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# A Note on the Index Bundle over the Moduli Space of Monopoles

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**Abstract:** Donaldson has shown that the moduli space of monopoles $M_k$ is diffeomorphic to the space $Rat_k$ of based rational maps from the two-sphere to itself. We use this diffeomorphism to give an explicit description of the bundle on $Rat_k$ obtained by pushing out the index bundle from $M_k$. This gives an alternative and more explicit proof of some earlier results of Cohen and Jones.

## 1. Introduction

In [4] Cohen and Jones study the topological type of the index bundle of various families of Dirac operators arising in the theory of monopoles and the relation between these index bundles and representations of the braid groups. The methods used in this general study were those of algebraic topology and index theory. For example, it was shown that using Donaldson's diffeomorphism between monopoles and based rational maps [5] and the relation between the space of based rational maps and the braid group that the $K$-theory class of the index bundle over the space of monopoles is completely determined by representations of the braid groups. In this note we show how Donaldson's diffeomorphism gives rise to a simple explicit characterisation of the bundle over the space of rational maps corresponding to this index bundle. The corresponding representation of the braid group is readily identified.

In more detail, let $M_k$ denote the moduli space of framed monopoles of charge $k$ over $\mathbf{R}^3$ with structure group $SU(2)$. Donaldson in [5] defines an explicit diffeomorphism

$$M_k \to Rat_k \tag{1.1}$$

from $M_k$ to the space of all based rational maps of degree $k$ from the two-sphere to itself. It can be shown [10] that if $(A, \Phi)$ is a monopole, then the space of $L^2$ solutions of the Dirac equation coupled to $(A, \Phi)$ has dimension $k$, where $k$ is the charge of the monopole. This defines a complex vector bundle over $M_k$ which, in fact, has a hermitian inner product and a real structure and hence has structure group $O(k)$, the group of real, orthogonal, $k$ by $k$ matrices. We denote by $Ind_k$, the corresponding principal $O(k)$ bundle on $M_k$. This is the index bundle and the point of this note is to show that we can explicitly describe the bundle over $Rat_k$ which is its "push-out" under the diffeomorphism (1.1). In fact this bundle arises quite naturally in Donaldson's work. We show further, that over the space $Rat_k^0$ of rational maps with distinct poles, this bundle has a reduction to the finite subgroup of $O(k)$ of signed permutations. This bundle therefore corresponds to a homomorphism of $\pi_1(Rat_k^0)$ into the group of signed permutations. This homomorphism can be readily calculated and it is, not surprisingly, the same as that in [4].

## 2. Monopoles and the Index Bundle

The index bundle $Ind_k$ is defined over $M_k$ the moduli space of framed monopoles of charge $k$. To define this moduli space consider first pairs $(A, \Phi)$ consisting of an $SU(2)$ connection $A$ on $\mathbf{R}^3$ and an $su(2)$ "Higgs field" $\Phi : \mathbf{R}^3 \to su(2)$, where $su(2)$ is the Lie algebra of $SU(2)$. The Bogomolny equations for such a pair are

$$\star F_A = \nabla_A \Phi \ , \tag{2.1}$$

where $F_A$ is the curvature of $A, \nabla_A \Phi$ the covariant derivative of $\Phi$ with respect to $A$ and $\star$ is the Hodge dual on forms. To be a monopole $(A, \Phi)$ has to satisfy the Bogomolny equations and also certain boundary conditions that we do not need to detail here. We refer to the book [2] as a good general reference for these and other details. However we do need to note that one of the boundary conditions is that the Higgs field gives rise to a map $\Phi^\infty$ from the two-sphere at infinity in $\mathbf{R}^3$ and that this takes values on the two-sphere inside $su(2)$. The degree of this map is called the magnetic charge of the monopole and we shall denote it by $k$. To be a framed monopole we require that

$$\lim_{t \to 0} \Phi(0,0,t) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ . \tag{2.2}$$

The group of gauge transformations, that is smooth maps $g : \mathbf{R}^3 \to SU(2)$, preserving the boundary conditions, acts on pairs $(A, \Phi)$ to give new pairs $(A^g, \Phi^g)$ defined by

$$A^g = g^{-1}Ag + g^{-1}dg \quad \text{and} \quad \Phi^g = g^{-1}\Phi g \ . \tag{2.3}$$

These gauge transformations preserve the set of solutions of the Bogomolny equations (2.1). They also preserve the framing condition (2.2) if the limit of $g(0,0,t)$ as $t$ goes to infinity is diagonal. A framed gauge transformation is defined to be one such that

$$\lim_{t \to 0} g(0,0,t) = 1 \ . \tag{2.4}$$

The group of framed gauge transformations acts freely on the set of framed monopoles and the quotient is $M_k$ the moduli space of framed monopoles. It is a manifold of dimension $4k$ [11].

Given a pair $(A, \Phi)$ we can form the coupled Dirac operator $D_0$ acting on sections of the trivial spinor bundle over $\mathbf{R}^3$ with fibre $\mathbf{C}^2 \otimes \mathbf{C}^2$ :

$$D_0 = \sum_{i=1}^{3} \sigma_i \otimes \nabla_{A_i} - 1 \otimes \Phi \ , \tag{2.5}$$

where the $\sigma_i$ are the matrices defining the action of the Clifford algebra of $\mathbf{R}^3$ on $\mathbf{C}^2$. It is known [10] that the space of $L^2$-solutions of the equation $D_0\psi = 0$, satisfying the given boundary conditions, has dimension $k$, where $k$ is the charge of the monopole. The group of framed gauge transformations acts on the spinor bundle and on the Dirac operator by conjugation hence quotienting gives rise to a vector bundle over $M_k$.

The Dirac operator $D_0$ acts on sections of a trivial bundle with fibre $\mathbf{C}^2 \otimes \mathbf{C}^2$, where one factor is acted on by the group $\mathrm{Spin}(3) = SU(2)$ and the other is acted on by the $SU(2)$ from the monopole bundle. In both cases the structure group is $SU(2) = Sp(1)$ and hence the individual bundles have quaternionic structures. Therefore there is a real structure on the tensor product, that is a conjugate linear map $r$ such that $r^2 = 1$. The space $L^2(\mathbf{R}^3, \mathbf{C}^2 \otimes \mathbf{C}^2)$ has a hermitian inner product defined by integrating the hermitian inner product on $\mathbf{C}^2 \otimes \mathbf{C}^2$ and this restricts to a hermitian inner product on the kernel of $D_0$. The real structure map $r$ preserves this natural hermitian inner product $\langle\, , \rangle$ and therefore defines an orthogonal form by $(v, w) = \langle v, r(w)\rangle$. It makes sense therefore to consider the space of all real, orthonormal frames for the kernel of $D_0$. Here real means fixed by the real structure. This space is acted on freely by $O(k)$ the group of real, orthogonal, $k$ by $k$ matrices. Hence we have constructed a principal $O(k)$ bundle over $M_k$ which we shall call the index bundle and denote by $Ind_k$.

## 3. The ADHMN Construction

The ADHM construction for instantons [1] as generalised by Nahm to monopoles [10] associates to every pair $(A, \Phi)$ satisfying the Bogomolny equations and the appropriate monopole boundary conditions a rank $k$ bundle $N$ over the interval $(-1, 1) \subset \mathbf{R}$ equipped with a connection $\nabla$ and three bundle endomorphisms $T_i$. If we trivialise the bundle with covariantly constant sections then the $T_i$ become matrices satisfying Nahm's equations:

$$\frac{dT_1}{dz} = [T_2, T_3] \ ,$$

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$$\frac{dT_3}{dz} = [T_1, T_2] \tag{3.1}$$

and some boundary conditions. Again it is not important precisely what the boundary conditions are and we refer the reader to [2] for details. We do need to note however that the $T_i$ are analytic and have simple poles at $\pm 1$. Let $t_i$ denote the residues of the $T_i$ at $-1$. It follows from Nahm's equations (3.1) that the residues are a representation of $su(2)$. It is one of the boundary conditions that they must in fact be irreducible, and hence $-it_3$ has eigenvalues $-(k-1), \ldots, (k-1)$. Although it was not explicit in Nahm's work one can follow through the constructions in [7] to see that the framing of the monopole means that we also have given a unit vector $v$ in the $(k-1)$ eigenspace of $-iT_3$.

The connection with the index bundle follows from the fact that fibre of the bundle $N$ over the point $t \in (-1, 1)$ is the $L^2$ kernel of the coupled Dirac operator

$$D_t = \sum_{i=1}^{3} \sigma_i \otimes \nabla_{A_i} - 1 \otimes (\Phi + it) \ . \tag{3.2}$$

The hermitian and real structures of the $\mathbf{C}^2 \otimes \mathbf{C}^2$ bundle over $\mathbf{R}^3$ pass to the bundle $N$ as follows. Firstly the vector space $N_t$ is a subspace of the hermitian inner product space $L^2(\mathbf{R}^3, \mathbf{C}^2 \otimes \mathbf{C}^2)$ so it inherits a hermitian inner product by restriction. Secondly the real structure is conjugate linear so it maps an element of the kernel of $D_t$ to an element in the kernel of $D_{-t}$. Hence it defines a conjugate linear map from $N_t$ to $N_{-t}$. There is an orthogonal projection

$$\pi_t : L^2(\mathbf{R}^3, \mathbf{C}^2 \otimes \mathbf{C}^2) \to N_t \ , \tag{3.3}$$

and this is used to define the connection and the endomorphisms $T_i$ by

$$\nabla(\chi) = \pi\left(\frac{d\chi}{dt}\right) \quad \text{and} \quad T_i(\chi) = \pi\left(x^i\chi\right) \ . \tag{3.4}$$

We can identify $N_0$ with $\mathbf{C}^k$ by choosing an orthonormal frame consisting of real vectors and using parallel transport to extend this to a frame at every point of $(-1,1)$ and hence make the $T_i$ into matrices. So to every triple $(A, \Phi, \{\psi_a\})$, where $(A, \Phi)$ is a monopole and $\{\psi_a\}$ is an orthonormal basis of real vectors in the space $N_0$ of $L^2$ solutions of the Dirac equation coupled to $(\nabla, \Phi)$, we have associated three matrix valued functions $T_i$ on the interval $(-1,1)$ satisfying Nahm's equations (3.1) and a $v$ as above. From the definition (3.4) we see that if we gauge transform the triple $(A, \Phi, \{\psi_a\})$, then the $T_i$ are left unchanged. The same is also true of $v$ because we are using framed gauge transformations. If the basis $\{\psi_a\}$ is multiplied by an element of $O(k)$ then the $T_i$ are conjugated by this same element and the $v$ is multiplied by it. So the image of the index bundle after pushing out with the ADHMN construction is all $(T_1, T_2, T_3, v)$, where the $T_i$ are a solution to Nahm's equations with appropriate boundary conditions and the $v$ is defined as above. The action of $O(k)$ on this principal bundle is conjugation of the $T_i$ and multiplication of the $v$.

## 4. The Index Bundle over $Rat_k$

Donaldson shows that the space $N_k$, and hence $M_k$, is diffeomorphic to the space $Rat_k$ of based rational maps from the two sphere to the two sphere of degree $k$. The diffeomorphism is defined as follows. Let us first combine Nahm's matrices as

$$A_0 = T_1 + iT_2, \quad A_1 = -iT_3, \quad \text{and} \quad A_2 = T_1 - iT_2 \ . \tag{4.1}$$

Then consider solutions $u : (-1,1) \to \mathbf{C}^k$ of the differential equation

$$\frac{du}{ds} - \frac{1}{2}A_1 u = 0 \ . \tag{4.2}$$

There is a unique solution $u$ with the property that

$$\lim_{s \to -1} s^{-(k-1)/2} u(s) = v \ . \tag{4.3}$$

Define $B = -A_0(1)$ and $W = u(1)$. Then Donaldson shows that $B$ is a symmetric matrix and that $W$ is a cyclic vector for $B$; that is $\{W, BW, \ldots, B^{k-1}W\}$ are linearly independent. He also shows that the space $\widehat{Rat}_k$ of all such pairs $(B, W)$ with $B$ symmetric and $W$ cyclic for $B$ is a principal $O(k)$ bundle over $Rat_k$. The projection $\widehat{Rat}_k \to Rat_k$ is given by

$$(B, W) \mapsto f(z) = W^t(zI - B)^{-1}W \ , \tag{4.4}$$

and the $O(k)$ action is given by conjugation on $B$ and left multiplication on $W$. The map $(T_1, T_2, T_3, p) \mapsto (B, W)$ is equivariant with respect to the $O(k)$ actions and Donaldson's result shows that it descends to a diffeomorphism from $N_k$ to $Rat_k$. It follows that the principal $O(k)$ bundle $\widehat{Rat}_k \to Rat_k$ is the pushout of the index bundle under Donaldson's diffeomorphism from $M_k$ to $Rat_k$.

## 5. The Reduction over $Rat_k^0$

The space $Rat_k^0$ is the space of based rational maps with distinct poles. Consider a diagonal matrix $B$ with distinct entries $(b_1, \ldots, b_k)$ and a vector $W = (W_1, \ldots, W_k)$ with non-zero components. It follows from the Vandermonde determinant that $W$ is cyclic for $B$. Denote by $\widehat{Rat}_k^0$ the set of all such $(B, W)$. The rational map defined by the pair $(B, W)$ using the projection $\widehat{Rat}_k \to Rat_k$, is

$$f(z) = \sum_{i=1}^{k} \frac{W_i^2}{z - b_i} \tag{5.1}$$

so that $\widehat{Rat}_0$ covers the space $Rat_k^0$ of all rational maps with distinct poles. Moreover, this set is stable under the action of the group $\sum_k^{\pm}$ of all signed permutations matrices, that is the subgroup of $O(k)$ generated by the diagonal matrices with plus or minus one on the diagonal and the permutation matrices. Indeed $\widehat{Rat}_k^0$ is a principal $\sum_k^{\pm}$ bundle over $Rat_k^0$ and it defines a reduction of the restriction of the bundle $\widehat{Rat}_k$ to $Rat_k^0$ to a $\sum_k^{\pm}$ bundle. It follows that over $Rat_k^0$ the bundle $Ind_k$ has a reduction to $\sum_k^{\pm}$ and therefore it defines a homomorphism of $\pi_1(Rat_k^0)$ to $\sum_k^{\pm}$. In [3] it is shown that $\pi_1(Rat_k^0)$ is the semi-direct product of the braid group on $k$ strings $\beta_k$ and $\mathbf{Z}^k$, where $\beta_k$ acts on $\mathbf{Z}^k$ via the natural homomorphism to $\sum_k$. This group maps naturally onto $\sum_k^{\pm}$ which is the semi-direct product of the symmetric group $\sum_k$ and $(\mathbf{Z}/2)^k$. By considering generators of $\pi_1(Rat_k^0)$ it is possible to show that the reduction of the bundle above corresponds exactly to this homomorphism; for the details see [3]. The topological implications of this fact are given in [4].

## 6. The Dirac Equation

It is interesting to consider what this reduction means for the solutions of the Dirac equation. The reduction of the principal bundle $Ind_k$ to $\sum_k^{\pm}$ corresponds to extra structure on the vector bundle of solutions of the Dirac equation. To see what this is note that an orthonormal basis of solutions of the Dirac equation would correspond to a reduction to the identity subgroup. This is more than we actually have. However if we choose not a basis of vectors but an (unordered) set of $k$ orthogonal lines that span the space then it is easy to see that this gives rise to an orthonormal basis up to the action of $\sum_k^{\pm}$.

So the reduction we have constructed corresponds to being able to find a set of $k$ orthogonal lines in the space of all solutions of the Dirac equation. One way of doing this has already been noted in [9]. This uses the fact that for widely separated monopoles there are solutions of the Dirac equation concentrated about each of the monopole locations. The reduction that we have given appears to be different to this. It can be understood by reference to Hitchin's twistor construction of solutions of Nahm's equations [7]. We shall sketch here, without proof how this occurs, and refer the reader to [7] and [8] for details. Recall that Hitchin showed in [6] that a monopole is determined by a certain algebraic curve $S$ in $TS^2$ the tangent bundle of the two sphere. This is the so-called spectral curve. The role played by $TS^2$ is that it parametrises the set of all oriented lines (not necessarily through the origin) in $\mathbf{R}^3$. The points of the two sphere correspond to the direction of the line and the fibres of the projection $TS^2 \to S^2$ correspond to the families of parallel lines. Denote by $F$ the fibre of all lines in the $x^3$ direction. In subsequent work [7] Hitchin showed that the space of solutions of the Dirac equation, $N_z$, can be identified with the space

$$H^0(S, L(k-1)) \tag{6.1}$$

of holomorphic sections of a certain line bundle over $S$. Which line bundle is not important for this discussion and we refer to Hitchin's papers [6] and [7] for the details. Hitchin also proves that the restriction map of a section to the intersection of the fibre $F$ with the curve $S$ is an isomorphism. Generically this intersection is $k$ distinct points. In fact these $k$ points are the poles of the rational map [8]. In such a case we can define a line in $N_z$ by considering those sections that vanish on restriction to all but one of the points. By changing the point we generate $k$ lines and these span the space $N_z$. It was shown by Hurtubise in [8] that these lines are orthogonal and that if they are chosen as an orthogonal basis then they determine a $B$ that is diagonal and a $W$ with non-zero components. It follows that the reduction we have described in terms of the space of rational maps corresponds to sections whose restriction to the fibre $F$ are supported at just one point. It would be interesting to understand what this means for solutions of the Dirac equation in $\mathbf{R}^3$. This would mean unravelling the twistor correspondence in more detail.

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## References

1. Atiyah, M.F., Drinfeld, V.G., Hitchin, N.J., Manin, Yu.I.: Construction of instantons. Phys. Letts. **65A**, 185–187 (1978)
2. Atiyah, M.F., Hitchin, N.J.: The Geometry and Dynamics of Magnetic Monopoles. Princeton, NJ: Princeton University Press, 1988
3. Cohen, R.L., Jones, J.D.S.: Representations of braid groups and operators coupled to monopoles. In: Geometry of low-dimensional manifolds: 1, Donaldson S.K. and Thomas C.B. editors, London Mathematical Society Lecture Notes **150**, 191–205 (1990)

4. Cohen, R.L., Jones, J.D.S.: Representations of the braid group and operators coupled to monopoles. Commun. Math. Phys. **158**, 241–266 (1993)
5. Donaldson, S.K.: Nahm's equations and the classification of monopoles. Commun. Math. Phys. **96**, 387–407 (1984)
6. Hitchin, N.J.: Monopoles and geodesics. Commun. Math. Phys. **83**, 579–602 (1982)
7. Hitchin, N.J.: On the construction of monopoles. Commun. Math. Phys. **89**, 145–190 (1983)
8. Hurtubise, J.C.: Monopoles and rational maps: A note on a theorem of Donaldson. Commun. Math. Phys. **100**, 191–196 (1985)
9. Manton, N.S., Schroers, B.J.: Bundles over moduli spaces and the quantisation of BPS monopoles. Ann. Phys. **225**, 290–338 (1993)
10. Nahm, W.: The construction of all self-dual monopoles by the ADHM method. In: Monopoles in Quantum Field Theory, Craigie, N., et al. editors, Singapore: World Scientific, 1982
11. Taubes, C.H.: Monopoles and Maps from $S^2$ to $S^2$; the topology of the Configuration Space. Commun. Math. Phys. **95**, 345–391 (1984)


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