

Fusion in the W_3 Algebra

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Abstract: We develop the notions of fusion for representations of the WA_2 algebra along the lines of Feigin and Fuchs. We present some explicit calculations for a WA_2 minimal model.

1. Introduction

The concept of fusion is central in the application of algebraic techniques in two-dimensional conformal field theory. In conformal field theory one supposes the presence of an infinite dimensional symmetry algebra, and local fields which transform under the algebra. The local fields are operator valued distributions, and it is taken as an axiom that the product of two fields may be written as a sum of fields; this is the operator product expansion. In particular there is a particular class of fields called primary fields, and in its simplest form the fusion algebra describes which irreducible representations ρ_k of the symmetry algebra can occur in the operator product of two primary fields, which we write symbolically as

$$\Phi_i \times \Phi_j \mapsto N_{ij}^k \rho_k, \quad (1.1)$$

where N_{ij}^k are the Verlinde fusion algebra coefficients, and are integers or infinite. For algebras with a non-zero central extension the operator product of two fields cannot simply correspond to the tensor product of two highest weight representations, as in the former case the value of the central charge is unchanged, whereas it adds under tensor product.

The simplest non-trivial algebra with which one must deal is the Virasoro algebra. Belavin, Polyakov and Zamolodchikov showed how null vectors of the Virasoro algebra affected allowed fusions [1], but Feigin and Fuchs were the first to translate their ideas into mathematical language and were able to prove the conjectured fusion rules of the Virasoro minimal models, as well as providing an algebraic definition of a minimal model in terms of a quasi-finite-representation [2]. A standard treatment would be to consider the detailed structure of the representation ρ_i in Eq. (1.1) and

find the constraints on the allowed representations ρ_k . Feigin and Fuchs method differs from a standard approach in that they consider instead the representation ρ_k , and derive constraints on the pairs of fields Φ_i, Φ_j which can couple to it. Their method also naturally extends to any number of fields

$$\Phi_1 \times \cdots \times \Phi_n \rightarrow \rho_0 . \quad (1.2)$$

It is the aim of this paper to show how to extend these results to W -algebras, and in particular the W_3 or WA_2 algebra introduced by Zamolodchikov in [3]. We shall show how these can be adapted to the WA_2 algebra, for 3-pt and n -pt functions, we provide a couple of examples and conclude with suggestions how the main conjectures in this paper may be proven.

2. Quantum WA_2 Algebra and its Representations

We take the quantum WA_2 algebra to have generations L_m, Q_m with relations

$$\begin{aligned} [L_m, L_n] &= \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} + (m-n)L_{m+n} , \\ [L_m, Q_n] &= (2m-n)Q_{m+n} , \\ [Q_m, Q_n] &= \frac{(22+5c)}{48} \frac{c}{3 \cdot 5!} (m^2-4)(m^2-1)m\delta_{m+n} + \frac{1}{3}(m-n)A_{m+n} \\ &\quad + \frac{(22+5c)(m-n)}{48 \cdot 30} (2m^2 - mn + 2n^2 - 8)L_{m+n} , \end{aligned} \quad (2.1)$$

where

$$A_m = \sum_{p>-2} L_{m-p}L_p + \sum_{p\leq-2} L_pL_{m-p} - \frac{3}{10}(m+2)(m+3)L_m , \quad (2.2)$$

and c is a central element.

The representation theory of the W_3 algebra can be developed in analogy with that of the Virasoro algebra. A W_3 highest weight vector $|h, q\rangle$ satisfies

$$L_m|h, q\rangle = \delta_{m,0}h|h, q\rangle, Q_m|h, q\rangle = \delta_{m,0}q|h, q\rangle, \quad m \geq 0 . \quad (2.3)$$

The Verma module $V_{h,q,c}$ of the W_3 -algebra is spanned by states of the form

$$L_{i_1} \cdots L_{i_j} Q_{k_1} \cdots Q_{k_l} |h, q\rangle, \quad i_m \leq i_{m+1} \leq -1, k_m \leq k_{m+1} \leq -1 , \quad (2.4)$$

and by the usual abuse of notation the central element c takes the value c . If the Verma module is reducible, then the irreducible representation $L_{h,q,c}$ is the quotient of the Verma module by its maximal invariant submodule.

We can parametrise the weights of a W -highest weight vector as follows [5],

$$h = \frac{1}{3}(x^2 + xy + y^2 - 3a^2) , \quad q = \frac{1}{27}(x-y)(2x+y)(x+2y) , \quad (2.5)$$

where we define a, α_{\pm} by

$$c = 2 - 24a^2, \alpha_{\pm}^2 - \alpha_{\pm}a - 1 = 0 . \quad (2.6)$$

The condition that $V_{h,q,c}$ has a null vector with eigenvalues h', q' is that we can find some x, y such that h, q are given by Eq. (2.5) and x satisfies

$$x = r\alpha_+ + s\alpha_-, \quad r, x \in \mathbb{N}, \quad rs > 0, \quad (2.7)$$

in which case h', q' are given by Eqs. (2.5) with $x' = x - 2r\alpha_+, y' = y + r\alpha_+$. If

$$x = r\alpha_+ + s\alpha_-, \quad y = t\alpha_+ + u\alpha_-; \quad r, t, s, u \in \mathbb{N}^+, \quad (2.8)$$

then there are two independent null states in $V_{h,q,c}$ and we call such a representation doubly-degenerate. We write h and q as $h[rt;su], q[rt,su]$ and the highest weight state as $|rt,su\rangle$, or simply denote the representation by $[rt;su]$.

The W_3 minimal models are those which have

$$\alpha_+ = \sqrt{p/q}, \quad p, q \in \mathbb{N}, \quad p, q \text{ coprime}, \quad c = c(p, q) = 50 - 24\frac{p}{q} - 24\frac{q}{p}, \quad (2.9)$$

and the fields in these models are of the form $[rt;su]$ with $0 < r, s, t, u, r + t < q, s + u < p$ (see[6]). These representations have three independent null vectors in $V_{h,q,c}$. Those minimal models with $p = m + 1, q = m, m \geq 3$ are unitary since they can be constructed in the explicitly unitary coset construction [7].

For each state $|\psi\rangle$ in a highest weight representation, we can define a field, $\psi(z)$ such that $\psi(0)|0\rangle = |\psi\rangle$. For each operator X_m , where $X = L$ or Q we can define the field $\hat{X}_m\psi(z)$ by $\hat{X}_m\psi(0)|0\rangle = X_m|\psi\rangle$. Then it is possible to write the commutation relations of L_m and Q_m with an arbitrary field as a sum,

$$[L_m, \psi] = \sum_{j=-1}^{\infty} z^{m-j} \binom{m+1}{j+1} \hat{L}_j \psi, \quad (2.10)$$

$$[Q_m, \psi] = \sum_{j=-2}^{\infty} z^{m-j} \binom{m+2}{j+2} \hat{Q}_j \psi. \quad (2.11)$$

With a primary field $\Phi_{h,q}(z), L_m$ and Q_m have commutation relations

$$\begin{aligned} [L_m, \Phi_{h,q}(z)] &= (h(m+1)z^m + z^{m+1}\partial) \Phi_{h,q}(z), \\ [Q_m, \Phi_{h,q}(z)] &= \left(\frac{q}{2}(m+2)(m+1)z^m + (m+2)z^{m+1}\hat{Q}_{-1} + z^{m+2}\hat{Q}_{-2}\right) \Phi_{h,q}(z). \end{aligned} \quad (2.12)$$

Since $\hat{L}_{-1}\psi(z) = \partial\psi(z)$, this results in a representation of the Virasoro algebra on the modes of $\psi(z)$, whereas for the modes Q_m this is not possible, as the commutator (2.11) includes the new fields $\hat{Q}_{-1}\psi(z)$ and $\hat{Q}_{-2}\psi(z)$. This fact is responsible for many of the difficulties in the theory of the WA_2 algebra.

The approach of Feigin and Fuchs is to consider the correlation functions of the form (1.2) as a map φ from the irreducible representation $L_{h,q,c}$,

$$\varphi : |\psi\rangle \mapsto \langle h^\infty, q^\infty | \prod_{i=1}^n \Phi_{h_i, q_i}(w_i) | \psi \rangle. \quad (2.13)$$

Although the modes L_m and Q_m do not have nice commutation relations with primary fields, as for the Virasoro case [2] one can define combinations of these modes which have much simpler commutation relations. By cancelling all the poles in the operators product of $L(z)$ or $Q(z)$ with a primary field, we can arrange that

certain operators e_m and f_m commute with a field, and by cancelling all but the leading pole we can pick out the weights of the fields. If we wish to consider correlation functions of the form

$$\langle h^\infty, q^\infty | \prod_{i=1}^n \Phi_{h^i, q^i}(w_i) | \psi \rangle, \quad w_i \neq 0 \quad (2.14)$$

for arbitrary states $|\psi\rangle$, then it is sensible to consider the operators

$$e_m = \oint_0 \left(\prod_{i=1}^n \left(\frac{z - w_i}{zw_i} \right)^2 \right) z^{m+1} L(z) \frac{dz}{2\pi i} = L_{-2n+m} + \dots + L_m \prod_i w_i^{-2},$$

$$f_m = \oint_0 \left(\prod_{i=1}^n \left(\frac{z - w_i}{zw_i} \right)^3 \right) z^{m+2} Q(z) \frac{dz}{2\pi i} = (-1)^n Q_{-3n+m} + \dots + Q_m \prod_i w_i^{-3}. \quad (2.15)$$

The operators e_m and f_m clearly depend on the points w_i but we suppress the dependence if it is clear from context. If the points w_i are finite and distinct, then we can also consider the operators

$$e_0^{(w_j)} = -\oint_{i \neq j} \left(\frac{(z - w_i)w_j}{(w_j - w_i)z} \right)^2 \frac{(z - w_j)w_j}{z} L(z) \frac{dz}{2\pi i},$$

$$f_0^{(w_j)} = -\oint_{i \neq j} \left(\frac{(z - w_i)w_j}{(w_j - w_i)z} \right)^3 \left(\frac{(z - w_j)w_j}{z} \right)^2 Q(z) \frac{dz}{2\pi i}. \quad (2.16)$$

For these operators

$$\langle h^\infty, q^\infty | \prod_i \Phi_{h_i, q_i}(w_i) (e_m - \delta_{m,0} h^\infty \prod_i w_i^{-2}) = 0, \quad m \leq 0,$$

$$\langle h^\infty, q^\infty | \prod_i \Phi_{h_i, q_i}(w_i) (e_0^{(w_j)} - h_j) = 0,$$

$$\langle h^\infty, q^\infty | \prod_i \Phi_{h_i, q_i}(w_i) (f_m - \delta_{m,0} q^\infty \prod_i w_i^{-3}) = 0, \quad m \leq 0,$$

$$\langle h^\infty, q^\infty | \prod_i \Phi_{h_i, q_i}(w_i) (f_0^{(w_j)} - q_j) = 0. \quad (2.17)$$

Let us define for $w_i \neq 0$,

$$\mathcal{W}_{<}(w_1, \dots, w_n) = \text{span} (e_m, f_m, m < 0), \quad (2.18)$$

and in the case w_i finite and distinct,

$$\mathcal{W}^0(w_1, \dots, w_n) = \text{span} (e_0^{(w_1)}, f_0^{(w_1)}, \dots; e_0^{(\infty)}, f_0^{(\infty)}). \quad (2.19)$$

For any highest weight representation space N , it is easy to see by direct calculation of their commutators that the element of \mathcal{W}^0 act as an Abelian algebra on $N/\mathcal{W}_{<}N$.

The space of maps φ which satisfy (2.17) is given as $L/\mathcal{W}_{<}L$, and the space of maps $\varphi_{\{h^i, q^i\}}$ with fixed values of weights is given as the quotient of $L/\mathcal{W}_{<}L$ by the relations $\{e_0^{(w_i)} - h^i, f_0^{(w_i)} - q^i, \}$. The structure of $L/\mathcal{W}_{<}L$ as a representation of \mathcal{W}^0 may lead to restrictions on the allowed values of $\{h^i, q^i\}$. In the worst case there are no restrictions on the weights of the fields and the space of maps $\varphi_{\{h^i, q^i\}}$ is infinite dimensional for each choice of weights. In the best case, the space of maps φ is

finite dimensional and hence there is only a finite set of allowed weights $\{h^i, q^i\}$ which give non-zero correlation functions with the irreducible representation $L_{h,q,c}$. If the points w_i are distinct then in this case we expect \mathcal{W}^0 to be diagonalisable and the eigenvalues of $\{e_0^{(w_i)} f_0^{(w_i)}\}$ give the allowed weights.

Conjecture. $\dim L/\mathcal{W}_{<}(w_1 \dots w_n)L$ is independent of the points w_i , for all $c = c(p, q)$ with $n = 2$ (p, q coprime, $p, q > 2$), and for all n with $c = c(3, p)$, (p coprime to 3, $p > 3$), provided these points are all non-zero. In particular the dimension does not change if w_i are coincident or infinite.

Clearly the dimension of this space is constant for generic points w_i , and can only increase at special points. We conjecture that this does not happen. This has been shown to be the case for the Virasoro algebra with the special series of central charges $c(2, p) = -(p - 3)(3p - 4)/p$ by Feigin and Frenkel in [19].

If we are only interested in the dimensionality of $L/\mathcal{W}_{<}L$ then we can choose the w_i all infinite and restrict attention to the simpler space L/\mathcal{W}_nL , where

$$\begin{aligned} \mathcal{W}_n &= \mathcal{W}_{<}(\infty \dots \infty) \\ &= \text{span}(L_{-2n-1}, \dots; Q_{-3n-1}, \dots) . \end{aligned} \tag{2.21}$$

Whereas for the w_i finite and distinct $L/\mathcal{W}_{<}L$ carries a representation of \mathbb{C}^{2n+2} generated by \mathcal{W}^0 , the space L/\mathcal{W}_nL carries a representation of \mathbb{C}^{2n+2} generated by

$$\text{span}(L_{-n}, \dots, L_{-2n}; Q_{-n}, \dots, Q_{-3n}) . \tag{2.22}$$

2.1. Three-Point Functions. Let us consider a three point function

$$\langle h_1, q_1 | \Phi_{h_2, q_2}(w) | \psi \rangle . \tag{2.23}$$

From the preceding discussion, we need only consider the space

$$L_{h,q,c} / \mathcal{W}_{<}(1)L_{h,q,c} , \tag{2.24}$$

and if we are interested only in the dimensionality of this space we can restrict attention to the somewhat simpler space

$$\tilde{L}_{h,q,c} = L_{h,q,c} / \mathcal{W}_1 L_{h,q,c} . \tag{2.25}$$

$\tilde{L}_{h,q,c}$ is a quotient of the space $\tilde{V}_{h,q,c}$,

$$\tilde{V}_{h,q,c} = V_{h,q,c} / \mathcal{W}_1 V_{h,q,c} , \tag{2.26}$$

where $\tilde{V}_{h,q,c}$ has a canonical basis

$$U(\mathcal{W}_{<})|h, q, c\rangle, \quad \mathcal{W}_{<} = \text{span}(L_{-1}, L_{-2}, Q_{-1}, Q_{-2}, Q_{-3}) . \tag{2.27}$$

If $M_{h,q,c}$, the maximal invariant submodule of $V_{h,q,c}$, is generated by a finite set of highest weight null states N_t , then

$$\tilde{L} = \tilde{V} / \tilde{M}, \quad \tilde{M} = U(\mathcal{W}_{<})\text{span}(N_s) . \tag{2.28}$$

As mentioned above, the modes $\{L_{-1}, L_{-2}, Q_{-2}, Q_{-3}\}$ act as an Abelian algebra on \tilde{L} , but we can also consider the whole of \mathcal{W}_- , which (acting on \tilde{L}) has relations

$$\begin{aligned} [Q_{-1}, L_{-1}] &= Q_{-2} & [Q_{-1}, L_{-2}] &= 3Q_{-3} , \\ [Q_{-1}, Q_{-2}] &\sim \frac{2}{3}L_{-2}L_{-1} & [Q_{-1}, Q_{-3}] &\sim \frac{2}{3}L_{-2}L_{-2} . \end{aligned} \quad (2.29)$$

We can represent this in terms of a differential polynomial ring, with generators

$$\begin{aligned} L_{-1} &= x, L_{-2} = y, Q_{-2} = z, Q_{-3} = u , \\ Q_{-1} = D &= z \frac{\partial}{\partial x} + 3u \frac{\partial}{\partial y} + \frac{2}{3}xy \frac{\partial}{\partial z} + \frac{2}{3}y^2 \frac{\partial}{\partial u} . \end{aligned} \quad (2.30)$$

D has a non-trivial kernel, containing, amongst other things, $-27u^2 + 4y^3$ and $-9uxz + x^2y^2 + 3yz^2$. This is particularly useful in the vacuum representation as we need only consider the action of D on the null highest-weight states to find all the restrictions on $\tilde{L}_{0,0,c}$.

2.2. Classes of WA_2 Representations. There is not yet a complete classification of WA_2 Verma module structures, and so we can present only some partial results. In ref. [8] we considered four classes of WA_2 algebra representations. Of these the classes 2c(i) and 1(c) of ref. [4] are of special interest, as ‘‘quasi-rational’’ and ‘‘quasi-finite’’ representations respectively.

We say an irreducible highest weight representation is quasi-finite if

$$\dim(L_{h,q,c}/\mathcal{W}_{<}(w_1, \dots, w_n)L_{h,q,c}) < \infty \quad (2.31)$$

for any set of non-zero points w_i . We have

Conjecture. *A representation $L_{h,q,c}$ is quasi-finite if and only if it is a minimal model representation. If it is quasi-finite, then for any set of distinct points $\{w_i\}$, $\mathcal{W}^0(\{w_i\})$ is diagonalisable on $L_{h,q,c}/\mathcal{W}_{<}(\{w_i\})L_{h,q,c}$.*

The term quasi-rational has been used by Nahm in [9] for those representations for which in Eq. (2.23) for given irreducible representation ρ_1 there are only a finite number of allowed representations ρ_2 . In the context of the WA_2 models, if we fix $\{h_1, q_1\}$ then the number of allowed values d of $\{h_2, q_2\}$ is given by

$$\begin{aligned} d &= \dim L_{h,q,c}/\langle e_0^{(\infty)}, f_0^{(\infty)}, \mathcal{W}_{<}(1) \rangle L_{h,q,c} \\ &= \dim L_{h,q,c}/\langle L_{-2}, L_{-3}, \dots; Q_{-3}, Q_{-4} \dots \rangle L_{h,q,c} . \end{aligned} \quad (2.32)$$

In Nahm’s terminology, the representation is quasi-rational if d is finite, and Eq. (2.32) is his requirement of quasi-rationality. In [8] we conjectured that the doubly degenerate representations are quasi-rational.

At the minimal values for c , the quasi-rational representations acquire a third independent null vector. This will reduce the allowed fusions of the form (2.23) to a finite set of pairs $\{h_1, q_1\}, \{h_2, q_2\}$. The restrictions this imposes on the representations which arise in the minimal models can be derived from the vacuum representation alone, as Feigin et al. described in [10]. The fusion rules of the other $L_{h,q,c}$ can be derived from the structure of $\tilde{L}_{h,q,c}$.

We now present two simple examples, the vacuum representation and the doubly-degenerate representation [11;12], for generic c -values, and then in the case of the minimal model $c = c(7,3) = -114/7$.

2.3. *The vacuum Representation.* The first representation in which we might be interested is the vacuum representation. This has weights $h = q = 0$ and has null vectors

$$L_{-1}|0\rangle, Q_{-1}|0\rangle, Q_{-2}|0\rangle$$

for all c values. As a result, $L_{0,0,c}$ must factor through the space with basis

$$e_{i_1} \dots e_{i_p} f_{j_1} \dots f_{j_q} L_{-2}^a Q_{-3}^b |h, q\rangle, \\ i_m \leq i_{m+1} \leq -1, j_m \leq j_{m+1} \leq -1. \quad (2.33)$$

If $w \neq \infty$, then we can define $e_0^{(\infty)}, e_0^{(w)}, f_0^{(\infty)}$ and $f_0^{(w)}$ and it is straightforward to see that $e_0^{(\infty)} = e_0^{(w)}$ and $f_0^{(\infty)} = f_0^{(w)}$ on the space $\tilde{L}_{0,0,c}$ so that the only possible fusions with the vacuum sector are of the form

$$\Phi_{h,q} \times \Phi_{h,q} \rightarrow L_{0,0,c},$$

so from Eq. (2.32) the vacuum representation is quasi-rational. For generic c values, $\tilde{L}_{0,0,c}$ is infinite dimensional, as we can see by looking at the restriction of the Shapovalov form (inner product matrix) to the space spanned by the states (2.33). The determinant of this form is non-zero, as the leading contributions come from the diagonal, and so $\tilde{L}_{0,0,c}$ is reducible for a countable set of c values only.

If there is another independent null vector in the vacuum sector, then there may be a restriction on the allowed values of h and q in a field theory. This is expected to be the case for the minimal models, for which we expect that the number of allowed representations is given by

$$N = \sum_{i,j} N_{ij}^0 = \dim L_{0,0,c} / \mathcal{W}_1 L_{0,0,c}. \quad (2.34)$$

We present a calculation of the space $L_{0,0,c} / \mathcal{W}_{<}(1) L_{0,0,c}$ for $c = -114/7$ in Sect. 3.

2.4. *The Representation [11;12].* This representation has h and q as given in (2.5, 2.8), with $\alpha = \alpha_+$,

$$h(11; 12) = \frac{4}{3\alpha^2} - 1, \quad q(11; 12) = \frac{(5 - 3\alpha^2)(4 - 3\alpha^2)}{27\alpha^3}.$$

There are the following null vectors in the Verma module $V_{h[11;12],q[11;12],c}$ for all c values

$$|N_1\rangle = \left(Q_{-1} + \left(\frac{\alpha}{2} - \frac{5}{6\alpha} \right) L_{-1} \right) |11; 12\rangle, \\ |N_2\rangle = \left(Q_{-2} + \frac{2}{3\alpha} L_{-2} - \alpha L_{-1}^2 \right) |11; 12\rangle, \\ |N_3\rangle = \left(Q_{-3} - \alpha^3 L_{-1}^3 + \left(\frac{1}{6\alpha} + \frac{\alpha}{2} \right) L_{-3} + \alpha L_{-2} L_{-1} \right) |11; 12\rangle. \quad (2.35)$$

The state $|N_3\rangle$ is a descendant of $|N_1\rangle$ and $|N_2\rangle$. As a result, $\tilde{L}_{h[11;12],q[11;12],c}$ factors through the space with basis

$$L_{-2}^m L_{-1}^n |11; 12\rangle.$$

For generic c values, there are no more identities as we can again see by looking at the $c \rightarrow \infty$ limit. On \tilde{L} we see $e_0^{(\infty)}$ and $e_0^{(1)}$ may be taken as independent, and from the null vectors (2.35) $f_0^{(\infty)}$ and $f_0^{(1)}$ are given as

$$\begin{aligned} f_0^{(\infty)} &= \frac{1}{27\alpha^3} (9\alpha^4 (e_0^{(\infty)})^2 - 18\alpha^4 e_0^{(\infty)} e_0^{(1)} + 9\alpha^4 (e_0^{(1)})^2 - 9\alpha^4 - 3\alpha^2 e_0^{(\infty)} \\ &\quad - 6\alpha^2 e_0^{(1)} + 18\alpha^2 - 8) \times (3\alpha^2 e_0^{(x)} - 3\alpha^2 e_0^{(1)} + 1) , \\ f_0^{(1)} &= \frac{1}{27\alpha^3} (9\alpha^4 (e_0^{(\infty)})^2 - 18\alpha^4 e_0^{(\infty)} e_0^{(1)} + 9\alpha^4 (e_0^{(1)})^2 - 9\alpha^4 \\ &\quad - 6\alpha^2 e_0^{(\infty)} - 3\alpha^2 e_0^{(1)} + 18\alpha^2 - 8) \times (3\alpha^2 e_0^{(x)} - 3\alpha^2 e_0^{(1)} + 1) . \end{aligned} \quad (2.36)$$

We find that \tilde{L} is equivalent to $\mathbf{C}[e_0^{(1)}, e_0^{(\infty)}, f_0^{(1)}, f_0^{(\infty)}]$ modulo the relations (2.36), so that the representation [11;12] is quasi-rational. We shall again consider this representation and the restrictions which arise from the extra null vector in the minimal model $c = -114/7$ in Sect. 3.

3. The Model $c = -114/7$

The minimal models of the WA_2 algebra are parameterised by coprime integers p, q greater than 2. There are at least two series of special interest, $(p, q) = (m, m + 1)$ and $(p, q) = (3, q)$. The first is the unitary series, and the second is a non-unitary series. For the Virasoro algebra the corresponding non-unitary series of models $c = c_{Vir}(2, q)$ lead to relations with Gordon identities [10] and the fusion rings and representations have special properties. There is every reason to believe that the $(3, q)$ series of the WA_2 algebra will also have interesting properties. Here we shall limit ourselves to the model $(3, 7)$ which has 5 representations and central charge $c = -114/7$. We choose $\alpha = \sqrt{7/3}$, in which case the model's representations are [11; ab] with $1 \leq a, 1 \leq b, a + b \leq 6$ with each representation occurring three times in this list. We shall focus in particular on the vacuum representation [11;11] and the [11;12] representation, and calculate the spaces $L_{h,q,c}/\mathcal{W}_1 L_{h,q,c}$ and $L_{h,q,c}/\mathcal{W}_{<(1)} L_{h,q,c}$ for these two representations.

As a point of notation, we shall \equiv to denote equivalence in the irreducible representation L and \sim to denote equivalence in the space $L/\mathcal{W}_1 L$.

3.1. The Vacuum Representation. The vacuum representation has three equivalent parameterisations,

$$[11; 11], [11; 15], [11; 51] ,$$

which implies that there is a null vector at level 5, which is not a descendant of those at level 1. From Sect. 2.3 we need only consider as a basis of $\tilde{L}_{0,0,c}$ the states

$$L_{-2}^a Q_{-3}^b |0\rangle . \quad (3.1)$$

We can find the explicit form of the additional null state at level 5 from ref. [11]. After reduction modulo the generic vacuum sector relations, we find the expression

$$(7L_{-2}Q_{-3} - 3Q_{-5})|0\rangle \equiv 0 , \quad (3.2)$$

which tells us that there is another relation in $L/\mathcal{W}_1 L$,

$$L_{-2}Q_{-3}|0\rangle \sim 0 . \quad (3.3)$$

We can derive extra information from considering the repeated action of Q_{-1} on the state (3.2), or alternatively from the action of D on (3.3), from which we deduce that

$$(2L_{-2}^3 + 9Q_{-3}^2)|0\rangle \sim 0, L_{-2}^4|0\rangle \sim 0, \quad (3.4)$$

which leaves us with a 5 dimensional basis of $L_{0,0,-114/7}/\mathcal{W}_{2,3}L_{0,0,-114/7}$, viz.

$$|0\rangle, L_{-2}|0\rangle, Q_{-3}|0\rangle, L_{-2}L_{-2}|0\rangle, L_{-2}L_{-2}L_{-2}|0\rangle,$$

from which we deduce that there are 5 fields in this theory. There can be no more restrictions on $\tilde{L}_{0,0,-114/7}$ as null states at higher levels clearly cannot reduce the dimension further.

We can consider the extra relations in $L/\mathcal{W}_{<L}$ from the null vectors at levels 5 and 6, which are

$$f_0(7e_0 + 3)|0\rangle \sim 0, (2e_0(7e_0 + 4)(7e_0 + 5) + 441f_0^2)|0\rangle \sim 0.$$

There are five solutions to these equations as expected,

$$(e_0, f_0) = (0, 0), \left(-\frac{4}{7}, 0\right), \left(-\frac{5}{7}, 0\right), \left(-\frac{3}{7}, \pm \frac{2}{\sqrt{21}}\right). \quad (3.5)$$

We can also find a basis of $L_{0,0,-114/7}/\mathcal{W}_{<(1)}L_{0,0,-114/7}$ and diagonalise the algebra \mathcal{W}^0 on this space. We find that, as expected,

$$L/\mathcal{W}_{<L} = \bigoplus_{i=1}^5 \mathbb{C}v_i, \quad (3.6)$$

where

$$e_0^{(1)}v_a = h_a v_a, f_0^{(1)}v_a = q_a v_a.$$

Explicitly we can find expressions for representatives of the v_a ,

$$\begin{aligned} v_1 &= (e_0 + 3/7)(e_0 + 4/7)(e_0 + 5/7)|0\rangle, \\ v_2 &= e_0(e_0 + 3/7)(e_0 + 5/7)|0\rangle, \\ v_3 &= e_0(e_0 + 3/7)(e_0 + 4/7)|0\rangle, \\ v_4 &= f_0(f_0 + 2/(7\sqrt{21}))|0\rangle, \\ v_5 &= f_0(f_0 - 2/(7\sqrt{21}))|0\rangle, \end{aligned} \quad (3.7)$$

for which we find the eigenvalues,

	v_1	v_2	v_3	v_4	v_5
$e_0^{(1)}$	0	$-\frac{4}{7}$	$-\frac{5}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$
$f_0^{(1)}$	0	0	0	$\frac{2}{7\sqrt{21}}$	$-\frac{2}{7\sqrt{21}}$

(3.8)

which are exactly the allowed representations in the $c = -114/7$ minimal model. This is very clearly related to the procedure of Feigin et al. in [10], and we obtain the same equations from the null vectors at levels 5 and 6 as they obtain in ref. [10] Eq. (4.10). We may also consider other representations and obtain their fusion rules in an analogous fashion.

3.2. The [11;12] Representation. The [11;12] representation can also be parameterised as the [11;41] and [11;24] representations and there are independent null

vectors at levels 1, 2, and 4. The highest weight state has L_0 and Q_0 eigenvalues $-3/7$ and $\omega = 2/(7\sqrt{21})$ respectively. In L/\mathcal{W}_1L the null vectors at levels 1, 2 and 3 from Eq. (2.35) which are generic to representations of type [11;12] imply

$$\begin{aligned} Q_{-1}|11; 12\rangle &\sim -\frac{1}{\sqrt{21}}L_{-1}|11; 12\rangle, \quad Q_{-2}|11; 12\rangle \sim -\frac{1}{\sqrt{21}}(7L_{-1}^2 - 2L_{-2})|11; 12\rangle, \\ Q_{-3}|11; 12\rangle &\sim \sqrt{\frac{7}{3}}\left(\frac{7}{3}L_{-1}^3 - L_{-2}L_{-1}\right)|11; 12\rangle, \end{aligned} \quad (3.9)$$

which leaves us with a possible basis of $L/\mathcal{W}_{2,3}L$ of the form

$$L_{-2}^a L_{-1}^b |11; 12\rangle.$$

We can now use the independent null vector at level 4,

$$\left(L_{-2}^2 - \frac{49}{6}L_{-1}^4\right)|11; 12\rangle \sim 0$$

and the repeated action of Q_{-1} on this state to obtain

$$\left(L_{-2}L_{-1}^3 - \frac{3}{7}L_{-1}^5\right)|11; 12\rangle \sim 0, \quad L_{-1}^7|11; 12\rangle \sim 0$$

and reduce the possible basis states to

$$\begin{aligned} &|11; 12\rangle, \quad L_{-2}|11; 12\rangle, \quad L_{-2}L_{-1}|11; 12\rangle, \quad L_{-2}L_{-1}^2|11; 12\rangle, \\ &L_{-1}|11; 12\rangle, \quad L_{-1}^2|11; 12\rangle, \quad L_{-1}^3|11; 12\rangle, \quad L_{-1}^4|11; 12\rangle, \quad L_{-1}^5|11; 12\rangle, \quad L_{-1}^6|11; 12\rangle. \end{aligned} \quad (3.10)$$

We can now try to find ‘‘fusion basis,’’ that is a basis of $L/\mathcal{W}_{<}(1)L$. From the null vectors at levels 1 to 7 we obtain equations which lead to a total of 10 solutions for the fusion,

$$L_{-3/7, \omega, -114/7} \mathcal{W}_{<}(1) L_{-3/7, \omega, -114/7} = \mathbb{C}^{10},$$

with the following eigenvalues of \mathcal{W}^0 on the basis states:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
$e_0^{(\infty)}$	$-3/7$	$-3/7$	$-3/7$	$-3/7$	$-4/7$	$-4/7$	$-5/7$	$-5/7$	$-5/7$	0
$f_0^{(\infty)}$	$-\omega$	$-\omega$	ω	ω	0	0	0	0	0	0
$e_0^{(1)}$	$-3/7$	$-4/7$	$-5/7$	0	$-3/7$	$-5/7$	$-3/7$	$-4/7$	$-5/7$	$-3/7$
$f_0^{(1)}$	ω	0	0	0	ω	0	$-\omega$	0	0	$-\omega$

and so in this case we find that

$$\dim L_{-3/7, \omega, -114/7} / \mathcal{W}_1 L_{-3/7, \omega, -114/7} = \dim L_{-3/7, \omega, -114/7} / \mathcal{W}_{<}(1) L_{-3/7, \omega, -114/7}.$$

4. Conclusions

Building on our work in [8], we have outlined an algebraic way to extend the work of Feigin and Fuchs in ref. [2] to the WA_2 algebra. Clearly this will extend to all the

algebras Wg_n , and probably all the algebras which can be obtained by generalised Drinfel'd-Sokolov construction.

How could one prove the conjectures we have made here? The proofs in Feigin and Fuchs relied on their calculation of the embedding structure of Verma modules of the Virasoro algebra in [12]. The corresponding calculation has not yet been performed for the WA_2 algebra representations, and there are new problems such as the presence of subsingular vectors, and the fact the action of Q_0 is on many occasions not diagonalisable on doubly-degenerate Verma module representations. The consideration of these problems is work in progress.

Interesting developments which might help are the work on the structure of finite W algebra modules by de Vos and van Driel [13] and Bajnok's construction of null vectors of the WA_2 algebra using complex powers of generators [14]. Certainly there are some interesting results for the $c = c(3, p)$ models [15].

This method is most suitable for the study of the minimal models, but for the study of quasi-rational models as suggested by Nahm in [9], it is necessary to consider some more general ideas, and attempt to construct some form of tensor product of WA_2 representations, as proposed by Gaberdiel in [16].

The fusion rules for the representations of W -algebras obtained by quantum Hamiltonian reduction were obtained in [17] from the modular properties of the characters obtained on the basis of conjectured resolutions by Wakimoto modules. It would be nice if one could obtain a direct connection with this work.

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Note added in proof. The construction in Sect. 2.1 is the same as that of Zhu's algebra $A(V)$ in [18], with $V = L_{0,0,c}$, $A(V) = L_{0,0,c}/\mathcal{W}_{<}(-1)L_{0,0,c}$, and with the multiplications $L_* = e^{(\infty)}$ and $Q_* = q^{(\infty)}$.

References

1. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Nucl. Phys. **B241**, 333 (1984)
2. Feigin, B.L., Fuchs, D.B.: J. Geom. Phys. **5**, 209 (1988)
3. Zamolodchikov, A.B.: Theor. Math. Phys. **65**, 347 (1985)
4. Watts, G.M.T.: Nucl. Phys. **B326**, 648 (1989), erratum Nucl. Phys. **B336**, 720 (1989)
5. Fateev, V.A., Zamolodchikov, A.B.: Nucl. Phys. **B280** [FS18], 644 (1987)
6. Fateev, V.A., Luk'yanov, S.L.: Int. J. Mod. Phys. **A3**, 507 (1988), Sov. Sci. Rev. **A15**, 1 (1990)
7. Bais, F.A., Bouwknegt, P., Schoutens, K., Surridge, M.: Nucl. Phys. **B304**, 348 (1988); Nucl. Phys. **B304**, 37 (1988)
8. Bowcock, P., Watts, G.M.T.: Theor. Math. Phys. **98**, 350 (1994)
9. Nahm, W.: Quasi-rational fusion products. Bonn University Preprint hep-th/9402039
10. Feigin, B.L., Nakanishi, T., Ooguri, H.: Int. J. Mod. Phys. **A7** (S1A), 217 (1992)
11. Bowcock, P., Watts, G.M.T.: Phys. Lett. **297B**, 282 (1992)
12. Feigin, B.L., Fuchs, D.B.: Representations of the Virasoro algebra. Repts. Dept. Math. Stockholm Univ. (1986), published in 'Representations of infinite-dimensional Lie groups and Lie algebras', eds. Vershik, A., Zhelobenko, D., London: Gordon and Breach, 1989

13. de Vos, K., van Driel, P.: The Kazhdan-Lusztig conjecture for finite W-algebras. DAMTP Preprint DAMTP-93-66, UCLA/93/TEP/46, hep-th/9312016
14. Bajnok, Z.: Phys. Lett. **329B**, 225 (1994)
15. Frenkel, E.V.: Private communication
16. Gaberdiel, M.: Nucl. Phys. **B417**, 130 (1994)
17. Frenkel, E.V., Kac, V., Wakimoto, M.: Commun. Math. Phys. **147**, 295 (1992)
18. Zhu, Y.: Vertex operator algebras, elliptic functions and modular forms, Ph.D. dissertation, Yale Univ., (1990)
19. Feigin, F., Frenkel, E.V.: Adv. Sov. Math. **16**, 139–148 (1993)

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