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Modifying the KP, the n^{th} Constrained KP Hierarchies and their Hamiltonian Structures

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Abstract: The Kadomtsev–Petviashvili (KP) hierarchy has infinitely many Hamiltonian pairs, the n^{th} pair of them is associated with L^n, where L is the pseudodifferential operator (PDO) [3, 4]. In this paper, by the factorization $L^n = L_n \cdots L_1$ with L_j, $j = 1, \ldots, n$ being the independent PDOs, we construct the Miura transformation for the KP, which leads to a decomposition of the second Hamiltonian structure in the n^{th} pair to a direct sum. Each term in the sum is the second structure in the initial pair associated with L_j. When we impose a constraint (1.9) (i.e. a new type of reduction) to the KP hierarchy, we obtain the similar results for the constrained KP hierarchy. In particular the second Hamiltonian structure for this hierarchy is transformed to a vastly simpler one.

1. Introduction

It has been known that the n^{th} Korteweg–de Vries (KdV) type (also called the Gelfand–Dickey) hierarchy associated with a scalar n^{th} order differential operator

$$L = \partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0, \quad \partial = \partial/\partial x \tag{1.1}$$

has many remarkable properties, among them we are specially interested in the following.

1. Equations in the hierarchy have the bi-Hamiltonian structures and infinitely many conserved quantities (see [1, 2, 4]).

2. There exists a Miura transformation relating the equations to the modified equations. By the Miura transformation the second Hamiltonian structure of the n^{th} KdV type equations is transformed to a vastly simpler one (essentially just $\partial/\partial x$) on an appropriate space of the modified variables. This is what we call the Kupershmidt–Wilson (KW) theorem [5]. A short proof of this theorem was then given by Dickey [4, 6].

3. There exists a remarkable connection between the second Poisson brackets of the KdV type equations and the so-called W_n algebra in the conformal field theory. The Miura transformation plays an important role in the construction of the free field realization of the W algebra (see [7, 8] and references therein).

As we understand, one of the main steps in the proof of the KW theorem is to factorize the operator L in a multiplication form [4, 5, 6]

$$L = \sum_{j=0}^{n} u_j \partial^j = (\partial + v_n)(\partial + v_{n-1}) \cdots (\partial + v_1), \tag{1.2}$$

with $u_n = 1$, $u_{n-1} = 0$ and $\sum v_j = 0$. This yields an expression for each u_j as a differential polynomial in v_j, $j = 1, \ldots, n$,

$$u_j = Q_j(v_1, \ldots v_n), \quad j = 0, 1, \ldots, n-2. \tag{1.3}$$

The substitution (1.3) given by the identity (1.2) is called the Miura transformation.

It is noted that the n^{th} KdV type hierarchy can be reduced from the Kadomtsev–Petviashvili (KP) hierarchy. The latter is based on the pseudodifferential operator (PDO) [9,10,11]

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots, \tag{1.4}$$

and is written in the Lax representation

$$L_{t_m} = [B_m, L], \quad m = 1, 2, \ldots, \tag{1.5}$$

where $B_m = L^m_+$ and $L^m_\pm$ denote respectively the differential and the residual parts of L^m. The coefficients u_j, $j = 1, 2, \ldots$ are understood as the functions of infinitely many variables $t = (t_1, t_2, \ldots)$ with $t_1 = x$.

The n^{th} reduction is the constraint condition

$$L^n = B_n, \quad \text{or} \quad (L^n)_- = 0 \tag{1.6}$$

on coefficients of L^n. The reduction leads the KP hierarchy to the n^{th} KdV type hierarchy

$$B_{n,t_m} = [B_m, B_n], \quad m \not\equiv 0 \,(\text{mod } n). \tag{1.7}$$

Recently we have shown that the KP hierarchy admits another type of reduction (called the constraint) [12–17]. According to [16, 17], such a constraint can be put into compact form,

$$L^n = B_n + q\partial^{-1} r, \quad \text{or} \quad L^n_- = q\partial^{-1} r \tag{1.8}$$

on coefficients of L^n and the KP hierarchy is constrained to

$$(L^n)_{t_m} = [B_m, L^n], \tag{1.9a}$$

$$q_{t_m} = (B_m q), \tag{1.9b}$$

$$r_{t_m} = -(B^*_m r), \tag{1.9c}$$

$$m = 1, 2, 3, \ldots,$$

where for an operator $A = \sum a_j \partial^j$, $A^* = \sum (-\partial)^j a_j$ denotes the formal adjoint to A. From now on, for any operator F and a function f, (Ff) has the meaning that the operator F acts on f, this is not the product of two operators, F and the multiplication operator f.

The constrained KP hierarchy (1.9) includes many well-known systems, such as the AKNS hierarchy ($n = 1$), the hierarchy of the Yajima–Oikawa (YO) equations and many others [12–17]. They also share many features with other integrable systems; for example (1.9) possesses the bi-Hamiltonian structures and infinitely many conserved quantities (see [15, 18, 19]). However, there exist some essential differences between the KdV type equations and the constrained equations. It has been shown in [14, 15, 16] that the solution to the constrained KP hierarchy provides the explicit solution to KP itself, while the solution to the KdV type hierarchy does not.

The idea of factorization has been generalized to the KP hierarchy by Kupershmidt [20], where L^n is factorized in a multiplication form

$$L^n = L_n L_{n-1} \cdots L_1 \ , \tag{1.10}$$

with L_j, $j = 1, \ldots, n$ being the independent first order PDOs. This factorization leads us to a Miura transformation connecting the whole KP hierarchy and a modified KP hierarchy which was proved to be a family of commuting flows [20]. However, ref. [20] does not provide any information about the properties of the Hamiltonian structures (particularly the second one) of the KP hierarchy under the Miura transformation and only a special case of the factorization was discussed in detail.

The main purpose of this paper is to discuss the general factorization and the resulting Miura transformations for the KP hierarchy, its restriction to the constrained KP hierarchy and the properties of the correspondent Hamiltonian structures under the Miura transformation. After a brief review of the Hamiltonian structures for the KP and constrained KP hierarchies, we show, in Sect. 3, that the Miura transformation which resulted from the factorization (1.10) leads a decomposition of the second Hamiltonian structure in the n^{th} Hamiltonian pair of the KP hierarchy to a direct sum. Each term in the sum is the second Hamiltonian structure of the initial pair which corresponds to PDO L_j, where $j = 1, \ldots, n$. We will also show the generalized modified KP hierarchy as well as the comparison of these results with that for the KdV type hierarchy.

In Sect. 4, we impose a constraint condition such that the constrained operator in (1.9) is also factorizable and the KW theorem can be established for the constrained KP hierarchy, namely we first have a Miura transformation and a modified hierarchy. Secondly we prove that the very complicated second Hamiltonian structure for the constrained KP hierarchy can be transformed to a much simpler one on an appropriate space of the modified variables.

We have noticed that the second Hamiltonian structures for the KP hierarchy is connected to the W-infinity algebra [21], while for an example of constrained KP hierarchy it is identical to the $W_3^{(2)}$ algebra [19, 23]. We hope the Miura transformation discussed in this paper is helpful to the free field realization of these algebras.

2. The KP and Constrained KP Hierarchies

2.1. The KP Hierarchy. In this subsection we briefly review some necessary results for the KP hierarchy and its Hamiltonian structures. The construction of the KP hierarchy can be found in [9, 10, 11] and its Hamiltonian structures are from [1–4]. We refer readers to these literatures for further information.

The KP hierarchy (1.5) is a class of nonlinear evolution equations for infinitely many dynamical variables $u_1, u_2, \ldots$. For example the first two flows read

$$
\begin{aligned}
u_{1,t_2} &= u_{1,xx} + 2u_{2,x} \ , \\
u_{2,t_2} &= u_{2,xx} + 2u_{3,x} + 2u_1 u_{1,x} \ , \\
u_{3,t_2} &= u_{3,xx} + 2u_{4,x} - 2u_1 u_{1,xx} + 4u_{1,x}u_2 \ , \\
&\ldots
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
u_{1,t_3} &= u_{1,xxx} + 3u_{2,xx} + 3u_{3,x} + 6u_1 u_{1,x} \ , \\
u_{2,t_3} &= u_{2,xxx} + 3u_{3,xx} + 4u_{4,x} + 6(u_1 u_2)_x \ , \\
&\ldots
\end{aligned}
\tag{2.2}
$$

If we eliminate u_2 and u_3 from (2.1) and (2.2) and rename $t_2 = y$, $t_3 = t$ and $u = u_1$, we obtain the KP equation

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0 \ . \tag{2.3}$$

To construct the Hamiltonian structures for the whole KP hierarchy we introduce a more general PDO as

$$L = \partial + u_0 + u_1\partial^{-1} + u_2\partial^{-2} + \cdots \ , \tag{2.4}$$

and then L^n can be written as

$$L^n = \partial^n + v_{n-1}\partial^{n-1} + v_{n-2}\partial^{n-2} + \cdots \ . \tag{2.5}$$

When $u_0 = 0$ (i.e. $v_{n-1} = 0$), L is reduced to the operator for the KP hierarchy.

We denote by $\mathscr{A}$ the differential algebra of polynomials in formal symbols $\{v_i^{(j)}\}, v_i^{(j)} = \partial^j v_i/\partial x^j$. For the Lie algebra $\mathscr{G}$, we take the Lie algebra of the following vector fields:

$$\partial_a = \sum_{i=-\infty}^{n-1} \sum_{j=0}^{\infty} a_i^{(j)} \frac{\partial}{\partial v_i^{(j)}} \ , \tag{2.6}$$

in $\mathscr{A}$ with the commutator

$$[\partial_a, \partial_b] = \partial_{\partial_a b - \partial_b a} \ , \tag{2.7}$$

where $a = (a_{n-1}, a_{n-2}, \ldots)$, etc. Each vector field ∂_a is known to be in one-to-one correspondence with the operator

$$a = \sum_{-\infty}^{n-1} a_j \partial^j \ . \tag{2.8}$$

The dual space consists of operators

$$X = \sum_{-\infty}^{n-1} \partial^{-j-1} X_j \ , \tag{2.9}$$

and the coupling is

$$\langle \partial_a, X \rangle = \int \mathrm{res}(aX)dx = \int \sum a_j X_j dx \ , \tag{2.10}$$

where for any operator $A = \sum A_j \partial^j$, $\mathrm{res} A = A_{-1}$. The Hamiltonian mappings, $H^{n(\infty)}$ and $H^{n(0)}$ based on L^n are defined by

$$H^{n(\infty)}(X) = [L_-^n, X_+]_- - [L_+^n, X_-]_+ , \tag{2.11}$$

and

$$\begin{aligned} H^{n(0)}(X) &= (L^n X)_+ L^n - L^n (X L^n)_+ \\ &= L^n (X L^n)_- - (L^n X)_- L^n , \end{aligned} \tag{2.12}$$

for any operator X of the form (2.9).

For a functional $\tilde{f} = \int f(v) dx$, $v = (v_{n-1}, v_{n-2}, \ldots)$ and v_j are the coefficients of L^n. We define

$$d\tilde{f} = \frac{\delta f}{\delta L^n} = \sum_{-\infty}^{n-1} \partial^{-i-1} \frac{\delta f}{\delta v_i} . \tag{2.13}$$

With two Hamiltonian mappings, two Poisson brackets can be formulated which are

$$\{\tilde{f}, \tilde{g}\}^{(\infty)} = \int \mathrm{res}\, H^{n(\infty)} \left(\frac{\delta f}{\delta L^n} \right) \frac{\delta g}{\delta L^n} dx , \tag{2.14}$$

and

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \mathrm{res}\, H^{n(0)} \left(\frac{\delta f}{\delta L^n} \right) \frac{\delta g}{\delta L^n} dx . \tag{2.15}$$

Since n is arbitrary, therefore we have an infinite series of the Hamiltonian pairs. The Hamiltonians of the KP hierarchy corresponding to two Hamiltonian structures in the n^{th} pair are

$$\tilde{h}_m = -\frac{n}{m+n} \int \mathrm{res}\, L^{m+n} dx , \tag{2.16}$$

and

$$\tilde{g}_m = \frac{n}{m} \int \mathrm{res}\, L^m dx . \tag{2.17}$$

It should be noted that, on one hand the first Hamiltonian structure reduces to $v_{n-1} = 0$ automatically, but the second one is reducible if and only if the coefficients of (2.13) satisfy

$$\mathrm{res} \left[L^n, \frac{\delta f}{\delta L^n} \right] = 0 . \tag{2.18}$$

On the other hand, when the condition of n^{th} reduction (1.6) is assumed the Hamiltonian structures in the n^{th} Hamiltonian pair for the KP are reduced to these for the n^{th} KdV type hierarchy, but for the second Hamiltonian structure the equation

$$\mathrm{res} \left[B_n, \frac{\delta f}{\delta L^n} \right] = 0 \tag{2.19}$$

must be taken into account if the second leading coefficient of B_n is restricted to be zero.

2.2. The Constrained KP Hierarchy. The so-called constrained KP equations were first proposed in [12–17]. By the identification of a symmetry of KP to its symmetry generator, the Lax pair and the adjoint Lax pair were constrained to integrable nonlinear equations. It was then observed in [16, 17] that such a constraint can be put into a compact form of (1.8) on the PDO in the framework of Sato theory. The simplest example is for $n = 1$, the constrained system (1.9) coincides with the AKNS hierarchy, while for $n = 2$ we obtain the YO hierarchy which includes the equation of [24, 25]

$$\begin{aligned} q_{t_2} &= q_{xx} + 2u_1 q \,, \\ u_{1,t_2} &= (qr)_x \,, \\ r_{t_2} &= -r_{xx} - 2u_1 r \,; \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} q_{t_3} &= q_{xxx} + 3u_1 q_x + \frac{3}{2}u_{1,x}q + \frac{3}{2}q^2 r \,, \\ u_{1,t_3} &= \frac{1}{4}u_{1,xxx} + 3u_1 u_{1,x} + \frac{3}{4}(q_x r - q r_x)_x \,, \\ r_{t_3} &= r_{xxx} + 3u_1 r_x + \frac{3}{2}u_{1,x} r - \frac{3}{2}q r^2 \,, \end{aligned} \tag{2.21}$$

as the first two nontrivial flows. More examples in the constrained KP hierarchy can be found in [12–17]. The Hamiltonian structures for the first few constrained systems were given in [16] and in general were discussed in [18, 19]. By a direct calculation the Hamiltonian structures for the n^{th} constrained KP hierarchy can be obtained from results of [3, 4] for the KP, namely by restricting the operator L^n of (2.5) to be

$$L^n = \partial^n + v_{n-1}\partial^{n-1} + \cdots + v_0 + q\partial^{-1}r \,, \tag{2.22}$$

such that the other coefficients are expressed in terms of q, r and their derivatives $v_{-j} = (-1)^j q(\partial^{j-1}r)/(\partial x^{j-1})$. The Poisson brackets (2.14) and (2.15) are then reduced to [18, 19]

$$\{\tilde{f},\tilde{g}\}^{(\infty)} = \int \operatorname{res}\left[\frac{\delta f}{\delta B_n}, B_n\right]_+ \frac{\delta g}{\delta B_n}dx + \int\left(\frac{\delta g}{\delta r}\frac{\delta f}{\delta q} - \frac{\delta f}{\delta r}\frac{\delta g}{\delta q}\right)dx \,, \tag{2.23}$$

and

$$\begin{aligned} \{\tilde{f},\tilde{g}\}^{(0)} = &\int \operatorname{res}\left(\left(B_n\frac{\delta f}{\delta B_n}\right)_+ B_n - B_n\left(\frac{\delta f}{\delta B_n}B_n\right)_+\right)\frac{\delta g}{\delta B_n}dx \\ &+ \int\left(\frac{\delta g}{\delta q}\left(B_n\frac{\delta f}{\delta r}\right) - \frac{\delta g}{\delta r}\left(B_n^*\frac{\delta f}{\delta q}\right)\right)dx \\ &+ \int\left(r\left(\frac{\delta g}{\delta B_n}B_n\right)_+\frac{\delta f}{\delta r}\right) - q\left(\left(B_n\frac{\delta g}{\delta B_n}\right)_+^*\frac{\delta f}{\delta q}\right)\Big)dx \\ &+ \int\left(\frac{\delta g}{\delta q}\left(\left(B_n\frac{\delta f}{\delta B_n}\right)_+ q\right) - \frac{\delta g}{\delta r}\left(\left(\frac{\delta f}{\delta B_n}B_n\right)_+^* r\right)\right)dx \\ &+ \int\left(r\left(\left(\frac{\delta g}{\delta B_n}B_n\frac{\delta f}{\delta B_n}\right)_+ q\right) - r\left(\left(\frac{\delta f}{\delta B_n}B_n\frac{\delta g}{\delta B_n}\right)_+ q\right)\right)dx \\ &+ \int\left(q\frac{\delta g}{\delta q} - r\frac{\delta g}{\delta r}\right)\left(\left(\partial^{-1}r\frac{\delta f}{\delta r}\right) - \left(\partial^{-1}q\frac{\delta f}{\delta q}\right)\right)dx \,, \end{aligned} \tag{2.24}$$

for any functionals $\tilde{f} = \int f(v_{n-1}, \dots, v_0, q, r)dx$, etc. We emphasize here again that for an operator F and a function f, (Ff) is understood as the operator F acts on f, (while Ff means the product of two operators F and f).

Once $v_{n-1} = 0$ is imposed, we find that (2.23) is automatically reduced while for (2.24), the coefficients $\delta f/\delta v_{n-1}$ and $\delta g/\delta v_{n-1}$ of $\delta f/\delta L^n$ and $\delta g/\delta L^n$ should be expressed in terms of other coefficients through [12],

$$\operatorname{res}\left[B_n + q\partial^{-1}r, \frac{\delta f}{\delta L^n}\right] = \operatorname{res}\left[B_n + q\partial^{-1}r, \frac{\delta g}{\delta L^n}\right] = 0 \ . \tag{2.25}$$

The following gives two explicit examples which were first derived in [16] by using the Lax approach.

Example 1: *n* = *1*. In this case

$$L = \partial + v_0 + q\partial^{-1}r \ , \tag{2.26}$$

and the two Poisson brackets are given by

$$\{\tilde{f}, \tilde{g}\}^{(j)} = \int \left\langle \frac{\delta f}{\delta V}, A^{(j)} \frac{\delta g}{\delta V} \right\rangle dx \quad j = \infty, 0 \ , \tag{2.27}$$

with $\delta f/\delta V = (\delta f/\delta v_0, \delta f/\delta q, \delta f/\delta r)^T$, etc. and $\langle\, ,\rangle$ is denoted by $\langle a, b\rangle = \sum a_j b_j$ for any $a = (a_1, \dots, a_{n+2}), b = (b_1, \dots, b_{n+2})$. The two Hamiltonian operators read

$$A^{(\infty)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ , \tag{2.28}$$

and

$$A^{(0)} = \begin{pmatrix} \partial & q & -r \\ -q & q\partial^{-1}q & -\partial_1 - q\partial^{-1}r \\ r & \partial_1^* - r\partial^{-1}q & r\partial^{-1}r \end{pmatrix} \ , \tag{2.29}$$

where $\partial_1 = \partial + v_0$, $\partial_1^* = -\partial + v_0$.

If we reduce the Poisson brackets to the submanifold $v_0 = 0$, we have the Poisson brackets still in the form (2.27) but with $\delta f/\delta V = (\delta f/\delta q, \delta f/\delta r)^T$, etc. and

$$A^{(\infty)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{2.30}$$

$$A^{(0)} = \begin{pmatrix} 2q\partial^{-1}q & -\partial - 2q\partial^{-1}r \\ -\partial - 2r\partial^{-1}q & r\partial^{-1}r \end{pmatrix} \ . \tag{2.31}$$

In the reduction of second Poisson bracket one should use

$$\operatorname{res}\left[\partial + q\partial^{-1}r, \frac{\delta f}{\delta L}\right] = 0$$

to express $\delta f/\delta v_0$ in terms of $\delta f/\delta q$ and $\delta f/\delta r$ through

$$\left(\frac{\delta f}{\delta v_0}\right)_x = r\frac{\delta f}{\delta r} - q\frac{\delta f}{\delta q} \ , \tag{2.32}$$

and a similar expression for $\delta g/\delta v_0$.

One finds that (2.30) and (2.31) coincide with the Hamiltonian operators in the AKNS hierarchy.

Example 2: $n = 2$. In this case we only give the results on the submanifold $v_1 = 0$. The operator L^2 then takes the form

$$L = \partial^2 + v + q\partial^{-1}r \ , \tag{2.33}$$

and the Poisson brackets are in the form of (2.27) but with the Hamiltonian operators [16]

$$A^{(\infty)} = \begin{pmatrix} 2\partial & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \tag{2.34}$$

and

$$A^{(0)} = -\frac{1}{2}\begin{pmatrix} J & 3q\partial + q_x & 3r\partial + r_x \\ 3q\partial + 2q_x & -3q\partial^{-1}q & 2(\partial^2 + v) + 3q\partial^{-1}r \\ 3r\partial + 2r_x & -2(\partial^2 + v) + 3r\partial^{-1}q & -3r\partial^{-1}r \end{pmatrix} , \tag{2.35}$$

where

$$J = \partial^3 + 4v\partial + 2v_x \ . \tag{2.36}$$

3. Modifying the KP Hierarchy and the Second Hamiltonian Structure

Let us start our discussion on the modification of the KP hierarchy. We still take the PDO in the form of (2.4) so that

$$L^n = \partial^n + v_{n-1}\partial^{n-1} + v_{n-2}\partial^{n-2} + \cdots \ , \tag{3.1}$$

and finally consider the reduction to $v_{n-1} = 0$. For a fixed integer n, we factorize L^n to the following multiplication form

$$L^n = L_n L_{n-1} \cdots L_1 \ , \tag{3.2}$$

where

$$L_j = \partial + v_{j,0} + v_{j,1}\partial^{-1} + v_{j,2}\partial^{-2} + \cdots, \quad j = 1, \ldots, n \ , \tag{3.3}$$

are PDOs having the same form as L in (2.4) and independent of each other. Compare coefficients of the same powers in both sides of (3.2); all coefficients v_j of L^n can be expressed as differential polynomials in $v_{i,k}$ [20]:

$$\begin{aligned} v_{n-1} &= \sum_{j=1}^{n} v_{j,0} \ , \\ v_j &= F_j(v_{i,k}), \quad j = n-2, n-3, \ldots \ . \end{aligned} \tag{3.4}$$

We call this expression the Miura transformation for the KP hierarchy. We also note that from (3.4) the condition of $v_{n-1} = 0$ is equivalent to $\sum_{j=1}^{n} v_{j,0} = 0$.

Proposition 3.1. *The second Poisson bracket in the* $\mathrm{n^{th}}$ *pair for the* KP *hierarchy is expressed by*

$$\{\tilde{f},\tilde{g}\}^{(0)} = \sum_{j=1}^{n} \int \mathrm{res}\left(\left(L_j\frac{\delta f}{\delta L_j}\right)_+ L_j - L_j\left(\frac{\delta f}{\delta L_j}L_j\right)_+\right)\frac{\delta g}{\delta L_j}dx\,. \tag{3.5}$$

Proof. We first express $\delta f/\delta L_j$ in terms of $\delta f/\delta L^n$ by

$$\begin{aligned}\delta\tilde{f} &= \int \mathrm{res}\left(\frac{\delta f}{\delta L^n}\right)\delta L^n dx = \sum \int \mathrm{res}\,\frac{\delta f}{\delta L_j}\delta L_j \\ &= \int \sum \mathrm{res}\,\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1}\delta L_j L_{j-1}\cdots L_1 dx \\ &= \int \sum \mathrm{res}\,L_{j-1}\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1}\delta L_j dx\,.\end{aligned}$$

This expression implies that

$$\frac{\delta f}{\delta L_j} = L_{j-1}\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1},\ \bmod R(-\infty,-2) \tag{3.6}$$

where $R(-\infty,-k)$ contains all of the operators of the form $\sum_{-\infty}^{-k} a_j\partial^j$. From (3.6) we find

$$\begin{aligned}L_j\frac{\delta f}{\delta L_j} &= \frac{\delta f}{\delta L_{j+1}}L_{j+1} \\ &= L_j\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1},\ \bmod R(-\infty,-1)\,.\end{aligned} \tag{3.7}$$

By using this equation, the right hand side of (3.5) is

$$\begin{aligned}&\sum_{j=1}^{n}\int \mathrm{res}\left[\left(L_j\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1}\right)_+ L_j - L_j\left(L_{j-1}\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_j\right)_+\right]\frac{\delta g}{\delta L_j}dx \\ &= \sum_{j=1}^{n}\int \mathrm{res}\left[L_j\left(L_{j-1}\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_j\right)_- - \left(L_j\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1}\right)_- L_j\right]\frac{\delta g}{\delta L_j}dx \\ &= \sum_{j=1}^{n}\int \mathrm{res}\left[\left(L_{j-1}\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_j\right)_-\left(\frac{\delta g}{\delta L_j}L_j\right)\right)_+ \\ &\quad - \left(L_j\cdots L_1\frac{\delta f}{\delta L^n}L_n\cdots L_{j+1}\right)_-\left(L_j\frac{\delta g}{\delta L_j}\right)_+\Bigg]dx\end{aligned}$$

Use (3.7) respect to g, and take the sum we can reach the left hand side of (3.5).

Note that each term on the right-hand side of (3.5) represents the second Poisson bracket associated with the operator L_j in (3.3) with the correspondent second Hamiltonian mapping

$$H_j^{(0)}(X) = (L_jX)_+L_j - L_j(XL_j)_+\,, \tag{3.8}$$

defined for any operator of the form $X = \sum_{-\infty}^{0}\partial^{-j-1}X_j$.

Thus we conclude that the factorization (3.2) leads to a decomposition that decomposes the second Hamiltonian structure in the n^{th} Hamiltonian pair of the KP hierarchy to the direct sum of the second Hamiltonian structures in the initial pairs which are associated with PDOs $L_j, j = 1, \ldots, n$.

Restricting the above Proposition to the n^{th} KdV type hierarchy, we find that the factorization (1.2) leads to the same decomposition but each term is the second Hamiltonian structure corresponding to the operator $L_j = \partial + v_j$. However the second Hamiltonian operator corresponding to this type of trivial operator $L = \partial + v$ is simply $\partial/\partial x$, which is nothing but the first Hamiltonian operator of the KdV type hierarchy.

Proposition 3.2. *The condition* (2.18) *is equivalent to*

$$\sum_{j=1}^{n} \operatorname{res}\left[L_j, \frac{\delta f}{\delta L_j}\right] = 0 \tag{3.9}$$

if L^n is factorized by (3.2), *namely* (3.5) *is reducible to $v_{n-1} = 0$ (i.e. $\sum_{j=1}^{n} v_{j,0} = 0$) if* (3.9) *is taken into account.*

Proof. By using (3.6) and (3.7) we have

$$\begin{aligned}\sum_{j=1}^{n} \operatorname{res}\left(L_j \frac{\delta f}{\delta L_j} - \frac{\delta f}{\delta L_j} L_j\right) &= \operatorname{res}\left(L_1 \frac{\delta f}{\delta L_1} - \frac{\delta f}{\delta L_n} L_n\right) \\ &= -\operatorname{res}\left[L^n, \frac{\delta f}{\delta L^n}\right] .\end{aligned}$$

Proposition 3.3. *If the Hamiltonian $\tilde{g}_m = \frac{n}{m} \int \operatorname{res} L^m dx$ in* (2.17) *of the* KP *hierarchy $L_{t_m} = [B_m, L] = H^{n(0)}(\delta g/\delta L^n)$ with respect to the second structure is expressed in terms of the modified variables $\{v_{k,j}\}$ by the Miura transformation, then the correspondent modified equations will be*

$$L_{j,t_m} = H_j^{(0)}\left(\frac{\delta g_m}{\delta L_j}\right), \quad j = 1, 2, \ldots, n , \tag{3.10}$$

where $H_j^{(0)}$ is in (3.8).

Proof. This is a corollary of Proposition 3.1.

Since $\delta g_m/\delta L^n = L^{m-n}$ (see Dickey's book [4]) and (3.6), we find

$$\frac{\delta g_m}{\delta L_j} = L_{j-1} \cdots L_1 L^{m-n} L_n \cdots L_{j+1} , \tag{3.11}$$

with $L = (L_n L_{n-1} \cdots L_1)^{1/n}$, so we have

Proposition 3.4. *Equations in* (3.10) *can be written as*

$$L_{j,t_m} = P_{j,m} L_j - L_j P_{j-1,m}, \quad j = 1, \ldots, n , \tag{3.12}$$

with

$$P_{j,m} = \left(L_j \cdots L_1 L^{m-1} L_n \cdots L_{j+1}\right)_+ , \tag{3.13}$$

namely (3.10) *can be written in the Lax form*

$$\tilde{L}_{t_m} = \left[\tilde{P}, \tilde{L}\right] \ , \tag{3.14}$$

where

$$\tilde{L} = \begin{pmatrix} 0 & L_n & & \\ & \ddots & \ddots & \\ & & \ddots & L_2 \\ L_1 & & & 0 \end{pmatrix}; \qquad \tilde{P} = \begin{pmatrix} P_{n,m} & & \\ & \ddots & \\ & & P_{1,m} \end{pmatrix} . \tag{3.15}$$

Define a cyclic permutation

$$\Omega : L_1 \to L_2, \ L_2 \to L_3, \ldots, L_n \to L_1 \tag{3.16}$$

so $L^n_\Omega = L_1 L_n \ldots L_2$, and similarly we have $L^n_{\Omega^j}(L^n_{\Omega^n} = L^n); P_{j,m}$ is then given by

$$P_{j,m} = \left(L^n_{\Omega^j}\right)^{\frac{m}{n}}_+ \ . \tag{3.17}$$

Thus the Lax equations coincide with these in [20]. Equation (3.17) can easily been proved because

$$\begin{aligned} \left(L^n_{\Omega^j}\right)^{\frac{m}{n}} &= \left(L_j \cdots L_1 L_n \cdots L_{j+1}\right)^{m\frac{1}{n}} \\ &= \left(L_j \cdots L_1 \left(L_n \cdots L_1\right)^{m-1} L_n \cdots L_{j+1}\right)^{\frac{1}{n}} \\ &= \left(L_j \cdots L_1 \left(L_n \cdots L_1\right)^{\frac{m-n}{n}} L_n \cdots L_{j+1}\right)^{n\frac{1}{n}} \\ &= \left(L_j \cdots L_1 \left(L_n \cdots L_1\right)^{\frac{m-n}{n}} L_n \cdots L_{j+1}\right) . \end{aligned} \tag{3.18}$$

It should also be noted that the KP hierarchy and the Hamiltonian $\tilde{g}_m$ are invariant under the cyclic permutation (3.16) because of (3.12) and (3.18). The latter means

$$\tilde{g}_m = \frac{n}{m} \int \mathrm{res}\, L^n dx = \frac{n}{m} \int \mathrm{res}\left(L^n_{\Omega^j}\right)^{\frac{m}{n}} dx \, . \tag{3.19}$$

Let us see an explicit example. Let

$$L^2 = L_2 L_1 \, , \tag{3.20}$$

where the PDO L is in (1.4) (i.e. the operator without zero order term) and so L_1, L_2 are in the form

$$\begin{aligned} L_1 &= \partial - v + w_1 \partial^{-1} + w_2 \partial^{-2} + \cdots \, , \\ L_2 &= \partial + v + v_1 \partial^{-1} + v_2 \partial^{-2} + \cdots \, . \end{aligned} \tag{3.21}$$

Compare coefficients of ∂^j from both sides of (3.20). We express u_j in terms of v_k and w_k

$$
\begin{aligned}
2u_1 &= -v_x - v^2 + v_1 + w_1 \,, \\
2u_2 &= v_2 + w_2 - \frac{1}{2}\left(v_{1,x} - w_{1,x}\right) - v\left(v_1 - w_1\right) + \frac{1}{2}v_{xx} + vv_x \,, \\
2u_3 &= v_3 + w_3 - \frac{1}{2}\left(v_2 - w_2\right)_x - v\left(v_2 - w_2\right) + \frac{1}{4}\left(v_1 - w_1\right)_{xx} \\
&\quad + \frac{1}{2}u\left(v_1 - w_1\right)_x + 2v_x v_1 - \frac{1}{4}\left(v_1 - w_1\right)^2 + \frac{1}{2}v^2\left(v_1 + w_1\right) \\
&\quad - \frac{1}{4}v_{xxx} - \frac{1}{2}vv_{xx} - \frac{3}{4}v_x^2 - \frac{1}{4}v^2 - \frac{1}{2}v^2 v_x \,,
\end{aligned}
$$

etc. (3.22)

The first few operators $P_{0,m} = B_m$ and $P_{1,m}$ are list below

$$
\begin{aligned}
P_{0,2} &= (L_2L_1)_+ = \partial^2 + 2u_1 \,, \\
P_{1,2} &= (L_1L_2)_+ = \partial^2 + 2\bar{u}_1 \,, \\
P_{0,3} &= (L_2L_1)_+^{\frac{3}{2}} = \partial^3 + 3u_1\partial + 3\left(u_2 + u_{1,x}\right) \,, \\
P_{1,3} &= (L_1L_2)_+^{\frac{3}{2}} = \partial^3 + 3\bar{u}_1\partial + 3\left(\bar{u}_2 + \bar{u}_{1,x}\right) \,,
\end{aligned}
\tag{3.23}
$$

where u_1, u_2, u_3 are given by (3.22) and $\bar{u}_1$, $\bar{u}_2, \bar{u}_3$ are obtained from u_j by the cyclic permutation $v \to -v, v_1 \to w_1, v_2 \to w_2$. For example $\bar{u}_1$ reads $2\bar{u}_1 = -v_x - v^2 + v_1 + w_1$. The second and the third flows can be calculated. They are

$$
\begin{aligned}
v_{t_2} &= (v_1 - w_1)_x \,, \\
v_{1,t_2} &= v_{1,xx} + 2v_{2,x} - 2v_x v_1 \,, \\
w_{1,t_2} &= w_{1,xx} + 2w_{2,x} + 2v_x w_1 \,,
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
v_{t_3} &= \frac{1}{4}v_{xxx} - \frac{3}{2}v^2 v_x + \frac{3}{4}\left(v_1 - w_1\right)_{xx} + \frac{3}{2}\left(v_2 - w_2\right)_x \,, \\
v_{1,t_3} &= v_{1,xxx} + 3v_{2,xx} + 3v_{3,x} - \frac{3}{2}v_x v_{1,x} + \frac{3}{2}v_{1,x}\left(v_1 + w_1\right) \\
&\quad - 3vv_x v_1 + 3v_1 w_{1,x} - 3v_x v_2 - \frac{3}{2}v^2 v_{1,x} \,, \\
w_{1,t_3} &= w_{1,xxx} + 3w_{2,xx} + 3w_{3,x} + \frac{3}{2}v_x w_{1,x} + \frac{3}{2}w_{1,x}\left(w_1 + v_1\right) \\
&\quad - 3vv_x w_{1,x} + 3w_1 v_{1,x} + 3v_x w_2 - \frac{3}{2}v^2 w_{1,x} \,.
\end{aligned}
\tag{3.25}
$$

Eliminating v_2, w_2, v_3, w_3 from (3.24) and (3.25) and renaming $t_2 = y$, $t_3 = t$, we find

$$v_t = \frac{1}{4}v_{xxx} - \frac{3}{2}v^2 v_x + \frac{3}{4}\partial^{-1}v_{yy} + \frac{3}{2}v_x(v_1 + w_1),$$

$$v_{1,t} = \frac{1}{4}v_{1,xxx} + \frac{3}{4}\partial^{-1}v_{1,yy} + \frac{3}{2}\partial^{-1}(v_x v_1)_y + \frac{3}{2}(v_1 w_1)_x - \frac{3}{2}v^2 v_{1,x},$$

$$w_{1,t} = \frac{1}{4}w_{1,xxx} + \frac{3}{4}\partial^{-1}w_{1,yy} - \frac{3}{2}\partial^{-1}(v_x w_1)_y + \frac{3}{2}(w_1 v_1)_x - \frac{3}{2}v^2 w_{1,x}. \quad (3.26)$$

One can check directly that when (v, v_1, w_1) solves (3.26), both

$$u = v_x - v^2 + v_1 + w_1,$$

$$\bar{u} = -v_x - v^2 + v_1 + w_1, \quad (3.27)$$

solve the KP equation. We call the system (3.26) the generalized modified KP equation.

If we restrict $L_1 = \partial - v$, then the modified KP equation (3.26) is reduced to the usual modified KP equation given by Konopelchenko and Dubrovsky [22],

$$v_t = \frac{1}{4}v_{xxx} + \frac{3}{4}\partial^{-1}v_{yy} - \frac{3}{2}v^2 v_x + \frac{3}{2}v_x\partial^{-1}v_y, \quad (3.28)$$

and the correspondent Miura transformation is reduced to the well-known one of [22],

$$2u = v_x - v^2 + \partial^{-1}v, \quad (3.29)$$

namely the Miura transformation of [22] can be obtained by factorizing the PDO in a special case [20]

$$L^2 = \left(\partial + v + v_1\partial^{-1} + v_2\partial^{-2} + \cdots\right)(\partial - v). \quad (3.30)$$

It would be very interesting to solve the generalized modified KP equation in (3.26) and derive the correspondent solution to the KP equation. We leave this problem for later investigation.

4. The KW Theorem for the Constrained KP Hierarchy

In this section we concentrate on the modification of the constrained KP hierarchy and establish the KW theorem for this hierarchy.

4.1. The Miura Transformations and Second Hamiltonian Structure. We factorize the constrained PDO L^n of (2.22) in the following multiplication form:

$$L^n = \partial^n + v_{n-1}\partial^{n-1} + \cdots + v_0 + q\partial^{-1}r$$

$$= (\partial + p_n)(\partial + p_{n-1})\cdots(\partial + p_2)(\partial + p_1 + \hat{q}\partial^{-1}\hat{r}). \quad (4.1)$$

This factorization yields an expression for all v_j, q and r as differential polynomials in $p_j, \hat{q}$ and $\hat{r}$

$$v_j = Q_j(p_n, \dots, p_1, \hat{q}, \hat{r}), \quad j = 1, \dots, n-1 \,,$$
$$q = ((\partial + p_n) \cdots (\partial + p_2)\hat{q}), \quad r = \hat{r} \,, \tag{4.2}$$

which is called the first level Miura transformation.

It is noted that the restriction condition $v_{n-1} = 0$ is equivalent to

$$\sum_{j=1}^{n} p_j = 0 \,, \tag{4.3}$$

because of the factorization. The problem we are going to investigate is first to simplify the second Hamiltonian structure of (2.24) by using the Miura transformation and then to derive the motion of equations for the modified variables.

Proposition 4.1. *By the first level Miura transformation* (4.2), *the second Poisson bracket* (2.24) *is transformed to*

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \left[\sum_{j=1}^{n} \frac{\delta f}{\delta p_j} \left(\frac{\delta g}{\delta p_j} \right)_x + \frac{\delta f}{\delta p_1} \left(\hat{q} \frac{\delta g}{\delta \hat{q}} - \hat{r} \frac{\delta g}{\delta \hat{r}} \right) \right.$$
$$- \left(\hat{q} \frac{\delta f}{\delta \hat{q}} - \hat{r} \frac{\delta f}{\delta \hat{r}} \right) \frac{\delta g}{\delta p_1} - \frac{\delta f}{\delta \hat{q}} \partial_1 \frac{\delta g}{\delta \hat{r}} + \frac{\delta f}{\delta \hat{r}} \partial_1^* \frac{\delta g}{\delta \hat{q}}$$
$$\left. + \left(\hat{q} \frac{\delta f}{\delta \hat{q}} - \hat{r} \frac{\delta f}{\delta \hat{r}} \right) \partial^{-1} \left(\hat{q} \frac{\delta g}{\delta \hat{q}} - \hat{r} \frac{\delta g}{\delta \hat{r}} \right) \right] dx \,, \tag{4.4}$$

which is in terms of variables $p_j, \hat{q}$ and $\hat{r}$, where $\partial_1 = \partial + p_1$, $\partial_1^* = -\partial + p_1$.

Proof. Equation (4.4) can be derived simply by the application of Proposition 3.1 and the result of Example 1 in Sect. 2 by letting

$$L_n = \partial + p_n, \dots, \quad L_2 = \partial + p_2, \quad L_1 = \partial + p_1 + \hat{q}\partial^{-1}\hat{r} \,. \tag{4.5}$$

Proposition 4.2. *The condition that admits the restriction to the submanifold* (4.3) *is given by*

$$\sum_{j=1}^{n} \left(\frac{\delta f}{\delta p_j} \right)_x + \hat{q} \frac{\delta f}{\delta \hat{q}} - \hat{r} \frac{\delta f}{\delta \hat{r}} = 0 \,, \tag{4.6}$$

namely, by the condition (4.3), *$\delta f/\delta p_n, \dots, \delta f/\delta p_1$ $\delta f/\delta \hat{q}$ and $\delta f/\delta \hat{r}$ are no longer independent.*

To write down the Poisson bracket (4.4) explicitly under the condition (4.3), we introduce the new variables by

$$\mathbf{w} = \Lambda \mathbf{p} \,, \tag{4.7}$$

where $\mathbf{w} = (w_n, w_{n-1}, \dots, w_1)^T, \mathbf{p} = (p_n, p_{n-1}, \dots, p_1)^T$ and

$$\Lambda = \frac{1}{n} (\lambda^{(i-1)j})_{n \times n} \tag{4.8}$$

is an $n \times n$ matrix with $\lambda = \exp(\frac{2i\pi}{n})$ (i.e. the elementary root of $\lambda^n = 1$). This matrix satisfies

$$\Lambda\Lambda^T = \frac{1}{n}\begin{pmatrix} 1 & \\ & E_{n-1} \end{pmatrix}, \qquad \Lambda^T\Lambda = \frac{1}{n}\begin{pmatrix} E_{n-1} & \\ & 1 \end{pmatrix}, \tag{4.9}$$

because $\sum_{j=1}^{n} \lambda^{kj} = n$ for $k \equiv 0 \pmod n$ and $= 0$ for other cases, where

$$E_{n-1} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}. \tag{4.10}$$

From the transformation (4.7), we have

$$w_n = \frac{1}{n}\sum_{j=1}^{n} p_j; \quad p_1 = \sum_{j=1}^{n} w_j\,, \tag{4.11}$$

$$\frac{\delta f}{\delta \mathbf{p}} = \Lambda \frac{\delta f}{\delta \mathbf{w}}\,, \tag{4.12}$$

in particular

$$\frac{\delta f}{\delta w_n} = \sum_{j=1}^{n} \frac{\delta f}{\delta p_j}; \qquad \frac{\delta f}{\delta p_1} = \frac{1}{n}\sum_{j=1}^{n} \frac{\delta f}{\delta w_j}\,. \tag{4.13}$$

From Eqs. (4.11) and (4.12) we find that condition (4.3) is equivalent to $w_n = 0$ and (4.6) becomes

$$\left(\frac{\delta f}{\delta w_n}\right)_x + \hat{q}\frac{\delta f}{\delta \hat{q}} - \hat{r}\frac{\delta f}{\delta \hat{r}} = 0\,. \tag{4.14}$$

In conclusion if we restrict the second Poisson bracket to the submanifold (4.3) or equivalently to $w_n = 0$, the term $\delta f/\delta w_n$ must be expressed in terms of $\delta f/\delta \hat{q}$ and $\delta f/\delta \hat{r}$ through (4.14) and similarly for $\delta g/\delta w_n$.

The transformation (4.7) has been used for the modification of the n^{th} KdV type hierarchy in [5]. In the present case we have

Proposition 4.3. *The condition* (4.3) (*or* $w_n = 0$) *leads the second Poisson bracket* (4.4) *to the following*:

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \left\langle \frac{\delta f}{\delta V}, A_1^{(0)} \frac{\delta g}{\delta V} \right\rangle dx\,, \tag{4.15}$$

which is in terms of variables $w_{n-1}, \ldots, w_1, \hat{q}, \hat{r}$, *where*

$$\frac{\delta f}{\delta V} = \left(\frac{\delta f}{\delta w_{n-1}}, \ldots, \frac{\delta f}{\delta w_1}, \frac{\delta f}{\delta \hat{q}}, \frac{\delta f}{\delta \hat{r}}\right), \text{ etc.}\,, \tag{4.16}$$

and

$$A_1^{(0)} = \begin{pmatrix} \frac{1}{n}E_{n-1}\partial & -B^T \\ B & C \end{pmatrix}, \tag{4.17}$$

with

$$B = -\frac{1}{n}\begin{pmatrix} \hat{q} & \ldots & \hat{q} \\ -\hat{r} & \ldots & -\hat{r} \end{pmatrix}_{2\times(n-1)}, \tag{4.18}$$

and

$$C = \begin{pmatrix} \frac{n+1}{n}\hat{q}\partial^{-1}\hat{q} & -\partial_1 - \frac{n+1}{n}\hat{q}\partial^{-1}\hat{r} \\ \partial_1^* - \frac{n+1}{n}\hat{r}\partial^{-1}\hat{q} & \frac{n+1}{n}\hat{r}\partial^{-1}\hat{r} \end{pmatrix}_{2\times 2}, \tag{4.19}$$

$$\partial_1 = \partial + p_1 = \partial + \sum_{j=1}^{n-1} w_j . \tag{4.20}$$

Let us see an explicit example of $n = 2$. The correspondent factorization reads

$$L^2 = \partial^2 + v_1\partial + v_0 + q\partial^{-1}r = (\partial + p_2)\left(\partial + p_1 + \hat{q}\partial^{-1}\hat{r}\right) , \tag{4.21}$$

and the new variables w_1, w_2 are given by

$$w_1 = \frac{1}{2}(p_1 - p_2), \quad w_2 = \frac{1}{2}(p_2 + p_1) , \tag{4.22}$$

because $\lambda = 1$. Restricting to the submanifold $v_1 = 0$, or $p_1 + p_2 = 0$, or $w_2 = 0$ we have the first level Miura transformation

$$\begin{aligned} v &= -w_x - w^2 + \hat{q}\hat{r} , \\ q &= \hat{q}_x + w\hat{q}, \ r = \hat{r} , \end{aligned} \tag{4.23}$$

where $v = v_0$, $w = w_1$. By (4.23) the correspondent Hamiltonian operator $A^{(0)}$ in (2.35) is transformed to

$$A_1^{(0)} = \frac{1}{2}\begin{pmatrix} \partial & \hat{q} & -\hat{r} \\ -\hat{q} & 3\hat{q}\partial^{-1}\hat{q} & -2\partial_1 - 3\hat{q}\partial^{-1}\hat{r} \\ \hat{r} & 2\partial_1^* - 3\hat{r}\partial^{-1}\hat{q} & 3\hat{r}\partial^{-1}\hat{r} \end{pmatrix} , \tag{4.24}$$

which is in terms of the modified variables $w, \hat{q}, \hat{r}$, where $\partial_1 = \partial - w$. Namely $A^{(0)}$ in (2.35) and $A_1^{(0)}$ in (4.24) are related by [23]

$$A^{(0)} = M_1' A_1^{(0)} (M_1')^{\dagger} , \tag{4.25}$$

where M_1' is the Jacobian

$$M_1' = \frac{\delta(v, q, r)}{\delta(w, \hat{q}, \hat{r})} . \tag{4.26}$$

In the following, we show the second level Miura transformation which simplifies further the second Hamiltonian structure (4.15).

Proposition 4.4. *Define the second level Miura transformation by*

$$(w_{n-1}, \ldots, w_1, \hat{q}, \hat{r}) \to (w_{n-1}, \ldots, w_1, J_+, J_-) , \tag{4.27}$$

with

$$J_+ = \hat{q}\exp(-\theta), \quad J_- = \hat{r}\exp(\theta), \quad \theta_x = \alpha \sum_{j=1}^{n-1} w_j , \tag{4.28}$$

and α being a constant, the Poisson bracket (4.15) *is then transformed to*

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \left\langle \frac{\delta f}{\delta V}, A_2^{(0)} \frac{\delta g}{\delta V} \right\rangle dx , \tag{4.29}$$

with $\delta f/\delta V = (\delta f/\delta w_{n-1}, \ldots, \delta f/\delta w_1, \delta f/\delta J_+, \delta f/\delta J_-)$. *The Hamiltonian operator* $A_2^{(0)}$ *depends on the parameter* α *which has two interesting choices. In the case of* $\alpha = -1$, $A_2^{(0)}$ *takes the form*

$$A_2^{(0)} = \begin{pmatrix} \frac{1}{n}E_{n-1}\partial & 0 \\ 0 & \hat{C} \end{pmatrix}, \tag{4.30}$$

with

$$\hat{C} = \begin{pmatrix} 2J_+\partial^{-1}J_+ & -\partial - 2J_+\partial^{-1}J_- \\ -\partial - 2J_-\partial^{-1}J_+ & 2J_-\partial^{-1}J_- \end{pmatrix}. \tag{4.31}$$

In the case of $\alpha = \sqrt{\frac{2n}{n-1}} - 1$, *we have*

$$A_2^{(0)} = \begin{pmatrix} \frac{1}{n}E_{n-1}\partial & \frac{1}{n}\sqrt{\frac{2n}{n-1}}J^T \\ -\frac{1}{n}\sqrt{\frac{2n}{n-1}}J & \hat{C} \end{pmatrix}, \tag{4.32}$$

with

$$J = \begin{pmatrix} J_+ & \ldots & J_+ \\ -J_- & \ldots & -J_- \end{pmatrix}. \tag{4.33}$$

$$\hat{C} = \begin{pmatrix} 0 & -\partial_1 \\ \partial_1^* & 0 \end{pmatrix}, \tag{4.34}$$

$$\partial_1 = \partial + \sqrt{\frac{2n}{n-1}}\sum_{j=1}^{n-1} w_j\,. \tag{4.35}$$

The proof is simply a direct calculation.

If we take $n = 2$, the Hamiltonian operators in (4.30) and (4.32) are reduced to

$$A_2^{(0)} = \begin{pmatrix} \frac{1}{2}\partial & 0 & 0 \\ 0 & 2J_+\partial^{-1}J_+ & -\partial - 2J_+\partial^{-1}J_- \\ 0 & -\partial - 2J_-\partial^{-1}J_+ & 2J_-\partial^{-1}J_- \end{pmatrix}, \tag{4.36}$$

in the first case and

$$A_2^{(0)} = \begin{pmatrix} \frac{1}{2}\partial & -J_+ & J_- \\ J_+ & 0 & -\partial - 2w_1 \\ -J_- & -\partial + 2w_1 & 0 \end{pmatrix}. \tag{4.37}$$

in the second case.

We have seen that by the second level Miura transformation (4.27) and (4.28), the Hamiltonian operator $A_2^{(0)}$ in (4.32) for $\alpha = \sqrt{2n/(n-1)} - 1$ becomes very simple, while for $\alpha = -1$, it is in the form of the block diagonal matrix. The first block is exactly the Hamiltonian operator for the modified n^{th} KdV type hierarchy, while the second block represents the second Hamiltonian operator for the AKNS hierarchy. The following proposition indicates that in the case of $\alpha = -1$, the Hamiltonian operator $A_2^{(0)}$ in (4.30) can still be simplified.

Proposition 4.5. *For* $\alpha = -1$, *define the transformation*

$$J_+ = \phi_+ \exp \xi, \quad J_- = \phi_- \exp(-\xi), \quad \xi_x = \phi_+ + \phi_- , \tag{4.38}$$

the Poisson bracket in terms of the variables $(w_{n-1}, \ldots, w_1, \phi_+, \phi_-)$ *are given by*

$$\{\tilde{f}, \tilde{g}\}^{(0)} = \int \left\langle \frac{\delta f}{\delta V}, A_3^{(0)} \frac{\delta g}{\delta V} \right\rangle dx , \tag{4.39}$$

with $\delta f / \delta V = (\delta f / \delta w_{n-1}, \ldots, \delta f / \delta w_1, \delta f / \delta \phi_+, \delta f / \delta \phi_-)$ *and the Hamiltonian operator*

$$A_3^{(0)} = \begin{pmatrix} \frac{1}{n} E_{n-1} \partial & \\ & -E_2 \partial \end{pmatrix} , \tag{4.40}$$

where E_{n-1} *and* E_2 *are the matrices of the form of* (4.10) *with respectively the order of* $n-1$ *and* 2.

4.2. Modified Equations. In the following we derive the modified constrained KP hierarchy and the correspondent Hamiltonians. The motion of modified equations in terms the modified variables $\{p_j, \hat{q}, \hat{r}\}$ can be derived from the general discussion in (3.12) by restricting the operators L_j in the form of (4.5). An explicit example is shown below.

Let $L^2 = L_2 L_1$ with $L_2 = \partial - w$, $L_1 = \partial + w + \hat{q} \partial^{-1} \hat{r}$ as in (4.5). We then have the first few operators $P_{0,m}$ and $P_{1,m}$ in the form of (3.23) but with

$$\begin{aligned} 2u_1 &= w_x - w^2 + \hat{q}\hat{r} , \\ 2u_2 &= -\frac{1}{2} w_{xx} + w w_x - w \hat{q}\hat{r} + \frac{1}{2}(\hat{q}_x \hat{r} - \hat{q}\hat{r}_x) , \\ &\text{etc.} , \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} 2\bar{u}_1 &= -w_x - w^2 + \hat{q}\hat{r} , \\ 2\bar{u}_2 &= \frac{1}{2} w_{xx} + w w_x - w \hat{q}\hat{r} - \frac{1}{2} \hat{q}_x \hat{r} - \frac{3}{2} \hat{q}\hat{r}_x , \\ &\text{etc.} \end{aligned} \tag{4.42}$$

From the equation

$$L_{j, t_m} = P_{j,m} L_j - L_j P_{j-1,m}, \quad j = 1, \ldots, n , \tag{4.43}$$

we have

$$\begin{aligned} w_{t_2} &= (\hat{q}\hat{r})_x , \\ \hat{q}_{t_2} &= \hat{q}_{xx} - (w_x + w^2 - \hat{q}\hat{r})\hat{q} , \\ \hat{r}_{t_2} &= -\hat{r}_{xx} - (w_x - w^2 + \hat{q}\hat{r})\hat{r} , \end{aligned} \tag{4.44}$$

and

$$
\begin{aligned}
w_{t_3} &= \left[\frac{1}{4}w_{xx} - \frac{1}{2}w^3 + \frac{3}{4}\left(\hat{q}_x\hat{r} - \hat{q}\hat{r}_x\right)\right]_x , \\
\hat{q}_{t_3} &= \hat{q}_{xxx} - \frac{3}{4}\left(2w_x + 2w^2 - 3\hat{q}\hat{r}\right)\hat{q}_x - \frac{3}{4}\left(w_{xx} + 2ww_x + 2w\hat{q}\hat{r} + \hat{q}\hat{r}_x\right)\hat{q} , \\
\hat{r}_{t_3} &= \hat{r}_{xxx} + \frac{3}{4}\left(2w_x - 2w^2 + 3\hat{q}\hat{r}\right)\hat{r}_x + \frac{3}{4}\left(w_{xx} - 2ww_x + 2w\hat{q}\hat{r} - \hat{q}_x\hat{r}\right)\hat{r} .
\end{aligned}
\tag{4.45}
$$

Equations (4.44) and (4.45) are the modified equations of (2.20) and (2.21) respectively. The first level Miura transformation connecting them are given by

$$
2u_1 = w_x - w^2 + \hat{q}\hat{r}, \quad q = \hat{q}_x - w\hat{q}, \quad r = \hat{r} . \tag{4.46}
$$

The second level Miura transformation in this case is given by

$$
J_+ = \hat{q}\exp(-\theta), \quad J_- = \hat{r}\exp(\theta), \quad \theta_x = \alpha w , \tag{4.47}
$$

where α is either -1 or 1 because $n = 2$.

When $\alpha = -1$ the correspondent motion of equation reads

$$
\begin{aligned}
w_{t_2} &= (J_+J_-)_x , \\
J_{+,t_2} &= J_{+,xx} - 2wJ_{+,x} - 2w_xJ_+ + 2J_+^2J_- , \\
J_{-,t_2} &= -J_{-,xx} - 2wJ_{-,x} - 2w_xJ_- - 2J_+J_-^2 ,
\end{aligned}
\tag{4.48}
$$

and

$$
\begin{aligned}
w_{t_3} &= \left[\frac{1}{4}w_{xx} - \frac{1}{2}w^3 + \frac{3}{4}\left(J_{+,x}J_- - J_+J_{-,x}\right) - \frac{3}{2}wJ_+J_-\right]_x , \\
J_{+,t_3} &= J_{+,xxx} - 3wJ_{+,xx} + \frac{3}{2}\left(-3w_x + w^2 + 2J_+J_-\right)J_{+,x} \\
&\quad - \frac{3}{2}\left(w_{xx} - 2ww_x + 4wJ_+J_- + J_+J_{-,x}\right)J_+ , \\
J_{-,t_3} &= J_{-,xxx} + 3wJ_{-,xx} + \frac{3}{2}\left(3w_x + w^2 + 2J_+J_-\right)J_{-,x} \\
&\quad + \frac{3}{2}\left(w_{xx} - 2ww_x + 4wJ_+J_- - J_{+,x}J_-\right)J_- .
\end{aligned}
\tag{4.49}
$$

The Miura transformation from (2.20) and (2.21) to the above equations is

$$
\begin{aligned}
2u_1 &= w_x - w^2 + J_+J_- , \\
q &= (J_{+,x} - 2wJ_+)\exp(\theta), \quad r = J_-\exp(-\theta), \quad \theta_x = -w .
\end{aligned}
\tag{4.50}
$$

When $\alpha = 1$ we have

$$
\begin{aligned}
w_{t_2} &= (J_+ J_-)_x \,, \\
J_{+,t_2} &= J_{+,xx} + 2wJ_{+,x} \,, \\
J_{-,t_2} &= -J_{-,xx} + 2wJ_{-,x} \,,
\end{aligned} \tag{4.51}
$$

and

$$
\begin{aligned}
w_{t_3} &= \left[\frac{1}{4}w_{xx} - \frac{1}{2}w^3 + \frac{3}{4}\left(J_{+,x}J_- - J_+J_{-,x}\right) + \frac{3}{2}wJ_+J_-\right]_x \,, \\
J_{+,t_3} &= J_{+,xxx} + 3wJ_{+,xx} + \frac{3}{2}\left(w_x + w^2 + J_+J_-\right)J_{+,x} \,, \\
J_{-,t_3} &= J_{-,xxx} - 3wJ_{-,xx} - \frac{3}{2}\left(w_x - w^2 - J_+J_-\right)J_{-,x} \,.
\end{aligned} \tag{4.52}
$$

The Miura transformation from (2.20) and (2.21) to (4.51) and (4.52) is given by

$$
\begin{aligned}
2u_1 &= w_x - w^2 + J_+J_- \,, \\
q &= J_{+,x}\exp(\theta), \quad r = J_-\exp(-\theta), \quad \theta_x = w \,.
\end{aligned} \tag{4.53}
$$

Let us show the Hamiltonian for our modified equations. From the general construction of the n^{th} constrained KP hierarchy [18, 19], the Hamiltonians are given by $H_m = \int \mathrm{res}\, L^m\, dx$ for the constrained PDO L. The first few of them for the second constrained KP hierarchy are listed below

$$
H_0 = \int qr\, dx, \quad H_1 = \frac{1}{2}\int (q_x r - q r_x + u^2)\, dx, \text{ etc.} \tag{4.54}
$$

By the first level Miura transformation (4.46) these Hamiltonians are transformed to

$$
H_0 = \int (\hat{q}_x - w\hat{q})\hat{r}\, dx \,, \tag{4.55}
$$

$$
H_1 = \frac{1}{2}\int \left[\hat{q}_{xx}\hat{r} + \hat{q}\hat{r}_{xx} - \frac{1}{2}w_x\hat{q}\hat{r} - w(\hat{q}_x\hat{r} - \hat{q}\hat{r}_x) + \frac{1}{4}\left(w_x^2 + w^4 - 2w^2\hat{q}\hat{r}\right)\right] dx \,, \tag{4.56}
$$

while by the second Miura transformation (4.47) we have for $\alpha = -1$,

$$
H_0 = \int (J_{+,x} - 2wJ_+)J_-\, dx \,, \tag{4.57}
$$

$$
\begin{aligned}
H_1 = \frac{1}{2}\int \Big(& J_{+,xx}J_- + J_+J_{-,xx} + \frac{1}{4}J_+^2J_-^2 - \frac{5}{2}wJ_{+,x} + \frac{7}{2}wJ_+J_{-,x} \\
& + \frac{9}{2}w^2J_+J_- + \frac{1}{4}w_x^2 + \frac{1}{4}w^4\Big)\, dx \,.
\end{aligned} \tag{4.58}
$$

For $\alpha = 1$,

$$H_0 = \int J_{+,x} J_- dx \,, \tag{4.59}$$

$$H_1 = \int \left[J_{+,xx} J_- + J_+ J_{-,xx} - w J_{+,x} J_+ \frac{1}{4} \left(w_x - w^2 + J_+ J_- \right)^2 \right] dx \,. \tag{4.60}$$

These are the Hamiltonians for the modified equations corresponding to each of the Hamiltonian operators.

5. Concluding Remarks

In this paper we find that the factorization (3.2) of the n^{th} power of the PDO L^n to a multiplication form leads to the generalized Miura transformation for the KP hierarchy which decomposes the second Hamiltonian structure in the n^{th} pair to a direct sum of the second one in the initial pair associated with the factors L_j, $j = 1, \ldots, n$. By the constraint (1.9), we obtained similar results for the constrained KP hierarchy, in particular the second Hamiltonian structure for the constrained KP hierarchy becomes very simple by our Miura transformation.

It has been known that the second Hamiltonian structures for the integrable systems, such as the KdV type and the KP hierarchies, provide the structures of the so-called classical W-algebras in conformal field theory (see e.g. [7, 8, 21] and references therein). The correspondent Miura transformations play an important role in the construction of the free field realizations of the W-algebras. It would be expected that the Miura transformations intensively discussed in this paper have significance to the free field realizations of the W-algebras associated with the KP and the constrained KP hierarchies.

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