Commun. Math. Phys. 170, 117–150 (1995)

Communications in
Mathematical
Physics

Non-Local Matrix Generalizations of W-Algebras

Adel Bilal★
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received: 6 April 1994

Abstract: There is a standard way to define two symplectic (hamiltonian) structures, the first and second Gelfand–Dikii brackets, on the space of ordinary m^{th}-order linear differential operators $L = -d^m + U_1 d^{m-1} + U_2 d^{m-2} + \cdots + U_m$. In this paper, I consider in detail the case where the U_k are $n \times n$-matrix-valued functions, with particular emphasis on the (more interesting) second Gelfand–Dikii bracket. Of particular interest is the reduction to the symplectic submanifold $U_1 = 0$. This reduction gives rise to matrix generalizations of (the classical version of) the *non-linear* W_m-algebras, called $V_{n,m}$-algebras. The non-commutativity of the matrices leads to *non-local* terms in these $V_{n,m}$-algebras. I show that these algebras contain a conformal Virasoro subalgebra and that combinations W_k of the U_k can be formed that are $n \times n$-matrices of conformally primary fields of spin k, in analogy with the scalar case $n = 1$. In general however, the $V_{m,n}$-algebras have a much richer structure than the W_m-algebras as can be seen on the examples of the *non-linear* and *non-local* Poisson brackets $\{(U_2)_{ab}(\sigma), (U_2)_{cd}(\sigma')\}$, $\{(U_2)_{ab}(\sigma), (W_3)_{cd}(\sigma')\}$ and $\{(W_3)_{ab}(\sigma), (W_3)_{cd}(\sigma')\}$ which I work out explicitly for all m and n. A matrix Miura transformation is derived, mapping these complicated (second Gelfand–Dikii) brackets of the U_k to a set of much simpler Poisson brackets, providing the analogue of the free-field representation of the W_m-algebras.

1. Introduction

Since their discovery by Zamolodchikov [1], W-algebras have been an active field of investigation in theoretical and mathematical physics (see refs. [2, 3] for reviews). They are extensions of the conformal Virasoro algebra by higher spin fields W_k. The commutator of two such higher spin fields is a *local* expression involving *non-linear* differential polynomials of the W_l. W-algebras were found to arise naturally in the context of the 1 + 1-dimensional Toda field theories [4] where the higher spin fields

---

★ On leave of absence from Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France (unité propre du CNRS). e-mail: bilal@puhep 1.princeton.edu

appeared as coefficients of a linear differential operator $L = -\partial^m + \sum_{k=2}^m u_k \partial^{m-k}$ that annihilates the first Toda field $e^{-\phi_1}$. At a more formal level, the classical (i.e. Poisson bracket) version of W-algebras were shown [5, 6, 7] to be given by the second Gelfand–Dikii bracket associated with the linear differential operator L. This also implied close connections with the generalized KdV hierarchies [8].

Recently, in the study of the simplest 1 + 1-dimensional *non-abelian* Toda field theory [9], a related but more general structure was discovered. This non-abelian Toda theory has 3 conserved left-moving currents T, V^+, V^- and 3 conserved right-moving ones $\bar{T}, \bar{V}^+, \bar{V}^-$. The study of ref. 9 was purely at the classical level (Poisson brackets), and it was found that T, V^+ and V^- form a *non-linear* and *non-local* Poisson bracket algebra[1]:

$$\begin{aligned}
\gamma^{-2}\{T(\sigma), T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\big[T(\sigma')\delta(\sigma - \sigma')\big] - \frac{1}{2}\delta'''(\sigma - \sigma')\,, \\
\gamma^{-2}\{T(\sigma), V^\pm(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})\big[V^\pm(\sigma')\delta(\sigma - \sigma')\big]\,, \\
\gamma^{-2}\{V^\pm(\sigma), V^\pm(\sigma')\} &= \varepsilon(\sigma - \sigma')V^\pm(\sigma)V^\pm(\sigma')\,, \\
\gamma^{-2}\{V^\pm(\sigma), V^\mp(\sigma')\} &= -\varepsilon(\sigma - \sigma')V^\pm(\sigma)V^\mp(\sigma') \\
&\quad + (\partial_\sigma - \partial_{\sigma'})\big[T(\sigma')\delta(\sigma - \sigma')\big] - \frac{1}{2}\delta'''(\sigma - \sigma')\,. \quad (1.1)
\end{aligned}$$

One sees that T generates the conformal algebra. More precisely, if σ takes values on the unit circle S^1, then $L_r = \gamma^{-2}\int_{-\pi}^{\pi} \mathrm{d}\sigma[T(\sigma) + \frac{1}{4}]e^{ir\sigma}$ generates a Poisson bracket Virasoro algebra $i\{L_r, L_s\} = (r - s)L_{r+s} + \frac{c}{12}(r^3 - r)\delta_{r+s,0}$ with central charge $c = 12\pi\gamma^{-2}$, where γ^2 is a (classically arbitrary) scale factor. The bracket (1.1) also shows that V^+ and V^- are spin-2 conformal primary fields. The Poisson brackets of V^+ and V^- with themselves contain the new non-local terms involving[2] $\varepsilon(\sigma - \sigma') \sim 2\partial^{-1}\delta(\sigma - \sigma')$. The complete mode expansion of the algebra (1.1) was written in ref. 9 (Eq. (3.24)). To emphasize the similarities (non-linearity) and differences (non-locality) with the W-algebras, this algebra was called V-algebra [9].

It was conjectured in ref. 9 and confirmed in ref. 10 that the V-algebra (1.1) is again associated with a linear differential operator $L = -\partial^2 + U$ (where $\partial \equiv \partial_\sigma = \partial/\partial\sigma$), but this time U being a matrix, namely the 2×2 matrix $U = \begin{pmatrix} T & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T \end{pmatrix}$. The non-locality of the algebra turned out to be related to the non-commutativity of matrices. More precisely, there is a standard way [11–17] to associate two symplectic structures (Poisson brackets), the first and second Gelfand–Dikii bracket, to any linear differential operator with scalar coefficients u_k. It was shown in ref. 9 that the analogous construction of the second Gelfand–Dikii bracket for $L = -\partial^2 + U$, with U the above 2×2-matrix, precisely is the V-algebra (1.1). Using the 2×2-matrix U written above, the algebra (1.1) can be written

[1] The $\bar{T}, \bar{V}^+, \bar{V}^-$ form an isomorphic algebra, while the Poisson bracket of a left-moving with a right-moving current vanishes.

[2] The precise definition of $\varepsilon(\sigma - \sigma')$ depends on the boundary conditions. For example, on the space of functions on S^1 without constant Fourier mode, ∂^{-1} is well-defined and $\varepsilon(\sigma - \sigma') = \frac{1}{i\pi}\sum_{n\neq 0}\frac{1}{n}e^{in(\sigma-\sigma')}$.

more compactly. Since U is constrained by $\operatorname{tr}\sigma_3 U = 0$, it is convenient to introduce 2×2-matrix-valued (test-) functions F and G subject to the same constraint: $\operatorname{tr}\sigma_3 F = \operatorname{tr}\sigma_3 G = 0$. Then

$$\gamma^{-2}\{\int\operatorname{tr} FU, \int\operatorname{tr} GU\} = \int\operatorname{tr}\,(GF''' + [F,U]\partial^{-1}[G,U] \\ - (F'G - FG' + GF' - G'F)U)\,, \qquad (1.2)$$

where the commutators $[F,U]$ and $[G,U]$ on the r.h.s. are simply the commutators of the 2×2-matrices.

It is the purpose of this paper to generalize these developments to a linear differential operator $L = -\partial^m + \sum_{k=1}^m U_k \partial^{m-k}$ of arbitrary order $m \geqq 1$ with coefficients U_k that are $n \times n$-matrices. The corresponding algebras will be called $V_{n,m}$-algebras[3]. Symplectic structures (Poisson brackets) associated with $n \times n$-matrix m^{th}-order differential operators have been studied in the mathematical literature by Gelfand and Dikii [12]. The Poisson bracket they define is now called the first Gelfand–Dikii bracket, and leads to a *linear* algebra. In their paper [12] they also give the asymptotic expansion of the resolvent of L and a recursion relation for an infinite sequence of hamiltonians in involution with respect to this first Gelfand–Dikii bracket. Here, I am mainly interested in the second Gelfand–Dikii bracket [15], and to the best of my knowledge this bracket has never been worked out so far for the matrix case. For $m = 2$ and $L = -\partial^2 + U$, it was shown in ref. 10 that the models are bihamiltonian, and that the second Gelfand–Dikii bracket actually follows from the recursion relations for the resolvent, or those for the hamiltonians[4]. For $m \geqq 3$, these recursion relations are much more complex [12], and it is not clear whether the second Gelfand–Dikii bracket given in the present paper could also be extracted from the formulas of ref. 12. In any case, the construction given here is a straightforward generalization of the scalar case. Let me remark that all the developments of the present paper can be generalized by replacing the $n \times n$-matrices U_k by operators $\hat{U}_k$ acting in some Hilbert space, provided the products of these operators and their traces, as well as the functional derivatives $\delta/\delta\hat{U}_k$ are well-defined.

Let me note that for $m = 2$, it was shown in refs 10, 18, 20 how to construct an infinite sequence of hamiltonians with respect to both Gelfand–Dikii brackets. This led to matrix KdV hierarchies. The present developments are connected with matrix versions of the generalized KdV hierarchies (i.e. matrix Gelfand–Dikii hierarchies) or matrix KP hierarchies. The latter were recently studied from a somewhat different point of view by Kac and van de Leur [21], and the present paper is quite complementary to theirs.

In Sect. 2, I will work out the second Gelfand–Dikii bracket[5], first for generic $U_1, \ldots U_m$ and then for differential operators L with $U_1 = 0$. The reduction from the general case to the case $U_1 = 0$ is non-trivial, as usual. In particular, it introduces the integral (∂^{-1}) of a commutator of two matrices. It is thus the reduction to the symplectic submanifold $U_1 = 0$ together with the non-commutativity of matrices that leads to the non-localities in the second Gelfand–Dikii bracket. I note that for $m = 1$ (and $U_1 \neq 0$) one simply recovers a $gl(n)$ Kac–Moody algebra (cf. ref. 22).

[3] More precisely, I reserve the name $V_{n,m}$-algebra for the reduction to the submanifold where $U_1 = 0$, see below.

[4] Earlier related studies for $m = 2$ can be found in refs. 18, 19 and 20.

[5] The first Gelfand–Dikii bracket will also be given but it seems to be less interesting.

In Sect. 3, I prove that a Miura-type transformation $L = -(\partial - P_1)(\partial - P_2)\dots(\partial - P_m)$ maps the (relatively) complicated symplectic structure given by the Poisson brackets of the U_k to a much simpler symplectic structure given by the Poisson brackets of a set of m decoupled fields P_i, each P_i being a $n \times n$-matrix. This provides the analogue of the usual free field realization: whereas in the scalar case ($n = 1$) the P_i are just m free fields, i.e. m collections of harmonic oscillators, or in other words they form m copies of a (Poisson bracket) $U(1)$ current algebra, here, due to the matrix character, the P_i form m copies of a (Poisson bracket) $gl(n)$ current algebra. I also discuss the reduction to $\sum_i P_i = 0$ corresponding to $U_1 = 0$. Since the Jacobi identity is obviously satisfied by the Poisson bracket of the P_i, as an important corollary, the Miura transformation immediately implies the Jacobi identity for the second Gelfand–Dikii bracket of the U_k, which was not obvious a priori.

In Sect. 4, I will discuss the conformal properties. It turns out that $T \sim \mathrm{tr}\ U_2$ generates the conformal (Virasoro) algebra, and the Poisson brackets of $\mathrm{tr}\ U_2$ with any matrix element of any U_k is relatively easy to spell out. They turn out to be exactly the same as in the scalar case, and I conclude that appropriately symmetrized combinations W_k can be formed so that all of their matrix elements are conformal primary fields of weight (spin) k. This analogy with the scalar case is due to the fact that the conformal properties are determined by $T \sim \mathrm{tr}\ U_2 \equiv \mathrm{tr}\ \mathbf{1}U_2$ and the unit matrix $\mathbf{1}$ always commutes.

In Sect. 5, I spell out the Poisson bracket algebra of the matrix elements of U_2 and U_3 for any $m \geqq 2$ (with the restriction $U_1 = 0$ which is the more interesting case). Again, this is done more compactly by considering $\int \mathrm{tr}\ FU_2$ and $\int \mathrm{tr}\ FU_3$. For $m = 3$, and in the primary basis U_2, W_3, the result reads (a is related to the scale factor γ^2 by $a = -2\gamma^2$):

$$\left\{\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GU_2\right\} = a\int \mathrm{tr}\ \left(-\frac{1}{3}[F,U_2]\partial^{-1}[G,U_2] - [F,G]W_3 + \frac{1}{2}(F'G + GF' - FG' - G'F)U_2 - 2GF'''\right), \quad (1.3)$$

$$\left\{\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GW_3\right\} = a\int \mathrm{tr}\ \left(-\frac{1}{3}[F,U_2]\partial^{-1}[G,W_3] - \frac{1}{6}[F,G]U_2^2 + \left(-\frac{1}{4}[F',G'] + \frac{1}{2}[F'',G] + \frac{1}{12}[F,G'']\right)U_2 + \left(F'G + GF' - \frac{1}{2}FG' - \frac{1}{2}G'F\right)W_3\right), \quad (1.4)$$

$$\begin{aligned}\left\{\int \mathrm{tr}\ FW_3, \int \mathrm{tr}\ GW_3\right\} = a\int \mathrm{tr}\ \Big(&-\frac{1}{3}[F,W_3]\partial^{-1}[G,W_3] \\ &-\frac{1}{6}[F,G](W_3U_2 + U_2W_3) + \frac{2}{3}(FU_2GW_3 - GU_2FW_3) \\ &+\frac{5}{12}(F'U_2GU_2 - G'U_2FU_2) + \frac{1}{12}(FG' - GF')U_2^2 + \frac{1}{12}[F,G]U_2'U_2 \\ &+\frac{7}{12}[F',G']W_3 - \frac{1}{6}[F,G]''W_3 + \frac{1}{12}(FG''' + G'''F - F'''G - GF''')U_2 \\ &+\frac{1}{8}(F''G' + G'F'' - F'G'' - G''F')U_2 + \frac{1}{6}GF^{(5)}\Big)\,. \end{aligned} \quad (1.5)$$

One remarks that in the scalar case ($n=1$) this reduces to the Poisson bracket version of Zamolodchikov's W_3-algebra, as it should. In the matrix case however, even if $F=f\mathbf{1}, G=g\mathbf{1}$ (with scalar f,g) this is a different algebra, i.e. $\{\int \operatorname{tr} W_3(\sigma), \int \operatorname{tr} W_3(\sigma')\}$ does not reduce to the W_3-algebra, since the r.h.s. contains non-linear terms and $\operatorname{tr} U_2^2 \neq (\operatorname{tr} U_2)^2$. In other words, the scalar ($n=1$) W_m-algebras are not subalgebras of the matrix $V_{n,m}$-algebras. The only exception is $m=2$, since one always has a Virasoro subalgebra.

In the final Sect. 6, I first discuss various restrictions, like imposing hermiticity conditions on the W_k, or for $n=2, m=2$ how the restriction $\operatorname{tr} \sigma_3 U_2 = 0$ is imposed. I also mention restrictions like setting all U_k with odd k to zero, etc. Some other problems like free field representations and quantization are briefly addressed, before I conclude.

Appendix A gives some results on pseudo-differential operators, while in Appendix B, I evaluate certain sums of products of binomial coefficients needed in Sect. 2.

2. The Gelfand–Dikii Brackets and the $V_{n,m}$-Algebras

2.1. Preliminaries. In this section, I will compute the (first and) second Gelfand–Dikii bracket of two functionals f and g of the $n\times n$ matrix coefficients $U_k(\sigma)$ of the linear m^{th}-order differential operator

$$L = -\partial^m + \sum_{k=1}^{m} U_k \partial^{m-k} \equiv \sum_{k=0}^{m} U_k \partial^{m-k} , \tag{2.1}$$

where $\partial = \frac{d}{d\sigma}$. To make the subsequent formula more compact, I formally introduced $U_0 = -\mathbf{1}$. The functionals f and g one considers are of the form $f = \int \operatorname{tr} P(U_k)$, where P is some polynomial in the $U_k, k=1,\dots m$, and their derivatives (i.e. a differential polynomial in the U_k). P may also contain other constant or non-constant numerical matrices so that these functionals are fairly general. (Under suitable boundary conditions, *any* functional of the U_k and their derivatives can be approximated to arbitrary "accuracy" by an f of the type considered.) The integral can either be defined in a formal sense as assigning to any function an equivalence class by considering functions only up to total derivatives (see e.g. Sect. 1 of ref. 12), or in the standard way if one restricts the integrand, i.e. the U_k, to the class of e.g. periodic functions or sufficiently fast decreasing functions on $\mathbf{R}$, etc. All that matters is that the integral of a total derivative vanishes and that one can freely integrate by parts.

To define the Gelfand–Dikii brackets, it is standard to use pseudo-differential operators [13, 14] involving integer powers of ∂^{-1}. Again, ∂^{-1} can be defined in a formal sense by $\partial\partial^{-1} = \partial^{-1}\partial = 1$, but one can also give a concrete definition on appropriate classes of functions. For example for C^∞-functions h on $\mathbf{R}$ decreasing exponentially fast as $\sigma \to \pm\infty$, one can simply define $(\partial^{-1}h)(\sigma) = \int_{-\infty}^{\sigma} d\sigma' h(\sigma') = \int_{-\infty}^{\infty} d\sigma' \theta(\sigma-\sigma') h(\sigma')$. Alternatively any of the choices $(\partial^{-1}h)(\sigma) = \int_{-\infty}^{\infty} d\sigma' (\lambda\theta(\sigma-\sigma') - (1-\lambda)\theta(\sigma'-\sigma)) h(\sigma')$ is equally valid, the most symmetric choice being $\lambda = \frac{1}{2}$: $(\partial^{-1}h)(\sigma) = \int_{-\infty}^{\infty} d\sigma' \frac{1}{2}\varepsilon(\sigma-\sigma') h(\sigma')$. For periodic functions

[6] Throughout this paper, m will denote the order of L which is a positive integer.

on the circle, ∂^{-1} is well defined on functions with no constant Fourier mode. Then $(\partial^{-1}h)(\sigma) = \int_{-\pi}^{\pi} d\sigma' \frac{1}{2}\varepsilon(\sigma - \sigma')h(\sigma')$ with $\varepsilon(\sigma - \sigma') = \frac{1}{i\pi}\sum_{k\neq 0}\frac{1}{k}e^{ik(\sigma-\sigma')}$. From the definition of ∂^{-1} as inverse of ∂ one deduces the basic formula

$$\partial^{-r}h = \sum_{s=0}^{\infty}(-)^s\binom{r+s-1}{s}h^{(s)}\partial^{-r-s} . \tag{2.2}$$

For a pseudo-differential operator $A = \sum_{i=-\infty}^{l} a_i\partial^i$ one denotes

$$A_+ = \sum_{i=0}^{l} a_i\partial^i \quad , \quad A_- = \sum_{i=-\infty}^{-1} a_i\partial^i \quad , \quad \text{res } A = a_{-1} , \tag{2.3}$$

so that $A = A_+ + A_-$. A well-known important property [13] is that for any two pseudo-differential operators A and B the residue of the commutator is a total derivative and hence $\int \text{res } [A,B] = 0$. This is true if the a_i, b_j commute with each other. In the case of present interest the a_i, b_j are matrix-valued functions and one has instead

$$\int \text{tr res } [A,B] = 0 . \tag{2.4}$$

For completeness, this and other properties of pseudo-differential operators are proven in Appendix A.

2.2. The Gelfand–Dikii Brackets for General $U_1, \dots U_m$. In analogy with the scalar case (i.e. $n = 1$) [13, 14, 15, 17], I define the first and second Gelfand–Dikii brackets associated with the $n \times n$-matrix m^{th}-order differential operator L as[7]

$$\begin{aligned} \{f,g\}_{(1)} &= a\int \text{tr res } \left([L,X_f]_+ X_g\right) , \\ \{f,g\}_{(2)} &= a\int \text{tr res } \left(L(X_f L)_+ X_g - (LX_f)_+ LX_g\right) , \end{aligned} \tag{2.5}$$

where a is an arbitrary scale factor and X_f, X_g are the pseudo-differential operators

$$\begin{aligned} X_f &= \sum_{l=1}^{m}\partial^{-l}X_l, \quad X_g = \sum_{l=1}^{m}\partial^{-l}Y_l , \\ X_l &= \frac{\delta f}{\delta U_{m+1-l}}, \quad Y_l = \frac{\delta g}{\delta U_{m+1-l}} . \end{aligned} \tag{2.6}$$

The functional derivative of $f = \int d\sigma \text{ tr } P(U)$ is defined as usual by

$$\left(\frac{\delta f}{\delta U_k}(\sigma)\right)_{ij} = \sum_{r=0}^{\infty}\left(-\frac{d}{d\sigma}\right)^r\left(\frac{\partial \text{tr } P(u)(\sigma)}{\partial (U_k^{(r)})_{ji}}\right) , \tag{2.7}$$

where $(U_k^{(r)})_{ji}$ denotes the (j,i) matrix element of r^{th} derivative of U_k. It is easily seen, that for $n = 1$, Eqs. (2.5)–(2.7) reduce to the standard definitions of the Gelfand–Dikii brackets [13, 14, 15, 17]. For $m = 2$, $n = 2$ and with the extra restrictions $U_1 = 0$, tr $\sigma_3 U_2 = 0$, the second equation (2.5) was shown in ref. 10 to reproduce the original V-algebra (1.1) (with $a = -2\gamma^2$).

---

[7] To avoid confusion, let me insist that $[L,X_f]_+$ means the differential operator part of the commutator, and has nothing to do with an anticommutator.

To start with, I compute the first Gelfand–Dikii bracket. Inserting the definitions of L, X_f and X_g into the first equation (2.5), and using formula (2.2) and the cyclic commutativity (2.4) under $\int$ tr res, it is rather straightforward to obtain

$$(X_f L)_+ = \sum_{l=1}^{m}\sum_{k=l}^{m}\sum_{s=0}^{k-l}(-)^s \binom{l+s-1}{s} (X_l U_{m-k})^{(s)} \partial^{k-l-s} ,$$

$$(LX_f)_+ = \sum_{l=1}^{m}\sum_{k=l}^{m}\sum_{s=0}^{k-l} \binom{k-l}{s} U_{m-k} X_l^{(s)} \partial^{k-l-s} , \tag{2.8}$$

and after changing the summation indices and integrating by parts

$$\{f,g\}_{(1)} = a \int \mathrm{tr} \sum_{p=0}^{m-1}\sum_{q=0}^{m-1-p}\sum_{s=0}^{m-1-p-q} \binom{p+s}{s} \times \left(Y_{q+1}^{(s)} X_{p+1} - X_{q+1}^{(s)} Y_{p+1}\right) U_{m-1-p-q-s} , \tag{2.9}$$

where X_l and Y_k are given by (2.6). After renaming $X_{s+1} \to i^s F_s, Y_{r+1} \to i^r G_r$, $U_{m-k} \to (-i)^k U_k$, Eq. (2.9) coincides, up to an irrelevant overall factor, with the Poisson bracket defined by Eq. 8 of ref. 12. It is obvious from the r.h.s. of (2.9) that it is antisymmetric under $f \leftrightarrow g$. It is however non-trivial to prove the Jacobi identity. This was done in ref. 12.

Next, I will consider the second Gelfand–Dikii bracket as defined by the second equation (2.5). Contrary to the first bracket (2.9) which is linear in the U_k, the second bracket, in general, will turn out to be non-linear in the U_k and will show a richer structure. First, one has the following

Lemma 2.1. *Let*

$$V_{X_f}(L) = L(X_f L)_+ - (LX_f)_+ L \tag{2.10}$$

which is a differential operator of order $2m-1$ *at most. Define its coefficients* $\mathscr{V}(f)_j$ *by*

$$V_{X_f}(L) = \sum_{j=0}^{2m-1} \mathscr{V}(f)_j \partial^j . \tag{2.11}$$

Then

$$\{f,g\}_{(2)} = a \int \mathrm{tr} \sum_{k=0}^{m-1} \mathscr{V}(f)_k Y_{k+1} . \tag{2.12}$$

Proof. The lemma follows obviously from the definitions (2.5) of the second Gelfand–Dikii bracket and (2.6) of X_j together with Lemma A.2 of the appendix.

Note that only the $\mathscr{V}(f)_k$ with $k = 0, \ldots m-1$ appear. I will show below (Lemma 2.5) that all $\mathscr{V}(f)_k$ with $k \geqq m$ actually vanish, so that V_{X_f} is a mapping from the space of m^{th}-order differential operators onto itself. Next one has the

Lemma 2.2.

$$L(X_f L)_+ = \sum_{j=0}^{2m-1}\sum_{l=1}^{m}\sum_{p=0}^{2m-j-l}\sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} S_{q-p,l}^{q,j} U_{m-q}(X_l U_{m-p-j-l+q})^{(p)} \partial^j , \tag{2.13}$$

where for $l \geqq 1$

$$S^{q,j}_{r,l} = \sum_{s=\max(0,r)}^{\min(q,j)} (-)^{s-r} \binom{s-r+l-1}{l-1} \binom{q}{s} \tag{2.14}$$

with $S^{q,j}_{r,l} = 0$ *if* $\max(0,r) > \min(q,j)$.

Proof. Starting with (2.8) for $(X_f L)_+$ and the (second) definition (2.1) of L one has

$$L(X_f L)_+ = \sum_{q=0}^{m} \sum_{l=1}^{m} \sum_{k=l}^{m} \sum_{s=0}^{k-l} (-)^s \binom{l+s-1}{s} \sum_{r=0}^{q} \binom{q}{r} \times U_{m-q}(X_l U_{m-k})^{(s+q-r)} \partial^{k-l-s+r} . \tag{2.15}$$

Reshuffling the summation indices ($\tilde{s} = k - l - s$ and then $j = r + \tilde{s}$) gives for the r.h.s of (2.15),

$$\sum_{q=0}^{m} \sum_{k=0}^{m} \sum_{l=1}^{k} \sum_{j=0}^{k+q-l} S^{q,j}_{j-k+l,l} U_{m-q}(X_l U_{m-k})^{(k-l+q-j)} \partial^j . \tag{2.16}$$

Since $S^{q,j}_{j-k+l,l} = 0$ for $k < l$ one can extend the sum over l from 1 to k while $k = 0$ does not contribute. Also, the sum over j can be rewritten as $\sum_{j=0}^{2m-1}$ at the expense of restricting the sum over q. Introducing $p = k + q - j - l$ one finally arrives at (2.13).

Lemma 2.3.

$$(LX_f)_+ L = \sum_{j=0}^{2m-1} \sum_{l=1}^{m} \sum_{p=0}^{2m-j-l} \sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} \binom{q-l}{p} \times U_{m-q}(X_l U_{m-p-j-l+q})^{(p)} \partial^j . \tag{2.17}$$

Proof. Once again one starts with Eq. (2.8), this time for $(LX_f)_+$. However, it is more convenient to rewrite it as

$$(LX_f)_+ = \sum_{l=1}^{m} \sum_{k=l}^{m} U_{m-k} \partial^{k-l} X_l , \tag{2.18}$$

so that

$$(LX_f)_+ L = \sum_{l=1}^{m} \sum_{k=l}^{m} \sum_{q=0}^{m} \sum_{r=0}^{k-l} \binom{k-l}{r} U_{m-k}(X_l U_{m-q})^{(k-l-r)} \partial^{q+r} . \tag{2.19}$$

Throughout this paper, I define a binomial coefficient $\binom{a}{b}$ to vanish unless $a \geqq b \geqq 0$. Hence, the sum over k can be written $\sum_{k=0}^{m}$, and one then interchanges the roles of k and q. Introducing then $j = k + r$ and $p = q - l - r$ one finally obtains (2.17).

Whereas (2.17) just contains a simple binomial coefficient, (2.13) contains the $S^{q,j}_{r,l}$ which are sums over products of two binomial coefficients. Unfortunately, for general r, l, q, j, I was not able to derive a simpler expression for $S^{q,j}_{r,l}$. However, one has the following lemma, proven in Appendix B.

Lemma 2.4. *For $q \leqq j$ and $r \geqq l \geqq 1$ one has*

$$S^{q,j}_{r,l} = \binom{q-l}{q-r}, \tag{2.20}$$

and for $q \leqq j$ and $l > r \geqq 0$ one has

$$S^{q,j}_{r,l} = (-)^{q-r}\binom{l-1-r}{q-r}. \tag{2.21}$$

Lemma 2.5. *The coefficients of ∂^j in $(LX_f)_+L$ and $L(X_fL)_+$ are equal for $j \geqq m$. In other words,*

$$\mathscr{V}(f)_j = 0 \quad \text{for } j \geqq m. \tag{2.22}$$

Proof. Since for any pseudo-differential operator A one has $A_+ = A - A_-$, one can rewrite (2.10) as $V_{X_f}(L) = -L(X_fL)_- + (LX_f)_-L$ from which it is obvious that $V_{X_f}(L)$ is at most of degree $m-1$ and (2.22) follows. Alternatively, as a consistency check, it can also be easily proven directly by considering the terms with $j \geqq m$ in Eq. (2.13) for $L(X_fL)_+$. Then $q \geqq \max(0, p+j+l-m) = p+j+l-m \geqq p+l$, and $q \leqq \min(m, p+j+l) = m \leqq j$. Hence $q \leqq j$ and $q-p \geqq l$, so that by the previous lemma $S^{q,j}_{q-p,l} = \binom{q-l}{q-(q-p)} = \binom{q-l}{p}$. Comparing with Eq. (2.17) for $(LX_f)_+L$ proves (2.22) again.

Collecting the results of Lemmas 2.1, 2.2 and 2.3 one has the

Proposition 2.6. *The second Gelfand–Dikii bracket associated with the $n \times n$-matrix m^{th}-order differential operator L as defined by Eqs.* (2.5) *and* (2.6) *equals*

$$\begin{aligned}
\{f, g\}_{(2)} &= a\int \operatorname{tr} \sum_{j=0}^{m-1} \mathscr{V}(f)_j Y_{j+1}, \\
\mathscr{V}(f)_j &= \sum_{l=1}^{m} \sum_{p=0}^{2m-j-l} \sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} \left(S^{q,j}_{q-p,l} - \binom{q-l}{p}\right) \\
&\quad \times U_{m-q}(X_l U_{m-p-j-l+q})^{(p)}.
\end{aligned} \tag{2.23}$$

Remark. 2.7. It will follow from the results of the next section on the Miura transformation that the previous proposition gives a well-defined symplectic structure, i.e. that (2.23) is a well-defined Poisson bracket obeying antisymmetry and the Jacobi identity. More precisely, I will show that the U_k and thus also the functionals f and g are expressible in terms of certain $n \times n$-matrices $P_i, i = 1, \ldots m$, and that the bracket (2.23) equals the Poisson bracket of f with g when computed using the much simpler P_i-Poisson bracket which is manifestly antisymmetric and obeys the Jacobi identity.

Remark. 2.8. For $m = 1$, the first Gelfand–Dikii bracket vanishes, while the second Gelfand–Dikii bracket is very simple. From (2.23) one has $\{f, g\}_{(2)} = a\int \operatorname{tr} \mathscr{V}(f)_0 Y$, where $Y \equiv Y_1 = \delta g/\delta U_1$, and also $X \equiv X_1 = \delta g/\delta U_1$. Now, for

$m=1, \mathscr{V}(f)_0$ only contains 3 terms, involving $S_{0,1}^{0,0}=1$, $S_{1,1}^{1,0}=0$ and $S_{0,1}^{1,0}=1$, so that $\mathscr{V}(f)_0=-U_1X+XU_1+X'$ and

$$\{f,g\}_{(2)}=a\int \mathrm{tr}\ (-[X,Y]U_1+X'Y)\,. \tag{2.24}$$

This is a Poisson bracket version of the $gl(n)$ Kac–Moody algebra as considered by Drinfeld and Sokolov [22]. To make this even clearer, introduce a basis $\{T^b\}, b=1,\dots n^2$ of the Lie algebra $gl(n)$ of $n\times n$-matrices with $[T^a,T^b]=f^{abc}T^c$ and define for σ on the unit circle $J_r^b=\frac{1}{(-a)}\int_{-\pi}^{\pi}\mathrm{d}\sigma e^{ir\sigma}\ \mathrm{tr}\ T^bU_1(\sigma)$. Then Eq. (2.24) becomes

$$i\{J_r^a,J_s^b\}_{(2)}=if^{abc}J_{r+s}^c+\left(\frac{2\pi}{-a}\right)r\delta_{r+s,0}\ \mathrm{tr}\ T^aT^b\,. \tag{2.25}$$

2.3. The Gelfand–Dikii Brackets Reduced to $U_1=0$. The problem of consistently restricting a given symplectic manifold (phase space) to a symplectic submanifold by imposing certain constraints $\phi_i=0$ has been much studied in the literature. The basic point is that for a given phase space one cannot set a coordinate to a given value (or function) without also eliminating the corresponding momentum. More generally, to impose a constraint $\phi=0$ consistently, one has to make sure that for any functional f the bracket $\{\phi,f\}$ vanishes if the constraint $\phi=0$ is imposed *after* computing the bracket. In general this results in a modification of the original Poisson bracket.

For the first Gelfand–Dikii bracket it is easy to see that (2.9) does not contain any terms containing $X_m=\frac{\delta f}{\delta U_1}$ or $Y_m=\frac{\delta g}{\delta U_1}$. Hence the first Gelfand–Dikii bracket of U_1 with any functional automatically vanishes. As a consequence, one may consistently restrict it to the submanifold of vanishing U_1 simply by taking (2.9) and setting $U_1=0$ on the r.h.s.

For the second Gelfand–Dikii bracket the situation is slightly less trivial. One has to impose $\{f,U_1\}|_{U_1=0}=0$ for all f. Since $Y_m=\frac{\delta g}{\delta U_1}$, one sees from (2.23) that this requires $\mathscr{V}(f)_{m-1}|_{U_1=0}=0$. In practice this determines X_m which otherwise would be undefined if one starts with $U_1=0$. In the scalar case it is known [15] that X_m should be determined by res $[L,X_f]=0$. The following two lemmas show that this is still true in the matrix case.

Lemma 2.9. *One has*

$$\mathscr{V}(f)_{m-1}=-\mathrm{res}\ [L,X_f]\,. \tag{2.26}$$

Proof. On the one hand, from the definitions of L and X_f one easily obtains

$$\mathrm{res}\ [L,X_f]=\sum_{l=1}^{m}[U_{m+1-l},X_l]+\sum_{l=1}^{m}\sum_{k=l}^{m}(-)^{k-l}\binom{k}{l-1}(X_lU_{m-k})^{(k-l+1)}\,. \tag{2.27}$$

Note the commutator term which is a new feature of the present matrix case as opposed to the scalar case. On the other hand, from Eq. (2.23) one has

$$\mathscr{V}(f)_{m-1}=\sum_{l=1}^{m}\sum_{p=0}^{m-l+1}\sum_{q=p+l-1}^{m}\left(S_{q-p,l}^{q,m-1}-\binom{q-l}{p}\right)\times U_{m-q}(X_lU_{q-p+1-l})^{(p)}\,. \tag{2.28}$$

From Lemma 2.4 one knows that all terms with $p+l \leqq q \leqq m-1$ vanish. The remaining terms have either $q = p+l-1$ or $q = m$. Unless $p = m-l+1$, one has $p+l-1 \leqq m-1$, so that using Eq. (2.21) and Eq. (B.7) one has

$$\begin{aligned}\mathscr{V}(f)_{m-1} =& \sum_{l=1}^{m} U_{m+1-l} X_l U_0 - \sum_{l=1}^{m} \sum_{p=0}^{m-l} (-)^p \binom{p+l-1}{l-1} \\ & \times U_0 (X_l U_{m-p-l+1})^{(p)} \\ & + \sum_{l=1}^{m} (-)^{m-l} \binom{m}{l-1} U_0 (X_l U_0)^{(m-l+1)} . \end{aligned} \tag{2.29}$$

The last two terms combine into $\sum_{p=0}^{m-l+1}(\ldots)$. Upon relabelling $k = p+l-1$, and recalling $U_0 \equiv -1$, the r.h.s. of this equation is seen to coincide, up to a minus sign, with the r.h.s. of Eq. (2.27). This proves (2.26).

Lemma 2.10. *For* $U_1 = 0$, res $[L, X_f] = 0$ *is equivalent to*

$$X_m = \frac{1}{m} \sum_{l=1}^{m-l} \left(\partial^{-1} [U_{m+1-l}, X_l] + \sum_{k=l}^{m} (-)^{k-l} \binom{k}{l-1} (X_l U_{m-k})^{(k-l)} \right) . \tag{2.30}$$

Proof. In Eq. (2.27), separate the terms with $l = m$ from those with $l \neq m$. For $U_1 = 0$, the terms with $l = m$ are simply $-mX_m'$ while those with $l \neq m$ coincide with m times the derivative of the r.h.s. of (2.30).

One sees that the non-local term $\partial^{-1}[U_{m+1-l}, X_l]$ originates from the non-commutativity of matrices and the necessity of solving for X_m when reducing to the symplectic submanifold with $U_1 = 0$.

The main result of this section is the following

Theorem 2.11. *The second Gelfand–Dikii bracket for* $n \times n$*-matrix* m^{th}*-order differential operators* L *with vanishing* U_1 *is given by*

$$\begin{aligned}\{f, g\}_{(2)} =& a \int \operatorname{tr} \sum_{j=0}^{m-2} \tilde{\mathscr{V}}(f)_j Y_{j+1} , \\ \tilde{\mathscr{V}}(f)_j =& \frac{1}{m} \sum_{l=1}^{m-1} [U_{m-j}, \partial^{-1}[X_l, U_{m-l+1}]] \\ & + \frac{1}{m} \sum_{l=1}^{m-1} \left\{ \sum_{k=0}^{m-l} (-)^k \binom{k+l}{l-1} (X_l U_{m-k-l})^{(k)} U_{m-j} \right. \\ & \left. - \sum_{k=0}^{m-j-1} \binom{k+j+1}{j} U_{m-k-j-1} (U_{m-l+1} X_l)^{(k)} \right\} \\ & + \sum_{l=1}^{m-1} \sum_{p=0}^{2m-j-1} \sum_{q=\max(0,p+j+l-m)}^{\min(m,p+j+l)} C_{q-p,l}^{q,j} U_{m-q} (X_l U_{m-p-j-l+q})^{(p)} , \\ C_{q-p,l}^{q,j} =& S_{q-p,l}^{q,j} - \binom{q-l}{p} - \frac{1}{m} (-)^{q-p+j} \binom{q}{j} \binom{p-q+j+l}{l-1} , \end{aligned} \tag{2.31}$$

where the $S^{q,j}_{q-p,l}$ *are defined by Eq.* (2.14), *and it is understood that* $U_0 = -1$ *and* $U_1 = 0$.

Proof. As discussed above, the bracket for $U_1 = 0$ is obtained from the unrestricted one, Eq. (2.23), by determining (the otherwise undetermined) X_m from $\mathscr{V}(f)_{m-1}|_{U_1=0} = 0$. This ensures that all brackets $\{f, U_1\}$ vanish when U_1 is set to zero after computing the bracket. By Lemma 2.9, $\mathscr{V}(f)_{m-1}|_{U_1=0} = 0$ implies $\text{res}\,[L, X_f]|_{U_1=0} = 0$ which by Lemma 2.10 determines X_m as given by (2.30). All one has to do is to insert (2.30) into (2.23). The terms $\sim X_l$ with $l < m$ are not affected and give those terms in (2.31) that do not have a factor $\frac{1}{m}$ in front. On the other hand, the terms $\sim X_m$ in $\mathscr{V}(f)_j$ are $(j < m)$

$$\sum_{p=0}^{m-j} \sum_{q=p+j}^{m} \left(S^{q,j}_{q-p,m} - \binom{q-m}{p} \right) U_{m-q}(X_m U_{q-p-j})^{(p)} . \tag{2.32}$$

Now, $S^{q,j}_{q-p,m}$ is non-zero only if $q - p \leqq \min(q, j)$. Since $q \geqq p + j$ this is only possible if $q = p + j$ in which case $S^{p+j,j}_{j,m} = \binom{p+j}{j}$. Hence (2.32) equals

$$[X_m, U_{m-j}] - \sum_{p=1}^{m-j} \binom{p+j}{j} U_{m-p-j} X_m^{(p)} . \tag{2.33}$$

Inserting Eq. (2.30) for X_m it is straightforward to see that, upon rearranging the summation indices, one obtains exactly all the terms in Eq. (2.31) for $\tilde{\mathscr{V}}(f)_j$ that contain a factor $\frac{1}{m}$. One can check again that $\tilde{\mathscr{V}}(f)_{m-1}$ vanishes.

Remark. 2.12. If one takes $m = 2$, $L = -\partial^2 + U$, so that $U_2 \equiv U$ and $X_1 \equiv X$, only $\tilde{\mathscr{V}}(f)_0$ is non-vanishing:

$$\tilde{\mathscr{V}}(f)_0 = -\frac{1}{2}[U, \partial^{-1}[U, X]] + \frac{1}{2}(XU + UX)' + \frac{1}{2}(X'U + UX') - \frac{1}{2}X''' , \tag{2.34}$$

and with $X = \frac{\delta f}{\delta U}$ and $Y = \frac{\delta g}{\delta U}$ one obtains (using $\int x\partial^{-1}y = -\int(\partial^{-1}x)y$)

$$\begin{aligned} \{f, g\}_{(2)} = a \int \text{tr} \Bigg(& -\frac{1}{2}[U, X]\partial^{-1}[U, Y] \\ & + \frac{1}{2}(X'Y + YX' - XY' - Y'X)U - \frac{1}{2}YX''' \Bigg) , \end{aligned} \tag{2.35}$$

which obviously is a generalization of the original V-algebra (1.2) to arbitrary $n \times n$-matrices $U \equiv U_2$. To appreciate the structure of the non-local terms, I explicitly write this algebra in the simplest case for $n = 2$ (but *without* the restriction [8] $\text{tr}\,\sigma_3 U = 0$ which is present for (1.1)).

Let

$$U = \begin{pmatrix} T + V_3 & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T - V_3 \end{pmatrix} . \tag{2.36}$$

[8] See Sect. 6.2 for a discussion of the reduction to tr $\sigma_3 U = 0$

Then one obtains from (2.35) (with $a = -2\gamma^2$) the algebra

$$\begin{aligned}
\gamma^{-2}\{T(\sigma), T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[T(\sigma')\delta(\sigma-\sigma')] - \frac{1}{2}\delta'''(\sigma-\sigma'),\\
\gamma^{-2}\{T(\sigma), V^{\pm}(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[V^{\pm}(\sigma')\delta(\sigma-\sigma')],\\
\gamma^{-2}\{T(\sigma), V_3(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'})[V_3(\sigma')\delta(\sigma-\sigma')],\\
\gamma^{-2}\{V^{\pm}(\sigma), V^{\pm}(\sigma')\} &= \varepsilon(\sigma-\sigma')V^{\pm}(\sigma)V^{\pm}(\sigma'),\\
\gamma^{-2}\{V^{\pm}(\sigma), V^{\mp}(\sigma')\} &= -\varepsilon(\sigma-\sigma')(V^{\pm}(\sigma)V^{\mp}(\sigma') + V_3(\sigma)V_3(\sigma'))\\
&\quad + (\partial_\sigma - \partial_{\sigma'})[T(\sigma')\delta(\sigma-\sigma')] - \frac{1}{2}\delta'''(\sigma-\sigma'),\\
\gamma^{-2}\{V_3(\sigma), V^{\pm}(\sigma')\} &= \varepsilon(\sigma-\sigma')V^{\pm}(\sigma)V_3(\sigma'),\\
\gamma^{-2}\{V_3(\sigma), V_3(\sigma')\} &= \varepsilon(\sigma-\sigma')(V^{+}(\sigma)V^{-}(\sigma') + V^{-}(\sigma)V^{+}(\sigma'))\\
&\quad + (\partial_\sigma - \partial_{\sigma'})[T(\sigma')\delta(\sigma-\sigma')] - \frac{1}{2}\delta'''(\sigma-\sigma').
\end{aligned} \tag{2.37}$$

Once again, one sees that T generates the conformal algebra with a central charge, while V^+, V^- and V_3 are conformally primary fields of weight (spin) two. It is easy to check on the example (2.35), that antisymmetry and the Jacobi identity are satisfied. For general m this follows from the Miura transformation to which I now turn.

3. The Miura Transformation

In this section, I will give the matrix Miura transformation. By this transformation, all $U_k(\sigma)$ can be expressed as differential polynomials in certain $n \times n$-matrices $P_j(\sigma), j = 1,\dots m$, and hence every functional f of the U_k will also be a functional $\tilde{f}$ of the P_j. I will define a very simple Poisson bracket for functionals of the P_j. Using this Poisson bracket one can compute $\{\tilde{f}(P), \tilde{g}(P)\} \equiv \{f(U(P)), g(U(P))\}$. I will show that this Poisson bracket coincides with the second Gelfand–Dikii bracket $\{f(U), g(U)\}_{(2)}$ defined by Eq. (2.23) in the previous section. As a corollary, this proves the antisymmetry and Jacobi identity for the latter. The Poisson bracket of the P_j can be reduced to the submanifold where $\sum_{j=1}^{m} P_j = 0$. This implies $U_1 = 0$. Then I will show that this reduced Poisson bracket for the P_j gives the second Gelfand–Dikii bracket for the U_k reduced to $U_1 = 0$, i.e. Eq. (2.31).

3.1. The Case of General $U_1, \dots U_m$.

Definition and Lemma 3.1. *Introduce the $n \times n$-matrix-valued functions $P_j(\sigma), j = 1,\dots m$. Then for functionals f, g (integrals of traces of differential polynomials) of the P_j the following Poisson bracket is well-defined*

$$\{f, g\} = a\sum_{i=1}^{m} \int \mathrm{tr}\left(\left(\frac{\delta f}{\delta P_i}\right)' \frac{\delta g}{\delta P_i} - \left[\frac{\delta f}{\delta P_i}, \frac{\delta g}{\delta P_i}\right] P_i\right) \tag{3.1}$$

or equivalently for $n \times n$-matrix-valued (numerical) functions F and G

$$\{\int \mathrm{tr}\ F P_i, \int \mathrm{tr}\ G P_j\} = a\,\delta_{ij} \int \mathrm{tr}\ (F'G - [F,G]P_i)\ . \tag{3.2}$$

Proof. Definition (3.1) is obviously bilinear in f and g and reduces to (3.2) for the special functionals chosen. When (3.2) is chosen as a starting point, bilinearity is implicitly understood, so that (3.1) follows. Antisymmetry is obvious for both (3.1) and (3.2). It is easy to see that the Jacobi identity for the bracket (3.2) amounts to the Jacobi identity for the matrix commutator $[F,G]$ which, of course, is satisfied.

Note that due to the δ_{ij} in (3.2) one has m decoupled Poisson brackets. In the scalar case ($n = 1$), (3.2) simply gives $\{P_i(\sigma), P_j(\sigma')\} = (-a)\delta_{ij}\delta'(\sigma - \sigma')$. These are m free fields or m $U(1)$ current algebras. In the matrix case, comparing (3.1) or (3.2) with (2.24) one sees that each P_j actually gives a $gl(n)$ current algebra. So one has no longer free fields but m completely decoupled current algebras. This is still much simpler than the bracket (2.23). To connect both brackets one starts with the following obvious

Lemma 3.2. *Let $P_j, j = 1,\ldots m$ be as in Lemma 3.1. Then*

$$L = -(\partial - P_1)(\partial - P_2)\ldots(\partial - P_m) \tag{3.3}$$

is a m^{th}-order $n \times n$-matrix linear differential operator and can be written $L = \sum_{k=0}^{m} U_{m-k}\partial^k$ with $U_0 = -1$ as before. This identification gives all $U_k, k = 1,\ldots m$ as k^{th}-order differential polynomials in the P_j, i.e. it provides an embedding of the algebra of differential polynomials in the U_k into the algebra of differential polynomials in the P_j. One has in particular

$$U_1 = \sum_{j=1}^{m} P_j \quad , \quad U_2 = -\sum_{i<j}^{m} P_i P_j + \sum_{j=2}^{m} (j-1)P_j'\ . \tag{3.4}$$

I will call the embedding given by (3.3) a (matrix) Miura transformation. The most important property of this Miura transformation is given by the following matix-generalization of a well-known theorem [16, 17].

Theorem 3.3. *Let $f(U)$ and $g(U)$ be functional of the U_k, $k = 1,\ldots m$. By Lemma 3.2 they are also functionals of the P_j, $j = 1,\ldots m$: $f(U) = \tilde{f}(P)$, $g(U) = \tilde{g}(P)$, where $\tilde{f}(P) = f(U(P))$ etc. One then has*

$$\{\tilde{f}(P), \tilde{g}(P)\} = \{f(U), g(U)\}_{(2)}\ , \tag{3.5}$$

where the bracket on the l.h.s is the Poisson bracket (3.1) *and the bracket on the r.h.s. is the second Gelfand–Dikii bracket* (2.23).

Proof. In the scalar case, the simplest proof is probably the one given by Dikki [17]. Here, I repeat this proof, adapting it to the matrix case where necessary. The l.h.s. of (3.5) is given by the r.h.s. of (3.1) which can be rewritten as

$$\{\tilde{f}(P), \tilde{g}(P)\} = a\sum_i \int \mathrm{tr}\ \frac{\delta\tilde{g}}{\delta P_i}\left[\partial - P_i, \frac{\delta\tilde{f}}{\delta P_i}\right]\ , \tag{3.6}$$

whereas the r.h.s of (3.5) is given by (2.23) which I recall is the explicit form of $a \int \mathrm{tr}\ \mathrm{res}\ (L(X_f L)_+ X_g - (LX_f)_+ LX_g)$. Since X_f and X_g contain $\frac{\delta f}{\delta U_k}$ and $\frac{\delta g}{\delta U_k}$ one needs to obtain $\frac{\delta \tilde{f}}{\delta P_i}$ in terms of $\frac{\delta f}{\delta U_k}$, and similarly for g. This is done as follows. One has, using $\delta L = \sum_{k=1}^{m} \delta U_k \partial^{m-k}$, and $X_f = \sum_{l=1}^{m} \partial^{-l} \frac{\delta f}{\delta U_{m+1-i}}$, as well as Lemma A.1,

$$\int \text{tr res } X_f \delta L = \int \text{tr } \sum_{k=1}^{m} \frac{\delta f}{\delta U_k} \delta U_k = \delta f(U) . \tag{3.7}$$

On the other hand, denoting $\partial_i = \partial - P_i$, so that $L = -\partial_1 \ldots \partial_m$, one has $\delta L = \sum_{i=1} \partial_1 \ldots \partial_{i-1} \delta P_i \partial_{i+1} \ldots \partial_m$ so that using Lemmas A.1 and A.3,

$$\int \text{tr res } X_f \delta L = \int \text{tr } \sum_{i=1}^{m} \delta P_i \text{ res } (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1}) . \tag{3.8}$$

Writing $\delta f(U) = \partial \tilde{f}(P) = \int \text{tr } \sum_{i=1}^{m} \delta P_i \frac{\delta \tilde{f}}{\delta P_i}$ one obtains from (3.7) and (3.8) the desired expression of $\frac{\delta \tilde{f}}{\delta P_i}$ in terms of $\frac{\delta f}{\delta U_k}$:

$$\frac{\delta \tilde{f}}{\delta P_i} = \text{res } (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1}) . \tag{3.9}$$

Then Eq. (3.6) becomes

$$\begin{aligned} &\{\tilde{f}(P), \tilde{g}(P)\} \\ &= a \sum_i \int \text{ tr res } (\partial_{i+1} \ldots \partial_m X_g \partial_1 \ldots \partial_{i-1}) [\partial_i, \text{ res } (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1})] . \end{aligned} \tag{3.10}$$

At this point one has an important difference with the scalar case. In the scalar case, Eq. (3.6) reads a $\sum_i \int \frac{\delta \tilde{g}}{\delta P_i} (\frac{\delta \tilde{f}}{\delta P_i})'$ so that only ∂ not $\partial_i \equiv \partial - P_i$ appears in the commutator with res $(\ldots)$ in (3.10). But in the scalar case, P_i commutes with res $(\ldots)$, so that one could replace ∂ by ∂_i and also obtain (3.10). In the present matrix case, however, one cannot simply replace ∂ by ∂_i, and one needs ∂_i from the beginning in (3.6). The rest of the proof goes as in the scalar case [17] and I only give it here for completeness. Using first Lemmas A.3 and A.1, and then Lemma A.4, the r.h.s. of (3.10) becomes

$$\begin{aligned} &a \sum_i \int \text{ tr res } (\partial_{i+1} \ldots \partial_m X_g \partial_1 \ldots \partial_i \text{ res } (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1}) \\ &\qquad - \partial_i \ldots \partial_m X_g \partial_1 \ldots \partial_{i-1} \text{ res } (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1})) \\ &\quad = a \sum_i \int \text{ tr res } (\partial_{i+1} \ldots \partial_m X_g \partial_1 \ldots \partial_i ((\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1}) - \partial_i)_+ \\ &\qquad - \partial_i \ldots \partial_m X_g \partial_1 \ldots \partial_{i-1} (\partial_i (\partial_{i+1} \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1})_-)_+) . \end{aligned} \tag{3.11}$$

If one would replace the $-$ subscripts by $+$ subscripts, this expression would vanish (since then the external $+$ subscripts could be dropped and both terms cancel). Since $(\ldots)_- = (\ldots) - (\ldots)_+$, one can thus simply drop the $-$ subscripts to obtain

$$\begin{aligned} &\{\tilde{f}(P), \tilde{g}(P)\} = a \sum_{i=1}^{m} (S_{i+1} - S_i) , \\ &S_i = \int \text{ tr res} \, \partial_i \ldots \partial_m X_g \partial_1 \ldots \partial_{i-1} (\partial_i \ldots \partial_m X_f \partial_1 \ldots \partial_{i-1})_+ . \end{aligned} \tag{3.12}$$

Performing the sum, all S_i, cancel except for

$$S_{m+1} = \int \text{tr res } X_g \partial_1 \dots \partial_m (X_f \partial_1 \dots \partial_m)_+ = \int \text{tr res } X_g L (X_f L)_+ \,,$$
$$S_1 = \int \text{tr res } \partial_1 \dots \partial_m X_g (\partial_1 \dots \partial_m X_f)_+ = \int \text{tr res } L X_g (L X_f)_+ \,, \quad (3.13)$$

so that

$$\{\tilde{f}(P), \tilde{g}(P)\} = a(S_{m+1} - S_1) = a \int \text{ tr res } (L(X_f L)_+ X_g - (L X_f)_+ L X_g) \,, \quad (3.14)$$

which completes the proof.

The previous proposition states that one can either compute $\{U_k, U_l\}$ using the complicated formula (2.23) or using the simple Poisson bracket (3.1) for more or less complicated functionals $U_k(P)$ and $U_k(P)$. In particular Lemma 3.1 implies the

Corollary 3.4. *The second Gelfand–Dikii bracket* (2.23) *obeys antisymmetry and the Jacobi identity. Bilinearity in f and g being evident, it is well defined Poisson bracket.*

3.2. The Case $U_1 = 0$. As seen from (3.4), $U_1 = 0$ corresponds to $\sum_{i=1}^m P_i = 0$. In order to describe the reduction to $\sum_i P_i = 0$ it is convenient to go from the $P_i, i = 1, \dots m$ to a new "basis": $Q = \sum_{i=1}^m P_i$ and $\mathscr{P}_a, a = 1, \dots m-1$, where all $\mathscr{P}_a$ lie in the hyperplane $Q = 0$. Of course, Q and each $\mathscr{P}_a$ are still $n \times n$-matrices. More precisely:

Definition and Lemma 3.5. *Consider a $(m-1)$-dimensional vector space, and choose an overcomplete basis of m vectors h_j, $j = 1, \dots m$. Denote the components of each h_j by h_j^a, $a = 1, \dots m-1$. Choose the h_j be such that*

$$\sum_{j=1}^m h_j = 0 \,,$$
$$h_i \cdot h_j = \delta_{ij} - \frac{1}{m} \,,$$
$$\sum_{i=1}^m h_i^a h_i^b = \delta_{ab} \,, \quad (3.15)$$

and define the completely symmetric rank-3 tensor D_{abc} by

$$D_{abc} = \sum_{i=1}^m h_i^a h_i^b h_i^c \,. \quad (3.16)$$

Define Q and $\mathscr{P}_a, a = 1, \dots m-1$ to be the following linear combinations of the P_j:

$$\mathscr{P}_a = \sum_{j=1}^m h_j^a P_j \,, \quad Q = \sum_{j=1}^m P_j \,. \quad (3.17)$$

If one considers the P_j as an orthogonal basis in a m-dimensional vector-space, then the $\mathscr{P}_a$ are an orthogonal basis in a $(m-1)$-dimensional hyperplane orthogonal to the line spanned by Q. Equation (3.17) *is inverted by*

$$P_i = \frac{1}{m} Q + \sum_{a=1}^{m-1} h_i^a \mathscr{P}_a \,. \quad (3.18)$$

Proof. First, note that the vectors h_j with the desired properties (3.15) exist, since one can choose them to be the weight vectors of the vector representation of $SU(m) \sim A_{m-1}$. Next, considering the P_i as orthogonal means that one formally introduces some inner product $(P_i, P_j) = \delta_{ij}$. Then obviously from the definitions (3.17) and the properties (3.15) of the h_j one has $(\mathscr{P}_a, \mathscr{P}_b) = \delta_{ab}$, as well as $(Q, \mathscr{P}_a) = 0$. Finally, Eq. (3.18) also is an immediate consequence of (3.17) and (3.15).

Thus it is convenient to use Q and the $\mathscr{P}_a$ to discuss the reduction to $Q = 0$. Note that the occurence of the weights of the vector representation of A_{m-1} is not surprising since in the scalar case, $n = 1$, the reduction to $Q = 0$ is well-known to be related to A_{m-1} via the Drinfeld-Sokolov reduction [22].

By Lemma 3.5 any functional of the P_i can be written as a functional of the $\mathscr{P}_a$ and Q. One has

Lemma 3.6. *If for functionals $f(\mathscr{P}, Q), g(\mathscr{P}, Q)$ one denotes*

$$V_a = \frac{\delta f}{\delta \mathscr{P}_a} \quad , \quad V_0 = \frac{\delta f}{\delta Q} \quad , \quad W_a = \frac{\delta g}{\delta \mathscr{P}_a} \quad , \quad W_0 = \frac{\delta g}{\delta Q} \, , \tag{3.19}$$

then the Poisson bracket (3.1) *of f with g is* [9]

$$\begin{aligned} \{f, g\} = a \int \mathrm{tr} \bigg(& m W_0 V_0' + W_b V_b' - [V_0, W_0] Q - \frac{1}{m} [V_b, W_b] Q \\ & - [V_0, W_b] \mathscr{P}_b - [V_b, W_0] \mathscr{P}_b - D_{bcd} [V_b, W_c] \mathscr{P}_d \bigg) \, , \end{aligned} \tag{3.20}$$

where here and in the following repeated vector indices (b, c, d) are to be summed over.

Proof. From (3.17) one has $\frac{\delta f}{\delta P_i} = \frac{\delta f}{\delta Q} + h_i^a \frac{\delta f}{\delta \mathscr{P}_a} = V_0 + h_i^a V_a$ and similarly $\frac{\delta g}{\delta P_i} = W_0 + h_i^b W_b$. Inserting these relations into (3.1) and using relations (3.15), (3.16) and (3.18) yields (3.20).

Proposition 3.7. *The Poisson bracket* (3.1) *can be reduced to the symplectic submanifold with $Q \equiv \sum_{j=1}^{m} P_j = 0$. The reduced Poisson bracket is*

$$\{f, g\} = a \int \mathrm{tr} \left(W_b V_b' - [V_b, W_c] D_{bcd} \mathscr{P}_d - \frac{1}{m} [V_b, \mathscr{P}_b] \partial^{-1} [W_c, \mathscr{P}_c] \right) \, , \tag{3.21}$$

where V_b, W_b are defined in Eq. (3.19), *or equivalently*

$$\left\{ \int \mathrm{tr}\, F \mathscr{P}_b, \int \mathrm{tr}\, G \mathscr{P}_c \right\} = a \int \mathrm{tr} \left(G F' \delta_{bc} - [F, G] D_{bcd} \mathscr{P}_d - \frac{1}{m} [F, \mathscr{P}_b] \partial^{-1} [G, \mathscr{P}_c] \right) . \tag{3.22}$$

Proof. The reduced bracket is obtained from the unreduced one, Eq. (3.20), by determining V_0 and W_0 such that the Poisson bracket of Q with any functional g

[9] One should not confuse the scale factor a in front of the integral sign with the vector indices b, c, d.

vanishes on the submanifold $Q = 0$. From (3.20) one has

$$\{\int \operatorname{tr} V_0 Q, g\} = a \int \operatorname{tr} V_0(-mW_0' + [Q, W_0] + [\mathscr{P}_b, W_b]) . \tag{3.23}$$

The vanishing of the r.h.s. for $Q = 0$ gives

$$W_0 = \frac{1}{m}\partial^{-1}[\mathscr{P}_b, W_b] \tag{3.24}$$

and similarly $V_0 = \frac{1}{m}\partial^{-1}[\mathscr{P}_b, W_b]$. Inserting these relations into (3.20) and setting $Q = 0$ yields (3.21). Equation (3.22) follows obviously from (3.21).

Of course, the result (3.22) can also be expressed in terms of the P_i directly. Using (3.18) for $Q = 0$, i.e. $P_i = h_i^a \mathscr{P}_a$, one immediately obtains from (3.22), using the relations (3.15) and (3.16), the

Corollary 3.8. *The Poisson bracket* (3.2) *when reduced to* $Q = \sum_{j=1}^m P_j = 0$ *can be equivalently written as*

$$\{f, g\} = a\int \operatorname{tr}\left(GF'\left(\delta_{ij} - \frac{1}{m}\right) - [F, G]\left(\delta_{ij} - \frac{2}{m}\right)\frac{1}{2}(P_i + P_j) - \frac{1}{m}[F, P_i]\partial^{-1}[G, P_j]\right) . \tag{3.25}$$

To prove the main result of this section one needs the following

Lemma 3.9. *Let as before* $X_f = \sum_{l=1}^m \partial^{-l} X_l$, $\partial_i = \partial - P_i$, *so that* $L = -\partial_1 \dots \partial_m$, *and let*

$$\pi_i = \partial_{i+1} \dots \partial_m X_f \partial_1 \dots \partial_{i-1} , \tag{3.26}$$

then

$$\sum_{i=1}^m [\partial_i, res\ \pi_i] = res\,[X_f, L] . \tag{3.27}$$

Proof. By Lemma A.5, one has $[\partial_i, \operatorname{res} \pi_i] = \operatorname{res}\ [\partial_i, \pi_i]$ so that the l.h.s. of (3.27) is

$$\begin{aligned} \operatorname{res} \sum_{i=1}^m (\partial_i \pi_i - \pi_i \partial_i) &= \operatorname{res} \sum_{i=1}^m (\partial_i \dots \partial_m X_f \partial_1 \dots \partial_{i-1} - \partial_{i+1} \dots \partial_m X_f \partial_1 \dots \partial_i) \\ &= \operatorname{res}\ (\partial_1 \dots \partial_m X_f - X_f \partial_1 \dots \partial_m) = \operatorname{res}\ [X_f, L] . \end{aligned} \tag{3.28}$$

Now one has the

Theorem 3.10. *Let* $U_1 = 0$ *and hence* $Q = \sum_{j=1}^m P_j = 0$. *By the Miura transformation of Lemma* 3.2 *any functionals* $f(U), g(U)$ *of the* $U_k, k = 2, \dots m$ *only are also functionals* $\tilde{f}(\mathscr{P}) = f(U(\mathscr{P}))$, $\tilde{g}(\mathscr{P}) = g(U(\mathscr{P}))$ *of the* $\mathscr{P}_a, a = 1, \dots m$ *only. The reduced second Gelfand–Dikii bracket* (2.31) *of* f *and* g *equals the reduced Poisson bracket* (3.21) *of* $\tilde{f}$ *and* $\tilde{g}$.

Proof. Recall that for $U_1 = 0$ the reduced bracket (2.31) equals the unreduced one $\{f, g\}_{(2)} = a \int \operatorname{tr}\ \operatorname{res}\ (L(X_f L)_+ X_g - (LX_f)_+ LX_g)$ supplemented by the constraint $\operatorname{res}\ [X_f, L] = 0$ which determines X_m. Recall also that for $\sum_{j=1}^m P_j = 0$ the reduced bracket (3.21) equals the unreduced one, Eq. (3.1), which can be written in the form (3.6), supplemented by the constraint that the functional derivative $\frac{\delta g}{\delta Q}$ does not appear, or in other words, as seen from (3.6), that $\sum_{i=1}^{m}[\partial_i, \frac{\delta \tilde{f}}{\delta P_i}] = 0$. By Theorem 3.3, the unreduced brackets are equal. Hence the reduced ones will be equal if the reducing constraints are equivalent. But since from (3.9) one has $\frac{\delta \tilde{f}}{\delta P_i} = \text{res } \pi_i$, the previous lemma shows that res $[X_f, L] = 0$ is equivalent to $\sum_{i=1}^{m}[\partial_i, \frac{\delta \tilde{f}}{\delta P_i}] = 0$, and the theorem follows.

Corollary 3.11. *The second Gelfand–Dikii bracket* (2.31) *obeys antisymmetry and the Jacobi identity. Bilinearity in f and g being evident, it is a well-defined Poisson bracket.*

Example 3.12. Consider the example $m = 2$. The h_j can be taken to be the negative of the weight vectors of $SU(2)$ which are one-dimensional: $h_1 = -\frac{1}{\sqrt{2}}, h_2 = \frac{1}{\sqrt{2}}$. Then there is only one $\mathscr{P}$ which by (3.18) equals $\mathscr{P} = \sqrt{2}P_2 = -\sqrt{2}P_1$, and $U \equiv U_2 = \frac{1}{\sqrt{2}}\mathscr{P}^2 + \frac{1}{\sqrt{2}}\mathscr{P}'$. Since $D_{bcd} = 0$ for $m = 2$, the Poisson bracket (3.22) becomes

$$\{\textstyle\int \text{tr } F\mathscr{P}, \int \text{tr } G\mathscr{P}\} = a \int \text{tr}\left(GF' - \frac{1}{2}[F,\mathscr{P}]\partial^{-1}[G,\mathscr{P}]\right). \tag{3.29}$$

It can be easily checked directly that this implies the second Gelfand–Dikii bracket (2.35).

4. The Conformal Properties

In the scalar case, i.e. for $n = 1$, the second Gelfand–Dikii bracket (with $U_1 = 0$) gives the W_m-algebras [5, 6, 7]. The interest in the W-algebras stems from the fact that they are extensions of the conformal Virasoro algebra, i.e. they contain the Virasoro algebra as a subalgebra. Furthermore, in the scalar case, it is known that certain combinations of the U_k and their derivatives yield primary fields of integer spins $3, 4 \ldots m$. It is the purpose of this section to establish the same results for the matrix case, $n > 1$. Throughout this section, I only consider the second Gelfand–Dikii bracket (2.31) for the case $U_1 = 0$. I will simply write $\{f, g\}$ instead of $\{f, g\}_{(2)}$. Also, it is often convenient to replace the scale factor a by γ^2 related to a by

$$a = -2\gamma^2 . \tag{4.1}$$

(Note that γ^2 need not be positive.)

4.1. The Virasoro Subalgebra. For the V-algebra (1.1) given in the introduction (corresponding to $m = 2, n = 2$ and an additional constraint tr $\sigma_2 U_2 = 0$) one sees that $T = \frac{1}{2}\text{tr } U_2$ generates the conformal algebra. I will now show that for general m, n the generator of the conformal algebra is still given by this formula.

Lemma 4.1 *For arbitrary $m \geqq 2$ one has*

$$\left\{\int \operatorname{tr} FU_2, \int \operatorname{tr} GU_2\right\} = a\int \operatorname{tr}\left(-\frac{1}{m}[F,U_2]\partial^{-1}[G,U_2] - [F,G]\left(U_3 - \frac{m-2}{2}U_2'\right) + \frac{1}{2}(F'G - G'F + GF' - FG')U_2 - \frac{1}{2}\binom{m+1}{3}GF'''\right). \tag{4.2}$$

Note that for $m = 2$ one has to set $U_3 = 0$.

Proof. From Eq. (2.31) one knows that the l.h.s. of (4.2) equals $a \int \operatorname{tr} \tilde{\mathscr{V}}(f)_{m-2}G$ with $f = \int \operatorname{tr} FU_2$, i.e. $X_l = F\delta_{l,m-1}$. Still from (2.31) one has (recalling $U_1 = 0$, $U_0 = -1$)

$$\tilde{\mathscr{V}}(f)_{m-2}\big|_{X_l = F\delta_{l,m-1}} = \frac{1}{m}[U_2, \partial^{-1}[F,U_2]] + \frac{1}{m}\binom{m}{2}(F'U_2 + (U_2F)') + \sum_{p=0}^{3}\sum_{q=\max(0,p+m-3)}^{\min(m,p+2m-3)} C^{q,m-2}_{q-p,m-1} U_{m-q}(FU_{q-p+3-m})^{(p)} \tag{4.3}$$

At this point one needs explicit expressions for the coefficients $C^{q,m-2}_{q-p,m-1}$, i.e. of the $S^{q,m-2}_{q-p,m-1}$. Note that the latter are non-vanishing only if $q - p \leqq \min(q, m-2)$, i.e. $q \leqq p + m - 2$. Some of them follow from Lemma 2.4. The others have to be obtained directly from the definition (2.14) which is not difficult since the sum involves at most $(m-2) - (q-p) + 1 \leqq (m-2) - (m-3) + 1 = 2$ terms. Since $U_1 = 0$ only five C-coefficients are needed. They are $C^{m-3,m-2}_{m-3,m-1} = 1$, $C^{m,m-2}_{m,m-1} = -1$, $C^{m-2,m-2}_{m-3,m-1} = C^{m,m-2}_{m-1,m-1} = \frac{m-3}{2}$, $C^{m,m-2}_{m-3,m-1} = -\frac{(m+1)m(m-1)}{12}$. Inserting this into (4.3) and performing some simple algebra gives (4.2).

Proposition 4.2 *Let $T(\sigma) = \frac{1}{2}tr\ U_2(\sigma)$. Then*

$$\gamma^{-2}\{T(\sigma_1), T(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})(T(\sigma_2)\delta(\sigma_1 - \sigma_2)) - \frac{n}{4}\binom{m+1}{3}\delta'''(\sigma_1 - \sigma_2). \tag{4.4}$$

Equivalently, if, for $\sigma \in S^1$, one defines for integer r,

$$L_r = \gamma^{-2}\int_{-\pi}^{\pi} d\sigma\, T(\sigma)e^{ir\sigma} + \frac{c}{24}\delta_r, 0\,, \tag{4.5}$$

where

$$c = \frac{6\pi}{\gamma^2}n\binom{m+1}{3} = \frac{12\pi}{(-a)}n\binom{m+1}{3}, \tag{4.6}$$

then the L_r form a Poisson bracket version of the Virasoro algebra with (classical) central charge c:

$$i\{L_r, L_s\} = (r-s)L_{r+s} + \frac{c}{12}(r^3 - r)\delta_{r+s,0}\,. \tag{4.7}$$

Also if $\{A_\mu\}_{\mu=1,\ldots n^2-1}$ is a basis for the traceless $n \times n$-matrices, then each $S_\mu\{\sigma\} =$ tr $A_\mu U_2(\sigma), \mu = 1, \ldots n^2 - 1$ is a conformally primary field of conformal dimension *(spin) 2 :*

$$\gamma^{-2}\{T(\sigma_1), S_\mu(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})(S_\mu(\sigma_2)\delta(\sigma_1 - \sigma_2)), \tag{4.8}$$

or for the modes $(S_\mu)_r = \gamma^{-2}\int_{-\pi}^{\pi} d\sigma\, S_\mu(\sigma)e^{ir\sigma}$ *one has equivalently* $i\{L_r, (S_\mu)_s\} = (r-s)(S_\mu)_{r+s}$. *Equations* (4.4) *and* (4.8) *can be written in matrix notation as* (**1** *denotes the* $n \times n$ *unit matrix)*

$$\gamma^{-2}\{T(\sigma_1), U_2(\sigma_2)\} = (\partial_{\sigma_1} - \partial_{\sigma_2})(U_2(\sigma_2)\delta(\sigma_1 - \sigma_2)) - \frac{1}{2}\binom{m+1}{3}\mathbf{1}\delta'''(\sigma_1 - \sigma_2). \tag{4.9}$$

Proof. Consider first (4.2) with $F(\sigma) = \frac{1}{2}\delta(\sigma - \sigma_1)\mathbf{1}$ and $G(\sigma) = \frac{1}{2}\delta(\sigma - \sigma_2)\mathbf{1}$. Then the l.h.s. of (4.2) is $\{T(\sigma_1), T(\sigma_2)\}$ while on the r.h.s. all commutator terms vanish. Recall $a = -2\gamma^2$ and tr $\mathbf{1} = n$ and Eq. (4.4) follows. It is then standard (and straightforward) to show that (4.5), (4.6) imply (4.7). Finally let $F(\sigma) = \frac{1}{2}\delta(\sigma - \sigma_1)\mathbf{1}$ and $G(\sigma) = \delta(\sigma - \sigma_2)A_\mu$. Again, all commutator terms vanish in (4.2) and (4.8) follows. Equation (4.9) then is obvious.

Note that (4.6) is a classical central charge. If one can implement a free-field quantization, the central charge receives additional normal-ordering contributions expected to be $(m-1)n^2$ so the $c_{\text{tot}} = n(m-1)\left(n + \frac{\pi}{\gamma^2}m(m+1)\right)$. One could speculate about series of unitary representation, etc., but I will not do so here.

4.2. The Conformal Properties of the U_k *for* $k \geqq 3$. In the previous subsection, I have computed the conformal properties of the matrix elements of U_2. The aim of this subsection is to give those for all other U_k i.e. compute $\{T(\sigma_1), U_k(\sigma_2)\}$ or equivalently, for any (test-) function $\varepsilon(\sigma)$, $\{\int \varepsilon T, U_k(\sigma_2)\}$ for all $k \geqq 3$. I will find that this Poisson bracket is linear in the U_l and their derivatives and is formally identical to the result of the scalar case. It then follows that appropriately symmetrized combinations W_k can be formed that are $n \times n$-matrices, each matrix element of W_k being a conformal primary field of dimension (spin) k.

Lemma 4.3. *For a scalar function* ε *and a* $n \times n$*-matrix-valued function* F, *one has*

$$\gamma^{-2}\{\int \varepsilon T, \int \operatorname{tr} FU_k\} = \int \varepsilon \operatorname{tr} \tilde{\mathscr{V}}(f)_{m-2}\big|_{X_l = F\delta_{l,\,m+1-k}} \tag{4.10}$$

with

$$\operatorname{tr} \tilde{\mathscr{V}}(f)_{m-2}\big|_{X_l = F\delta_{l,\,m+1-k}} = \frac{1}{m}\sum_{q=0}^{k-1}(-)^q\binom{m-k+q+1}{m-k}\operatorname{tr}(FU_{k-1-q})^{(q)}U_2 + \frac{m-1}{2}\operatorname{tr}(U_kF)' + \sum_{p=0}^{k+1}\ \sum_{q=\max(0,\,k+1-p-m)}^{\min(m,\,k+1-p)} C^{m-q,\,m-2}_{m-q-p,\,m+1-k}\operatorname{tr} U_q(FU_{k+1-p-q})^{(p)}. \tag{4.11}$$

Proof. The l.h.s. of (4.10) equals $-\frac{1}{2\gamma^2}\{\int \operatorname{tr} FU_k, \int \varepsilon\mathbf{1}T\}$ which by (2.31) and (4.1) equals $\int \operatorname{tr}\sum_{j=0}^{m-2}\tilde{\mathscr{V}}(f)_j Y_{j+1}$ with $Y_{j+1} = \varepsilon\mathbf{1}\delta_{j,\,m-2}$ and $X_l = F\delta_{m+1-l,\,k}$ which proves (4.10). Equation (4.11) then follows upon inserting $j = m-2$ and X_l into Eq. (2.31) for $\tilde{\mathscr{V}}(f)_j$ and changing the summation index in the last sum from q to $m-q$.

Lemma 4.4. *When evaluating the sums in the third term on the r.h.s of* (4.11) *one has to distinguish three cases: The coefficients* $C^{m-q,m-2}_{m-q-p,m+1-k}$ *defined in Eq.* (2.31), *are given by* $(2 \leqq k \leqq m)$[10]
a) $q \geqq 2$ *and* $q \leqq k-1-p$

$$C^{m-q,m-2}_{m-q-p,m+1-k} = \frac{1}{m}(-)^{p+1}\delta_{q,2}\binom{m-k+p+1}{m-k}\,, \tag{4.12}$$

b) $q \geqq 2$ *and* $q = k+1-p$

$$C^{m-q,m-2}_{m-q-p,m+1-k} = (-)^p\binom{1}{p} + \frac{1}{m}(-)^k\delta_{p,k-1}\binom{m}{k}\,, \tag{4.13}$$

c) $q = 0$

$$C^{m-q,m-2}_{m-q-p,m+1-k} = (-)^p\frac{m+1}{2}\binom{m-k+p-1}{m-k} - (-)^p\binom{m-k+p}{m-k}\,. \tag{4.14}$$

Proof. First note that since $U_1 = 0$ no terms with $q = 1$ or $q = k-p$ are present in the sum considered in (4.11). Thus one only has to distinguish $q = 0$ or $q \geqq 2$ on the one hand, and $q \leqq k-1-p$ or $q = k-p+1$ on the other hand. Hence one has the cases a) and b), while in case c) one should distinguish $p \leqq k-1$ and $p = k+1$. For cases a) and b), $q \geqq 2$ and $m-q \leqq m-2$, so that one can use Lemma 2.4 to evaluate $S^{m-q,m-2}_{m-q-p,m+1-k}$. In case a) Eq. (2.20) applies and one gets Eq. (4.12). In case b), Eq. (2.21) applies if also $q \leqq m-p$, i.e. $k < m$, so that $S^{m-q,m-2}_{m-q-p,m+1-k} = (-)^p\binom{q+p-k}{p} = (-)^p\binom{1}{p}$. If however $k = m$, so that $q = m - p + 1$ (which is only possible for $p \geqq 1$), then $S^{m-q,m-2}_{m-q-p,m+1-k} = S^{m-q,m-2}_{-1,1}$ which is easily evaluated directly from the definition (2.14) to be $-\delta_{q,m} = -\delta_{p,1} = (-)^p\binom{1}{p}$ since $p \geqq 1$. Hence (4.13) follows. In case c), one has $q = 0$, hence $m-q > m-2$ and Lemma 2.4 does not apply directly. However, using Eq. (B.8) first, one can still use Lemma 2.4, i.e. Eq. (2.20) if $p \leqq k-1$ and Eq. (2.21) if $p = k+1$ ($p = k$, $q = 0$ gives a U_1 and does not contribute). For $p \leqq k-1$ it is straightforward to obtain (4.14). For $p = k+1$ one has still to distinguish $k = m$ and $k < m$, but the result is the same in both situations, and it differs from (4.14) only by a term $\binom{k-1}{p}$ which vanishes since $p = k+1$. Hence one obtains (4.14) again.

Proposition 4.5. *The conformal properties of all matrix elements of all* U_k, $k = 2, \ldots m$ *are given by*

$$\begin{aligned}\gamma^{-2}\{\textstyle\int\varepsilon T, U_k\} = & -\varepsilon U_k' - k\varepsilon' U_k + \frac{k-1}{2}\binom{m+1}{k+1}\varepsilon^{(k+1)} \\ & + \sum_{l=2}^{k-1}\left[\binom{m-l}{k+1-l} - \frac{m-1}{2}\binom{m-l}{k-l}\right]\varepsilon^{(k-l+1)}U_l,\end{aligned} \tag{4.15}$$

which is formally the same equation as in the scalar case $n = 1$.

[10] The equations are actually also valid for $k = 1$.

Proof. That (4.15) is the same equation as in the scalar case is readily seen by comparing with Eq. (2.10) of ref. 7, setting $\gamma^2 = 1$ and observing that the a_l of ref. 7 correspond to the present $-U_l$ (for $n = 1$), and thus also a_2 corresponds to $-T$. Let me now prove (4.15) in the matrix case. Note that for $k = 2$, Eq. (4.15) is equivalent to (4.9). Hence one only needs to consider $k \geqq 3$. One starts with Eq. (4.10) and inserts the results of Lemma 4.4 into Eq. (4.11). Case a) can be realized if $2 \leqq q \leqq k - 1 - p$, i.e. for $p \leqq k - 3$. Case b) can be realized if $p \leqq k - 1$, while case c) can be realized if $k + 1 - m \leqq p$. After some simple algebra one gets

$$\operatorname{tr} \tilde{\mathscr{V}}(f)_{m-2}\big|_{X_l = F\delta_{l,\,m+1-k}} = \operatorname{tr}\left(\frac{m-1}{2}(U_kF)' - U_{k+1}F + U_kF'\right) - \sum_{p=\max(0,\,k+1-m)}^{k+1} (-)^p \left[\frac{m+1}{2}\binom{m-k+p-1}{m-k} - \binom{m-k+p}{m-k}\right] \operatorname{tr}\,(FU_{k+1-p})^{(p)} \tag{4.16}$$

(where for $k = m$ one sets $U_{k+1} = U_{m+1} = 0$). Note that terms like $\operatorname{tr}(FU_{k-1-p})^{(p)}U_2$ cancelled against terms $\operatorname{tr} U_2(FU_{k-1-p})^{(p)}$ which would not have been the case without taking the trace. Let first $k < m$ so that the sum over p is from 0 to $k + 1$. Separate the $p = k + 1, p = 0$ and $p = 1$ terms from the sum (the $p = k$ term vanishes since $U_1 = 0$). Using the identity

$$\frac{m+1}{2}\binom{m-k+p-1}{m-k} - \binom{m-k+p}{m-k} = \frac{m-1}{2}\binom{m-k+p-1}{p-1} - \binom{m-k+p-1}{p}, \tag{4.17}$$

one obtains

$$\begin{aligned}\operatorname{tr} \tilde{\mathscr{V}}(f)_{m-2}\big|_{X_l = F\delta_{l,\,m+1-k}} &= \operatorname{tr}\Bigg(U_kF' + (k-1)(U_kF)' \\ &\quad + (-)^{k+1}\frac{k-1}{2}\binom{m+1}{k+1}F^{(k+1)}\Bigg) \\ &\quad + \sum_{p=2}^{k-1}(-)^p\Bigg[\binom{m-k+p-1}{p} \\ &\quad - \frac{m-1}{2}\binom{m-k+p-1}{p-1}\Bigg]\operatorname{tr}\,(FU_{k+1-p})^{(p)}\,.\end{aligned} \tag{4.18}$$

It is easily seen that for $k = m$ one also obtains the same equation (4.18). Now multiply by ε and integrate to obtain $\gamma^{-2}\{\int \varepsilon T, \int \operatorname{tr} FU_k\}$ (cf. Eq. (4.10)). Upon taking the functional derivative with respect to F and relabelling the summation index $p = k + 1 - l$ one obtains (4.15).

Since the conformal properties (4.15) are formally the same as in the scalar case, and in the latter case it was possible to form combinations W_k that are spin-k conformally primary fields, one expects a similar result to hold in the matrix case. Indeed, one has the

Theorem 4.6. *For matrices $A_1, A_2, \ldots A_r$ denote by $S[A_1, A_2, \ldots, A_r]$ the completely symmetrized product normalized to equal A^r if $A_s = A$ for all $s = 1, \ldots, r$. Let*

$$W_k = \sum_{l=2}^{k} B_{kl} U_l^{(k-l)} + \sum_{\substack{0 \leqq p_1 \leqq \cdots \leqq p_r \\ \sum p_i + 2r = k}} (-)^{r-1} C_{p_1 \ldots p_r} S[U_2^{(p_1)}, \ldots, U_2^{(p_r)}]$$
$$+ \sum_{\substack{0 \leqq p_1 \leqq \cdots \leqq p_r \\ s \leqq l \leqq k - \sum p_i - 2r}} (-)^r D_{p_1 \ldots p_r, l} S[U_2^{(p_1)}, \ldots, U_s^{(p_r)}, U_l^{(k-l-\sum p_i - 2r)}] , \quad (4.19)$$

where the coefficients $B_{kl}, C_{p_1 \ldots p_r}$ and $D_{p_1 \ldots p_{r,l}}$ are the same as those given in ref. 7 for the scalar case, in particular

$$B_{kl} = (-)^{k-l} \frac{\binom{k-1}{k-l}\binom{m-l}{k-l}}{\binom{2k-2}{k-l}} . \quad (4.20)$$

Then the W_k are spin-k conformally primary $n \times n$-matrix-valued fields, i.e.

$$\gamma^{-2}\{\textstyle\int \varepsilon T, W_k\} = -\varepsilon W_k' - k\varepsilon' W_k . \quad (4.21)$$

For $\sigma \in S^1$ one can define the modes $(W_k)_s = \gamma^{-2} \int_{-\pi}^{\pi} \mathrm{d}\sigma\, W_k(\sigma) e^{is\sigma}$ and the Virasoro generators L_r as in (4.5). *Then one has equivalently*

$$i\{L_r, (W_k)_s\} = ((k-1)r - s)(W_k)_{r+s} , \quad (4.22)$$

where each $(W_k)_s$ is a $n \times n$-matrix.

Proof. Note that in the scalar case Eq. (4.19) is identical to the formula (2.11a) of ref. 7 if one identifies $U_l = -a_l, W_k = -w_k$. The crucial property to prove the theorem is that (4.15) is at most linear in the U_l. It follows that

$$\gamma^{-2}\{\textstyle\int \varepsilon T, S[U_{k_1}^{(p_1)}, U_{k_2}^{(p_2)}, \ldots, U_{k_r}^{(p_r)}]\}$$
$$= \sum_{i=1}^{r} S\left[U_{k_1}^{(p_1)}, \ldots, U_{k_{i-1}}^{(p_{i-1})}, \gamma^{-2}\{\textstyle\int \varepsilon T, U_{k_i}\}^{(p_i)}, U_{ki+1}^{(p_{i+1})}, \ldots, U_{kr}^{(p_r)}\right], \quad (4.23)$$

and since matrices commute under the symmetrization $S[\ldots]$ one may manipulate them just as in the scalar case. Hence Eq. (4.19) can be proven exactly as in the scalar case and thus follows from Eq. (4.15) and the results of ref. 7.

Examples. From the previous theorem and the results of ref. 7 (Table I) one has explicitly:

$$W_3 = U_3 - \frac{m-2}{2} U_2' ,$$
$$W_4 = U_4 - \frac{m-3}{2} U_3' + \frac{(m-2)(m-3)}{10} U_2'' + \frac{(5m+7)(m-2)(m-3)}{10m(m^2-1)} U_2^2 ,$$
$$W_5 = U_5 - \frac{m-4}{2} U_4' + \frac{3(m-3)(m-4)}{28} U_3'' - \frac{(m-2)(m-3)(m-4)}{84} U_2'''$$
$$+ \frac{(7m+13)(m-3)(m-4)}{14m(m^2-1)} (U_2 W_3 + W_3 U_2) . \quad (4.24)$$

Note that all coefficients are such that $W_k = 0$ for $k > m$ if one sets $U_l = 0$ for $l > m$. These relations can be inverted to give

$$
\begin{aligned}
U_3 &= W_3 + \frac{m-2}{2} U_2' , \\
U_4 &= W_4 + \frac{m-3}{2} W_3' + \frac{3(m-2)(m-3)}{20} U_2'' - \frac{(5m+7)(m-2)(m-3)}{10m(m^2-1)} U_2^2 , \\
U_5 &= W_5 + \frac{m-4}{2} W_4' + \frac{(m-3)(m-4)}{7} W_3'' + \frac{(m-2)(m-3)(m-4)}{30} U_2''' \\
&\quad - \frac{(7m+13)(m-3)(m-4)}{14m(m^2-1)} (U_2 W_3 + W_3 U_2) \\
&\quad - \frac{(5m+7)(m-2)(m-3)(m-4)}{20m(m^2-1)} (U_2^2)' . \qquad (4.25)
\end{aligned}
$$

5. The Poisson Bracket Algebra of the U_2 and W_3 for Arbitrary m

From the previous subsection one might have gotten the impression that the matrix case is not very different from the scalar case. This is however not true. In the previous subsection only the conformal properties, i.e. the Poisson brackets with $T = \frac{1}{2}\text{tr } \mathbf{1} U_2$ were studied, and since the unit-matrix $\mathbf{1}$ always commutes, most of the new features due to the non-commutativity of matrices were not seen. Technically speaking, only tr $\tilde{\mathscr{V}}(f)$ was needed, not $\tilde{\mathscr{V}}(f)$ itself. In this section, I will give the Poisson brackets, for the (more interesting) reduction to $U_1 = 0$, of any two matrix elements of U_2 or U_3, or equivalently of U_2 or W_3, for arbitrary m. In the case $m = 3$ this is the complete algebra, giving a matrix generalization of Zamolodchikov's W_3-algebra.

The Poisson bracket algebra will again be obtained from (2.31). Since $\{\int \text{tr } FU_2, \int \text{tr } GU_2\}$ was already computed in the previous section, Eq. (4.2), here I will need to compute $\{\int \text{tr } FU_3, \int \text{tr } GU_2\}$ and $\{\int \text{tr } FU_3, \int \text{tr } GU_3\}$ only. Thus all one needs is $\tilde{\mathscr{V}}(f)_j$ for $j = m-2, m-3$ and with $X_l = F\delta_{l,m-2}$. For $j = m-2$ all relevant coefficients $C^{m-q,m-2}_{m-q-p,m-2}$ are given in Lemma 4.4 (with $k = 3$), and the computation of $\tilde{\mathscr{V}}(f)_{m-2}|_{X_l = F\delta_{l,m-2}}$ proceeds as in the proof of Proposition 4.5 (but without discarding terms that vanish upon taking the trace. It is then straightforward to obtain $\{\int \text{tr } FU_3, \int \text{tr } GU_2\} = a \int \text{tr } \tilde{\mathscr{V}}(f)_{m-2}|_{X_l = F\delta_{l,m-2}} G$, and using the antisymmetry of the bracket also

$$
\begin{aligned}
\left\{\int \text{tr } FU_2, \int \text{tr } GU_3\right\} &= a \int \text{tr } \Big(-\frac{1}{m}[F, U_2]\partial^{-1}[G, U_3] - \frac{m-2}{m} F[G, U_2]U_2 \\
&\quad - [F, G]U_4 + \frac{(m-1)(m-2)}{6}[F, G'']U_2 - \frac{m-1}{2}[F', G]U_3 \\
&\quad + (2F'G - G'F)U_3 + (m-2)F''GU_2 - \binom{m+1}{4} GF^{(4)} \Big) . \qquad (5.1)
\end{aligned}
$$

Of course, if $F = \frac{1}{2}\varepsilon \mathbf{1}$ all commutator terms vanish and upon taking the functional derivative with respect to G one recovers Eq. (4.15) for $k = 3$ (recall $a = -2\gamma^2$).

Consider now $\tilde{\mathscr{V}}(f)_j$ for $j = m-3$ and with $X_l = F\delta_{l,m-2}$. The relevant sum in (2.31) containing the C-coefficients is

$$\sum_{p=0}^{5} \sum_{q=\max(0,p+m-5)}^{\min(m,p+2m-5)} C^{q,m-3}_{q-p,m-2} U_{m-q}(FU_{q-(p+m-5)})^{(p)} . \tag{5.2}$$

For $m \geqq 5$ the sum over q is simply $\sum_{q=p+m-5}^{m}$. For $m < 5$, one can still formally write $\sum_{q=p+m-5}^{m}$ if one defines $U_k = 0$ for $k < 0$ or $k > m$. In any case one has $q - p \geqq m - 5$. From the definition (2.14) of $S^{q,m-3}_{q-p,m-2}$ one sees that it vanishes for $q - p > m - 3$, and that it is a sum of at most three terms for $m - 3 \geqq q - p \geqq m - 5$. Thus all relevant $S^{q,m-3}_{q-p,m-2}$ can be easily computed directly to obtain the $C^{q,m-3}_{q-p,m-2}$ appearing in (5.2). I will not give all the details here. The result is

$$\begin{aligned}\left\{\textstyle\int \mathrm{tr}\ FU_3, \int \mathrm{tr}\ GU_3\right\} = a\textstyle\int \mathrm{tr}\ \Big(&-\frac{1}{m}[F,U_3]\partial^{-1}[G,U_3] + \frac{2}{m}(FU_2GU_3 - FU_3GU_2)\\ &- \frac{m-2}{m}[F,G]U_2U_3 + \frac{2(m-2)}{m}(FU_2)'GU_2\\ &- [F,G]U_5 + 2(F'G - G'F)U_4 + (G''F - F''G)U_3\\ &+ \frac{(m-1)(m-2)}{6}([F'',G] + [F,G''])U_3\\ &- \frac{(m-1)(m-2)}{3}(F'''G - G'''F)U_2\\ &+ \frac{4m^2-3m-7}{30}\binom{m}{3} GF^{(5)}\Big) ,\end{aligned} \tag{5.3}$$

where $U_5 = 0$ for $m = 4$ and $U_5 = U_4 = 0$ for $m = 3$. Note the obvious antisymmetry under exchange of F and G.

One may also reexpress the Poisson brackets (4.2), (5.1) and (5.3) using the primary fields W_k. Substituting Eqs. (4.25) it is straightforward algebra to obtain

$$\begin{aligned}\left\{\textstyle\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GU_2\right\} = a\textstyle\int \mathrm{tr}\ \Big(&-\frac{1}{m}[F,U_2]\partial^{-1}[G,U_2] - [F,G]W_3\\ &+ \frac{1}{2}(F'G - G'F + GF' - FG')U_2\\ &- \frac{1}{2}\binom{m+1}{3} GF'''\Big) ,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\left\{\textstyle\int \mathrm{tr}\ FU_2, \int \mathrm{tr}\ GW_3\right\} = a\textstyle\int \mathrm{tr}\ \Big(&-\frac{1}{m}[F,U_2]\partial^{-1}[G,W_3] - \frac{4(m^2-4)}{5m(m^2-1)}[F,G]U_2^2\\ &- [F,G]W_4 + (F'G + GF' - \frac{1}{2}FG' - \frac{1}{2}G'F)W_3\\ &+ \frac{m^2-4}{5}\Big(-\frac{1}{4}[F',G'] + \frac{1}{12}[F,G''] + \frac{1}{2}[F'',G]\Big)U_2\Big)\end{aligned} \tag{5.5}$$

[11] One uses the obvious relations $\{f, \int \mathrm{tr}\ GU_2'\} = -\{f, \int \mathrm{tr}\ G'U_2\}$ and also the fact that $\int A\partial^{-1}B = -\int(\partial^{-1}A)B$.

$$
\begin{aligned}
\{\textstyle\int \mathrm{tr}\, FW_3, \int \mathrm{tr}\, GW_3\} &= a \textstyle\int \mathrm{tr}\, \Big(-\frac{1}{m}[F, W_3]\partial^{-1}[G, W_3] \\
&+ \frac{2}{m}(FU_2GW_3 - GU_2FW_3) - \frac{11m^2-71}{7m(m^2-1)}[F,G](W_3U_2 + U_2W_3) \\
&+ \frac{m^2-4}{4m}(F'U_2GU_2 - G'U_2FU_2) + \frac{(m-2)^2}{4m}([F,G]U_2'U_2 + (FG' - GF')U_2^2) \\
&+ \frac{(5m+7)(m-2)(m-3)}{10m(m^2-1)}(FG' - GF' + G'F - F'G)U_2^2 \\
&- [F,G]W_5 - (FG' - GF' + G'F - F'G)W_4 \\
&+ \frac{m^2-16}{84}(2[F,G]'' - 7[F',G'])W_3 \\
&+ \frac{m^2-4}{120}(2(FG''' + G'''F - F'''G - GF''') \\
&+ 3(F''G' + G'F'' - F'G'' - G''F'))U_2 \\
&+ \frac{1}{6}\binom{m+2}{5} GF^{(5)} \Big) . \qquad (5.6)
\end{aligned}
$$

Note again, as a consistency check, that (5.4) and (5.6) are obviously antisymmetric under $F \leftrightarrow G$. Of course, one has to set $W_5 = 0$ for $m = 4$, and $W_5 = W_4 = 0$ for $m = 3$. For $m = 3$, Equations (5.4)–(5.6) are the complete algebra and reproduce Eqs. (1.3)–(1.5) written in the introduction. Remark that for $F = \frac{1}{2}\varepsilon\mathbf{1}$, the r.h.s. of Eq. (5.5) reduces to $-\gamma^2 \int \mathrm{tr}\,(2\varepsilon' G - \varepsilon G')W_3$ (recall $a = -2\gamma^2$), confirming once again that every matrix element of W_3 is a conformal primary field of dimension 3. If both F and G are proportional to the unit matrix $F = f\mathbf{1}, G = g\mathbf{1}$, with scalar f, g, most of the terms in (5.6) disappear and one has

$$
\begin{aligned}
\{\textstyle\int f\, \mathrm{tr}\, W_3, \int g\, \mathrm{tr}\, W_3\} = a \textstyle\int \mathrm{tr}\, \Big(&(m^2-4)(f'g - g'f) \\
&\times \left(\frac{8}{5m(m^2-1)} \mathrm{tr}\, U_2^2 - \frac{1}{15}T'' \right) \\
&+ 2(f'g - g'f)\mathrm{tr}\, W_4 \\
&+ \frac{m^2-4}{6}(f''g' - g''f')T \\
&+ \frac{n}{6}\binom{m+2}{5} g f^{(5)} \Big) , \qquad (5.7)
\end{aligned}
$$

where I used $\mathrm{tr}\, \mathbf{1} = n$ and $\mathrm{tr}\, U_2 = 2T$. This looks similar to the corresponding bracket in the scalar case ($n = 1$). It is different, however, since $\mathrm{tr}\, U_2^2 \neq (\mathrm{tr}\, U_2)^2$ in the matrix case. Thus the standard (scalar) W-algebra is *not* a subalgebra of the $n \neq 1$-algebras.

[12] There is one trivial exception: for $m = 2$ one only has $\{\mathrm{tr}\, FU_2, \int \mathrm{tr}\, GU_2\}$ which is linear, and hence $T = \frac{1}{2}\mathrm{tr}\, U_2$ forms a closed subalgebra, the Virasoro algebra discussed in section 4.

6. Other Algebras, Restrictions and Concluding Remarks

6.1. Hermiticity of the W_k. The (quantum) Virasoro algebra $[L_r, L_s] = (r-s)L_{r+s} + \frac{c}{12}(r^3 - r)\delta_{r+s,0}$ is compatible with the hermiticity condition $L_r^+ = L_{-r}$ (where the hermitian conjugation refers to the inner product on the Hilbert space on which the Virasoro generators act). Similarly, the Poisson bracket version (4.7) is compatible with the reality condition $L_r^* = L_{-r}$. For real γ^2, i.e. real scale factor a this is equivalent to $T^* = T$. The natural extension of this condition to the matrix case is the hermiticity condition $U_2^+ = U_2$ (where now hermitian conjugation is simply the hermitian conjugation of the $n \times n$-matrix). Assuming the matrix U_2 to be hermitian is also natural when studying (for $m = 2$) the resolvent of $L = -\partial^2 + U_2$ [10]. More generally one has the

Conjecture 6.1. For real scale factor a, the second Gelfand–Dikii bracket, Eq. (2.31), is compatible with the hermiticity conditions

$$U_2^+ = U_2\,, \quad W_k^+ = (-)^k W_k\,, \quad k \geqq 3\,. \tag{6.1}$$

Lemma 6.2. *A sufficient condition for the bracket*

$$\{\textstyle\int \mathrm{tr}\; FW_k, \int \mathrm{tr}\; GW_l\} = \int \mathrm{tr}\; P_{kl}(F, G, W_r) \tag{6.2}$$

to be compatible with the conditions (6.1) *is*

$$\textstyle\int \mathrm{tr}\; P_{kl}(F, G, W_r)^+ = (-)^{k+l} \int \mathrm{tr}\; P_{kl}(F^+, G^+, (-)^r W_r^+)\,. \tag{6.3}$$

Proof. Compatibility means that if one takes the hermitian conjugate of (6.2) and uses (6.1) one gets back the *same* bracket (6.2) with the same functional P_{kl} and F and G replaced by F^+ and G^+. Taking the hermitian conjugate of (6.2) yields, using (6.1) and (6.3),

$$\begin{aligned} (-)^{k+l}\{\textstyle\int \mathrm{tr}\; F^+W_k, \int \mathrm{tr}\; G^+W_l\} &= \textstyle\int \mathrm{tr}\; P_{kl}(F, G, W_r)^+ \\ = (-)^{k+l} \textstyle\int \mathrm{tr}\; P_{kl}(F^+, G^+, (-)^r W_r^+) &= (-)^{k+l} \textstyle\int \mathrm{tr}\; P_{kl}(F^+, G^+, W_r)\,, \end{aligned} \tag{6.4}$$

which is again Eq. (6.2) with F and G replaced by F^+ and G^+. Thus (6.3) is a sufficient condition.

Lemma 6.3. *All the second Gelfand–Dikii brackets of the U_2 or the W_k given explicitly in this paper, namely* (4.21) *and Eqs.* (5.4) *to* (5.6), *are compatible with the hermiticity conditions* (6.1).

Proof. By the previous lemma one has to check whether (6.3) is satisfied. For Eq. (4.21) this is trivial. In general however, the condition (6.3) is non-trivial. Consider e.g. Eq. (5.6). Since (6.3) is a linear condition it can be checked on groups of terms separately. For $[F, G]W_5$ one has e.g. $\mathrm{tr}\;([F,G]W_5)^+ = \mathrm{tr}\;([F,G]^+W_5^+) = -\mathrm{tr}\;([F^+,G^+]W_5^+) = \mathrm{tr}\;([F^+,G^+](-)^5W_5^+) = (-)^{3+3}\mathrm{tr}\;([F^+,G^+](-)^5W_5^+)$, while for $(F'G + GF')W_4$ one has $\mathrm{tr}\;((F'G+GF')W_4)^+ = (-)^{3+3}\mathrm{tr}\;\big((F^{+\prime}G^+ + G^+F^{+\prime})(-)^4W_4^+\big)$ and the condition (6.3) is satisfied. On the other hand, a term like

[13] Recall the example of the Virasoro algebra where complex conjugation using $L_r^* = L_{-r}$ gives the same algebra upon relabelling $r \to -r$, $s \to -s$ which corresponds to replacing $f = e^{ir\sigma} \to f^* = e^{-ir\sigma}$ and $g = e^{is\sigma} \to g^* = e^{-is\sigma}$.

tr $[F,G]W_4'$ would lead to the wrong sign, and indeed it does not appear (although it has the correct antisymmetry properties under $F \leftrightarrow G$ and the correct "naive" dimension). One can easily check that all terms on the r.h.s. of (5.4)–(5.6) have the required properties. The only slightly non-trivial terms in (5.6) are $\int \text{tr}\,([F,G]U_2'U_2 + (FG' - GF')U_2^2)$. Here one needs to integrate by parts to show that (6.3) is satisfied.

6.2. Restrictions and Other Algebras. In Sect. 2, I already discussed the restriction (reduction) to $U_1 = 0$, and in Sect. 3 the corresponding reduction $\sum_{i=1}^m P_i = 0$. The original V-algebra (1.1) corresponds to $m = 2, n = 2, U_1 = 0$ and furthermore tr $\sigma_3 U_2 = 0$. Recall that the reduction to $U_1 = 0$ was implemented by determining X_m (which formerly was $\frac{\delta f}{\delta U_1}$) such that $\{U_1, f\}|_{U_1=0}$ vanishes. Similarly if one decomposes the 2×2-matrices,

$$U_2 \equiv U = \begin{pmatrix} T + V_3 & -\sqrt{2}V^+ \\ -\sqrt{2}V^- & T - V_3 \end{pmatrix}, \quad \frac{\delta f}{\delta U} = \begin{pmatrix} (F_0 + F_3)/2 & -F^-/\sqrt{2} \\ -F^+/\sqrt{2} & (F_0 - F_3)/2 \end{pmatrix}, \tag{6.5}$$

the reduction to $V_3 = 0$ is achieved by determining F_3 (which formerly was $\sim \frac{\delta f}{\delta V_3}$) such that $\{V_3, f\}|_{V_3=0} = 0$. It can be seen from (2.35) with $n = 2$ or from (2.37) that this implies $F_3 = 0$. (This contrasts with the reduction to $U_1 = 0$, where X_m was a non-trivial function of the $X_1, \dots, X_{m-1}$.) Thus to obtain the reduction to tr $\sigma_3 U = 2V_3 = 0$ it is enough to simply set $V_3 = F_3 = 0$ in Eq. (2.35) or (2.37). The result is equivalent to the original V-algebra (1.1).

Other reductions can be achieved by not only taking $U_1 = 0$, but by reducing to the symplectic submanifold where several U_k vanish, e.g. $U_k = 0$ for all odd k. Another, and probably more fruitful approach is to take advantage of the Miura transformation and impose conditions on the P_i. Here, I studied the A_{m-1}-type reduction $\sum_{i=1}^m P_i = 0$. But one can study other reductions, like $P_{m+1-i} = -P_i$ that correspond to the Lie algebras $B_{\frac{m-1}{2}}$ if m is odd, and to $C_{\frac{m}{2}}$ if m is even. They should lead to matrix generalizations of the $WB_{\frac{m-1}{2}}$ and $WC_{\frac{m}{2}}$-algebras.

6.3. Concluding Remarks. In this paper, I have given matrix generalizations of the well-known W_m-algebras by constructing the second Gelfand–Dikii bracket associated with a matrix linear differential operator of order m. Upon reducing to $U_1 = 0$, the non-commutativity of matrices implies the presence of non-local terms in the algebra.

One always has a Virasoro subalgebra generated by $T = \frac{1}{2}\text{tr}\, U_2$, and all other ("orthogonal") combinations of matrix elements of U_2 (i.e. tr $A_\mu U_2$ with tr $A_\mu = 0$) are spin-two conformally primary fields, while the $U_k, k \geqq 3$ can be combined into matrices $W_k, k \geqq 3$ which are (matrices of) spin-k conformal primary fields. I have given the complete Poisson bracket algebra for $m = 3$, and, for all m, the Poisson brackets involving U_2, W_3.

A Miura transformation relates these Poisson brackets of the U_k to much simpler ones of a set of $n \times n$-matrices P_i. Contrary to the case $n = 1$, the P_i are not free fields. However, for $m = n = 2$, $U_1 = 0$ and tr $\sigma_3 U_2 = 0$ a simple free-field realization for $P_1 = -P_2$ was given in refs. 9 and 10 in terms of vertex operator-like fields. In the general case, it is not clear how to give such a free-field realization. The main difficulty is to realize the non-local terms. Comparing the more general case (2.37) with (1.1) one sees the origin of the difficulty: $\{V^+(\sigma), V^-(\sigma')\} \sim \varepsilon(\sigma-\sigma')V^+(\sigma)V^-(\sigma')+\dots$ can be realized by $V^\pm(\sigma) \sim e^{\mp i\sqrt{2}\varphi(\sigma)}R(\partial\varphi)$, where $R(\partial\varphi)$ is some differential polynomial in $\partial\varphi$ and φ is a free field. On the other hand, $\{V_3(\sigma), V^\pm(\sigma')\} \sim \varepsilon(\sigma-\sigma')V^\pm(\sigma)V_3(\sigma')+\dots$ cannot be realized by this type of vertex operator construction, since the arguments σ and σ' of $V^\pm$ and V_3 have been exchanged. This kind of relation is however very reminiscent of the braiding relations of chiral screened vertex operators in conformal quantum field theories [23, 24] and it might well be that a free field realization, involving screening type integrals, can be given. Once a free field realization is found, one can try to quantize the structures described in this paper. This will certainly lead to most interesting developments.

Appendix A

In this appendix, I recall some well-known properties of pseudo-differential operators and adapt them to the matrix case.

Lemma A.1. *Let A,B be some matrix-valued pseudo-differential operators, i.e. let $A=\sum_{i=-\infty}^{l} a_i\partial^i$, $B=\sum_{j=-\infty}^{k} b_j\partial^j$ with a_i, b_j some matrix-valued functions. Then there exists a matrix-valued function h such that*

$$\text{tr res } [A,B] = (\partial h) \equiv h' \,. \tag{A.1}$$

In particular, if the integral of a total derivative vanishes one has $\int \text{tr res } AB = \int \text{tr res } BA$.

Proof. The proof is a straightforward generalization of the scalar case (see e.g. p.9 of ref. 25). By linearity, it is sufficient to prove this for monomials $A=a_i\partial^i, B=b_j\partial^j$. If $i,j \geqq 0$ or $i+j<1$ the residue obviously vanishes and $h=0$. Hence let $i \geqq 0$ and $j<0$ with $i+j \geqq 1$. Then using Eq. (2.2):

$$\begin{aligned}\text{res } [A,B] &= \text{res } (a_i\partial^i b_j\partial^j - b_j\partial^j a_i\partial^i) \\ &= \text{res } \left(\sum_{p=0}^{i} \binom{i}{p} a_i b_j^{(i-p)}\partial^{j+p} - \sum_{s=0}^{\infty}(-)^s \binom{-j+s-1}{s} b_j a_i^{(s)}\partial^{j-s+i}\right) \\ &= \binom{i}{i+j+1}\left(a_i b_j^{(i+j+1)} + (-)^{i+j} b_j a_i^{(i+j+1)}\right) .\end{aligned} \tag{A.2}$$

Let

$$h = \binom{i}{i+j+1} \sum_{p=0}^{i+j} (-)^p a_i{}^{(p)} b_j{}^{(i+j-p)} \,. \tag{A.3}$$

Upon taking h' one sees that there are cancellations between two successive p terms, and only part of the terms with $p=0$ and $p=i+j$ survive. The result equals the r.h.s. of (A.2), except for the order of the matrices. Upon taking the trace, Eq. (A.1) follows.

Lemma A.2. *Let $A=\sum_{i=-\infty}^{l} a_i\partial^i$. One can always rewrite $A=\sum_{i=-\infty}^{l}\partial^i\tilde{a}_i$. One has $\tilde{a}_{-1}=a_{-1}$.*

Proof. Decompose $A = \sum_{i=0}^{l} \partial^i \tilde{a}_i + \partial^{-1}\tilde{a}_{-1} + \sum_{i=-\infty}^{-2} \partial^i \tilde{a}_i$ and use (2.2) to see that $\tilde{a}_{-1}\partial^{-1}$ is the only term containing ∂^{-1}.

Lemma A.3. *If h is a matrix-valued function and A a matrix-valued pseudo-differential operator then*

$$\begin{aligned} \text{res}\ (Ah) &= (\text{res}\ A)h\,, \\ \text{res}\ (hA) &= h\,\text{res}\ A\,. \end{aligned} \tag{A.4}$$

Proof. The first identity is shown as for Lemma A.2 and the second is obvious.

Lemma A.4. *Let h and A be as before. Then*

$$\text{res}\ A = (\partial A_-)_+ = (A_-\partial)_+ = ((\partial - h)A_-)_+ = (A_-(\partial - h))_+\,. \tag{A.5}$$

Proof. Writing $A_- = a_{-1}\partial^{-1} + A_{--}$, where $A_{--} = \sum_{i=-\infty}^{-2} a_i\partial^i$, the proof is obvious.

Lemma A.5. *Let h and A be as before. Then*

$$[\partial - h, \text{res}\ A] = \text{res}\ [\partial - h, A]\,. \tag{A.6}$$

Proof. Let $\hat{\partial} = \partial - h$. Then from the previous lemma, and since $A_- = A - A_+$,

$$\begin{aligned} [\partial - h, \text{res}\ A] &= [\hat{\partial}, \text{res}\ A] = \hat{\partial}(A_-\hat{\partial})_+ - (\hat{\partial}A_-)_+\hat{\partial} \\ &= \hat{\partial}(A\hat{\partial})_+ - \hat{\partial}(A_+\hat{\partial})_+ - (\hat{\partial}A)_+\hat{\partial} + (\hat{\partial}A_+)_+\hat{\partial} \\ &= \hat{\partial}(A\hat{\partial})_+ - \hat{\partial}A_+\hat{\partial} - (\hat{\partial}A)_+\hat{\partial} + \hat{\partial}A_+\hat{\partial} \\ &= \hat{\partial}(A\hat{\partial})_+ - (\hat{\partial}A)_+\hat{\partial}\,, \end{aligned} \tag{A.7}$$

while

$$\begin{aligned} \text{res}\ [\partial - h, A] &= \text{res}\ \hat{\partial}A - \text{res}\ A\hat{\partial} = \left((\hat{\partial}A)_-\hat{\partial}\right)_+ - \left(\hat{\partial}(A\hat{\partial})_-\right)_+ \\ &= \left(\hat{\partial}A\hat{\partial}\right)_+ - \left((\hat{\partial}A)_+\hat{\partial}\right)_+ - \left(\hat{\partial}A\hat{\partial}\right)_+ + \left(\hat{\partial}(A\hat{\partial})_+\right)_+ \\ &= -(\hat{\partial}A)_+\hat{\partial} + \hat{\partial}(A\hat{\partial})_+\,. \end{aligned} \tag{A.8}$$

Comparing (A.7) with (A.8) completes the proof.

Appendix B

In this appendix, I prove Lemma 2.4 and some other useful formulas for the $S_{r,l}^{q,j}$. In both cases considered in Lemma 2.4 one has $r \geqq 0$ and $q \leqq j$ so that

$$S_{r,l}^{q,j} = \sum_{s=r}^{q} (-)^{s-r} \binom{s-r+l-1}{l-1} \binom{q}{s} = \sum_{s=0}^{q-r} (-)^s \binom{s+l-1}{l-1} \binom{q}{s+r}\,. \tag{B.1}$$

Note that for $q < r, S^{q,j}_{r,l}$ was defined to vanish, as do the r.h.s. of (2.20) and (2.21). So let's suppose $q - r \geqq 0$. Then the r.h.s. of (B.1) is

$$\sum_{s=0}^{q-r}(-)^s \frac{(s+l-1)!}{(l-1)!s!}\frac{q!}{(s+r)!(q-r-s)!}$$
$$= \frac{q!}{(l-1)!(q-r)!}\sum_{s=0}^{q-r}(-)^s\frac{(s+l-1)!}{(s+r)!}\binom{q-r}{s}. \tag{B.2}$$

Let first $l > r$, so that $\frac{(s+l-1)!}{(s+r)!} = (s+l-1)(s+l-2)\dots(s+r+1)$ and

$$\sum_{s=0}^{q-r}(-)^s\frac{(s+l-1)!}{(s+r)!}\binom{q-r}{s} = \left(\frac{d}{dx}\right)^{l-1-r}\sum_{s=0}^{q-r}x^{s+l-1}(-)^s\binom{q-r}{s}\Big|_{x=1}$$
$$= \left(\frac{d}{dx}\right)^{l-1-r}x^{l-1}(1-x)^{q-r}\Big|_{x=1}. \tag{B.3}$$

This vanishes if $l-1 < q$ while for $l-1 \geqq q$ it equals $(-)^{q-r}\frac{(l-1)!(l-1-r)!}{q!(l-1-q)!}$. Inserting this result into (B.2) one has

$$S^{q,j}_{r,l} = (-)^{q-r}\binom{l-1-r}{q-r}, \tag{B.4}$$

which proves (2.21)

Let now $1 \leqq l \leqq r$, so that $\frac{(s+l-1)!}{(s+r)!} = \frac{1}{(s+r)(s+r-1)\dots(s+l)}$ and

$$\sum_{s=0}^{q-r}(-)^s\frac{(s+l-1)!}{(s+r)!}\binom{q-r}{s} = (\partial_x^{-1})^{r-l+1}\sum_{s=0}^{q-r}(-)^s x^{s+l-1}\binom{q-r}{s}\Big|_{x=1}$$
$$= (\partial_x^{-1})^{r-l+1}x^{l-1}(1-x)^{q-r}\Big|_{x=1}, \tag{B.5}$$

where here $(\partial_x^{-1})^p h(x) = \int_0^x \mathrm{d}x_1 \int_0^{x_1}\mathrm{d}x_2 \dots \int_0^{x_{p-1}}\mathrm{d}x_p h(x_p)$. For non-negative integers a, b one has

$$\int_0^1 \mathrm{d}x_1 \int_0^{x_1}\mathrm{d}x_2 \dots \int_0^{x_{p-1}}\mathrm{d}x_p\, x_p^a(1-x_p)^b = \frac{a!(b+p-1)!}{(a+b+p)!(p-1)!} \tag{B.6}$$

so that expression (B.5) equals $\frac{(l-1)!(q-l)!}{q!(r-l)!}$. Inserting this into the r.h.s. of (B.2) yields $\binom{q-l}{q-r}$ which proves (2.20).

Another immediate consequence of the definition (2.14) of $S^{q,j}_{r,l}$ is, for $p \geqq 0$ and $l \geqq 1$

$$S^{m,m-1}_{m-p,l} = \sum_{s=\max(0,m-p)}^{m-1}(\dots) = \sum_{s=\max(0,m-p)}^{m}(\dots) - (\dots)|_{s=m}$$
$$= S^{m,m}_{m-p,l} - (-)^p\binom{p+l-1}{l-1}. \tag{B.7}$$

Furthermore, for $p \geqq 1, l \geqq 1$ one has

$$S_{m-p,l}^{m,m-2} = \sum_{s=\max(0,m-p)}^{m-2} (\ldots) = \sum_{s=\max(0,m-p)}^{m} (\ldots) - (\ldots)|_{s=m} - (\ldots)|_{s=m-1}$$

$$= S_{m-p,l}^{m,m} - (-)^p \begin{pmatrix} p+l-1 \\ l-1 \end{pmatrix} + (-)^p \, m \begin{pmatrix} p+l-2 \\ l-1 \end{pmatrix} . \qquad \text{(B.8)}$$

If now $p = 0$, one has $S_{m,l}^{m,m} = 1, S_{m,l}^{m,m-2} = 0$ and $\binom{l-2}{l-1} = 0$, so that Eq. (B.8) remains true also for $p = 0$.

References

1. Zamolodchikov, A.B.: Infinite additional symmetries in two-dimensional conformal field theory. Theor. Math. Phys. **65**, 1205 (1985)
2. Bilal, A.: Introduction to W-algebras. In: Proc. Trieste Spring School on String Theory and Quantum Gravity, April 1991, J. Harvey et al eds., Singapore: World Scientific, pp. 245–280
3. Bouwknegt, P., Schoutens, K.: W-symmetry in conformal field theory. Phys. Rep. **223**, 183 (1993)
4. Bilal, A., Gervais, J.-L.: Systematic approach to conformal systems with extended Virasoro symmetries. Phys. Lett. **B206**, 412 (1988); Systematic construction of $c = \infty$ conformal systems from classical Toda field theories. Nucl. Phys. **B314**, 646 (1989); Systematic construction of conformal theories with higher-spin Virasoro symmetries. Nucl. Phys. **B318**, 579 (1989)
5. Bakas, I.: Higher spin fields and the Gelfand–Dikii algebras. Commun. Math. Phys. **123**, 627 (1989)
6. Mathieu, P.: Extended classical conformal algebras and the second hamiltonian structure of Lax equations. Phys. Lett. **B208**, 101 (1988)
7. Di Francesco, P., Itzykson, C., Zuber, J.-B.: Classical W-algebras. Commun. Math. Phys. **140**, 543 (1991)
8. Segal, G., Wilson, G.: Loop groups and equations of KdV type. Inst. Hautes Etudes Sci. Publ. Math. **61**, 5 (1985)
9. Bilal, A.: Non-abelian Toda theory: a completely integrable model for strings on a black hole background. Nucl. Phys. **B422**, 258 (1994)
10. Bilal, A.: Multi-component KdV hierarchy, V-algebra and non-abelian Toda theory. Lett. Math. Phys. **32**, 103 (1994)
11. Gel'fand, I.M., Dikii, L.A.: Fractional powers of operators and hamiltonian systems. Funct. Anal. Applic. **10**, 259 (1976)
12. Gel'fand, I.M., Dikii, L.A.: The resolvent and hamiltonian systems. Funct. Anal. Applic. **11**, 93 (1977)
13. Adler, M.: On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg-de Vries type equations. Invent. Math. **50**, 219 (1979)
14. Lebedev, D.R., Manin, Yu.I.: Gel'fand-Dikii hamiltonian operator and the coadjoint representation of the Volterra group. Funct. Anal. Applic. **13**, 268 (1979)
15. Gel'fand, I.M., Dikii, L.A.: A family of hamiltonians structures connected with integrable non-linear differential equations. Inst. Appl. Math. Acad. Sci. USSR preprint n° 136, 1978 (in Russian)
16. Kupershmidt, B.A., Wilson, G.: Modifying Lax equations and the second hamiltonian structure. Invent. Math. **62**, 403 (1981)
17. Dikii, L.A.: A short proof of a Kupershmidt-Wilson theorem. Commun. Math. Phys. **87**, 127 (1982)
18. Olmedilla, E., Martinez Alonso, L., Guil, F.: Infinite-dimensional hamiltonian systems associated with matrix Schroedinger operators. Nuovo Cim. **61B**, 49 (1981)
19. Bruschi, M., Ragnisco, O.: The hamiltonian structure of the nonabelian Toda hierarchy. J. Math. Phys. **24**, 1414 (1983)

20. Calogero, F., Degasperis, A.: Nonlinear evolution equations solvable by the inverse spectral transform associated with the multichannel Schroedinger problem, and properties of their solution. Lett. Nuovo Cim. **15**, 65 (1976); Nonlinear evolution equations solvable by the inverse spectral transform-II. Nuovo Cim. **39B**, 1 (1977)
21. Kac, V.G., van de Leur, J.W.: The n-component KP hierarchy and representation theory. MIT preprint (August 1993), hep-th@xxx/9308137
22. Drinfel'd, V.G., Sokolov, V.V.: Lie algebras and equations of Korteweg-de Vries type. Sov. Math. **30**, 1975 (1985)
23. Felder, G., Froehlich, J., Keller, G.: Braid matrices and structure constants for minimal conformal models. Commun. Math. Phys. **124**, 647 (1989)
24. Bilal, A.: Fusion and braiding in extended conformal theories. Nucl. Phys. **B330**, 399 (1990); Fusion and braiding in extended conformal theories (II): Generalization to chiral screened vertex operators labelled by arbitrary Young tableaux. Int. J. Mod. Phys. **A5**, 1881 (1990)
25. Dickey, L.A.: Soliton equations and hamiltonian systems. Singapore: World Scientific, 1991

Communicated by G. Felder