbf{u} \otimes \mathbf{f} \to k$ due to $\mathrm{Ann}^{\mathrm{left}}\, \omega = \mathrm{Ann}^{\mathrm{right}}\, \omega$. Therefore, an element $x \in H$ is representable in the form $l^-(f)$ for some $f \in \mathbf{f}$ iff $x \in \mathbf{u}$.

Theorem 3.6.1. *A ribbon Hopf algebra (H, R, κ) is* 2-modular *(that is, H-mod is 2-modular) if and only if $\nu \in \mathbf{u}$, or equivalently, $\nu^{-1} \in \mathbf{u}$.*

Proof. If the axiom (M2′) holds, then $\nu = l^-(\theta(1))$ for some $\theta : k \to \mathbf{f}$. Hence, $\nu \in \mathbf{u}$ by the above discussion.

If $\nu \in \mathbf{u}$, then for some $f \in \mathbf{f}$ we have $\nu = l^-(f)$. Since ν is central, the subspace $k\nu \subset \mathbf{u}$ is a trivial H-submodule. Hence, its preimage $kf \subset \mathbf{f}$ by an isomorphism $l^- : \mathbf{f} \to \mathbf{u}$ is also a trivial H-submodule. Now $\theta : k \to \mathbf{f}$, $1 \mapsto f$ is the homomorphism required in the axiom (M2′).

The condition $\underline{\nu \otimes 1}(\mathrm{Ann}\, \omega) \subset \mathrm{Ann}\, \omega$ is equivalent to $\underline{\nu^{-1} \otimes 1}(\mathrm{Ann}\, \omega) \subset \mathrm{Ann}\, \omega$ [37, Corollary 5.12]. Therefore, $\nu \in \mathbf{u}$ if and only if $\nu^{-1} \in \mathbf{u}$. □

Corollary 3.6.2. *If a ribbon Hopf algebra (H, R, κ) is 2-modular, then $\kappa \in \mathbf{h}$. Therefore, its minimal quasitriangular subalgebra $(\mathbf{h}, R)$ equipped with κ will be also a 2-modular ribbon Hopf algebra.*

Remark 3.6.1. Let H be a ribbon Hopf algebra. The category H-mod is perfect modular if and only if $F = \mathbf{f}$, or equivalently, $H = \mathbf{u}$, so H is called factorizable [49].

3.7. 3-modular Hopf algebras

Proposition 3.7.1. *Let $\alpha \in Fun$ be an element. Denote the corresponding linear functional $H \to k$ and the linear map $k \to Fun$, $1 \mapsto \alpha$, also by α. The following conditions are equivalent:*

(i) $\alpha : k \to F$ *is a morphism of H-modules;*
(ii) $\alpha_{(1)}\gamma(\alpha_{(3)}) \otimes \alpha_{(2)} = 1 \otimes \alpha$;
(iii) $\alpha(x_{(1)}u\gamma(x_{(2)})) = \varepsilon(x)\alpha(u)$ *for any* $x, u \in H$;
(iv) $\alpha : U \to k$ *is a morphism of H-modules;*
(v) $\alpha(xy) = \alpha(y\gamma^2(x))$ *for any* $x, y \in H$.

Proof is a straightforward check and it is left to the reader.

Theorem 3.7.2. *An element $\sigma \in F$ is a two-sided integral on the algebra U if and only if the equivalent conditions (i)–(v) of Proposition 3.7.1 are satisfied and σ is a left integral on the algebra H, that is, $(1 \otimes \sigma)\Delta u = \sigma(u)$ for any $u \in H$.*

Proof. Conditions (i) and (iv) are clearly necessary. Assume now that (iii) holds. $\sigma \in F$ is a two-sided integral if and only if

$$(\sigma \otimes 1)\nabla u = \sigma(u) = (1 \otimes \sigma)\nabla u \tag{3.7.1}$$

for any $u \in U$. Now we calculate

$$(\sigma \otimes 1)\nabla u = \sigma(\operatorname{ad} R''.u_{(2)})R'u_{(1)} = \sigma(u_{(2)})\varepsilon(R'')R'u_{(1)} = u_{(1)}\sigma(u_{(2)}),$$
$$(1 \otimes \sigma)\nabla u = u_{(1)}\gamma(R'')\sigma(\operatorname{ad} R'.u_{(2)}) = u_{(1)}\gamma(R'')\varepsilon(R')\sigma(u_{(2)}) = u_{(1)}\sigma(u_{(2)})$$

by Eq. (3.4.1) and Eq. (3.4.2). Thus, both Eqs. (3.7.1) hold if one equation $(1\otimes\sigma)\Delta u = \sigma(u)$ holds. □

Let the algebra F have a two-sided integral $\sigma : I \to F$. Then $\pi \circ \sigma : k \to \mathbf{f}$ is a two-sided integral, therefore it is proportional to μ. If the proportionality constant does not vanish, then σ can be rescaled to satisfy $\pi \circ \sigma = \mu$. Non-vanishing of $\pi \circ \sigma$ is equivalent to non-vanishing of $l^-(\sigma) \in \mathbf{u}$ or of $\gamma^{-1}(l^-(\sigma)) \in \mathbf{u}$. Formula (3.6.1) gives

$$\gamma^{-1}(l^-(\sigma)) = \sigma(R'_i R''_j)R''_i R'_j.$$

So we get

Theorem 3.7.3. *A finite dimensional ribbon Hopf algebra (H, R, κ) is* 3-modular *(that is, H-mod is 3-modular) if and only if the following conditions hold:*
(M2) $\nu \in \mathbf{u}$,
(M3) $\int(xy) = \int(y\gamma^2(x))$, $\int(R'_i R''_j)R''_i R'_j \neq 0$,
where $\int : H \to k$ is a left integral on the algebra H.

The property $\int(xy) = \int(y\gamma^2(x))$ above is equivalent to *unimodularity*[1] of H, which means that each left integral $\Lambda \in H$ is a right integral as well. This follows from the Radford's formula $\int(y\gamma^2(x)) = \int(x_{(1)}\alpha(x_{(2)})y)$ [46], where $\alpha \in G(H^*)$ is the modulus relating left and right integrals for H.

Proposition 3.7.4. *A factorizable ribbon Hopf algebra is unimodular.*

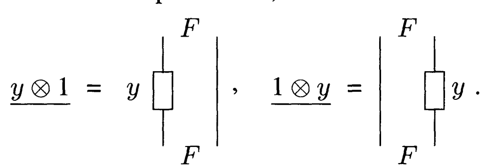
Proof. This a corollary of the above theorem. Or, notice simply that in the factorizable case the map $l^- : F \to U = H$ is an isomorphism of algebras, preserving the counit, and F is unimodular. □

For Drinfeld's doubles unimodularity was proven by Hennings [10] and Radford [45].

3.8. Some operators in F and U

3.8.1. 3-modular case. Assume that (H, R, κ) is 3-modular. We find explicitly the linear maps $S, T : F \to F$ and their transposed maps ${}^tS, {}^tT : U \to U$.

Let $y \in H$. There are maps $F \to F$,



The first is obtained as the projection of $(y \otimes 1)(l_a \otimes l^b) = \langle y, t_{La}{}^c\rangle l_c \otimes l^b$ in the form $\underline{y \otimes 1}(t_{La}{}^b) = \langle y, t_{La}{}^c\rangle t_{Lc}{}^b$. Therefore,

$$\underline{y \otimes 1}(f) = \langle y, f_{(1)}\rangle f_{(2)}$$

and similarly

$$\underline{1 \otimes y}(f) = \langle \gamma^{-1}(y), f_{(2)}\rangle f_{(1)}.$$

The transposed operators in U are

$$u.{}^t(\underline{y \otimes 1}) = uy, \qquad u.{}^t(\underline{1 \otimes y}) = \gamma^{-1}(y)u$$

for any $u \in U$. In particular, we have $T = \underline{\nu \otimes 1} = \underline{1 \otimes \nu}$ and its transpose ${}^tT$

(3.8.1) $$T(f) = \langle \nu, f_{(1)}\rangle f_{(2)} = f_{(1)}\langle \nu, f_{(2)}\rangle \qquad \text{for } f \in F,$$

(3.8.2) $${}^tT(u) = u\nu = \nu u \qquad \text{for } u \in U.$$

Similarly other maps are constructed by projection:

$$\Omega_l : X \otimes F \to X \otimes F$$

(3.8.3) $$\Omega_l(x \otimes f) = \sum_{i,j} R_j'' R_i' x \otimes \langle R_j' R_i'', f_{(1)}\rangle f_{(2)},$$

[1] I am grateful to the referee for this remark.

$$\Omega_r : F \otimes X \to F \otimes X$$
(3.8.4)
$$\Omega_r(f \otimes x) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f_{(2)} \rangle f_{(1)} \otimes R'_j R''_i x,$$

$$\Omega : F \otimes F \to F \otimes F$$
(3.8.5)
$$\Omega(h \otimes f) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), h_{(2)} \rangle h_{(1)} \otimes \langle R'_j R''_i, f_{(1)} \rangle f_{(2)},$$

$$\Omega^{-1} : F \otimes F \to F \otimes F$$
(3.8.6)
$$\Omega^{-1}(h \otimes f) = \sum_{i,j} \langle \gamma^{-2}(R''_j) R'_i, h_{(2)} \rangle h_{(1)} \otimes \langle R''_i R'_j, f_{(1)} \rangle f_{(2)}.$$

Assume according to (M3) that $\sigma \in F$ is a two-sided integral with $\pi \circ \sigma = \mu$. Then

$$S_-(\sigma) : F \to F$$
$$S_-(\sigma)(f) = \rho(\sigma_{(1)} \otimes \gamma(f_{(1)}))\rho(f_{(2)} \otimes \gamma(\sigma_{(3)}))\sigma_{(2)}$$
(3.8.7)
$$= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f \rangle \langle R'_j R''_i, \sigma_{(1)} \rangle \sigma_{(2)}$$
(3.8.8)
$$= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), \sigma_{(2)} \rangle \langle R'_j R''_i, f \rangle \sigma_{(1)},$$

$${}^t S_-(\sigma) : U \to U$$
$${}^t S_-(\sigma)(u) = \sum_{i,j} \sigma(\gamma^{-1}(R''_i) u \gamma(R'_j)) R'_i R''_j$$
(3.8.9)
$$= \sum_{i,j} \sigma(u \gamma(R''_i R'_j)) R'_i R''_j$$
(3.8.10)
$$= \sum_{i,j} \sigma(\gamma^{-1}(R''_i R'_j) u) R'_i R''_j,$$

$$S_+(\sigma) : F \to F$$
$$S_+(\sigma)(f) = \rho(f_{(1)} \otimes \sigma_{(1)})\rho(\sigma_{(3)} \otimes f_{(2)})\sigma_{(2)}$$
(3.8.11)
$$= \sum_{i,j} \langle \gamma^{-2}(R''_j) R'_i, f \rangle \langle R''_i R'_j, \sigma_{(1)} \rangle \sigma_{(2)}$$
(3.8.12)
$$= \sum_{i,j} \langle \gamma^{-2}(R''_j) R'_i, \sigma_{(2)} \rangle \langle R''_i R'_j, f \rangle \sigma_{(1)},$$

$$^tS_+(\sigma) : U \to U$$

(3.8.13) $$^tS_+(\sigma)(u) = \sum_{i,j} \sigma(R'_j u R''_i) R''_j R'_i$$

(3.8.14) $$= \sum_{i,j} \sigma(u R''_i R'_j) \gamma^{-2}(R''_j) R'_i$$

(3.8.15) $$= \sum_{i,j} \sigma(\gamma^{-2}(R''_i) R'_j u) R''_j R'_i.$$

We know by Theorem 2.1.9 that $P = S_+(\sigma)S_-(\sigma)$ is a projection in F with $\operatorname{Ker} P = \operatorname{Ann}\omega$. Therefore,

$$^tP = {}^tS_-(\sigma)^tS_+(\sigma) = {}^tS_+(\sigma)^tS_-(\sigma) = {}^tS_+(\sigma)^2\gamma_U^{-1}$$

is a projection in U with $\operatorname{Im}{}^tP = \mathbf{u}$.

Let us determine now the normalization for the integral σ. Using the pairing $\langle,\rangle : U \otimes F \to k$ one computes

$$\langle 1, \gamma_F^{-1} S_+(\sigma)^2(1)\rangle = \varepsilon(P(1)) = \varepsilon(1) = 1,$$

since $1 - P(1) \in \operatorname{Ann}\omega \subset \operatorname{Ker}\varepsilon$. On the other hand

$$\langle 1, \gamma_F^{-1} S_+(\sigma)^2(1)\rangle = \langle {}^tS_+(\sigma)^2\gamma_U^{-1}(1), 1\rangle = \langle {}^tS_+(\sigma)^2(1), 1\rangle,$$

so we have to satisfy $\varepsilon({}^tS_+(\sigma)^2(1)) = 1$. Using Eq. (3.8.13) twice we get

$$\varepsilon({}^tS_+(\sigma)(1)) = \sum_{i,j} \sigma(R'_j R''_i)\sigma(R''_j R'_i).$$

Therefore, σ is normalized so that

(3.8.16) $$(\sigma \otimes \sigma)(R^{12}R^{21}) = 1.$$

3.8.2. 2-modular case. Let us assume less now, namely, that (H, R, κ) is 2-modular. Then the formulae (3.8.3)–(3.8.12), (3.8.14), (3.8.15) with F replaced by $\mathbf{f}$, U replaced by $\mathbf{u}$ and σ replaced by $\mu \in \mathbf{f}$ or $\mu : \mathbf{u} \to k$ give the correct operators $\Omega^l_{X,\mathbf{f}}$, $\Omega^r_{\mathbf{f},X}$, $(\Omega_{\mathbf{f},\mathbf{f}})^{\pm 1}$, $S = S_-(\mu) : \mathbf{f} \to \mathbf{f}$, $S^{-1} = S_+(\mu) : \mathbf{f} \to \mathbf{f}$, ${}^tS : \mathbf{u} \to \mathbf{u}$ and ${}^tS^{-1} : \mathbf{u} \to \mathbf{u}$. Notice that

$$\sum_{i,j} \gamma^{-2}(R''_j)R'_i \otimes R''_i R'_j = (\gamma^{-1} \otimes 1)(R^{21}R^{12})^{-1} \in \mathbf{u} \otimes \mathbf{u}.$$

To determine the normalisation constant for μ we use Eq. (3.8.9). As above, using the perfect pairing $\langle,\rangle : \mathbf{u} \otimes \mathbf{f} \to k$ we get

$$1 = \langle 1, \gamma_{\mathbf{f}} S^2(1)\rangle = \varepsilon({}^tS^2(1)) = \sum_{i,j} \mu(R'_i R''_j)\mu(\gamma(R''_i R'_j)),$$

hence,

$$(\mu \otimes \mu\gamma)(R^{12}R^{21}) = 1.$$

In 3-modular case we can use either this formula or Eq. (3.8.16) in the form

$$\sum_{i,j} \mu(\gamma^{-2}(R''_i)R'_j)\mu(R''_j R'_i) = 1.$$

Proposition 3.8.1. *Let (H, R, κ) be a 2-modular Hopf algebra. Then*

$$ {}^tS(\nu^{\pm 1}) = \lambda^{\pm 1}\nu^{\mp 1}, \qquad {}^tS^{-1}(\nu^{\pm 1}) = \lambda^{\pm 1}\nu^{\mp 1}, $$
$$ \lambda = \mu(\nu), \qquad \lambda^{-1} = \mu(\nu^{-1}). $$

Moreover, if (H, R, κ) is 3-modular, then

$$ {}^tS_+(\sigma)(\nu^{\pm 1}) = \lambda^{\pm 1}\nu^{\mp 1}, \qquad {}^tS_-(\sigma)(\nu^{\pm 1}) = \lambda^{\pm 1}\nu^{\mp 1}, \tag{3.8.17} $$
$$ \lambda = \sigma(\nu), \qquad \lambda^{-1} = \sigma(\nu^{-1}). $$

The perfect modular case was considered in [40]. When H is a Drinfeld's double one can compute λ via contraction of the moduli (see Kerler [16]).

Proof. Both cases being similar, we prove the second statement. Equations (2.1.1), (2.1.2) mean that for any H-module X and any vector $x \in X$,

$$ (\varepsilon \otimes 1)\Omega_r(T^{\pm 1}(\sigma) \otimes x) = \lambda^{\pm 1}\nu^{\mp 1}x. $$

Using Eqs. (3.8.1) and (3.8.4) we get

$$ \begin{aligned} (\varepsilon \otimes 1)\Omega_r(T^{\pm 1}(\sigma) \otimes x) &= (\varepsilon \otimes 1)\Omega_r(\langle \nu^{\pm 1}, \sigma_{(1)}\rangle \sigma_{(2)} \otimes x) \\ &= \langle \nu^{\pm 1}, \sigma_{(1)}\rangle \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), \sigma_{(3)}\rangle \varepsilon(\sigma_{(2)}) R'_j R''_i x \\ &= \sum_{i,j} \langle \nu^{\pm 1}, \sigma_{(1)}\rangle \langle \gamma^{-1}(R''_j R'_i), \sigma_{(2)}\rangle R'_j R''_i x \\ &= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i)\nu^{\pm 1}, \sigma\rangle R'_j R''_i x \\ &= {}^tS_-(\sigma)(\nu^{\pm 1})x. \end{aligned} $$

Therefore, ${}^tS_-(\sigma)(\nu^{\pm 1}) = \lambda^{\pm 1}\nu^{\mp 1}$. Applying γ_U we get the same result for ${}^tS_+(\sigma)$. Applying ε to Eq. (3.8.17) we get $\sigma(\nu^{\pm 1}) = \lambda^{\pm 1}$ by Eq. (3.2.1). □

4. Representations of mapping class groups

According to [38] for any 2-modular category $\mathscr{C}$ there are projective representations of mapping class groups of $\mathscr{C}$-labled surfaces in vector spaces $\mathrm{Hom}_{\mathscr{C}}(-,-)$. Here we pay attention mainly to the categories $\mathscr{C} = H$-mod, where H is a 2-modular Hopf algebra, and we describe the representations explicitly. In the particular case of closed surfaces or surfaces with one hole these representations are closely related to the representations of mapping class groups in a category of tangles obtained by Matveev and Polyak [43].

By a *labeled surface* we shall understand the following: a compact oriented surface with a labeling of boundary circles $L : \pi_0(\partial\Sigma) \to \mathrm{Ob}\,\mathscr{C}$, $i \mapsto X_i$ and with a chosen point x_i on i^{th} boundary circle, i.e. a section $x : \pi_0(\partial\Sigma) \to \partial\Sigma$ of the projection $\partial\Sigma \to \pi_0(\partial\Sigma)$ is fixed. By a homeomorphism of labeled surfaces we mean an orientation and labeling preserving homeomorphism, which sends the set of chosen points to itself. A *mapping class group* $MCG(\Sigma)$ is defined as the group of homeomorphisms $\Sigma \to \Sigma$ modulo isotopy equivalence relation. An isotopy is supposed not to move the chosen points.

We use the convention $AB = A \cdot B = B \circ A$ in this section.

4.1. A central extension of the category of surfaces. Projective representations of the mapping class groups are constructed in [38] as follows. The category $Surf$ of labeled surfaces and their homeomorphisms is embedded (non-canonically) into a category ON of oriented nets, which are a sort of labeled oriented graphs with 1-,2-, or 3-valent vertices. Morphisms of ON are generated by natural maps of graphs and some extra generators called fusing, braiding, twists and switches. In particular, the homeomorphisms $S, T : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus go to generators S, T in ON. The generators of ON are subject to several relations, including

$$(ST)^3 = S^2. \tag{4.1.1}$$

A central extension EN of the category ON is defined [38] by adding one more generator C commuting with other generators and deforming the relation (4.1.1) to

$$(ST)^3 = CS^2.$$

Lift the central extension to $Surf$ as in the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & ESurf & \longrightarrow & Surf & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & EN & \xrightarrow{j} & ON & \longrightarrow & 1
\end{array}$$

The meaning of this diagram is the following. Objects of $ESurf$ are objects of $Surf$. For any two homeomorphic surfaces $\Sigma_1, \Sigma_2 \in \operatorname{Ob} Surf$ let $ESurf(\Sigma_1, \Sigma_2)$ be determined as a central extension of $Surf(\Sigma_1, \Sigma_2)$ as in pull-back

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & ESurf(\Sigma_1, \Sigma_2) & \longrightarrow & Surf(\Sigma_1, \Sigma_2) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & EN(\phi(\Sigma_1), \phi(\Sigma_2)) & \xrightarrow{j} & ON(\phi(\Sigma_1), \phi(\Sigma_2)) & \longrightarrow & 1
\end{array}$$

Simply set $ESurf(\Sigma_1, \Sigma_2) = \overset{-1}{j}(\phi(Surf(\Sigma_1, \Sigma_2)))$ and set $ESurf(\Sigma_1, \Sigma_2)$ empty if Σ_1 and Σ_2 are not homeomorphic in $Surf$.

The central extension $ESurf$ of the groupoid $Surf$ (as any other central extension by $\mathbb{Z}$) can be described by a cohomology class in $H^2(Surf, \mathbb{Z})$ uniquely up to equivalence of categories. Namely, choose arbitrarily a lifting $\tilde{f} \in ESurf$ for any morphism $f \in Surf$ and for any composable $f, g \in Surf$ set $\theta(f, g) = m$ if $\tilde{f}^{-1}\widetilde{(fg)}\tilde{g}^{-1} = C^m$. For any $f, g, h \in Surf$ such that fg and gh exist we have

$$\theta(f, g) + \theta(fg, h) = \theta(f, gh) + \theta(g, h).$$

Restricting to $f, g \in MCG(\Sigma)$ we get a 2-cocycle $\theta \in Z^2(MCG(\Sigma), \mathbb{Z})$, whose cohomology class $[\theta] \in H^2(MCG(\Sigma), \mathbb{Z})$ does not depend on the lifting.

A functor $EN \to k$-vect sending C to $\lambda \in k^\times$ was constructed in [38]. So a summary of this work can be given by the following commutative diagram of functors

$$\begin{array}{ccccc}
Z' : ESurf & \longrightarrow & EN & \longrightarrow & k\text{-vect} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle j} & & \\
Surf & \xrightarrow{\phi} & ON & &
\end{array}$$

Any section of the projection p would give a projective representation $Z : Surf \to k$-vect satisfying $Z(fg) = \theta(f,g)Z(f)Z(g)$ for any composable $f, g \in Surf$ with a 2-cocycle θ.

Here we shall describe the projective representation of $M'_{g,n} = MCG(\Sigma_{g,n})$ in $Z(\Sigma_{g,n}) = \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g})$ for various genera g and number of holes n. The group $M'_{g,n}$ depends on coincidence of labels $X_1, \ldots, X_n$. To describe the representations uniformly we allow for homeomorphisms which do not preserve the labels, obtaining a larger group $M_{g,n} \supset M'_{g,n}$. It has a projection $p : M_{g,n} \to \mathfrak{S}_n$ to the symmetric group. We represent elements $h \in M_{g,n}$ by operators

$$Z(h) : \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g}) \to \mathrm{Hom}(X_{p(h)^{-1}(1)} \otimes \ldots \otimes X_{p(h)^{-1}(n)}, \mathbf{f}^{\otimes g})$$

satisfying $Z(hf) = \lambda^k Z(h)Z(f)$ for some $k \in \mathbb{Z}$.

The reader might want to consider $\mathscr{C} = H$-mod as the category of labels (set $I = k$ in this case), although the results are valid for an arbitrary 2-modular category $\mathscr{C}$.

4.2. Sphere. Consider a sphere $\Sigma_{0,n}$ with n disks removed, boundary circles are labeled by $X_1, X_2, \ldots, X_n$. The mapping class group $M_{0,n}$ is generated by the braiding homeomorphisms ω_i interchanging the i^{th} and $i+1^{\text{st}}$ holes and the inverse Dehn twists R_i performed in a collar neighbourhood of the i^{th} boundary circle. The relations

$$R_i R_j = R_j R_i, \quad \omega_i R_i = R_{i+1}\omega_i, \quad \omega_i R_{i+1} = R_i \omega_i,$$

$$\omega_i R_j = R_j \omega_i \text{ if } j \neq i, i+1,$$

$$\omega_i \omega_j = \omega_j \omega_i \text{ if } |i-j| > 1,$$

$$\omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_i \omega_{i+1},$$

$$\omega_1 \omega_2 \ldots \omega_{n-1}^2 \ldots \omega_2 \omega_1 = R_1^{-2}, \quad (\omega_1 \ldots \omega_{n-1})^n = R_1^{-1} \ldots R_n^{-1}.$$

are defining relations of $M_{0,n}$.

The group $M_{0,n}$ is represented in

$$Z(\Sigma_{0,n}) = \mathrm{Hom}(X_1 \otimes X_2 \otimes \ldots \otimes X_n, I)$$

by the operators

$$Z(\omega_i) = \mathrm{Hom}(1 \otimes c_{X_{i+1},X_i} \otimes 1, I),$$
$$Z(R_i) = \mathrm{Hom}(1 \otimes \nu_{X_i} \otimes 1, I).$$

This is a usual representation not only projective.

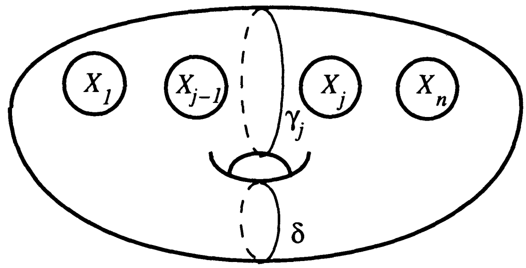


Fig. 1.

4.3. Torus. Consider a torus $\Sigma_{1,n}$ with n disks removed, boundary circles are labeled by $X_1, X_2, \dots, X_n$. The mapping class group $M_{1,n}$ is generated by the homeomorphisms S, $T = Tw_\delta$, R_i, $a_j = Tw_{\gamma_j}^{-1} Tw_\delta$, where the cycles γ_j separate $j-1^{\text{st}}$ and j^{th} holes as shown at Fig. 1, and the braiding homeomorphisms ω_i interchanging the i^{th} and $i+1^{\text{st}}$ holes. Denote $b_j = S^{-1} a_j S$ and for $j < k$ denote

$$B_{jk}^{-1} = (\omega_{k-1}\omega_{k-2}\dots\omega_j)(\omega_k\omega_{k-1}\dots\omega_{j+1})\dots(\omega_{n-1}\omega_{n-2}\dots\omega_{n+j-k})$$
$$(\omega_{j-k+n}\omega_{j-k+n-1}\dots\omega_j)(\omega_{j-k+n+1}\omega_{j-k+n}\dots\omega_{j+1})\dots(\omega_{n-1}\omega_{n-2}\dots\omega_{k-1}).$$

The relations

$$(ST)^3 = S^2, \qquad S^4 = B_{12}B_{23}B_{34}\dots B_{n-1,n}R_1^{-1}\dots R_n^{-1},$$
$$S^{-1}b_jS = b_ja_j^{-1}b_j^{-1}, \quad T^{-1}a_jT = a_j, \quad T^{-1}b_jT = b_ja_j,$$
$$\omega_iS = S\omega_i, \quad \omega_iT = T\omega_i, \quad \omega_iR_i = R_{i+1}\omega_i, \quad \omega_iR_{i+1} = R_i\omega_i,$$
$$\omega_iR_j = R_j\omega_i \text{ for } j \neq i, i+1,$$
$$\omega_i\omega_j = \omega_j\omega_i \text{ for } |i-j| > 2,$$
$$\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1},$$
$$a_1 = b_1 = 1, \quad a_ja_k = a_ka_j, \quad b_jb_k = b_kb_j,$$
$$a_ja_k^{-1}b_ka_j^{-1}a_kb_k^{-1} = B_{jk} \text{ for } j < k,$$
$$a_k^{-1}b_j^{-1}b_ka_kb_jb_k^{-1} = B_{jk} \text{ for } j < k,$$
$$a_iB_{jk} = B_{jk}a_i, \qquad b_iB_{jk} = B_{jk}b_i \text{ for } i \leqslant j < k$$

are defining relations of $M_{1,n}$ ([44], extending [1]).

Set

$$Z(\Sigma_{1,n}) = \mathrm{Hom}(X_1 \otimes \dots \otimes X_n, \mathbf{f}),$$
$$Z(S) = \mathrm{Hom}(X_1 \otimes \dots \otimes X_n, S),$$
$$Z(T) = \mathrm{Hom}(X_1 \otimes \dots \otimes X_n, T),$$
$$Z(R_i) = \mathrm{Hom}(X_1 \otimes \dots \otimes \nu_{X_i} \otimes \dots \otimes X_n, \mathbf{f}),$$
$$Z(\omega_i) = \mathrm{Hom}(X_1 \otimes \dots \otimes c_{X_{i+1},X_i} \otimes \dots \otimes X_n, \mathbf{f}),$$

$$
\begin{aligned}
Z(a_j) = \Big(&\mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes X_j \otimes \ldots \otimes X_n, \mathbf{f}) \\
&\xrightarrow{\sim} \mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1}, \mathbf{f} \otimes ({}^\vee X_n \otimes \ldots \otimes {}^\vee X_j)) \\
&\xrightarrow{\mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1}, \Omega_r^{-1})} \mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1}, \mathbf{f} \otimes ({}^\vee X_n \otimes \ldots \otimes {}^\vee X_j)) \\
&\xrightarrow{\sim} \mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes (X_j \otimes \ldots \otimes X_n), \mathbf{f}) \\
&\xrightarrow{\mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes \nu^{-1}_{X_j \otimes \ldots \otimes X_n}, \mathbf{f})} \mathrm{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes X_j \otimes \ldots \otimes X_n, \mathbf{f})\Big).
\end{aligned}
$$

Here S and T in the right-hand side mean $S : \mathbf{f} \to \mathbf{f}$, $T : \mathbf{f} \to \mathbf{f}$ described in Sect. 2.1. Equation (2.1.5) implies that

$$(Z(S)Z(T))^3 = \lambda Z(S)^2.$$

Moreover, the above is a projective representation of $M_{1,n}$ [38].

Notice that in the particular case $n = 0$ we have

$$Z(S)^4 = 1 : \mathrm{Hom}(I, \mathbf{f}) \to \mathrm{Hom}(I, \mathbf{f})$$

since $S^4 = \nu^{-1} : \mathbf{f} \to \mathbf{f}$.

Remark 4.3.1. The map $Z(b_j) \overset{\text{def}}{=} Z(S)^{-1} Z(a_j) Z(S)$ has another presentation, which is an important part of the proof [38]. Consider for simplicity the case $n = 2$. Introduce a map B_2 as the composition

$$
\begin{aligned}
&\mathrm{Hom}(X_1 \otimes X_2, F) \xrightarrow{\mathrm{Hom}(c_{X_2, X_1}, F)} \mathrm{Hom}(X_2 \otimes X_1, F) \\
&\quad \xrightarrow{\sim} \mathrm{Hom}(X_1, X_2{}^\vee \otimes \int^N N \otimes N^\vee) \\
&\quad \xrightarrow{-\otimes \mathrm{coev}_{X_1}} \mathrm{Hom}(X_1, \int^N X_2{}^\vee \otimes N \otimes N^\vee \otimes X_2{}^{\vee\vee} \otimes X_2{}^\vee) \\
&\xrightarrow{\sim} \mathrm{Hom}(X_1, \int^N (X_2{}^\vee \otimes N) \otimes (X_2{}^\vee \otimes N)^\vee \otimes X_2{}^\vee) \to \mathrm{Hom}(X_1, \int^P P \otimes P^\vee \otimes X_2{}^\vee) \\
&\quad \xrightarrow{\sim} \mathrm{Hom}(X_1 \otimes X_2{}^{\vee\vee}, F) \xrightarrow{\mathrm{Hom}(X_1 \otimes u_0^2, F)} \mathrm{Hom}(X_1 \otimes X_2, F).
\end{aligned}
$$

It covers $Z(b_2)$ in the sense that the diagram

$$
\text{(4.3.1)} \qquad
\begin{array}{ccc}
\mathrm{Hom}(X_1 \otimes X_2, F) & \xrightarrow{B_2} & \mathrm{Hom}(X_1 \otimes X_2, F) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X_1 \otimes X_2, \mathbf{f}) & \xrightarrow{Z(b_2)} & \mathrm{Hom}(X_1 \otimes X_2, \mathbf{f})
\end{array}
$$

is commutative. Notice that in the perfect modular case $F = \mathbf{f}$ and $Z(b_2)$ simply equals B_2.

If $\mathscr{C}$ is semisimple, the vertical arrows in (4.3.1) are surjective. The same holds if $\mathscr{C} = H$-mod for finite dimensional H and one of the H-modules X_1 or X_2 is projective (then $X_1 \otimes X_2$ is also projective). In these assumptions we can represent B_2 in a different way as a composition of bijections

$$\operatorname{Hom}(X_1 \otimes X_2, F) \xrightarrow{\operatorname{Hom}(c_{X_2,X_1},F)} \operatorname{Hom}(X_2 \otimes X_1, \int^N N \otimes N^\vee)$$

$$\simeq \operatorname{Hom}(X_2 \otimes X_1, \int^N N \otimes {}^\vee N) \simeq \int^N \operatorname{Hom}(X_2 \otimes (X_1 \otimes N), N)$$

$$\simeq \int^{N,P} \operatorname{Hom}(X_2 \otimes P, N) \otimes \operatorname{Hom}(X_1 \otimes N, P) \simeq \int^P \operatorname{Hom}(X_1 \otimes (X_2 \otimes P), P)$$

$$\simeq \int^P \operatorname{Hom}(X_1 \otimes X_2, P \otimes {}^\vee P) \simeq \operatorname{Hom}(X_1 \otimes X_2, \int^P P \otimes P^\vee) = \operatorname{Hom}(X_1 \otimes X_2, F).$$

Knowing $Z(b_2)$ by (4.3.1) for projective X_1 or X_2, one recovers this map for arbitrary X_1, X_2 using short projective resolutions. In the calculations above we used

Lemma 4.3.1 ([38]). *Let* $F : \mathscr{C} \to k\text{-Vect}$, $G : \mathscr{C}^{\mathrm{op}} \to k\text{-Vect}$ *be functors. Then*

$$\int^X F(X) \otimes \operatorname{Hom}(X, B) \to F(B), \qquad v \otimes f \mapsto F(f).v$$

$$\int^X \operatorname{Hom}(B, X) \otimes G(X) \to G(B), \qquad f \otimes v \mapsto G(f).v$$

are isomorphisms of vector spaces.

4.4. Closed surfaces and surfaces with one hole

4.4.1. Surfaces with one hole. Let $\Sigma = \Sigma_{g,1}$ be a surface of genus g with one disk removed and the boundary labeled by an H-module X. Lickorish [29] proved that its mapping class group $M_{g,1}$ is generated by inverse Dehn twists in a neighbourhood of the following cycles: a_k, b_k, d_k, e_k ($a_1 = d_1 = e_1$) (see Fig. 2). These Dehn twists are denoted by the same letters as cycles. Wajnryb [61] found a system of defining relations for $M_{g,1}$. An equivalent system of relations is the following:

(A) $a_i b_i a_i = b_i a_i b_i$, $a_{i+1} b_i a_{i+1} = b_i a_{i+1} b_i$, $d_i b_i d_i = b_i d_i b_i$, $e_i b_i e_i = b_i e_i b_i$ and all other pairs of generators commute.

(B) $(a_1 b_1 a_2)^4 = d_2 e_2$.

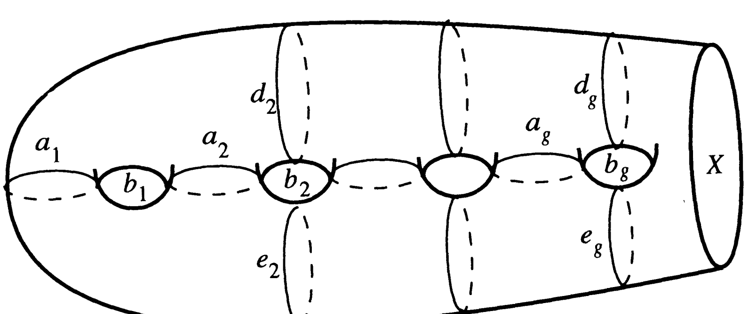


Fig. 2.

(C) $e_k G_k = G_k d_k$, where $G_k = b_k a_k \dots b_1 a_1 a_1 b_1 \dots a_k b_k$.

(D) $J_k d_{k+1} = d_k J_k$, where $J_k = b_k a_k a_{k+1} b_k b_{k+1} a_{k+1} a_k^{-1} b_k^{-1} d_k b_k a_k b_{k-1} d_{k-1}$ $(a_k^{-1} b_{k-1}^{-1} b_k^{-1} a_k^{-1} b_{k-1}^{-1} b_k^{-1}) d_k^{-1} b_k^{-1} a_{k+1}^{-1} b_{k+1}^{-1}$.

(E) $h b_1^{-1} a_1^{-1} a_2^{-1} b_1^{-1} h b_1 a_2 a_1 b_1 d_2 = d_3 a_3 a_2 a_1$, where

$$h = b_2^{-1} a_2^{-1} a_3^{-1} b_2^{-1} d_2 b_2 a_3 a_2 b_2.$$

To construct $Z(\Sigma)$ we follow the recipe of [38]. First of all we choose an *oriented net* (roughly this is an oriented graph, for precise definition see [38]) which encodes the structure of the surface. Several graphs can be chosen, for instance, Figs. 3–5.

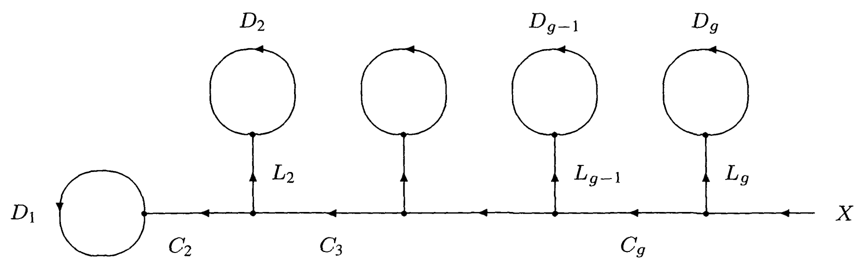


Fig. 3.

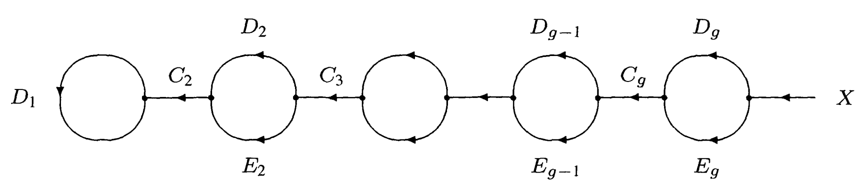


Fig. 4.

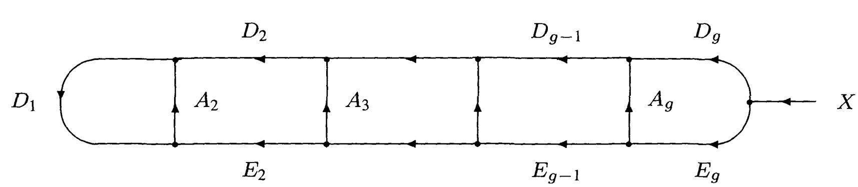


Fig. 5.

They are all isomorphic in the category of oriented nets [38]. Next step is to compute the functor $\mathscr{C}^{\text{op}} \to k$-vect corresponding to these graphs, obtained by taking coends over internal edges. The functor is obtained in different forms, which are isomorphic due to associativity isomorphisms in $\mathscr{C}$. By Lemma 4.3.1 we find that to Fig. 3 corresponds

$$\int^{C_i,D_i,L_i} \mathrm{Hom}(X, L_g \otimes C_g) \otimes \mathrm{Hom}(C_g, L_{g-1} \otimes C_{g-1}) \otimes \ldots \otimes \mathrm{Hom}(C_3, L_2 \otimes C_2) \otimes$$
$$\otimes \mathrm{Hom}(L_g \otimes D_g, D_g) \otimes \ldots \otimes \mathrm{Hom}(L_2 \otimes D_2, D_2) \otimes \mathrm{Hom}(C_2 \otimes D_1, D_1) \simeq$$

$$\simeq \int^{D_i,L_i} \mathrm{Hom}(X, L_g \otimes L_{g-1} \otimes \ldots \otimes L_2 \otimes C_2) \otimes \mathrm{Hom}(L_g, D_g \otimes {}^{\vee}D_g) \otimes \ldots \otimes$$
$$\otimes \mathrm{Hom}(L_2, D_2 \otimes {}^{\vee}D_2) \otimes \mathrm{Hom}(C_2, D_1 \otimes {}^{\vee}D_1) \simeq$$

$$\simeq \int^{D_i} \mathrm{Hom}(X, (D_g \otimes D_g{}^{\vee}) \otimes \ldots \otimes (D_2 \otimes D_2{}^{\vee}) \otimes (D_1 \otimes D_1{}^{\vee})). \tag{4.4.1}$$

To Fig. 4 corresponds

$$\int^{C_i,D_i,E_i} \mathrm{Hom}(C_2 \otimes D_1, D_1) \otimes \mathrm{Hom}(D_2 \otimes E_2, C_2) \otimes \mathrm{Hom}(C_3, D_2 \otimes E_2) \otimes \ldots \otimes$$
$$\otimes \mathrm{Hom}(C_g, D_{g-1} \otimes E_{g-1}) \otimes \mathrm{Hom}(D_g \otimes E_g, C_g) \otimes \mathrm{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{C_i,D_i} \mathrm{Hom}(C_2, D_1 \otimes D_1{}^{\vee}) \otimes \mathrm{Hom}(C_3, D_2 \otimes D_2{}^{\vee} \otimes C_2) \otimes \ldots \otimes$$
$$\otimes \mathrm{Hom}(C_g, D_{g-1} \otimes D_{g-1}{}^{\vee} \otimes C_{g-1}) \otimes \mathrm{Hom}(X, D_g \otimes D_g{}^{\vee} \otimes C_g) \simeq$$

$$\simeq \int^{D_i} \mathrm{Hom}(X, (D_g \otimes D_g{}^{\vee}) \otimes \ldots \otimes (D_2 \otimes D_2{}^{\vee}) \otimes (D_1 \otimes D_1{}^{\vee})). \tag{4.4.2}$$

To Fig. 5 corresponds

$$\int^{A_i,D_i,E_i} \mathrm{Hom}(E_2 \otimes D_1, A_2) \otimes \mathrm{Hom}(D_2 \otimes A_2, D_1) \otimes \mathrm{Hom}(E_3, A_3 \otimes E_2) \otimes \ldots \otimes$$
$$\otimes \mathrm{Hom}(E_g, A_g \otimes E_{g-1}) \otimes \mathrm{Hom}(D_g \otimes A_g, D_{g-1}) \otimes \mathrm{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{D_i,E_i} \mathrm{Hom}(E_2 \otimes D_1, D_2{}^{\vee} \otimes D_1) \otimes \mathrm{Hom}(E_3, D_3{}^{\vee} \otimes D_2 \otimes E_2) \otimes \ldots \otimes$$
$$\otimes \mathrm{Hom}(E_g, D_g{}^{\vee} \otimes D_{g-1} \otimes E_{g-1}) \otimes \mathrm{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{D_i} \mathrm{Hom}(X, (D_g \otimes D_g{}^{\vee}) \otimes \ldots \otimes (D_2 \otimes D_2{}^{\vee}) \otimes (D_1 \otimes D_1{}^{\vee})). \tag{4.4.3}$$

If X is projective this space is

$$\mathrm{Hom}(X, F \otimes \ldots \otimes F \otimes F).$$

The same answer will be obtained if we calculate the coend in the category of left exact functors $\mathscr{C}^{\mathrm{op}} \to k$-vect. The final step is taking a quotient and setting

$$Z(\Sigma_{g,1}) = \mathrm{Hom}(X, \mathbf{f}^{\otimes g}).$$

The generators a_k, d_k, e_k of $M_{g,1}$ are represented by applying a ribbon twist to internal variables A_k, D_k, E_k. Thus from any presentation (4.4.1)–(4.4.3) we get

$$Z(d_k) = \mathrm{Hom}(X, \mathbf{f}^{\otimes g-k} \otimes T \otimes \mathbf{f}^{\otimes k-1}). \tag{4.4.4}$$

From the third presentation (4.4.3) we get that a_k acts by applying a ribbon twist to $D_k{}^\vee \otimes D_{k-1}$, which induces $T \otimes T \cdot \Omega : F \otimes F \to F \otimes F$. Hence,

$$Z(a_k) = \mathrm{Hom}(X, \mathbf{f}^{\otimes g-k} \otimes (T \otimes T \cdot \Omega) \otimes \mathbf{f}^{\otimes k-2}). \tag{4.4.5}$$

From the second presentation (4.4.2) we get that e_k acts by applying a ribbon twist to $D_k{}^\vee \otimes C_k$ or, equivalently, to $D_k{}^\vee \otimes D_{k-1} \otimes D_{k-1}{}^\vee \otimes \ldots \otimes D_1 \otimes D_1{}^\vee$. This induces $\Omega^r \cdot T \otimes \nu : F \otimes F^{\otimes k-1} \to F \otimes F^{\otimes k-1}$, hence,

$$Z(e_k) = \mathrm{Hom}(X, \mathbf{f}^{\otimes g-k} \otimes \Omega^r_{\mathbf{f},\mathbf{f}^{\otimes k-1}}) \cdot \mathrm{Hom}(X, \mathbf{f}^{\otimes g-k} \otimes T \otimes \nu_{\mathbf{f}^{\otimes k-1}}). \tag{4.4.6}$$

Finally, b_k is conjugate to d_k by a homeomorphism of the type S, and we set

$$Z(b_k) = \mathrm{Hom}(X, \mathbf{f}^{\otimes g-k} \otimes STS^{-1} \otimes \mathbf{f}^{\otimes k-1}). \tag{4.4.7}$$

As shown in [38] these operators define a projective representation $M_{g,1} \to PGL(Z(\Sigma_{g,1}))$ with the 2-cocycle whose values are powers of λ. Therefore, this representation is induced by a projective representation $z : M_{g,1} \to \mathrm{Aut}_{\mathcal{C}}(\mathbf{f}^{\otimes g})/\{\lambda^n\}_{n\in\mathbb{Z}}$,

$$\begin{aligned}
a_k &\longmapsto \mathbf{f}^{\otimes g-k} \otimes (T \otimes T \cdot \Omega) \otimes \mathbf{f}^{\otimes k-2},\\
b_k &\longmapsto \mathbf{f}^{\otimes g-k} \otimes STS^{-1} \otimes \mathbf{f}^{\otimes k-1},\\
d_k &\longmapsto \mathbf{f}^{\otimes g-k} \otimes T \otimes \mathbf{f}^{\otimes k-1},\\
e_k &\longmapsto \mathbf{f}^{\otimes g-k} \otimes (\Omega^r_{\mathbf{f},\mathbf{f}^{\otimes k-1}} \cdot T \otimes \nu_{\mathbf{f}^{\otimes k-1}}).
\end{aligned}$$

Indeed, set $X = \mathbf{f}^{\otimes g}$ and apply both parts of relations (A)-(E) to the vector $\mathrm{id}_X \in Z(\Sigma_{g,1})$.

4.4.2. Closed surfaces. Now let $\Sigma_{g,0}$ be a closed surface of genus g. The mapping class group $M_{g,0}$ is the quotient of $M_{g,1}$ by two extra relations [61]:

(F) $H_g^2 = 1$, where $H_k = b_g a_g \ldots b_2 a_2 b_1 a_1 a_1 b_1 a_2 b_2 \ldots a_g b_g d_g e_g$.

(G) $d_g = e_g$.

Set $Z(\Sigma_{g,0}) = \mathrm{Hom}(\mathbb{C}, \mathbf{f}^{\otimes g})$ and define $Z(a_k)$, $Z(b_k)$, $Z(d_k)$, $Z(e_k)$ by Eqs. (4.4.4)–(4.4.7) with $X = \mathbb{C}$. We get a projective representation $M_{g,0} \to PGL(Z(\Sigma_{g,0}))$ [38].

4.4.3. Exercises. The reader might want to check some of the above results straightforwardly. Several exercises will help in doing this. They present minor generalizations of some of the quoted results and specify the power of λ involved in relations.

Easy exercise 1. All maps $z(a_i), z(d_j), z(e_k) : \mathbf{f}^{\otimes g} \to \mathbf{f}^{\otimes g}$ commute. $z(b_i)$ commutes with all maps except $z(a_i)$, $z(a_{i+1})$, $z(d_i)$, $z(e_i)$. Also $z(d_i)z(b_i)z(d_i) = z(b_i)z(d_i)z(b_i)$ (see relation (A)).

Exercise 2. For any $Y \in \mathscr{C}$ we have

$$\Omega^r_{\mathbf{f},Y} \cdot S^{-1} \otimes \nu_Y \cdot \Omega^r_{\mathbf{f},Y} = \lambda(T^{-1}S^{-1}T^{-1}) \otimes Y \cdot \Omega^r_{\mathbf{f},Y} \cdot (S^{-1}T^{-1}) \otimes Y : \mathbf{f} \otimes Y \to \mathbf{f} \otimes Y,$$

$$\Omega^l_{Y,\mathbf{f}} \cdot \nu_Y \otimes S^{-1} \cdot \Omega^l_{Y,\mathbf{f}} = \lambda Y \otimes (T^{-1}S^{-1}T^{-1}) \cdot \Omega^l_{Y,\mathbf{f}} \cdot Y \otimes (S^{-1}T^{-1}) : Y \otimes \mathbf{f} \to Y \otimes \mathbf{f}.$$

Easy exercise 3. Deduce from the previous exercise that $z(a_i)z(b_i)z(a_i) = z(b_i)z(a_i)$ $z(b_i)$, $z(a_{i+1})z(b_i)z(a_{i+1}) = z(b_i)z(a_{i+1})z(b_i)$, $z(e_i)z(b_i)z(e_i) = z(b_i)z(e_i)z(b_i)$ (see relation (A)).

Exercise 4. For any $Y \in \mathscr{C}$ we have

$$(\nu_Y \otimes S^{-1} \cdot \Omega^l_{Y,\mathbf{f}})^4 = \nu_Y \otimes \mathbf{f} \cdot \nu_{Y\otimes\mathbf{f}} : Y \otimes \mathbf{f} \to Y \otimes \mathbf{f}.$$

What does it mean for $Y = I$?

Easy exercise 5. Deduce from the previous exercise that

$$(z(d_1)z(b_1)z(a_2))^4 = \lambda^4 z(d_2)z(e_2)$$

(relation (B)).

Exercise 6. For any $Y \in \mathscr{C}$ we have

$$(T^{-1}S^{-1}) \otimes Y \cdot \Omega^r_{\mathbf{f},Y} \cdot (S^{-1}TS) \otimes Y \cdot (\Omega^r_{\mathbf{f},Y})^{-1} \cdot (ST) \otimes Y = \Omega^r_{\mathbf{f},Y} \cdot T \otimes \nu_Y : \mathbf{f} \otimes Y \to \mathbf{f} \otimes Y\ .$$

Exercise 7. Deduce from the previous exercise that $z(e_2) = z(G_2)z(d_2)z(G_2)^{-1}$, where $z(G_2) \overset{\text{def}}{=} z(b_2)z(a_2)z(d_1)^2 z(a_2)z(b_2)$.

Easy exercise 8. Show that

$$Z(d_g) = Z(e_g) : \mathrm{Hom}(\mathbb{C}, \mathbf{f}^{\otimes g}) \to \mathrm{Hom}(\mathbb{C}, \mathbf{f}^{\otimes g}).$$

4.5. General case. Let $\Sigma_{g,n}$ be a surface of genus g with n disks removed, boundary circles are labeled by $X_1, \ldots, X_n$. Figure 6 suggests an embedding $\Sigma_{g,1} \hookrightarrow \Sigma_{g,n}$, which induces a homomorphism $M_{g,1} \to M_{g,n}$. The mapping class group $M_{g,n}$ is generated by images a_k, b_k, d_k, e_k of generators of $M_{g,1}$, the braidings ω_i, the twists R_i, new generators S_l, which are homeomorphisms of the type S inside the $\mathbb{T}^2 - D^2$ region F_l and identity outside, and inverse Dehn twists $t_{j,k}$ in tubular neighbourhood of the cycles $t_{j,k}$ This is not a minimal system of generators, since S_k can be expressed through b_k and d_k. All of these generators are easily representable.

Set

$$Z(\Sigma_{g,n}) = \mathrm{Hom}(X_1 \otimes X_2 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g}),$$
$$Z(a_k) = \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g-k} \otimes (T \otimes T \cdot \Omega) \otimes \mathbf{f}^{\otimes k-2}),$$
$$Z(b_k) = \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g-k} \otimes STS^{-1} \otimes \mathbf{f}^{\otimes k-1}),$$

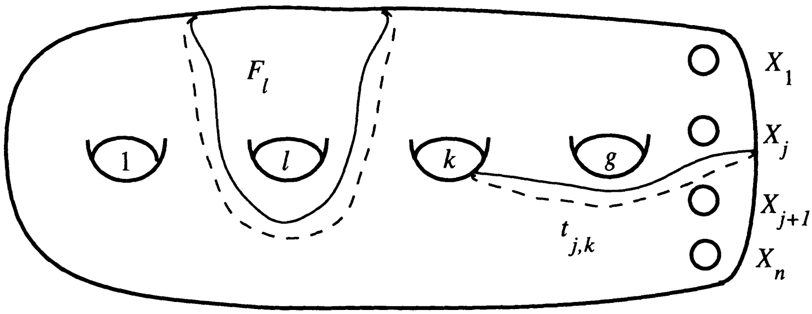


Fig. 6.

$$
\begin{aligned}
Z(d_k) &= \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g-k} \otimes T \otimes \mathbf{f}^{\otimes k-1}),\\
Z(e_k) &= \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g-k} \otimes (\Omega^r_{\mathbf{f},\mathbf{f}^{\otimes k-1}} \cdot T \otimes \nu_{\mathbf{f}^{\otimes k-1}})),\\
Z(\omega_i) &= \mathrm{Hom}(X_1 \otimes \ldots \otimes c_{X_{i+1},X_i} \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g}),\\
Z(R_i) &= \mathrm{Hom}(X_1 \otimes \ldots \otimes \nu_{X_i} \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g}),\\
Z(S_k) &= \mathrm{Hom}(X_1 \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g-k} \otimes S \otimes \mathbf{f}^{\otimes k-1}).
\end{aligned}
$$

Set $Z(t_{j,k})$ to be the composition

$$
\begin{array}{c}
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j \otimes X_{j+1} \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g})\\
\wr\downarrow\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j, \mathbf{f}^{\otimes g-k} \otimes \mathbf{f} \otimes \mathbf{f}^{\otimes k-1} \otimes {}^\vee X_n \otimes \ldots \otimes {}^\vee X_{j+1})\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j, \mathbf{f}^{\otimes g-k} \otimes \Omega^r_{\mathbf{f},\mathbf{f}^{\otimes k-1} \otimes {}^\vee X_n \otimes \ldots \otimes {}^\vee X_{j+1}}) \downarrow\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j, \mathbf{f}^{\otimes g-k} \otimes \mathbf{f} \otimes \mathbf{f}^{\otimes k-1} \otimes {}^\vee X_n \otimes \ldots \otimes {}^\vee X_{j+1})\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j, \mathbf{f}^{\otimes g-k} \otimes T \otimes \nu_{\mathbf{f}^{\otimes k-1} \otimes {}^\vee X_n \otimes \ldots \otimes {}^\vee X_{j+1}}) \downarrow\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j, \mathbf{f}^{\otimes g-k} \otimes \mathbf{f} \otimes \mathbf{f}^{\otimes k-1} \otimes {}^\vee X_n \otimes \ldots \otimes {}^\vee X_{j+1})\\
\wr\downarrow\\
\mathrm{Hom}(X_1 \otimes \ldots \otimes X_j \otimes X_{j+1} \otimes \ldots \otimes X_n, \mathbf{f}^{\otimes g}).
\end{array}
$$

Then the above is a projective representation of $M_{g,n}$ [38].

I don't know any defining system of relations of $M_{g,n}$. Perhaps, it was never written explicitly. Proof of the result [38] used the exact sequence [1, 56]

$$
\begin{array}{c}
1 \longrightarrow \bar{B}_{g,n} \longrightarrow M_{g,n} \longrightarrow M_{g,0} \longrightarrow 1,\\
1 \longrightarrow \mathbb{Z}^n \longrightarrow \bar{B}_{g,n} \longrightarrow B_{g,n} \longrightarrow 1,
\end{array}
$$

where $B_{g,n}$ is the braid group of a surface of genus g, whose presentation was given by Scott [56].

5. Invariants of closed 3-manifolds

I have nothing to add to the method of obtaining invariants of closed 3-manifolds via Kirby calculus invented by Reshetikhin and Turaev [51]. It was used afterwards by many authors including Lickorish [30, 31], Kirby and Melvin [19]. Turaev [60] has shown that a semisimple modular category serves well to define a 3-manifold invariant. In this paper I propose to use (eventually non-semisimple) 3-modular categories $\mathscr{C}$ as the starting data for the method. When $\mathscr{C} = H$-mod this invariant coincides with the one defined by Hennings [10] translated to an unoriented setting by Kauffman and Radford [14].

5.1. 3-manifolds from links in S^3. It is well known that any closed connected oriented 3-manifold can be obtained via surgery along a framed tame link in S^3 [28]. Namely, given a framed tame link $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m \subset S^3 = \partial B^4$ we construct a compact oriented 4-manifold W_L glueing m 2-handles $D^2 \times D^2$ (D^2 is a disk) to the ball B^4 along closed tubular neighbourhoods $U_i \subset S^3$ of L_i. The part of the boundary $S^1 \times D^2 = \partial D^2 \times D^2$ of the i^{th} 2-handle is identified with $U_i \simeq L_i \times D^2$ so that the linking number of $S^1 \times 1 \subset \partial D^2 \times \partial D^2$ with L_i coincides with the self-linking number of L_i. The boundary $M_L = \partial W_L$ is an oriented compact closed connected 3-manifold.

Kirby proved [17] that two manifolds M_L and $M_{L'}$ obtained by surgery from two framed links $L, L' \subset S^3$ are homeomorphic iff the link L' can be obtained from L by a finite sequence of transformations, which we call Kirby moves:

1. The elimination or insertion of an unknotted component labeled ± 1, unlinked with other components.
2. The band (or handle slide) move – making a connected sum of one of the components L_i with a parallel copy $\tilde{L}_j$ of another component L_j.

Fenn and Rourke [7] proved a similar result with a subset of transformations introduced by Kirby, which we call Kirby–Fenn–Rourke moves (see Figs. 7–8). Therefore, to give an invariant $\tau(M) \in k$ for 3-manifolds $M = M_L$ amounts to give an invariant $\tau(L)$ for framed links L in S^3, which would not change under Kirby or Kirby–Fenn–Rourke moves. We may and we shall consider framed links in $\mathbb{R}^3$ instead of S^3.

5.2. An invariant determined by a 3-modular category. Let $\mathscr{C}$ be a 3-modular category. The reader might want to assume that $\mathscr{C} = H$-mod for simplicity. First of all we construct an invariant of framed links in $\mathbb{R}^3$.

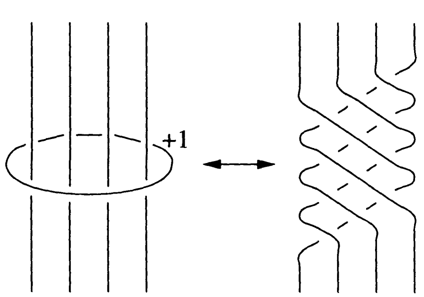


Fig. 7.

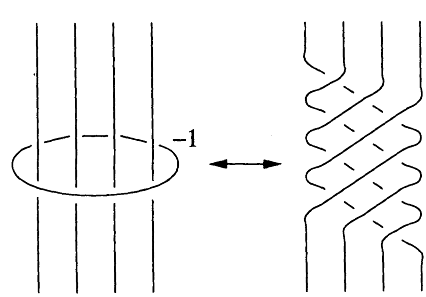


Fig. 8.

Let L be a framed tame link in $\mathbb{R}^3$. It can be deformed to a smooth link so that its projection $p(L)$ to the standard plane $\mathbb{R}^2$ is a smooth generic link diagram D_L and the normal vector field on L_i determining the framing is parallel to $\mathbb{R}^2$. Choose a straight line $\mathbb{R}^1$ in $\mathbb{R}^2$ so that D_L lies in one half-plane with respect to this line (we shall draw it as a lower half-plane). Choose a point $e_i \in L_i$ for each i and connect it in $\mathbb{R}^3$ with a point x_i of $\mathbb{R}^1$ by a curve γ_i, which projects to the lower half-plane. We assume $p(\gamma_i)$ smooth and transversal to $p(L_i)$ in the point $p(e_i)$. Make it generic, so that $x_i \neq x_j$ and the curves γ_j don't intersect each other and L except at the ends. Draw the corresponding plane diagram $\bar{D}_L$ assigning necessary signs of overcrossing or undercrossing to each double point of the projection to $\mathbb{R}^2$. Duplicate the projections of the curves γ_i, replacing them by two parallel curves γ_i^- and γ_i^+ (e.g. parts of the boundary of a small neighbourhood V_i of $p(\gamma_i)$). Remove the connected component of $L_i \cup V_i$ containing e_i. The result will be a diagram of a tangle looking like Fig. 9. Make this diagram into a $\mathscr{C}$-tangle T_L [37] assigning a color $X_i \in \operatorname{Ob}\mathscr{C}$ to the point x_i and $X_i{}^\vee$ to the point x_i' and inserting to γ_i^- such a morphism $u_0^{2a} : X_i \to X_i^{(2a\vee)}$, $a \in \mathbb{Z}$, as consistency requires. The tangle T_L represents a morphism in $\mathscr{C}$

$$\Phi(T_L; X_1, \ldots, X_m) : X_1 \otimes X_1{}^\vee \otimes X_2 \otimes X_2{}^\vee \otimes \ldots \otimes X_m \otimes X_m{}^\vee \to I.$$

Fix all objects $X_i, X_i{}^\vee$ except for $i = j$ and vary $X_j, X_j{}^\vee$. We see that the following diagram commutes for any morphism $f : Y_j \to Z_j \in \mathscr{C}$:

$$\begin{array}{ccc} X_1\otimes X_1{}^\vee\otimes\ldots\otimes Y_j\otimes Z_j{}^\vee\otimes\ldots\otimes X_m\otimes X_m{}^\vee & \xrightarrow{\ldots\otimes Y_j\otimes f^t\otimes\ldots} & X_1\otimes X_1{}^\vee\otimes\ldots\otimes Y_j\otimes Y_j{}^\vee\otimes\ldots\otimes X_m\otimes X_m{}^\vee \\ {\scriptstyle \ldots\otimes f\otimes Z_j{}^\vee\otimes\ldots}\downarrow & & \downarrow{\scriptstyle \Phi(T_L;X_1,\ldots,Y_j,\ldots,X_m)} \\ X_1\otimes X_1{}^\vee\otimes\ldots\otimes Z_j\otimes Z_j{}^\vee\otimes\ldots\otimes X_m\otimes X_m{}^\vee & \xrightarrow{\Phi(T_L;X_1,\ldots,Z_j,\ldots,X_m)} & I \end{array}$$

Indeed, insert a morphism f to a point of γ_j^- in Fig. 9 and push it along the curve through all crossings. It will appear on γ_j^+ as f^t.

Commutativity of the above diagram means that $\Phi(T_L; \ldots)$ factorizes through the canonical mappings $i_{X_j} : X_j \otimes X_j{}^\vee \to F$ on the j^{th} place (compare with (1.2.1)). Therefore, it defines a morphism

$$\Phi(T_L) : F^{\otimes m} \to I.$$

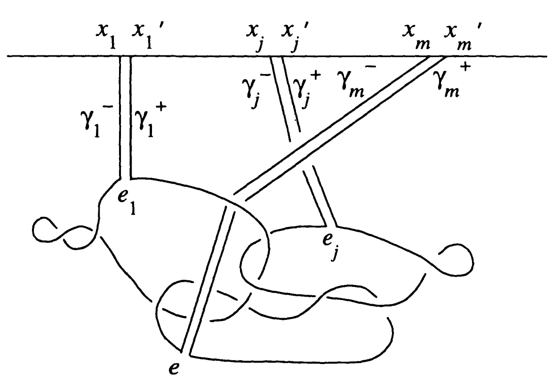


Fig. 9.

Take an invariant element of F, that is, a morphism $\alpha : I \to F \in \mathscr{C}$. Define a number in k

$$\phi(T_L, \alpha) : I = I^{\otimes m} \xrightarrow{\alpha^{\otimes m}} F^{\otimes m} \xrightarrow{\Phi(T_L)} I.$$

Proposition 5.2.1. *Assume that $\gamma_F(\alpha) = \alpha$. Then the number $\phi(T_L, \alpha) \in k$ depends only on L and α and not on choices made in constructing T_L.*

Notation. $\tau(L, \alpha) = \phi(T_L, \alpha)$.

Proof. Since the source of α is a unit object, we can change all overcrossings with γ_j to undercrossings and vice versa without changing the value of $\phi(T_L, \alpha)$. Since all tensor factors in $\alpha^{\otimes m}$ coincide, the result does not depend on the order of $x_j = \gamma_j(1)$ on the line. Indeed, an interchange of the two neighbour ends x_i and x_j is equivalent to adding a crossing which is interpreted as $c : I \otimes I \to I \otimes I$, and this is an identity morphism.

The tangent vector $p(\gamma_i)'(0)$ can be chosen normal to $p(L_i)$ at the point $p(e_i)$. The following diagram explains that the change from one normal vector to another is equivalent to composing α with γ_F or γ_F^{-1} on the j^{th} place (Fig. 10). Since $\gamma_F \alpha = \alpha = \gamma_F^{-1} \alpha$ the resulting $\phi(T_L, \alpha)$ will not change.

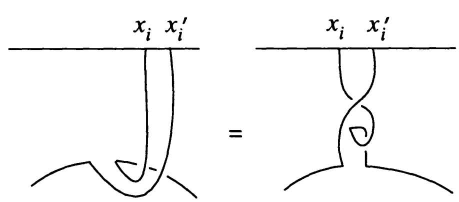
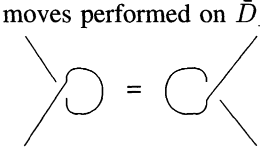


Fig. 10.

Also the value of $\phi(T_L, \alpha)$ does not depend on the choice of e_i. Indeed, make e_i slide along L_i and deform γ_i simultaneously. This isotopy will not change $\Phi(T_L; X_1, \ldots, X_m)$ and a forteriori $\phi(T_L, \alpha)$.

Therefore, $\phi(T_L, \alpha)$ depends only on the plane diagram $\bar{D}_L$ and α. It is invariant under the three Reidemeister moves performed on $\bar{D}_L$, namely, $\Omega 2$, $\Omega 3$ [47] and

Ω1F



Indeed, these moves are local, and we can always choose the points $e_i \in L_i$ outside the changed pieces, hence, the corresponding moves for $\mathscr{C}$-tangles apply without changing the morphism $\Phi(T_L; X_1, \ldots, X_m)$. For the Ω1F move notice that $u_1^2 : X \to X^{\vee\vee}$ (resp. $u_1^{-2} : X \to {}^{\vee\vee}X$) in the left (resp. right) hand side will be accompanied by a power of u_0^2. Thus, invariance under Ω1F follows from the equation

$$u_0^{-2} u_1^2 = \nu = u_0^2 u_1^{-2} : X \to X.$$

Since the set of equivalence classes of plane diagrams under Ω1F, $\Omega 2$, $\Omega 3$ is the same as the set of equivalence classes of framed links in $\mathbb{R}^3$ ([36] after [47]), we deduce that $\phi(T_L, \alpha)$ depends only on L and α. □

5.2.1. A 3-manifold invariant. By the very definition of a 3-modular category $\mathscr{C}$ the Hopf algebra $F \in \mathscr{C}$ has a two-sided integral $\sigma : I \to F$. In the following we shall consider the link invariant $\tau(L, \sigma) = \phi(T_L, \sigma)$.

Denote by $s(L)$ the signature of the intersection form in $H^2(W_L; \mathbb{R})$, where W_L is the 4-manifold obtained by surgery along L and $M_L = \partial W_L$. It is the same as the signature of the linking matrix of L [18].

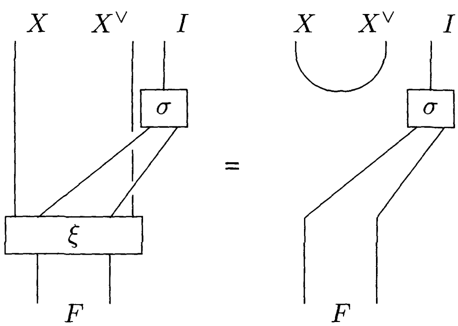
Theorem 5.2.2. *The number*

$$\tau(M_L) = \lambda^{-s(L)} \tau(L, \sigma)$$

is an invariant of 3-manifolds, that is, it depends only on the homeomorphism class of M_L.

Proof. Equations (2.1.1), (2.1.2) show that application of one of the Kirby–Fenn–Rourke moves multiplies $\tau(L, \sigma)$ by $\lambda^{\pm 1}$. Simultaneously $s(L)$ increases by ± 1. □

It is worth understanding why $\tau(M_L)$ is invariant under Kirby moves. The answer is because σ is an integral in F. Graphically this property is expressed by the following equation

(5.2.1) 

$: X \otimes X^\vee \otimes I \to F.$

Here the multiplication morphism ξ is determined from the commutative diagram

$$\begin{array}{ccc} X \otimes Y \otimes Y^\vee \otimes X^\vee & \xrightarrow{X \otimes \imath_Y \otimes X^\vee} & X \otimes F \otimes X^\vee \\ \wr \downarrow & & \downarrow \xi \\ (X \otimes Y) \otimes (X \otimes Y)^\vee & \xrightarrow{\imath_{X \otimes Y}} & F \end{array}$$

Let us give another proof of Theorem 5.2.2, using only the Kirby moves. For particular categories related with $SL(2)$ this was done by Lickorish [32] through a different approach. Choose a pair of components L_i, L_j in L and perform a Kirby band move as shown at Fig. 11. The points $a \in L_i$ and $b \in L_j$ become "connected" by a double line β. Take the point b as e_j and choose γ_j so that $\gamma_j'(0)$ pointed out in the direction of β. Duplicate all γ_k as required by the recipe. Since the curves L_j and a new-made part of L_i' go parallelly, we can insert the morphism ξ to this $\mathscr{C}$-F-tangle without changing the morphism $\ldots \otimes X \otimes X^\vee \otimes \ldots \otimes F \otimes \cdots \to k$ it represents (see Fig. 12). In the vicinity of e_j the picture looks like the left-hand side of Eq. (5.2.1). Therefore we can change it to the right-hand side without changing $\tau(L', \sigma)$. The resulting tangle at Fig. 13 is isotopic to the initial one, so its value $\tau(L', \sigma)$ equals to $\tau(L, \sigma)$. Finally, the signature $s(L)$ will not change under Kirby band move [18], whence Theorem 5.2.2 follows.

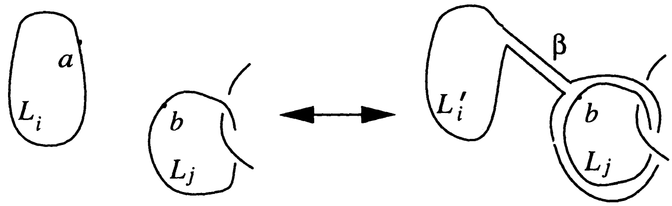


Fig. 11.

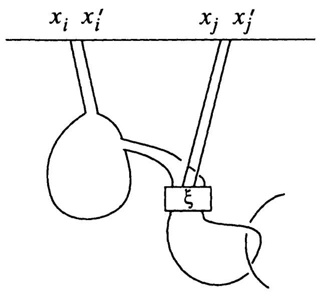


Fig. 12.

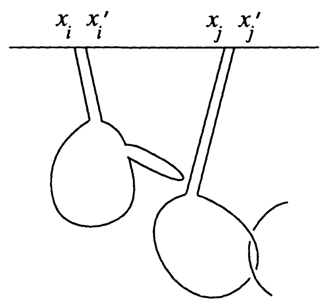
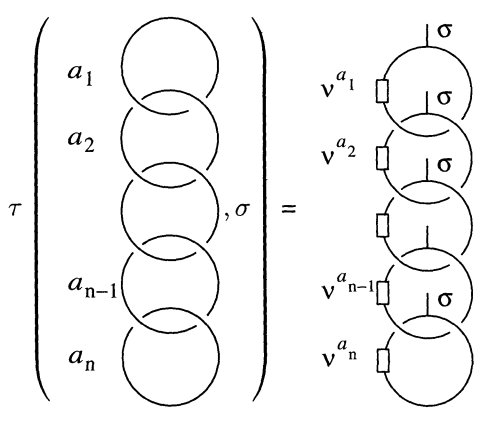


Fig. 13.

5.3. Lens spaces. We calculate as an example the invariant τ for lens spaces $L(p, q)$. Consider the n-component chain link $L \subset S^3$ and the value of its invariant

(5.3.1)



The surgery at L gives the lens space $L(p, q)$, where prime to each other $p, q \in \mathbb{Z}$ satisfy

$$\frac{p}{q} = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\ddots - \cfrac{1}{a_2 - \cfrac{1}{a_1}}}}$$

(see Rolfsen [52]). The right-hand side of Eq. (5.3.1) can be presented through the maps $S, T : F \to F$ as

$$
\begin{aligned}
\tau(L,\sigma) &= \varepsilon(T^{a_n}ST^{a_{n-1}}S\dots T^{a_2}ST^{a_1}S(1)) \\
&= \langle 1, T^{a_n}ST^{a_{n-1}}S\dots T^{a_2}ST^{a_1}S(1)\rangle \\
&= \langle {}^tS\,{}^tT^{a_1}\,{}^tS\,{}^tT^{a_2}\dots {}^tS\,{}^tT^{a_{n-1}}\,{}^tS(\nu^{a_n}), 1\rangle \\
&= \int' ({}^tT^{a_1}\,{}^tS\,{}^tT^{a_2}\dots {}^tS\,{}^tT^{a_{n-1}}\,{}^tS(\nu^{a_n})).
\end{aligned}
$$

The last formula uses only the maps ${}^tT$, ${}^tS = {}^tS_-(\sigma)$ in U, determined by Eqs. (3.8.2) and (3.8.9). We can also use ${}^tS_+(\sigma)$ (see Eq. (3.8.13)) instead of ${}^tS$.

In the particular case $n = 1$, $a_1 = 0$ we get $\tau(L,\sigma) = \varepsilon(\mu) = \int' 1$, which vanishes for all $u_q(\mathfrak{g})$, hence, $\tau(S^2 \times S^1) = 0$. This contrasts sharply with the case of a perfect modular semisimple $\mathscr{C}$, where $\varepsilon(\mu) = \sqrt{\sum_i (\dim_{\mathscr{C}} X_i)^2}$ is invertible (see [35, 60]). Here X_i runs over the set of isomorphism classes of simple objects of $\mathscr{C}$. Their categorical dimensions

$$
\dim_{\mathscr{C}} X_i = \left(I \xrightarrow{\text{coev}} X_i \otimes {}^\vee X_i \xrightarrow{1\otimes u_0^2} X_i \otimes X_i{}^\vee \xrightarrow{\text{ev}} I\right)
$$

are real numbers if the ground field is $\operatorname{End} I = \mathbb{C}$ [35], so $\varepsilon(\mu)$ is positive.

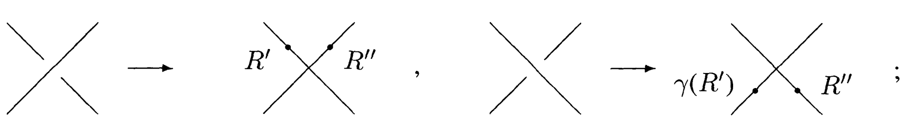
Finally, $s(L)$ is the signature of the linking matrix

$$
\begin{pmatrix}
a_1 & 1 & & & & \\
1 & a_2 & 1 & & 0 & \\
 & 1 & \ddots & \ddots & & \\
 & & \ddots & \ddots & 1 & \\
 & 0 & & 1 & a_{n-1} & 1 \\
 & & & & 1 & a_n
\end{pmatrix}
$$

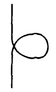
which gives $\tau(L(p,q))$.

5.4. The Hennings invariant. Let H be a finite dimensional 3-modular Hopf algebra (see Theorem 3.7.3). Construct a 3-manifold invariant $\tau(M)$ taking the 3-modular category $\mathscr{C} = H$-mod as data. Calculating $\tau(L,\sigma)$ for some link L we get an expression involving R-matrices (as many as there are crossings in $\bar{D}_L$), elements σ and the powers of κ (as many as there are components in L) under the sign of counit. It turns out that the result of calculation coincides with the Hennings invariant [10] defined in an unoriented setting by Kauffman and Radford [14] up to change of conventions. Indeed, the calculation of $\tau(L,\sigma)$ can be performed using the graphical rules of Hennings–Kauffman–Radford as follows:

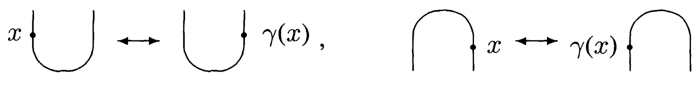
– change all crossings in $\bar{D}_L$ to the composite of R-matrix and the permutation



– add to each component such a power of the morphism $u_0^2 = v_0^2 \cdot \kappa = \kappa \cdot v_0^2$ or its inverse $u_0^{-2} = v_0^{-2} \cdot \kappa^{-1}$

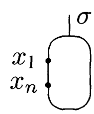
$v_0^2 =$  , $v_0^{-2} =$ 

that after cancellations of v_0^2's with v_0^{-2}'s the diagram becomes a disjoint union of unknots;
– slide all elements put on the diagram to the left of the chosen points e_i via the rule



, $x \in H$.

Now each component looks like:



with $x_1, \ldots, x_n \in H$ attached to it, and its contribution is

$$\varepsilon(\underline{x_n \ldots x_1 \otimes 1}(\sigma)) = \varepsilon(\langle x_n \ldots x_1, \sigma_{(1)} \rangle \sigma_{(2)}) = \langle x_n \ldots x_1, \sigma \rangle$$

(see Sect. 3.8.1). We can view σ as the left integral $\sigma : H \to k$ on the algebra H. So

$$\tau(L, \sigma) = \sum \sigma(x_n \ldots x_1) \ldots \sigma(z_m \ldots z_1),$$

where the arguments are either powers of κ or come from R-matrices acted upon by some powers of the antipode. This is the knot invariant used in the definition of the Hennings invariant [10] up to change of the sign of the braiding.

A. Quantum groups

A.1. R-matrices for quantum groups

Theorem A.1.1 ([20, 21, 26]). *The expression* $R = \tilde{R}\mathring{R} \in U_h(\mathfrak{g})\widehat{\otimes}U_h(\mathfrak{g})$ *is an R-matrix in the topological Hopf algebra* $U_h(\mathfrak{g})$, *where*

$$\text{(A.1.1)} \qquad \tilde{R} = \prod_{\beta \in (\beta_1, \ldots, \beta_N)} \exp_{q_\beta^{-2}}((q_\beta - q_\beta^{-1}) E_\beta \otimes F_\beta),$$

$$\mathring{R} = \exp\left(\frac{h}{2} \sum c_{ij} H_i \otimes H_j\right),$$

where $q_\beta = e^{h(\beta|\beta)/2}$ *and* (c_{ij}) *is the inverse matrix to* $(d_i a_{ij})$.

Proposition A.1.2 (cf. [59]). *The Cartan part of the R-matrix,* $\mathring{R} \in U_h(\mathfrak{h})\widehat{\otimes}U_h(\mathfrak{h})$ *satisfies*

$$\mathring{R} \cdot E_i \otimes 1 \cdot \mathring{R}^{-1} = E_i \otimes K_i, \quad \mathring{R} \cdot F_i \otimes 1 \cdot \mathring{R}^{-1} = F_i \otimes K_i^{-1},$$
$$\mathring{R} \cdot 1 \otimes E_i \cdot \mathring{R}^{-1} = K_i \otimes E_i, \quad \mathring{R} \cdot 1 \otimes F_i \cdot \mathring{R}^{-1} = K_i^{-1} \otimes F_i.$$

Consider an embedding of Hopf algebras $U_q(\mathfrak{g}) \hookrightarrow U_h(\mathfrak{g})\otimes_{\mathbb{C}[[h]]}\mathbb{C}[h^{-1},h]]$, $K_i \mapsto e^{hd_iH_i}$, with respect to a field extension $\mathbb{Q}(q) \hookrightarrow \mathbb{C}[h^{-1},h]]$, $q \mapsto e^h$. The image of Lusztig's divided power algebra $\Gamma(\mathfrak{g})$ is contained in $U_h(\mathfrak{g})$. Let

$$U_h(\mathfrak{g})^\alpha = \{x \in U_h(\mathfrak{g}) \mid [H_i, x] = \alpha(H_i)x\},$$
$$\Gamma(\mathfrak{g})^\alpha = \{x \in \Gamma(\mathfrak{g}) \mid K_i x K_i^{-1} = q^{(\alpha_i|\alpha)}x\}$$

be natural gradings, $\alpha \in Q$. Note that $\Gamma(\mathfrak{g})^\alpha \subset U_h(\mathfrak{g})^\alpha$. We have $R, \tilde{R} \in \prod_{\alpha\in Q_+} (U_h(\mathfrak{b}_+)^\alpha \widehat{\otimes} U_h(\mathfrak{b}_-)^{-\alpha})$. Denote $\tilde{R} = \sum_{\alpha\in Q_+} \tilde{R}_\alpha$, then $\tilde{R}_\alpha \in U_h(\mathfrak{b}_+)^\alpha \otimes U_h(\mathfrak{b}_-)^{-\alpha}$. Combining Theorem A.1.1 and Proposition A.1.2 we get

Proposition A.1.3 (cf. [59]). *In $U_h(\mathfrak{g})^{\widehat{\otimes}3}$ (or $U_h(\mathfrak{g})^{\widehat{\otimes}2}$) the following equations are satisfied:*

$$(\Delta \otimes 1)\tilde{R} = \tilde{R}^{13} \cdot \sum_{\alpha\in Q_+} K_{-\alpha} \otimes \tilde{R}_\alpha, \tag{A.1.2}$$

$$(1 \otimes \Delta)\tilde{R} = \tilde{R}^{13} \cdot \sum_{\alpha\in Q_+} \tilde{R}_\alpha \otimes K_\alpha, \tag{A.1.3}$$

$$\Delta^{\mathrm{op}}x \cdot \tilde{R} = \tilde{R} \cdot \tilde{\Delta}x, \tag{A.1.4}$$

where $x \in U_h(\mathfrak{g})$ and the new coproduct $\tilde{\Delta}x = \mathring{R}\Delta x\mathring{R}^{-1}$ satisfies

$$\tilde{\Delta}H_i = H_i \otimes 1 + 1 \otimes H_i, \quad \tilde{\Delta}E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \tilde{\Delta}F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

Moreover, $\tilde{R}_\alpha \in \Gamma(\mathfrak{b}_+)^\alpha \otimes_{\mathbb{Z}[q,q^{-1}]} \Gamma(\mathfrak{b}_-)^{-\alpha} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}[[h]]$ due to the formula

$$\exp_{q_\beta^{-2}}((q_\beta - q_\beta^{-1})E_\beta \otimes F_\beta) = \sum_{m\geqslant 0} q_\beta^{m(m-1)}(q_\beta - q_\beta^{-1})^m (m)_{q_\beta^{-2}}! E_\beta^{(m)} \otimes F_\beta^{(m)}, \tag{A.1.5}$$

where

$$E_{\beta_k}^{(m)} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}^{(m)}), \qquad F_{\beta_k}^{(m)} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k}^{(m)})$$

(cf. Sect. 1.1). Since $\Gamma(\mathfrak{g})$ is a free $\mathbb{Z}[q,q^{-1}]$-module [33], Eqs. (A.1.2)–(A.1.4) with $x \in \Gamma(\mathfrak{g})$ can be interpreted as follows. All terms are elements of the $\mathbb{Z}[q,q^{-1}]$-module $\prod_{\alpha,\beta,\gamma\in Q} \Gamma(\mathfrak{g})^\alpha \otimes \Gamma(\mathfrak{g})^\beta \otimes \Gamma(\mathfrak{g})^\gamma$ (or of $\prod_{\alpha,\beta\in Q} \Gamma(\mathfrak{g})^\alpha \otimes \Gamma(\mathfrak{g})^\beta \ni \tilde{R}$) and the equations state in particular that these elements can be multiplied.

Let $\varepsilon \in \mathbb{C}$ be a root of unity. Change the base by the homomorphism $\mathbb{Z}[q,q^{-1}] \to \mathbb{C}$, $q \mapsto \varepsilon$ and denote

$$\Gamma_\varepsilon(\mathfrak{g}) = \Gamma(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}.$$

Equations (A.1.2)–(A.1.4) still hold for $\Gamma_\varepsilon(\mathfrak{g})$ in place of $\Gamma(\mathfrak{g})$. But the sum (A.1.5) is finite, so $\tilde{R}_\alpha$ vanish for all α except a finite number and $\tilde{R} \in \Gamma_\varepsilon(\mathfrak{b}_+) \otimes \Gamma_\varepsilon(\mathfrak{b}_-)$. Therefore, Eqs. (A.1.2)–(A.1.4) hold in $\Gamma_\varepsilon(\mathfrak{g})^{\otimes 3}$ or $\Gamma_\varepsilon(\mathfrak{g})^{\otimes 2}$ with $x \in \Gamma_\varepsilon(\mathfrak{g})$.

Reversing the proof of Proposition A.1.3 we see that if we find a symmetric tensor $\mathring{R} \in \Gamma_\varepsilon(\mathfrak{h})^{\otimes 2}$ satisfying $(\Delta \otimes 1)\mathring{R} = \mathring{R}^{13}\mathring{R}^{23}$ and

$$\mathring{R} \cdot 1 \otimes E_i^{(p)} \cdot \mathring{R}^{-1} = K_i^p \otimes E_i^{(p)}, \quad \mathring{R} \cdot 1 \otimes F_i^{(p)} \cdot \mathring{R}^{-1} = K_i^{-p} \otimes F_i^{(p)} \tag{A.1.6}$$

for all $p \in \mathbb{Z}_{>0}$, we could construct an R-matrix $R = \tilde{R}\overset{\circ}{R}$ for $\Gamma_\varepsilon(\mathfrak{g})$. It turns out that such an element is easy to find not in $\Gamma_\varepsilon(\mathfrak{g})$ but in its quotient (which sometimes coincides with $\Gamma_\varepsilon(\mathfrak{g})$). Consider a symmetric bilinear (bimultiplicative) form

$$\pi : Q \times Q \to \mathbb{C}^\times, \qquad \pi(K_i, K_j) = q_i^{a_{ij}}, \qquad \pi(K_\alpha, K_\beta) = q^{(\alpha|\beta)}.$$

Let $\operatorname{Ann}\pi = \{g \in Q \mid \forall h \quad \pi(g,h) = 1\}$ be its annihilator. Since for $h \in Q$,

$$hE_j^{(p)} = \pi(K_j,h)^p E_j^{(p)} h, \qquad hF_j^{(p)} = \pi(K_j,h)^{-p} F_j^{(p)} h, \tag{A.1.7}$$

the elements of $\operatorname{Ann}\pi$ lie in the centre of $\Gamma_\varepsilon(\mathfrak{g})$. Introduce the Hopf $\mathbb{C}$-algebra

$$\Gamma'_\varepsilon(\mathfrak{g}) = \Gamma_\varepsilon(\mathfrak{g})/(g-1)_{g\in \operatorname{Ann}\pi}.$$

The subgroup generated by K_i in $\Gamma'_\varepsilon(\mathfrak{g})$ is denoted $\mathbb{G} = Q/\operatorname{Ann}\pi$. The form π factorizes through a non-degenerate form on $\mathbb{G}$ denoted also π by abuse of notations.

Clearly, $K_i^{2l_i} = 1$ in $\mathbb{G}$, where l_i are minimal positive integers such that $q_i^{2l_i} = 1$. More elements from $\operatorname{Ann}\pi$ can be found via the following

Lemma A.1.4. *For any $d = 1,2,3$ let $l(d)$ be the minimal positive integer l such that $q^{2dl} = 1$ and let $\rho(d) = \sum_{\substack{||\alpha||^2=2d\\ \alpha\in\Delta^+}} \alpha$. Then for any i*

$$q_i^{l(d)\langle \alpha_i, \rho(d)\rangle} = 1.$$

Proof. The equation to prove is $q^{l(d)(\alpha_i|\rho(d))} = 1$. Assume first that $||\alpha||^2 \neq 2d$. Let s_i be the simple reflection corresponding to α_i. The well known equation $s_i(\Delta^+ - \alpha_i) = \Delta^+ - \alpha_i$ implies $s_i(\Delta_d^+) = \Delta_d^+$, where $\Delta_d^+ = \{\alpha \in \Delta^+ \mid ||\alpha||^2 = 2d\}$. Hence, $s_i(\rho(d)) = \rho(d)$ and

$$(\alpha_i|\rho(d)) = (s_i(\alpha_i))|s_i(\rho(d))) = -(\alpha_i|\rho(d))$$

vanishes.

Assume now that $||\alpha||^2 = 2d$. Then $s_i(\Delta_d^+ - \alpha_i) = \Delta_d^+ - \alpha_i$ and $s_i(\rho(d) - \alpha_i) = \rho(d) - \alpha_i$. Hence,

$$(\alpha_i|\rho(d)) = (s_i(\alpha_i))|s_i(\rho(d))) = -(\alpha_i|\rho(d) - 2\alpha_i)$$

implies $(\alpha_i|\rho(d)) = (\alpha_i|\alpha_i) = 2d$ and the lemma follows. □

Corollary A.1.5. *For any $d = 1,2,3$ we have $K_{\rho(d)}^{l(d)} = 1$ in $\mathbb{G}$. In particular,*

$$\prod_{\alpha\in\Delta^+} K_\alpha^{l_\alpha} = 1.$$

The R-matrix from the following theorem was already obtained by Rosso [53] in the case of l relatively prime to $\det(d_i a_{ij})$ using Drinfeld's double.

Theorem A.1.6. *The Hopf algebra $H = \Gamma'_\varepsilon(\mathfrak{g})$ is quasitriangular with the R-matrix $R = \tilde{R}\overset{\circ}{R}$, where $\tilde{R}$ is given by Eq. (A.1.1) and*

$$\overset{\circ}{R} = \frac{1}{|\mathbb{G}|} \sum_{g,h\in\mathbb{G}} \pi(g,h)^{-1} g \otimes h.$$

Proof. Using Eqs. (A.1.7) one can check the property (A.1.6) of $\overset{\circ}{R} \in \Gamma'_\varepsilon(\mathfrak{g})^{\otimes 2}$. Non-degeneracy of the pairing π for $\Gamma'_\varepsilon(\mathfrak{h})$ implies that $\overset{\circ}{R}$ is an R-matrix for $\Gamma'_\varepsilon(\mathfrak{h})$. It follows from the above discussion that $\tilde{R}\overset{\circ}{R}$ is an R-matrix for $\Gamma'_\varepsilon(\mathfrak{g})$. □

A.2. The algebra $u_q(\mathfrak{g})$. Rewrite Theorem A.1.6 for $q = \varepsilon$ as

$$R = \sum_{0\leqslant m_\alpha<l_\alpha} \prod_\alpha \frac{(q_\alpha - q_\alpha^{-1})^{m_\alpha}}{(m_\alpha)_{q_\alpha^{-2}}!} \prod_\alpha E_\alpha^{m_\alpha} \otimes \prod_\alpha F_\alpha^{m_\alpha} \cdot \mathring{R} \tag{A.2.1}$$

(α runs over $\beta_1,\dots,\beta_N$). Non-degeneracy of $\pi : \mathbb{G}\times\mathbb{G} \to \mathbb{C}^\times$ implies that the minimal subspaces $A, B \subset \mathbb{C}[\mathbb{G}]$ such that $\mathring{R} \in A\otimes B$ are $A = B = \mathbb{C}[\mathbb{G}]$. Thus, Eq. (A.2.1) implies that the smallest subspaces $\mathbf{h}_+, \mathbf{h}_- \subset \Gamma'_\varepsilon(\mathfrak{g})$ such that $R \in \mathbf{h}_+ \otimes \mathbf{h}_-$ have bases $h\prod_{\beta\in(\beta_1,\dots,\beta_N)} E_\beta^{k_\beta}$, resp. $h\prod_{\beta\in(\beta_1,\dots,\beta_N)} F_\beta^{m_\beta}$, where $h\in\mathbb{G}$, $0\leqslant k_\beta < l_\beta$. As discussed in Sect. 3.1 (cf. [45]) $\mathbf{h}_+$ and $\mathbf{h}_-$ are Hopf subalgebras. They coincide with the subalgebras $u_q(\mathfrak{b}_+) = \mathbb{C}\langle K_i^{\pm1}, E_i\rangle_{1\leqslant i\leqslant n}$ and $u_q(\mathfrak{b}_-) = \mathbb{C}\langle K_i^{\pm1}, F_i\rangle_{1\leqslant i\leqslant n}$, since $E_i \in \mathbf{h}_+$, $E_\beta \in u_q(\mathfrak{b}_+)$ by e.g. [34, Proposition 40.1.3]. Therefore, by general theory $u_q(\mathfrak{g}) = \mathbb{C}\langle K_i^{\pm1}, E_i, F_i\rangle_{1\leqslant i\leqslant n} = \mathbf{h} = \mathbf{h}_+\mathbf{h}_- = \mathbf{h}_-\mathbf{h}_+$ is a quasitriangular Hopf subalgebra of $\Gamma'_\varepsilon(\mathfrak{g})$. It has the basis $\prod_{\alpha\in(\beta_1,\dots,\beta_N)} E_\alpha^{k_\alpha}\cdot h\cdot\prod_{\beta\in(\beta_1,\dots,\beta_N)} F_\beta^{m_\beta}$, where $h\in\mathbb{G}$, $0\leqslant k_\beta, m_\beta < l_\beta$. The R-matrix $R\in u_q(\mathfrak{b}_+)\otimes u_q(\mathfrak{b}_-)$ is the dual tensor to the non-degenerate Hopf pairing

$$\pi : \mathbf{h}_- \times \mathbf{h}_+^{\mathrm{op}} \to \mathbb{C}, \quad \pi(h, E_i) = \pi(F_i, h) = 0, \quad \pi(F_i, E_j) = \delta_{ij}(q_i - q_i^{-1})^{-1}$$

for $h\in\mathbb{G}$, $1\leqslant i,j\leqslant n$.

A.2.1. A presentation of $u_q(\mathfrak{g})$. A lemma of Levendorskii and Soibelman [27] states: for any positive roots $\alpha<\beta$ there are unique constants $c_{n_1,\dots,n_j}\in\mathbb{C}(q)$, such that in $U_q(\mathfrak{g})\otimes_{\mathbb{Q}(q)}\mathbb{C}(q)$

$$E_\alpha E_\beta - q^{-(\alpha|\beta)}E_\beta E_\alpha = \sum_n c_{n_1,\dots,n_j} E_{\gamma_1}^{n_1}\dots E_{\gamma_j}^{n_j}, \tag{A.2.2}$$

where j is the number of all positive roots lying between α and β, which are denoted $\alpha<\gamma_1<\dots<\gamma_j<\beta$. When this lemma is combined with Lusztig's basis theorem for $\Gamma(\mathfrak{g})$ [34, Propositions 41.1.4, 41.1.7] we see that in fact

$$c_{n_1,\dots,n_j} \in \mathbb{K} = \mathbb{Z}[q, q^{-1}, (q-q^{-1})^{-1}, [N(\mathfrak{g})]_q!^{-1}],$$

where the constant $N(\mathfrak{g}) \geqslant \max_i d_i$ is the minimal possible. An upper bound for it is given in Table 1. It is obtained as the product of $\max_i d_i$ with the maximal

Table 1. An estimate for the constants $N(\mathfrak{g})$

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$N(\mathfrak{g})\leqslant$	1	4	4	2	3	4	6	8	9

coefficient c_i in decomposition $\alpha_0 = \sum c_i\alpha_i$ of the highest root $\alpha_0\in\Delta$. (Apply $T_{i_k}^{-1}T_{i_{k-1}}^{-1}\dots T_{i_1}^{-1}$ to (A.2.2), if $E_\alpha = T_{i_1}\dots T_{i_{k-1}}E_{i_k}$.) This estimate is rather rough, and can be essentially improved. The conjectured value of $N(\mathfrak{g})$ is $\max_i d_i$.

Introduce a $\mathbb{K}$-subalgebra $\mathcal{U}_q(\mathfrak{g})\subset U_q(\mathfrak{g})$ generated by K_i, E_i, F_i. It is closed under the automorphisms T_i and all relations (A.2.2) make sense in $\mathcal{U}_q(\mathfrak{g})$, thus $\mathcal{U}_q(\mathfrak{g})$ has a Poincaré–Birkhoff–Witt basis $\prod_\alpha E_\alpha^{m_\alpha}\cdot K_\lambda\cdot\prod_\beta F_\beta^{n_\beta}$ (compare [25, 34]). Clearly, $\mathcal{U}_q(\mathfrak{g})$ is a subalgebra of $\Gamma(\mathfrak{g})\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{K}$.

Assume in this subsection that $\varepsilon^{2m} \neq 1$ for all $1 \leqslant m \leqslant N(\mathfrak{g})$. Then we have a homomorphism of Hopf algebras

(A.2.3) $$\phi : \mathscr{U}_q(\mathfrak{g}) \hookrightarrow \Gamma(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{K} \to \Gamma_\varepsilon(\mathfrak{g}) \to \Gamma'_\varepsilon(\mathfrak{g})$$

via the homomorphism $\mathbb{K} \to \mathbb{C}$, $q \mapsto \varepsilon$. Since $\mathbb{C}\operatorname{Im}\phi = u_q(\mathfrak{g})$ the relations (A.2.2) are valid also in $u_q(\mathfrak{g})$.

The algebra $U_\varepsilon(\mathfrak{g}) = \mathscr{U}_q(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{C}$ has the usual generators and relations of $U_q(\mathfrak{g})$ with q set to ε. The obvious homomorphism $\tilde{\phi} : U_\varepsilon(\mathfrak{g}) \to \Gamma'_\varepsilon(\mathfrak{g})$ induced by (A.2.3) has $u_q(\mathfrak{g})$ as its image. Its kernel contains $E_\alpha^{l_\alpha}$, $F_\alpha^{l_\alpha}$ and $h - 1$ for $h \in \operatorname{Ann}(\pi : Q \times Q \to \mathbb{C}^\times)$. Indeed, $E_\alpha^{l_\alpha} = [l_i]_{q_i}! T_{i_1} \dots T_{i_{k-1}}(E_i^{(l_i)})$ in $U_q(\mathfrak{g})$ and $\Gamma(\mathfrak{g})$ for some sequence $(i_1, \dots, i_{k-1}, i)$, and the factorial vanishes in $\Gamma'_\varepsilon(\mathfrak{g})$. Let $I \subset U_\varepsilon(\mathfrak{g})$ be a two-sided ideal generated by these elements. By the explicit form of bases we get $\dim U_\varepsilon(\mathfrak{g})/I \leqslant \dim u_q(\mathfrak{g})$. Since $\tilde{\phi}$ induces an epimorphism $U_\varepsilon(\mathfrak{g})/I \to u_q(\mathfrak{g})$, this is in fact an isomorphism, and a presentation of $u_q(\mathfrak{g})$ by generators and relations follows. Namely, the new relations

$$E_\alpha^{l_\alpha} = 0, \qquad F_\alpha^{l_\alpha} = 0, \qquad h = 1 \;\text{ for } \alpha \in \Delta^+,\; h \in \operatorname{Ann}(\pi : Q \times Q \to \mathbb{C}^\times)$$

are added to the standard presentation of $U_\varepsilon(\mathfrak{g})$.

A.3. Ribbon structure of $\Gamma'_\varepsilon(\mathfrak{g})$ and $u_q(\mathfrak{g})$. Let us find the grouplike element $g = u\gamma(u)^{-1}$ for the algebras H and $\mathbf{h}$, where $u = \sum_n \gamma(b^n)a_n$, $R = \sum_n a_n \otimes b^n$. Clearly,

$$\delta_+ = \sum_{h \in \mathbb{G}} h \cdot \prod_{\beta \in (\beta_1, \dots, \beta_N)} E_\beta^{l_\beta - 1} \in \mathbf{h}_+, \quad \delta_- = \sum_{h \in \mathbb{G}} h \cdot \prod_{\beta \in (\beta_1, \dots, \beta_N)} F_\beta^{l_\beta - 1} \in \mathbf{h}_-$$

are non-zero left integrals in the algebras $\mathbf{h}_+$ and $\mathbf{h}_-$. For any $x \in \mathbf{h}_-$, $y \in \mathbf{h}_+$ and $a_+^{-1} = a_- = \prod_\beta K_\beta^{-l_\beta + 1} = K_{2\rho} \in \mathbb{G}$ (cf. Corollary A.1.5) we have

$$\delta_- x = \pi(x, a_+)\delta_-, \qquad \delta_+ y = \pi(a_-, y)\delta_+.$$

By Drinfeld's theorem [6] (see Sect. 3.1) we find

$$g = a_+^{-1} a_- = K_{4\rho} \in \mathbf{h}.$$

By definition a ribbon structure is a choice of a group-like element κ such that $\kappa^2 = K_{4\rho}$ and $\kappa a = \gamma^2(a)\kappa$ for all $a \in \Gamma'_\varepsilon(\mathfrak{g})$. The only group-like elements of $\Gamma'_\varepsilon(\mathfrak{g})$ are $K_\alpha \in \mathbb{G}$. Commuting κ with E_i we get $\pi(\kappa, K_i) = q_i^2$, which holds for the only element $\kappa = K_{2\rho}$. Therefore, the quasitriangular Hopf algebra $\Gamma'_\varepsilon(\mathfrak{g})$ (or $u_q(\mathfrak{g})$) admits the unique ribbon structure $\kappa = K_{2\rho}$.

Let us find the ribbon twist element using the formula $\nu = \sum_n a_n K_{2\rho}^{-1} b^n$, $R = \sum_n a_n \otimes b^n$. Equation (A.2.1) can be rewritten as

$$R = \sum_{0 \leqslant m_\alpha < l_\alpha} \Big(\prod_\alpha \frac{(q_\alpha - q_\alpha^{-1})^{m_\alpha}}{(m_\alpha)_{q_\alpha^{-2}}!} E_\alpha^{m_\alpha} \Big) \mathring{R}'_a \Big(\prod_\alpha K_\alpha^{m_\alpha} \Big) \otimes \mathring{R}''_a \Big(\prod_\alpha F_\alpha^{m_\alpha} \Big),$$

where $\mathring{R} = \sum_a \mathring{R}'_a \otimes \mathring{R}''_a$. Therefore,

(A.3.1)
$$\nu = \sum_{0 \leqslant m_\alpha < l_\alpha} \Big(\prod_\alpha \frac{(q_\alpha - q_\alpha^{-1})^{m_\alpha}}{(m_\alpha)_{q_\alpha^{-2}}!} E_\alpha^{m_\alpha} \Big) \cdot \mathring{R}'_a \mathring{R}''_a K_{-2\rho} \Big(\prod_\alpha K_\alpha^{m_\alpha} \Big) \cdot \Big(\prod_\alpha F_\alpha^{m_\alpha} \Big).$$

A.3.1. The subalgebra **u**. Recall that the subalgebra $\mathbf{u} \subset \mathbf{h}$ is defined as the smallest subspace such that $R^{12}R^{21} \in \mathbf{u} \otimes \mathbf{h}$. Let us find it.

Theorem A.1.6 gives

$$\text{(A.3.2)} \quad R^{12}R^{21} = \sum_{n,m} \Big(\prod_{\alpha} \frac{(q_\alpha - q_\alpha^{-1})^{m_\alpha}}{(m_\alpha)_{q_\alpha^{-2}}!} E_\alpha^{m_\alpha} \Big) \mathring{R}'_a \mathring{R}''_b \Big(\prod_{\beta} (K_\beta^{-1} F_\beta)^{n_\beta} \Big) \otimes$$
$$\otimes \Big(\prod_{\alpha} F_\alpha^{m_\alpha} \Big) \mathring{R}''_a \mathring{R}'_b \Big(\prod_{\beta} \frac{(q_\beta - q_\beta^{-1})^{n_\beta}}{(n_\beta)_{q_\beta^{-2}}!} (E_\beta K_\beta)^{n_\beta} \Big).$$

Proposition A.3.1. *The algebra* **u** *has the basis*

$$\Big(\prod_{\alpha} E_\alpha^{m_\alpha}\Big) K_\lambda \Big(\prod_{\beta} F_\beta^{n_\beta}\Big),$$

where $K_\lambda = K_{2\mu} \cdot \prod_\beta K_\beta^{n_\beta}$ *in* $\mathbb{G}$ *for some* $\mu \in Q$.

This follows from the formula (A.3.2) above and

Lemma A.3.2. *The smallest subspace* $A \subset \mathbb{C}[\mathbb{G}]$ *such that* $\mathring{R}^2 \in A \otimes \mathbb{C}[\mathbb{G}]$ *is* $A = \mathbb{C}[2\mathbb{G}]$, *where* $2\mathbb{G} = \{x^2 \mid x \in \mathbb{G}\} \subset \mathbb{G}$ *is the subgroup of squares.*

Proof. Note that $\mathring{R}$ is a symmetric tensor and

$$\begin{aligned} \mathring{R}^2 &= \frac{1}{|\mathbb{G}|} \sum_{g,h \in \mathbb{G}} \pi(g,h)^{-1} g \otimes h \cdot \frac{1}{|\mathbb{G}|} \sum_{a,b \in \mathbb{G}} \pi(a,b)^{-1} a \otimes b \\ &= \frac{1}{|\mathbb{G}|^2} \sum_{c,d \in \mathbb{G}} c \otimes d \sum_{g,b \in \mathbb{G}} \pi(g, b^2 d^{-1}) \pi(c^{-1}, b) \\ &= \frac{1}{|\mathbb{G}|} \sum_{c,d \in \mathbb{G}} c \otimes d \sum_{b \in \mathbb{G}} \delta_{d,b^2} \pi(c,b)^{-1}. \end{aligned}$$

This implies that $\mathring{R}^2 \in \mathbb{C}[2\mathbb{G}] \otimes \mathbb{C}[2\mathbb{G}]$. Introduce a symmetric bilinear (bimultiplicative) pairing

$$\pi_2 : 2\mathbb{G} \times 2\mathbb{G} \to \mathbb{C}^\times, \qquad \pi_2(c,d) = \pi(c,b), \text{ where } b^2 = d.$$

Clearly, $\pi(c,b)$ has the same value for all $b \in \mathbb{G}$ such that $b^2 = d$. All such b differ by an element of $X = \{b \in \mathbb{G} \mid b^2 = 1\}$. Therefore,

(A.3.3)
$$\mathring{R}^2 = \frac{1}{|\mathbb{G}|} \sum_{c,d \in 2\mathbb{G}} \pi_2(c,d)^{-1} c \otimes d \sum_{b \in \mathbb{G}} \delta_{d,b^2} = \frac{1}{|2\mathbb{G}|} \sum_{c,d \in 2\mathbb{G}} \pi_2(c,d)^{-1} c \otimes d$$

and the statement follows. □

Corollary A.3.3. $u_q(\mathfrak{g})$ *is factorizable if and only if* $2\mathbb{G} = \mathbb{G}$. *In particular, it is factorizable if the degree* l *of the root* ε *is odd.*

A.4. 2-modular structure of $u_q(\mathfrak{g})$ and $\Gamma'_\varepsilon(\mathfrak{g})$

Theorem A.4.1. *The ribbon Hopf algebras $u_q(\mathfrak{g})$ and $\Gamma'_\varepsilon(\mathfrak{g})$ are 2-modular, that is $\nu \in \mathbf{u}$, if and only if for any $x \in \mathbb{G}$ such that $x^2 = 1$ we have*

$$\pi(x,x) = \pi(x, K_{2\rho}). \tag{A.4.1}$$

Remark A.4.1. Both sides of Eq. (A.4.1) are characters $X \to \{1,-1\}$, where X is the subgroup $\{x \in \mathbb{G} \mid x^2 = 1\}$.

Proof. Equation (A.3.1) together with Proposition A.3.1 imply that $\nu \in \mathbf{u}$ if and only if $\mathring{R}'_a \mathring{R}''_a K_{-2\rho} \in \mathbb{C}[2\mathbb{G}]$. We have

$$\mathring{R}'_a \mathring{R}''_a K_{-2\rho} = \frac{1}{|\mathbb{G}|} \sum_{g,h\in\mathbb{G}} \pi(g,h)^{-1} g h K_{-2\rho} = \frac{1}{|\mathbb{G}|} \sum_{b\in\mathbb{G}} \phi(b) b, \tag{A.4.2}$$

where the coefficient $\phi(b)$ is the Gaussian sum

$$\phi(b) = \sum_{g\in\mathbb{G}} \pi(g,g)\pi(g, b^{-1}K_{-2\rho}). \tag{A.4.3}$$

Its absolute value is found by the standard procedure

$$|\phi(b)|^2 = |\mathbb{G}| \sum_{x\in X} \pi(x,x)\pi(x, b^{-1}K_{-2\rho}). \tag{A.4.4}$$

This formula suggests to consider a function on $Y = \mathbb{G}/2\mathbb{G}$

$$\psi(y) = \sum_{x\in X} \pi(x,x)\pi(x,y).$$

Here the pairing π restricts to a non-degenerate pairing $\pi : X \times Y \to \{1,-1\}$. Thus the algebra $u_q(\mathfrak{g})$ is 2-modular iff $\psi(y) = \mathrm{const}\,\delta_{y,[K_{-2\rho}]}$. This condition means that the Fourier coefficients $\pi(x,x)$ of the function ψ are proportional to the Fourier coefficients $\pi(x, K_{2\rho})$ of the delta function $\delta_{y,[K_{-2\rho}]}$. Equivalently, $\pi(x,x) = \pi(x, K_{2\rho})$ for all $x \in X$ since the proportionality constant is 1. □

A.5. The integral on $u_q(\mathfrak{g})$

Proposition A.5.1. *The functional $\int : u_q(\mathfrak{g}) \to \mathbb{C}$*

$$\int \Big(\prod_\alpha E_\alpha^{m_\alpha} \cdot K_\lambda \cdot \prod_\beta F_\beta^{n_\beta}\Big) = \prod_\alpha \delta_{m_\alpha, l_\alpha - 1} \cdot \delta_{K_\lambda, K_{-2\rho}} \cdot \prod_\beta \delta_{n_\beta, l_\beta - 1}$$

is a left integral on the Hopf algebra $u_q(\mathfrak{g})$. It is independent of the choice of the reduced expression for w_0.

Proof. Notice that

$$\left(1\otimes\int\right)\Delta(\prod_\alpha E_\alpha^{l_\alpha-1}\cdot K_\lambda\cdot\prod_\beta F_\beta^{l_\beta-1})=K_{2\rho}K_\lambda\int(\prod_\alpha E_\alpha^{l_\alpha-1}\cdot K_\lambda\cdot\prod_\beta F_\beta^{l_\beta-1})$$

by Corollary A.1.5. This implies $(1\otimes\int)\Delta x=\int x$.

All left integrals are proportional to each other, whence all functionals $\int$ defined for different reduced expressions are proportional to each other. The elements of the highest grade $\prod_\alpha E_\alpha^{l_\alpha-1}\otimes\prod_\beta F_\beta^{l_\beta-1}$ are also proportional to each other. Since

$$\pi(\prod_\beta F_\beta^{l_\beta-1}\otimes\prod_\alpha E_\alpha^{l_\alpha-1})=\prod_\alpha\left((l_\alpha-1)_{q_\alpha^{-2}}!(q_\alpha-q_\alpha^{-1})^{1-l_\alpha}\right)$$

(see [20, 21, 26], compare with (A.2.1)) these elements are all equal. Therefore all integrals are equal. □

A.5.1. Invariance of the integral. When $u_q(\mathfrak{g})$ is factorizable, it is unimodular by Proposition 3.7.4. We will prove it also in non-factorizable case.

Proposition A.5.2. *The algebra $u_q(\mathfrak{g})$ is unimodular.*

Proof. A left integral $\delta_+\in u_q(\mathfrak{b}_+)$ and a right integral $\omega_-\in u_q(\mathfrak{b}_-)$ are given by

$$\delta_+=\sum_{h\in\mathbb{G}}h\cdot\prod_{\alpha)}E_\alpha^{l_\alpha-1},\qquad\omega_-=\prod_\beta F_\beta^{l_\beta-1}\cdot\sum_{h\in\mathbb{G}}h.$$

By a result of Hennings [10] and Radford [45] the double $D(u_q(\mathfrak{b}_+))$ is unimodular with the two-sided integral $\delta_+\omega_-$. Its projection to $u_q(\mathfrak{g})$ via the epimorphism $j: D(u_q(\mathfrak{b}_+))\to u_q(\mathfrak{g})$ is

$$\delta=j(\delta_+\omega_-)=|\mathbb{G}|\sum_{h\in\mathbb{G}}h\cdot\prod_{\alpha)}E_\alpha^{l_\alpha-1}\cdot\prod_\beta F_\beta^{l_\beta-1}.$$

This is a two-sided non-zero integral in $u_q(\mathfrak{g})$. □

A.6. 3-modular structure of $u_q(\mathfrak{g})$

Proposition A.6.1. *If $u_q(\mathfrak{g})$ is a 2-modular Hopf algebra then it is 3-modular as well.*

Proof. We have to check the remaining condition (M3) from Theorem 3.7.3, namely

$$\left(\int\otimes1\right)(R^{12}R^{21})\neq0.$$

Only maximal powers will contribute to this expression which can be found from Eq. (A.3.2),

$$\left(\int\otimes1\right)(R^{12}R^{21})=$$

$$\frac{1}{|2\mathbb{G}|}\left(\prod_\alpha\frac{(q_\alpha-q_\alpha^{-1})^{l_\alpha-1}}{(l_\alpha-1)_{q_\alpha^{-2}}!}\right)^2(\prod_\alpha F_\alpha^{l_\alpha-1})\sum_{d\in2\mathbb{G}}\pi(K_{2\rho},d)d(\prod_\beta E_\beta^{l_\beta-1})K_{-2\rho},$$

and this obviously does not vanish. □

Now we find the unique (up to a sign) normalization of the integral which will be used for constructing switching operators.

Proposition A.6.2. *Let* $\int' : u_q(\mathfrak{g}) \to \mathbb{C}$ *be the renormalized left integral*

$$\int' = \sqrt{|2\mathbb{G}|}\, q^{-2(\rho|\rho)} \prod_\alpha \frac{(l_\alpha - 1)_{q_\alpha^{-2}}!}{(q_\alpha - q_\alpha^{-1})^{l_\alpha - 1}} \int$$

considered also as an element $\mu \in fun_q(G)$*. If* $u_q(\mathfrak{g})$ *is 3-modular, then* $P = S^2\gamma_F : F \to F$ *is a projection for* $S = S_-(\mu)$*.*

Proof. The claim is equivalent to Eq. (3.8.16)

$$\left(\int' \otimes \int'\right)(R^{12}R^{21}) = 1.$$

Substituting the expression for $(\int \otimes 1)(R^{12}R^{21})$ from Proposition A.6.1 we get

$$\left(\int \otimes \int\right)(R^{12}R^{21}) = \left(\prod_\alpha \frac{(q_\alpha - q_\alpha^{-1})^{l_\alpha - 1}}{(l_\alpha - 1)_{q_\alpha^{-2}}!}\right)^2 \frac{1}{|2\mathbb{G}|} q^{(2\rho|2\rho)}$$

due to $(\prod_\alpha F_\alpha^{l_\alpha - 1})\cdot K_{-2\rho} = q^{(2\rho|2\rho)} K_{-2\rho} \prod_\alpha F_\alpha^{l_\alpha - 1}$. This fixes the normalization. □

Proposition A.6.3. *The exponential central charge* $\lambda = \int' \nu$ *for* 2*-modular* $u_q(\mathfrak{g})$ *is a root of unity*

$$\lambda = q^{-2(\rho|\rho)} \frac{\sqrt{|2\mathbb{G}|}}{|\mathbb{G}|} \sum_{g \in \mathfrak{G}} \pi(g, g)\pi(g, K_{-2\rho}) \tag{A.6.1}$$

$$= q^{-2(\rho|\rho)} \frac{\sqrt{|2\mathbb{G}|}}{l^n} \sum_{\alpha \in Q/lQ} q^{(\alpha|\alpha) - (\alpha|2\rho)}, \tag{A.6.2}$$

where n *is the rank of* $\mathfrak{g}$ *and* l *is the degree of the root of unity* $q = \varepsilon$*.*

Proof. From Eq. (A.3.1) we find

$$\nu = \prod_\alpha \frac{(q_\alpha - q_\alpha^{-1})^{l_\alpha - 1}}{(l_\alpha - 1)_{q_\alpha^{-2}}!} \int \Big(\prod_\alpha E_\alpha^{l_\alpha - 1}\Big) \cdot \mathring{R}'_a \mathring{R}''_a K_{-4\rho} \cdot \prod_\alpha F_\alpha^{l_\alpha - 1}\Big).$$

Substituting $\mathring{R}'_a \mathring{R}''_a K_{-2\rho} = \frac{1}{|\mathfrak{G}|} \sum_{b \in \mathfrak{G}} \phi(b) b$ by Eq. (A.4.2) we find

$$\lambda = \int' \nu = \sqrt{|2\mathbb{G}|}\, q^{-2(\rho|\rho)} \frac{1}{|\mathbb{G}|} \phi(1),$$

where $\phi(b)$ is defined by Eq. (A.4.3). Its absolute value is determined by Eq. (A.4.4)

$$|\phi(1)|^2 = |\mathbb{G}| \sum_{x \in X} \pi(x, x)\pi(x, K_{-2\rho}) = |\mathbb{G}| \cdot |X| \tag{A.6.3}$$

since all summands equal 1 by Theorem A.4.1. Therefore,

$$|\lambda|^2 = \frac{|2\mathbb{G}|}{|\mathbb{G}|}|X| = 1$$

due to exact sequence $0 \to X \to \mathbb{G} \xrightarrow{2} 2\mathbb{G} \to 0$.

By Eq. (A.6.1) λ satisfies an algebraic equation of degree $2l$. Since λ is an algebraic number of absolute value 1, it is a root of unity.

If we sum up in Eq. (A.6.1) not over $\mathbb{G}$, but over its covering group Q/lQ, the sum will multiply by $l^r/|\mathbb{G}|$. This proves Eq. (A.6.2). □

Remark A.6.1. One easily recognizes in Eqs. (A.6.1) and (A.6.2) generalized quadratic Gauss sums. Notice also that if $u_q(\mathfrak{g})$ is not 2-modular, then $\int \nu = 0$ by Eq. (A.6.3).

B. Examples of 3-modular algebras

We assume that (a_{ij}) is the Cartan matrix of a simple Lie algebra $\mathfrak{g}$ and $q = \varepsilon$ is a primitive root of unity of degree l such that $\varepsilon^{2p} \neq 1$ for $1 \leqslant p \leqslant N(\mathfrak{g})$. We explore case by case whether $u_q(\mathfrak{g})$ is 3-modular or equivalently 2-modular.

In general, $\mathbb{G} = Q/\operatorname{Ann}\pi$, Q being the root lattice and $\operatorname{Ann}\pi = Q \cap l\,\mathrm{Cow}$, where $\mathrm{Cow} = \mathbb{Z}\{\frac{1}{d_i}\omega_i\}$ is the coweight lattice and ω_i is the basis of the weight lattice P. This implies $\operatorname{Ann}\pi = Q \cap \mathbb{Z}\{l'_i\omega_i\}$, where $l'_i = l$ if $d_i \nmid l$ and $l'_i = l/d_i$ if $d_i | l$. The inclusion $Q \hookrightarrow P$, $\alpha_j = \sum_{i=1}^n a_{ij}\omega_i$, determines the annihilator:

$$\operatorname{Ann}\pi = \{u = \sum_j u_j\alpha_j \mid \forall i \quad \sum_j a_{ij}u_j \equiv 0 \pmod{l'_i}\}.$$

Similarly for $X = \operatorname{Ker}\{2 : \mathbb{G} \to \mathbb{G}\}$,

$$X = \{u = \sum_j u_j\alpha_j \mid \forall i \quad 2\sum_j a_{ij}u_j \equiv 0 \pmod{l'_i}\}/\operatorname{Ann}\pi.$$

2- or 3-modularity of $u_q(\mathfrak{g})$ is equivalent to the property

$$(u|u) - (2\rho|u) \equiv 0 \pmod{l}$$

for all $u \in X$.

If l is odd, $X = 0$ and $u_q(\mathfrak{g})$ is perfect modular (we shall see that this is not the only case).

B.1. The algebra $u_\varepsilon(\mathfrak{sl}(2))$. This example was already considered in [40] for odd l. Set $l_1 = l$ for l odd and $l_1 = l/2$ for l even, then it is the degree of ε^2. Since $\pi(K, K) = \varepsilon^2$ for $K = K_1$, the group $\mathbb{G}$ is isomorphic to $\mathbb{Z}/l_1\mathbb{Z}$. Several cases emerge.

1. $l_1 = 2m + 1$ *is odd.* Then $u_\varepsilon(\mathfrak{sl}(2))$ is perfect and

$$\lambda = \varepsilon^{-1}\frac{1}{\sqrt{l_1}}\sum_{k=1}^{l_1}\varepsilon^{2k^2-2k} = \left(\frac{2a}{l_1}\right)e^{-\frac{\pi i m}{2}}\varepsilon^{-1-2m^2},$$

where a is determined by $\varepsilon^2 = e^{2\pi i a/l_1}$. Here for any relatively prime integers c and odd positive b $\left(\frac{c}{b}\right) = \prod_j \left(\frac{a}{p_j}\right)$ denotes the Jacobi quadratic symbol, where $b = p_1 p_2 \dots p_r$ with prime p_j and $\left(\frac{a}{p_j}\right) = \pm 1$ is the Legendre symbol (see e.g. [24]).

2. $l_1 = 2(2m+1)$. Then $u_\varepsilon(\mathfrak{sl}(2))$ is 3-modular and

$$\lambda = \varepsilon^{-1} \frac{1}{\sqrt{2m+1}} \sum_{k=0}^{2m} \varepsilon^{2k^2-2k} = \left(\frac{a}{2m+1}\right) e^{-\frac{\pi i m}{2}} \varepsilon^{2m^2+2m-1},$$

where a is determined by $\varepsilon^2 = e^{2\pi i a/l_1}$. The value of the generalized Gauss quadratic sum was found here by a method of Lang [24] combined with results of Chandrasekharan [2].

3. $l_1 = 4m$. Then $u_\varepsilon(\mathfrak{sl}(2))$ is not 2-modular since $X = \{1, K^{2m}\}$ and

$$\pi(K^{2m}, K^{2m-1}) = \varepsilon^{4m(2m-1)} = \varepsilon^{\frac{l}{2}(2m-1)} = -1.$$

B.2. The Cartan matrix A_n. The determinant of the Cartan matrix is $n+1$. Let $p = (l, n+1)$ be the greatest common divisor. Then

$$\operatorname{Ann} \pi = \{\tfrac{l}{p} v \mid \forall i \quad \sum_j a_{ij} v_j \equiv 0 \pmod{p}\}.$$

Various possibilities appear here as we already have seen for $n = 1$.

B.3. The Cartan matrix B_n. Here $\mathfrak{g} = \mathfrak{o}(2n+1)$, $d_i = 2$ for $1 \leqslant i < n$ and $d_n = 1$. Index of connection $\det A = 2$. Let $l' = l$ for l odd and $l' = l/2$ for l even. Then

$$\operatorname{Ann} \pi = \{\sum_j u_j \alpha_j \mid \forall i < n \ \sum_j a_{ij} u_j \equiv 0 \pmod{l'}, \ \sum_j a_{nj} u_j \equiv 0 \pmod{l}\}.$$

In particular, $2u = (\det A)u \equiv 0 \pmod{l'}$ if $u \in \operatorname{Ann} \pi$. Solving the equations we find that $\operatorname{Ann} \pi = l'Q$ and $\mathbb{G} = Q/l'Q$.

1. *l' is odd.* Then $u_q(\mathfrak{g})$ is perfect modular. l might be even in this case.
2. *$l' = 2m$ is even.* Then $X = \mathbb{F}_2\{m\alpha_i\}$ and the algebra $u_\varepsilon(\mathfrak{g})$ is 2-modular iff m is odd.

B.4. The Cartan matrix C_n. Here $\mathfrak{g} = \mathfrak{sp}(2n)$, $d_i = 1$ if $1 \leqslant i < n$ and $d_n = 2$. The index of connection $\det A = 2$. Let $l' = l$ for l odd and $l' = l/2$ for l even. Then

$$\operatorname{Ann} \pi = \{\sum_j u_j \alpha_j \mid \forall i < n \ \sum_j a_{ij} u_j \equiv 0 \pmod{l}, \ \sum_j a_{nj} u_j \equiv 0 \pmod{l'}\}.$$

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular with $\mathbb{G} = Q/lQ$.

2. $l = 2(2m+1)$. Then $u_q(\mathfrak{g})$ is 3-modular and

$$\operatorname{Ann}\pi = \begin{cases} lQ + \mathbb{Z}(2m+1)\alpha_n & \text{for } n \text{ odd} \\ lQ + \mathbb{Z}(2m+1)\alpha_n + \mathbb{Z}(2m+1)(\alpha_1+\alpha_3+\cdots+\alpha_{n-1}) & \text{for } n \text{ even} \end{cases}.$$

In both cases $X = \mathbb{F}_2\{(2m+1)\alpha_i\}$.

3. $l = 4m$. Then

$$\operatorname{Ann}\pi = lQ + \mathbb{Z}2m\alpha_n + \mathbb{Z}m(2\alpha_1 + 2\alpha_3 + \cdots + 2\alpha_{2[\frac{n}{2}]-1} + n\alpha_n).$$

A) *n is odd.* Then $u_q(\mathfrak{g})$ is 3-modular and $X = \mathbb{F}_2\{2m\alpha_i, m\alpha_n\}$.
B) *$n = 2n'$ is even.* Then

$$X = \mathbb{F}_2\{2m\alpha_i, m\alpha_n, m(\alpha_1+\alpha_3+\cdots+\alpha_{n-1}+n'\alpha_n)\}$$

and $u_q(\mathfrak{g})$ is 3-modular iff n' even or m odd.

B.5. The Cartan matrix D_n. Here $\mathfrak{g} = \mathfrak{o}(2n)$ and $d_i = 1$. The index of connection $\det A = 4$. Since $Au \equiv 0 \pmod{l}$ implies $4u \equiv 0 \pmod{l}$, we have for $p = (4,l)$, $u = \frac{l}{p}b$, $b = (b_1,\ldots,b_n)$, $Ab \equiv 0 \pmod{p}$. This enables one to find all $u \in \operatorname{Ann}\pi = Q \cap lP$.

1. *l is odd.* $u_q(\mathfrak{g})$ is perfect modular with $\mathbb{G} = Q/lQ$.
2. $l = 2(2m+1)$. Then $u_q(\mathfrak{g})$ is 3-modular with

$$\mathbb{G} = Q/(lQ + \mathbb{Z}(2m+1)(\alpha_{n-1}+\alpha_n)) \quad \text{for } n \text{ odd or}$$
$$\mathbb{G} = Q/(lQ + \mathbb{Z}(2m+1)(\alpha_{n-1}+\alpha_n) + \mathbb{Z}(2m+1)(\alpha_1+\alpha_3+\cdots+\alpha_{n-1}))$$

for n even. In both cases $X = \mathbb{F}_2\{(2m+1)\alpha_i\}$.

3. *$l = 4m$ and n is odd.* $u_q(\mathfrak{g})$ is 3-modular with

$$\operatorname{Ann}\pi = lQ + \mathbb{Z}2m(\alpha_{n-1}+\alpha_n) + \mathbb{Z}m(2\alpha_1+2\alpha_3+\cdots+2\alpha_{n-2}-\alpha_{n-1}+\alpha_n),$$
$$X = \mathbb{F}_2\{2m\alpha_i, m(\alpha_{n-1}+\alpha_n)\}.$$

4. *$l = 4m$ and $n = 2n'$.* Here

$$\operatorname{Ann}\pi = lQ + \mathbb{Z}2m(\alpha_{n-1}+\alpha_n) + \mathbb{Z}2m(\alpha_1+\alpha_3+\cdots+\alpha_{n-3}+\alpha_{n-1}),$$
$$X = \mathbb{F}_2\{2m\alpha_i, m(\alpha_{n-1}+\alpha_n), m(\alpha_1+\alpha_3+\cdots+\alpha_{n-3}+\alpha_{n-1})\}.$$

The algebra $u_q(\mathfrak{g})$ is 3-modular iff n' even or m odd.

B.6. The Cartan matrix E_6. Here $d_i = 1$ and $\det A = 3$. The equation $Au \equiv 0 \pmod{l}$ implies $3u \equiv 0 \pmod{l}$.

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular with

$$\mathbb{G} = \begin{cases} Q/lQ, & \text{if } 3\nmid l \\ Q/(lQ + \mathbb{Z}p(\alpha_1-\alpha_3+\alpha_5-\alpha_6)), & \text{if } l = 3p \end{cases}.$$

2. *$l = 2m$ and $3\nmid m$.* Then $u_q(\mathfrak{g})$ is 3-modular with $\mathbb{G} = Q/lQ$ and $X = \mathbb{F}_2\{m\alpha_i\}$.
3. $l = 6m$. Again $u_q(\mathfrak{g})$ is 3-modular with

$$\mathbb{G} = Q/(lQ + \mathbb{Z}2m(\alpha_1-\alpha_3+\alpha_5-\alpha_6)),$$
$$X = \mathbb{F}_2\{3m\alpha_i, m(\alpha_1-\alpha_3+\alpha_5-\alpha_6)\}.$$

B.7. The Cartan matrix E_7. Here $d_i = 1$ and $\det A = 2$. The equation $Au \equiv 0 \pmod{l}$ implies $2u \equiv 0 \pmod{l}$.

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular with $\mathbb{G} = Q/lQ$.
2. $l = 2(2m+1)$. Then $u_q(\mathfrak{g})$ is 3-modular with $X = \mathbb{F}_2\{(2m+1)\alpha_i\}$

$$\mathbb{G} = Q/(lQ + \mathbb{Z}(2m+1)(\alpha_2 + \alpha_5 + \alpha_7)).$$

3. $l = 4m$. Then $X = \mathbb{F}_2\{2m\alpha_i, m(\alpha_2 + \alpha_5 + \alpha_7)\}$

$$\mathbb{G} = Q/(lQ + \mathbb{Z}2m(\alpha_2 + \alpha_5 + \alpha_7)).$$

The algebra $u_q(\mathfrak{g})$ is 3-modular iff m is odd.

B.8. The Cartan matrix E_8. Here $d_i = 1$ and $\det A = 1$, therefore, $\mathbb{G} = Q/lQ$.

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular.
2. $l = 2m$. Then $u_q(\mathfrak{g})$ is 3-modular with $X = \mathbb{F}_2\{m\alpha_i\}$.

B.9. The Cartan matrix F_4. Here $d_1 = d_2 = 2$, $d_3 = d_4 = 1$ and $\det A = 1$, which implies $Q = P$. Let $l' = l$ for l odd and $l' = l/2$ for l even. Then

$$\operatorname{Ann} \pi = \mathbb{Z}\{l'\omega_1, l'\omega_2, l\omega_3, l\omega_4\},$$

where

$$\begin{aligned} \omega_1 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, & \omega_2 &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4, \\ \omega_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4, & \omega_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \end{aligned}$$

are the fundamental weights.

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular with $\mathbb{G} = Q/lQ$.
2. $l = 2l'$. Then $u_q(\mathfrak{g})$ is 3-modular with $\mathbb{G} \simeq (\mathbb{Z}/l'\mathbb{Z})^2 \times (\mathbb{Z}/l\mathbb{Z})^2$

$$\mathbb{G} = Q/(l'\alpha_1, l'(3\alpha_2 + 4\alpha_3 + 2\alpha_4), l(\alpha_1 + 2\alpha_2 + 3\alpha_3), l\alpha_4),$$

$$X = \begin{cases} \mathbb{F}_2\{l'(\alpha_1 + 2\alpha_2 + 3\alpha_3), l'\alpha_4\} & \text{for } l' \text{ odd} \\ \mathbb{F}_2\{m\alpha_1, m(3\alpha_2 + 4\alpha_3 + 2\alpha_4), l'(\alpha_1 + 2\alpha_2 + 3\alpha_3), l'\alpha_4\} & \text{for } l' = 2m \end{cases}.$$

B.10. The Cartan matrix G_2. Here $d_1 = 1$, $d_2 = 3$ and $\det A = 1$, which implies $Q = P$. Let $l' = l$ if $3 \nmid l$ and $l' = l/3$ if $3|l$. Then $\operatorname{Ann} \pi = \mathbb{Z}\{l\omega_1, l'\omega_2\}$, where

$$\omega_1 = 2\alpha_1 + \alpha_2, \qquad \omega_2 = 3\alpha_1 + 2\alpha_2$$

are the fundamental weights. Therefore,

$$\mathbb{G} = Q/\mathbb{Z}\{l\omega_1, l'\omega_2\} \simeq \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l'\mathbb{Z}.$$

1. *l is odd.* Then $u_q(\mathfrak{g})$ is perfect modular.
2. $l = 2m$. Then $u_q(\mathfrak{g})$ is 3-modular with $X = \mathbb{F}_2\{m\omega_1, \frac{l'}{2}\omega_2\}$.

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