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On Universal Vassiliev Invariants

Daniel Altschuler[1]*, Laurent Freidel[2]

[1] Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland

[2] Laboratoire de Physique Théorique ENSLAPP†, Ecole Normale Supérieure de Lyon, 46, allée d'Italie, F-69364 Lyon Cedex 07, France

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Abstract: Using properties of ordered exponentials and the definition of the Drinfeld associator as a monodromy operator for the Knizhnik–Zamolodchikov equations, we prove that the analytic and the combinatorial definitions of the universal Vassiliev invariants of links are equivalent.

1. Introduction

Vassiliev's knot invariants [1] contain all the invariants, such as the Jones [2], HOMFLY [3] and Kauffman [4] polynomials, which can be obtained from a deformation $U_h(\mathscr{G})$, usually called quantum group [5], of the Hopf algebra structure of enveloping algebras $U(\mathscr{G})$, where $\mathscr{G}$ is a semisimple Lie algebra.

For a compact semisimple Lie group G with Lie algebra $\mathscr{G}$, observables of the quantized Chern–Simons model give knot invariants [6] associated to $U_h(\mathscr{G})$ at special values $h = 2\pi i k^{-1}$, k a positive integer. The coefficients of the expansion in powers of h of these observables are examples of Vassiliev invariants. This is a particular case of a general theorem [7], which states that for all h the coefficients of the power series expansion of the invariants associated with semisimple Lie algebras are Vassiliev invariants.

By treating the Chern–Simons model with the conventional methods of perturbation theory, the coefficients of the powers of h of the observables can be computed [8]. Feynman diagrams and Feynman rules are the main tools of the computation. Given a knot, or more generally a link L, and the degree n (order in perturbation theory) or power of h in which one is interested, the corresponding invariant $V_n(L)$ results from the application of a Feynman rule $W_{\mathscr{G}}$ to a finite linear combination $Z_n^{CS}(L)$ of diagrams. The vector space D_n of diagrams of degree n is of finite dimension, and $Z_n^{CS}(L) \in D_n$ depends on L and on the form of the Chern–Simons action. The Feynman rule $W_{\mathscr{G}}$ depends on $\mathscr{G}$ and the representations

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† URA 1436 du CNRS, associée à l'Ecole Normale Supérieure de Lyon et au laboratoire d'Annecy-le-Vieux de Physique des Particules

occurring in the definition of the observables. It is an element of the dual space D_n^*, and $V_n(L) = \langle W, Z_n^{CS}(L)\rangle$.

Here we have used Bar–Natan's way [9] of describing Feynman rules and diagrams. He found that the diagrams and rules of Chern–Simons theory obey a small number of fundamental properties, and this led him to define general diagrams and rules, the latter which he called weight systems, by these same properties. Kontsevich [10] discovered an integral formula for an invariant $Z_n(L) \in D_n$, which plays for generic h the same role as $Z_n^{CS}(L)$ does for the special values in the Chern–Simons case. The main ingredient hiding behind it is the flat connection associated with the Knizhnik–Zamolodchikov equations. The formal power series $Z(L) = \sum_{n \geqq 0} Z_n(L)h^n$ is called the universal Vassiliev invariant, since by varying W in $\langle W, Z(L)\rangle$ one gets all the invariants constructed from a deformation of the identity solution of the Yang–Baxter equation.

The deep questions remain: do the Vassiliev invariants form a complete set of knot invariants? Are there any other Feynman rules (weight systems) than those of the type $W_{\mathscr{G}}$ associated to Lie algebras? The following troubling result is related to the second question: Vassiliev invariants are invariants of oriented knots. However all Vassiliev invariants $\langle W_{\mathscr{G}}, Z_n(L)\rangle$ are independent of the orientation. Is there any weight system W which can distinguish the two orientations of a knot? The simplest example of a knot which is not isotopy-equivalent to the same knot with the reversed orientation can be found in [11]. It has 8 crossings.

The knot invariants constructed using the representations of $U_h(\mathscr{G})$ have been generalized to all quasi-triangular Hopf algebras by Reshetikhin and Turaev [12]. Their construction is purely combinatorial, the proof of invariance consists in verifying that the Reidemeister moves do not change the relevant expressions. Recently, similar combinatorial definitions of universal Vassiliev invariants have appeared [13, 14]. The aim of this paper is to show that the combinatorial and the analytic definition of Kontsevich are equivalent. More precisely, since the combinatorial approach leads naturally to invariants of framed knots, we will show that it is equivalent to a variant of the Kontsevich formula, which was written originally for unframed knots. The same notion of Kontsevich integral for framed knots appears in [15]. However here we will define it in a way which does not require the framed knot to be presented as a product of tangles with special properties.

As we were finishing this paper, we learned that the equivalence of the combinatorial and analytic definitions had been shown before in [16]. We believe that our methods make the proof more direct. While the authors of [16] work with the individual terms $Z_n(L)$ which are iterated integrals and are led to long computations in order to identify these terms with the corresponding terms of the combinatorial invariants, we essentially treat the whole series $Z(L)$ at once. It turns out that the main contribution to $Z(L)$ is a type of series called ordered exponential in the physics literature. Ordered exponentials satisfy many interesting, but not well-known, identities which makes them very powerful. These identities have been recently used in the context of quantum groups, in order to compute the universal quantum R-matrix from its classical counterpart [17]. A crucial step in the proof of equivalence is to identify an expression for the Drinfeld associator [18, 19] among the Kontsevich integrals. We do it quite naturally using only Drinfeld's definition of the associator as a monodromy operator between solutions of the Knizhnik–Zamolodchikov differential equations. We don't have to first find some expressions for the coefficients of the associator viewed as a power series, as is done in [16].

The contents of the paper are as follows. In Sect. 2, we define the ordered exponential and prove the properties which we use later in the proof. Sections 3 and 4 are devoted to the definitions of the combinatorial invariants and the Bar–Natan (Feynman) diagrams. In Sect. 5 we define the Kontsevich integral of framed links. The proof of the equivalence theorem occupies Sect. 6 and 7.

2. The Ordered Exponential

We give here without proofs some properties of the ordered exponential which we will need later in the paper. Let $A : \mathbb{R} \to \mathscr{A}$ be a function with values in the associative algebra $\mathscr{A}$. The ordered exponential of $\mathscr{A}$:

$$g(x,y) = \overleftarrow{\exp} \int_y^x du A(u) , \tag{2.1}$$

is the solution of the differential equation:

$$\frac{\partial}{\partial x} g(x,y) = A(x) g(x,y) \tag{2.2}$$

with the initial condition $g(y,y) = 1$. An equivalent definition is

$$g(x,y) = 1 + \sum_{n=1}^{+\infty} \int_y^x dt_1 \int_y^{t_1} dt_2 \cdots \int_y^{t_{n-1}} dt_n \, A(t_1) A(t_2) \cdots A(t_n) . \tag{2.3}$$

Proposition 1. *The ordered exponential is multiplicative*: $g(x,y) = g(x,z) g(z,y)$.

Corollary 1. $g(x,y)$ *is invertible*, $g^{-1}(x,y) = g(y,x)$ *and*

$$\frac{\partial}{\partial y} g(x,y) = -g(x,y) A(y) . \tag{2.4}$$

Proposition 2. *Let* $\delta \in \mathrm{der}\mathscr{A}$ *be a derivation of* $\mathscr{A}$, *then*:

$$\delta g(x,y) = \int_y^x dt g(x,t) \delta A(t) g(t,y) . \tag{2.5}$$

Proposition 3. *Behaviour with respect to gauge transformations*: *if* $h : \mathbb{R} \to \mathscr{A}$ *is a function such that* $h(t)$ *is invertible for* $x \geqq t \geqq y$, *then*

$$h(x) \left(\overleftarrow{\exp} \int_y^x dt A(t) \right) h^{-1}(y) = \overleftarrow{\exp} \int_y^x dt \left(h(t) A(t) h^{-1}(t) + \partial_t h \, h^{-1}(t) \right) . \tag{2.6}$$

Proposition 4. *Factorization identities*:

$$\overleftarrow{\exp} \int_y^x dt \, (A(t) + B(t)) = \overleftarrow{\exp} \left(\int_y^x dt A(t) \right) \overleftarrow{\exp} \left(\int_y^x dt \, {}^A B(t) \right) , \tag{2.7}$$

$$\overleftarrow{\exp}\int_y^x dt\,(A(t)+B(t)) = \overleftarrow{\exp}\left(\int_y^x dt B^A(t)\right)\overleftarrow{\exp}\left(\int_y^x dt\,A(t)\right)\,, \tag{2.8}$$

$$\left(\overleftarrow{\exp}\int_t^x du\,A(u)\right)B(t)\left(\overleftarrow{\exp}\int_t^x du A(u)\right)^{-1} = \overleftarrow{\exp}\left(\int_t^x du\,\mathrm{ad}A(u)\right)\cdot B(t)\,, \tag{2.9}$$

where

$${}^{A}B(t) = \left(\overleftarrow{\exp}\int_y^t du\,A(u)\right)^{-1} B(t)\left(\overleftarrow{\exp}\int_y^t du\,A(u)\right)\,, \tag{2.10}$$

$$B^A(t) = \left(\overleftarrow{\exp}\int_t^x du\,A(u)\right)B(t)\left(\overleftarrow{\exp}\int_t^x du\,A(u)\right)^{-1}\,, \tag{2.11}$$

and $\mathrm{ad} : \mathscr{A} \to \mathrm{der}\mathscr{A}$ *is given by* $\mathrm{ad}(a)\cdot b = [a,b] = ab - ba$ *for* $a, b \in \mathscr{A}$.

3. Ribbon Categories

In this section we recall the definition of the combinatorial invariants. We formulate it in the language of categories and we use the definitions of Cartier [14]. In the following, $\mathscr{C}$ is a monoidal category, and its product, which is a bifunctor $\mathscr{C}\times\mathscr{C}\to\mathscr{C}$ is denoted by $\otimes$. For any triple of objects X, Y, Z of $\mathscr{C}$, we have an isomorphism $\phi_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$, which is natural. The unit object is denoted by I, and we will assume that $X\otimes I = I\otimes X = X$ for simplicity.

When the object X is a left dual of Y, the duality morphisms are denoted $a : X\otimes Y\to I$, $b : I\to Y\otimes X$.

Let $X_1, X_2, \ldots, X_k$ be a (possibly empty) sequence of objects in $\mathscr{C}$, and consider $X = (((X_1((\otimes(X_2\otimes\cdots)\cdots))\otimes X_k))$, with a given distribution of parentheses. We shall say that X is standard if $X = I$, or if all its left parentheses are placed on the left of the first factor X_1. We also say that a morphism from X to X' is standard if both X and X' are standard. Every morphism from I to I can be written as the composition of standard morphisms. By Mac Lane's coherence theorem [20], for every object X there is a unique isomorphism ψ_X from X to a standard object X_{st}, and for every pair of morphisms $f : X\to X'$, $g : Y\to Y'$, the morphism $f\otimes_{\mathrm{st}} g = \psi_{X'\otimes Y'}(f\otimes g)\psi_{X\otimes Y}^{-1}$ is standard.

Following [14], a braiding in a monoidal category $\mathscr{C}$ is a function, which to any pair of objects X, Y of $\mathscr{C}$ associates a natural isomorphism $R_{X,Y} : X\otimes Y\to Y\otimes X$.

A ribbon category is a monoidal category equipped with a braiding, in which each object has a left dual, and such that for any object X there is a natural isomorphism $v_X : X\to X$, satisfying the relation

$$v_{X\otimes Y} = (R_{Y,X}R_{X,Y})^{-1}(v_X\otimes v_Y)\,, \tag{3.1}$$

for all objects X, Y, as well as other conditions [14].

An example of ribbon category is $\mathscr{R}$, the category of ribbon graphs [12,21]. To get the definition of objects and morphisms of this category, just replace in [21] the representations by symbols: a symbol is a pair $\langle x,\alpha\rangle$, where $x\in\mathbb{R}$ and $\alpha,\alpha'\in\{\uparrow,\downarrow\}$. To each extremity of an open ribbon is associated a symbol such that x is the middle point of the intersection of the ribbon with $\mathbb{R}\times\{0\}\times\{0,1\}$, and the arrow α is given by the direction of the ribbon. The product in $\mathscr{R}$ is denoted by $\square$, and we sometimes call words the objects of $\mathscr{R}$.

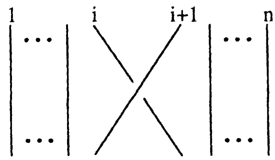

Fig. 1: $X_{i,n}^{+}$

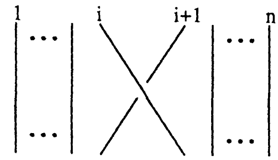

Fig. 2: $X_{i,n}^{-}$

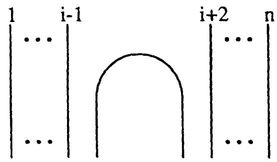

Fig. 3: $\bigcap_{i,n}$

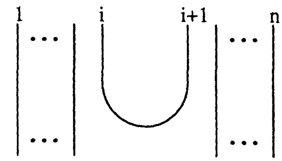

Fig. 4: $\bigcup_{i,n}$

It will be important for us that every closed graph is a composition of elementary standard graphs $X_{i,n}^{\pm}, \bigcap_{i,n}$ and $\bigcup_{i,n}$ (Figs. 1, 2, 3, 4).

Let K be a field of characteristic $0, \mathscr{T}$ a tensor category over K, in the sense of [14], and t an infinitesimal braiding in $\mathscr{T}$, which to any pair of objects X, Y associates the natural morphism $t_{X,Y} : X \otimes Y \to X \otimes Y$.

Let $\mathscr{B}_m$ be the associative graded algebra over K with generators $t_{ij} = t_{ji}, i \neq j$, $i, j \in \{1, \dots, m\}$ of degree 1, and relations

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \;, \tag{3.2}$$

$$[t_{ij}, t_{kl}] = 0 \;, \tag{3.3}$$

where i, j, k, l are distinct. Denote by $\widehat{\mathscr{B}_m}$ the completion with respect to the topology defined by the gradation. Similarly, let $\mathscr{F}_2$ be the free associative graded algebra on two generators A_1, A_2 of degree 1, and let $\widehat{\mathscr{F}_2}$ be its completion. A Drinfeld series, or associator, is a formal non-commutative series $\Phi(A_1, A_2) \in \widehat{\mathscr{F}_2}$ satisfying

$$\Phi(A_1, A_2) = \Phi(A_2, A_1)^{-1} \;, \tag{3.4}$$

$$\Phi(t_{12}, t_{23} + t_{34})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23}) \;, \tag{3.5}$$

$$\exp\frac{1}{2}(t_{13} + t_{23}) = \Phi(t_{13}, t_{12})\exp\frac{1}{2}(t_{13})\Phi(t_{13}, t_{23})^{-1}\exp\frac{1}{2}(t_{23})\Phi(t_{12}, t_{23}) \;, \tag{3.6}$$

where the last two relations hold in $\widehat{\mathscr{B}_4}$ and $\widehat{\mathscr{B}_3}$, respectively. Drinfeld has shown [18, 19] that such series exist for all K, in particular $K = \mathbb{Q}$. For $K = \mathbb{C}$ he gave an explicit construction of a solution Φ_{KZ} using the properties of the Knizhnik–Zamolodchikov equations: $\Phi_{KZ}(A_1, A_2) = G_2^{-1} G_1$ where $G_1, G_2 \in \widehat{\mathscr{F}_2}$ are two solutions of the differential equation

$$G'(x) = \frac{1}{2\pi i}\left(\frac{A_1}{x} + \frac{A_2}{x-1}\right) G(x) \tag{3.7}$$

defined in $0 < x < 1$ with the asymptotic behaviour $G_1(x) \sim x^{A_1/2\pi i}$ for $x \to 0$ and $G_2(x) \sim (1-x)^{A_2/2\pi i}$ for $x \to 1$. The coefficients of Φ_{KZ} are given by generalizations of Riemann's ζ function [15].

Note that $\widehat{\mathscr{F}_2}$ becomes a topological Hopf algebra with the comultiplication defined by $\Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i$, $i = 1, 2$. Let $\mathscr{L}$ be the Lie algebra of primitive elements in $\widehat{\mathscr{F}_2}$, and let $\mathscr{L}' = [\mathscr{L}, \mathscr{L}]$ be the derived subalgebra. Then $\log \Phi_{KZ}(A_1, A_2) \in \mathscr{L}'$, and if $\Phi(A_1, A_2)$ is a solution of (3.4–3.6) of the form $\exp P(A_1, A_2)$ with $P(A_1, A_2) \in \mathscr{L}$, then the decomposition $P = \sum_{n \geqq 1} P_n$ into homogeneous elements P_n of degree n satisfies $P_1 = 0$, i.e. $P \in \mathscr{L}'$, $P_2 = (1/24)[A_1, A_2], P_3 = a_3([A_1,[A_1,A_2]] - [A_2,[A_2,A_1]])$, where $a_3 \in K$ is arbitrary.

Given any tensor category $\mathscr{T}$ with an infinitesimal braiding t, we can construct a ribbon category $\mathscr{T}[[h]]$, as follows. It has the same objects as $\mathscr{T}$, but a morphism $X \to Y$ in $\mathscr{T}[[h]]$ is a formal series $\sum_{n \geqq 0} f_n h^n$, where $f_n : X \to Y$ is a morphism of $\mathscr{T}$. The tensor product of objects is the same as in $\mathscr{T}$, and the tensor product of morphisms is the extension to $K[[h]]$ of the one in $\mathscr{T}$, with the non-trival associator

$$\phi_{X,Y,Z} = \Phi(ht_{X,Y}, ht_{Y,Z}) \,. \tag{3.8}$$

The braiding is given by

$$R_{X,Y} = \sigma_{X,Y} \exp\left(\frac{h}{2} t_{X,Y}\right) , \tag{3.9}$$

where σ is the symmetric braiding of $\mathscr{T}$, and the ribbon structure can be found in [14].

Now we can define a generalized Reshetikhin–Turaev functor $F : \mathscr{R} \to \mathscr{T}[[h]]$, which restricted to closed ribbon graphs gives an invariant of oriented framed links. We choose first a fixed object $X_\downarrow$ in $\mathscr{T}[[h]]$, and a left dual $X_\uparrow$ of $X_\downarrow$. We put $F(\langle x, \downarrow \rangle) = X_\downarrow, F(\langle x, \uparrow \rangle) = X_\uparrow$ and extend the definition of F to all objects of $\mathscr{R}$ by requiring $F(w \Box w') = F(w) \otimes F(w')$. To define F on morphisms, we require that F preserves ϕ, the braiding R, the family of morphisms a and b which define duality, and the ribbon structure v.

4. The Category of Diagrams

A chord diagram D is a pair (X, C), where X is a set of lines and circles and C is a set of chords connecting pairs of points in X. The precise definition is as follows: X is a compact oriented, piecewise smooth one-dimensional submanifold of $\mathbb{R}^2 \times [0,1] = \{(x, y, t) | 0 \leqq t \leqq 1\}$ such that:

(i) $\partial X = N_0 \cup N_1$, where $N_i = \partial X \cap E_i$, $E_i = \{(x, 0, i) | x \in \mathbb{R}\}$,
(ii) for each $x \in N_i$, $i = 0, 1$, the function t restricted to the tangent line to X at x is not constant, i.e. the tangent line is not horizontal.

A chord on X is a subset $\{x, y\} \subset X \backslash \partial X$ of two elements $x \neq y$, and C is a finite set of disjoint chords on X, possibly empty. X is called the support of the diagram. If $\mathrm{Card}(C) = n$ we shall say that the diagram $D = (X, C)$ is of order n. A point $x \in X$ such that $\{x, y\} \in C$ for some $y \in X$ is called a vertex. Chord diagrams are represented as in Fig. 5.

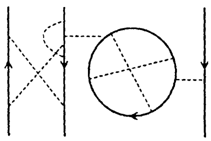

Fig. 5.

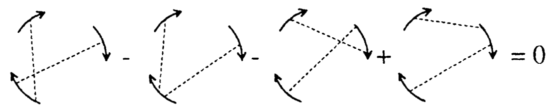

Fig. 6.

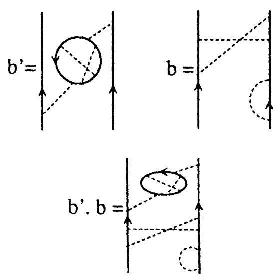

Fig. 7.

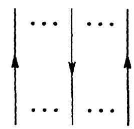

Fig. 8.

We identify two diagrams if they are related by a diffeomorphism which sends the lines E_0 and E_1 into themselves and conserves the orientation of X, E_0 and E_1 and we also identify over and under-crossings in chord diagrams.

Let K be a field of characteristic 0, and $D^{(n)}$ be the K-vector space spanned by the diagrams of order n. The space of Bar–Natan diagrams of order n is $B^{(n)} = D^{(n)}/R^{(n)}$, where $R^{(n)}$ is the subspace of $D^{(n)}$ spanned by the linear combinations of 4 diagrams defined in Fig. 6. Note that the 4 terms of Fig. 6 stand for arbitrary diagrams $D_1, \dots, D_4$ which are equal except for the parts shown there.

Now we are ready to define the category $\mathscr{D}_K$ of diagrams. Its objects are finite sequences $S = (\alpha_1, \alpha_2, \dots, \alpha_k)$, where $\alpha_j \in \{\uparrow, \downarrow\}$, $j = 1, \dots, k$, including the empty sequence $\emptyset$. A Bar–Natan diagram with support X is a morphism from S_0 to S_1, the S_i being defined as follows: use the fact that E_i is an ordered set, to build a sequence of unit tangent vectors $\tilde{S}_i = (u_1, u_2, \dots, u_k)$, where u_1 is the tangent vector to X at the smallest $x \in N_i$, u_2 is the next tangent vector to X along E_i, and so on. Then S_i is given by the projection of $\tilde{S}_i$ on the vertical axis t.

The composition of morphisms is defined by stacking the diagrams, as shown on Fig. 7.

The identity morphism is represented by the diagram of order 0 on Fig. 8.

This category of diagrams $\mathscr{D}_K$ is a tensor category. The monoidal structure $\otimes$ is defined on objects as concatenation of sequences, the unit object being $\emptyset$. For morphisms $b \otimes b'$ is juxtaposition, see Fig. 9.

For $\alpha, \alpha' \in \{\uparrow, \downarrow\}$, let $\Omega_{\alpha,\alpha'}$ be the diagram of Fig. 10, with α and α' given by the orientations (not shown on the figure) of the two lines. The category $\mathscr{D}_K$ has an infinitesimal braiding t defined by $t_{\alpha,\alpha'} = \pm\Omega_{\alpha,\alpha'}$, where the sign is $+$ or $-$ according to whether $\alpha = \alpha'$ or not. We denote by t_{ij} and Ω_{ij} with $i, j \in \{1, \dots, n\}$, the diagrams of degree one, whose supports have n connected components which are parallel vertical lines, such that they reduce to the diagrams $t_{\alpha,\alpha'}$, resp. $\Omega_{\alpha,\alpha'}$, if all lines but those labeled i and j are deleted.

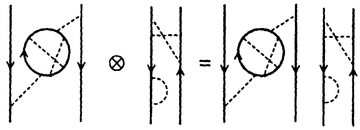

Fig. 9.

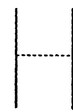

Fig. 10.

For any tensor category $\mathscr{T}$ over K with an infinitesimal braiding t, and any object $X_\downarrow$ in $\mathscr{T}$ with a left dual $X_\uparrow$, there is a functor of tensor categories $W_{\mathscr{T}} : \mathscr{D}_K \to \mathscr{T}$ defined by $W_{\mathscr{T}}(\downarrow) = X_\downarrow, W_{\mathscr{T}}(\uparrow) = X_\uparrow$ and $W_{\mathscr{T}}(t_{\alpha,\alpha'}) = t_{X_\alpha,X_{\alpha'}}$, for $\alpha, \alpha' \in \{\uparrow,\downarrow\}$. This functor defines a weight system on $\mathscr{D}_K$. The Reshetikhin–Turaev functor $F_U : \mathscr{R} \to \mathscr{D}_K[[h]]$ defined in the previous section is called the universal Vassiliev invariant. For any Reshetikhin–Turaev functor $F : \mathscr{R} \to \mathscr{T}[[h]]$, $F = W_{\mathscr{T}} \circ F_U$, because $\mathscr{D}_K$ is a free [20] tensor category with infinitesimal braiding.

From several sources [19, 21, 13] we can extract the next theorem, which enables the computation of $F_U(L)$ for any framed link (closed ribbon graph) L. Before stating it, we need a definition: let id_k be the identity morphism of an object S in $\mathscr{D}_K$ which is a sequence of k arrows. We consider id_k as a standard morphism in $\mathscr{D}_K[[h]]$. For any morphism f in $\mathscr{D}_K[[h]]$, define $f_{(k)}$ recursively by $f_{(0)} = f$, $f_{(k)} = f_{(k-1)} \otimes \mathrm{id}_1$, and put $f_{j,k} = (\mathrm{id}_j \otimes f)_{(k)}$.

Theorem 1. *The functor F_U takes on the elementary standard ribbon graphs the values*

$$F_U\left(X_{i,n}^{\pm}\right) = (\Phi^{(i)})^{-1}\sigma_{i,i+1}\exp\left(\pm\frac{h}{2}t_{i,i+1}\right)\Phi^{(i)}\,, \tag{4.1}$$

$$F_U\left(\bigcup_{i,n}\right) = a_{i-1,n-i-1}\Phi^{(i)}\,, \tag{4.2}$$

$$F_U\left(\bigcap_{i,n}\right) = (\Phi^{(i)})^{-1}b_{i-1,n-i-1}\,, \tag{4.3}$$

where

$$\Phi^{(i)} = \Phi\left(h\sum_{j=1}^{i-1}t_{ji}, ht_{i,i+1}\right)\,. \tag{4.4}$$

In fact F_U depends on the choice of a solution Φ to (3.4–3.6). What is remarkable is that for closed graphs L (framed links), $F_U(L)$ is independent of the choice of Φ [22]. This implies that the coefficients of the universal Vassiliev invariant of links are rational.

We denote by $\mathscr{A}(X)$ the space of all chord diagrams with support X. The space $\mathscr{A} = \mathscr{A}(S^1)$ has the structure of a commutative algebra, with the multiplication defined by means of the connected sum of the two circles [9, 10]. If X has m connected components $X_j, 1 \leqq j \leqq m$, then for each value of j we can define an $\mathscr{A}$-module $\mathscr{A}_j(X)$, which is isomorphic to $\mathscr{A}(X)$ as a vector space, and for which the action of $\mathscr{A}$ is given by the connected sum $S^1\#X_j$, where S^1 is the circle of $\mathscr{A} = \mathscr{A}(S^1)$. This is illustrated in Fig. 11. The 4-term relation of Fig. 6 implies that the result is independent of the point of insertion of the circle.

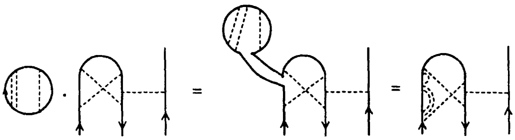

Fig. 11.

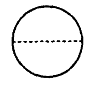

Fig. 12.

For further convenience we define $\Theta \in \mathscr{A}$ as the unique chord diagram on S^1 of degree 1, shown on Fig. 12.

5. The Kontsevich Integral

Let G be a standard ribbon graph, i.e. a morphism of $\mathscr{R}$ from w to w', where w and w' are two standard words. Remember that G is an equivalence class of ribbons which are related by an isotopy of $\mathbb{C} \times [0,1] = \{(z,t)|0 \leqq t \leqq 1\}$. We choose a particular element $\hat{G}$ in this class, such that t is a Morse function on $\hat{G}$, i.e. for a given value $t = t_0$ there is at most one extremum of $\hat{G}$. Each connected component $\hat{G}_j$ is the image of an embedding $\imath : G_j^0 \times [0,1] \to \mathbb{C} \times [0,1]$, where $G_j^0 = [0,1]$ or S^1. We assume that $\hat{G}$ is contained in the plane $\mathrm{Im}(z) = 0$ except for small neighbourhoods around the crossings of Figs. 1 and 2, where only the framing vector $\imath(x,1) - \imath(x,0)$ is contained in the plane $\mathrm{Im}(z) = 0$. In other words, we use the blackboard framing. Let $X_j(\hat{G})$ be the curve $\imath(G_j^0 \times \{0\})$, $X(\hat{G}) = \bigcup_j X_j(\hat{G})$. Let h be a formal variable, $\hbar = h(2\pi i)^{-1}, \varepsilon > 0$ and

$$Z_\varepsilon(\hat{G}) = \sum_{n=0}^{\infty} \hbar^n \int_{\substack{t_{\min} < t_1 < \cdots < t_n < t_{\max} \\ |z_i - z_i'| > \varepsilon}} \sum_{\text{pairings } P = \{z_i, z_i'\}} (-1)^{\# P_\uparrow} D(\hat{G}, P) \prod_{i=1}^{n} \frac{dz_i - dz_i'}{z_i - z_i'} . \tag{5.1}$$

Here $t_{\min}$ and $t_{\max}$ are the minimal and maximal value of t on $X(\hat{G})$, a pairing P is a choice of n unordered pairs (z_i, z_i'), such that for $1 \leqq i \leqq n$, (z_i, t_i) and (z_i', t_i) are distinct points on $X(\hat{G})$, $\#P_\uparrow$ is the number of vertices (z_i, z_i') of P at which $X(\hat{G})$ is oriented upwards. The coordinates z_i and z_i' are considered as functions of t_i, and integration is over the subset of the n-simplex $t_{\min} < t_1 < \cdots < t_n < t_{\max}$ defined by the conditions $|z_i(t_i) - z_i'(t_i)| > \varepsilon$. $D(\hat{G}, P)$ is the chord diagram of degree n with the support $X(\hat{G})$ and the set of chords C defined by the pairing P (Fig. 13).

Now we define the regularized Kontsevich integral as:

$$Z(\hat{G}) = \lim_{\varepsilon \to 0} \prod_{j=1}^{m} \varepsilon^{\hbar(n_j^+ - n_j^-)\Theta_j} \cdot Z_\varepsilon(\hat{G}) , \tag{5.2}$$

where $n_j^\pm$ are the number of critical points (maxima and minima) of the Morse function t on the component $X_j(\hat{G})$, and Θ_j denotes the diagram Θ acting on the latter. (Notice that $n_j^+ - n_j^- = 0$ if j corresponds to a circle.) Later we will show that a slightly modified version of $Z(\hat{G})$ is invariant under isotopies of ribbons.

We recover the usual definition of the Kontsevich invariant if we impose the additional relation $\Theta = 0$, i.e. if every diagram having an isolated cord is set equal to

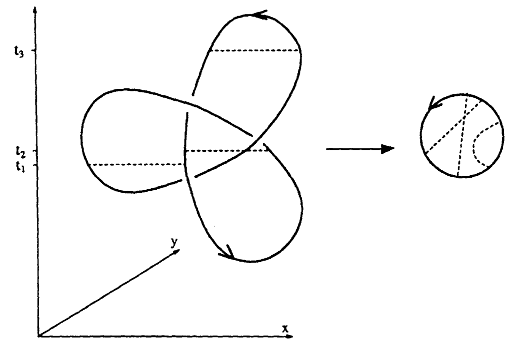

Fig. 13.

zero. This condition implements the framing independence in the original Kontsevich integral [10, 9].

Theorem 2. *The regularized Kontsevich integral is well-defined and multiplicative*:

$$Z(\hat{G}\hat{G}') = Z(\hat{G})Z(\hat{G}') . \tag{5.3}$$

Proof. A pairing P'' of $\hat{G}\hat{G}'$ can be partitioned into a disjoint union of pairings P, P' of $\hat{G}$ and $\hat{G}'$, such that $D(\hat{G}\hat{G}', P'') = D(\hat{G}, P)D(\hat{G}', P')$. Let

$$f = (-1)^{\#P_\uparrow} D(\hat{G}\hat{G}', P'') \prod_{i=1}^{n} f_i(t_i) , \tag{5.4}$$

where

$$f_i(t_i) = \begin{cases} \dfrac{\partial}{\partial t_i} \log(z_i - z_i'), & \text{if } |z_i - z_i'| > \varepsilon \\ 0 & \text{otherwise} . \end{cases} \tag{5.5}$$

The formula

$$\int_{a<t_1<\cdots<t_n<b} dt_1 \ldots dt_n f = \sum_{p=0}^{n} \int_{a<t_1<\cdots<t_p<c} dt_1 \ldots dt_p \int_{c<t_{p+1}<\cdots<t_n<b} dt_{p+1} \ldots dt_n f \tag{5.6}$$

implies that $Z_\varepsilon(\hat{G}\hat{G}') = Z_\varepsilon(\hat{G})Z_\varepsilon(\hat{G}')$. The action of each factor Θ_j in (5.2) through the connected sum is independent of its point of insertion on the diagram $D(\hat{G}, P)$, thus we can "move" it along the support until it reaches an extremum. By the definition of the Morse function t we can decompose $\hat{G}$ into a product of standard graphs each of which contains at most one extremum. Therefore the theorem is proved if we can show that $Z(\bigcap_{m,n})$ and $Z(\bigcup_{m,n})$ converge. This is done in the appendix. $\square$

The main ingredient in the construction of the Kontsevich invariant $Z(\hat{G})$, as we now explain, appears when $\hat{G}$ contains no extremum, so that it becomes a braid. Consider a path $\gamma : [0, 1] \to V_m$, where

$$V_m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m | z_i \neq z_j,\ i \neq j\} . \tag{5.7}$$

Let $\gamma_i(t) = z_i(t)$, $t \in [0, 1]$ be the i^{th} component of this path and construct a ribbon graph $\hat{G}^\gamma$ out of γ.

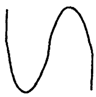

Fig. 14.

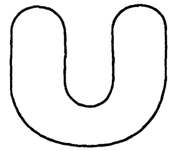

Fig. 15.

Define the abstract Knizhnik–Zamolodchikov connection with values in the algebra $\mathscr{B}_m$, as the 1-form on V_m:

$$\omega = \hbar \sum_{i<j} t_{ij} \frac{dz_i - dz_j}{z_i - z_j} \,. \tag{5.8}$$

It is easy to see that

$$Z(\hat{G}^{\gamma}) = D_\gamma \overleftarrow{\exp} \int_0^1 \gamma^* \omega \,, \tag{5.9}$$

where D_γ is the diagram of order zero whose support is given by γ. Let us quickly recall some well-known properties [23] of $Z(\hat{G}^{\gamma})$. Let γ' be another path in V_m, with $\gamma'(1) = \gamma(0)$, and let $\gamma' \cdot \gamma$ denote their product. Then we have $Z(\hat{G}^{\gamma' \cdot \gamma}) = Z(\hat{G}^{\gamma'})Z(\hat{G}^{\gamma})$. The four-term relation satisfied by diagrams (Fig. 6) means that the generators t_{ij} of $\mathscr{B}_m$ satisfy (3.2). The KZ connection ω is closed, $d\omega = 0$, and the relation (3.2) is equivalent to $\omega \wedge \omega = 0$, so that ω is flat: $F_\omega = d\omega + \omega \wedge \omega = 0$. This implies that $Z(\hat{G}^{\gamma})$ only depends on the homotopy class of γ relative to its endpoints. Thus the formula (5.9) gives, when $\gamma(1) = \sigma \cdot \gamma(0)$, where σ is a permutation of the coordinates, a representation of the full braid group B_m in the semidirect product of $\mathscr{B}_m$ with S_m, where the symmetric group S_m is identified with the group of order zero diagrams with m components, each of which is homeomorphic to $[0, 1]$.

The flatness of the KZ connection implies that Z is invariant under an isotopy which fixes the extrema. But one cannot remove the parts of a ribbon graph which look like Fig. 14, without changing the value of Z. However, if we define $\hat{Z}(\hat{G})$ by

$$\hat{Z}\left(\bigcap_{m,n}\right) = \mu^{-1} \cdot Z\left(\bigcap_{m,n}\right) \,, \tag{5.10}$$

where $\bigcap_{m,n}$ is the elementary graph of Fig. 3, $\mu = Z(U), U$ is the diagram of Fig. 15 acting on the component of $\bigcap_{m,n}$ carrying the maximum, and $\hat{Z}(\hat{G}) = Z(\hat{G})$ for the other elementary graphs $\hat{G}$, then $\hat{Z}$ is invariant under insertion or removal of the subgraph of Fig. 14. This shows:

Theorem 3. *$\hat{Z}(\hat{G})$ is an isotopy invariant of ribbon graphs.*

6. The Distancing Operator

In this section we introduce the distancing operator, which is the renormalized Kontsevich integral of a trivial braid with a pair of consecutive strands moving away from each other to infinity. Let $a \in \mathbb{R}, \lambda > 0$. We consider a path $\eta_i^{(n)} : [0,1] \to \Delta_n$, $\Delta_n = \{(z_1,\dots,z_n)|, z_1 < z_2 < \cdots < z_n\}$ such that

$$\eta_i^{(n)}(1) = (z_1,\dots,z_n)\,,$$
$$\eta_i^{(n)}(0) = (z_1 - a,\dots,z_i - a, z_{i+1} + \lambda - a,\dots,z_n + \lambda - a)\,. \tag{6.1}$$

(The strands i and $i+1$ are moving away from each other.) Due to the zero curvature condition satisfied by the KZ connection and the fact that Δ_n is simply-connected we can choose any path with these endpoints to calculate $Z(\eta_i^{(n)})$. Let us take the following one:

$$\eta_i^{(n)}(t) = \begin{cases} z_j - a(1-t) & 1 \leqq j \leqq i \\ z_j + (1-t)(\lambda - a) & i+1 \leqq j \leqq n, \lambda > 0\,. \end{cases} \tag{6.2}$$

It is useful to remark that $Z(\eta_i^{(n)})$ does not depend on the global translation a because the KZ connection depends only on the differences $z_j(t) - z_i(t)$. We put $g_i^{(n)}(\lambda, z_1,\dots,z_n) = Z(\eta_i^{(n)})$ and in the following we will not mention the arguments $(z_1,\dots,z_n)$ if there is no ambiguity. Using the definition of the Kontsevich integral we have:

$$g_i^{(n)}(\lambda) = \overleftarrow{\exp}\left(\hbar \int_0^1 dt\, \omega(t)\right)\,, \tag{6.3}$$

where

$$\omega(t) = \sum_{j=1}^{i} \sum_{k=i+1}^{n} \frac{\lambda t_{jk}}{(t-1)\lambda + z_j - z_k}\,. \tag{6.4}$$

The distancing operator is obtained by sending λ to infinity, but to do this a regularization is needed.

Theorem 4. *Set* $X_i^{(n)} = \sum_{j=1}^{i} \sum_{k=i+1}^{n} t_{jk}$.

1. *The distancing operator*

$$D_i^{(n)}(z_1,\dots,z_n) = \lim_{\lambda \to +\infty} g_i^{(n)}(\lambda)\lambda^{\hbar X_i^{(n)}} \tag{6.5}$$

is a well-defined function on V_n with values in $\mathcal{B}_n[[h]]$, the algebra of formal power series in h with coefficients in $\mathcal{B}_n$.

2. *It has the asymptotic behaviour $D_i^{(n)} \sim (z_n - z_1)^{-\hbar X_i^{(n)}}$ in the region*

$$\begin{cases} z_n - z_1 \gg z_n - z_k, & k > i \\ z_n - z_1 \gg z_j - z_1, & j \leqq i \end{cases}\,. \tag{6.6}$$

3. *It satisfies the differential equations*

$$\frac{\partial D_i^{(n)}}{\partial z_j} = \hbar \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{t_{jk}}{z_j - z_k} \right) D_i^{(n)} - \hbar D_i^{(n)} K_{ij}^{(n)}\,, \tag{6.7}$$

where

$$K_{ij}^{(n)} = \begin{cases} \sum\limits_{\substack{k=1 \\ k \neq j}}^{i} \frac{t_{jk}}{z_j - z_k}, & j \leqq i \\ \sum\limits_{\substack{k=i+1 \\ k \neq j}}^{n} \frac{t_{jk}}{z_j - z_k}, & j > i\,. \end{cases} \tag{6.8}$$

Proof. 1. We factorize from $g_i^{(n)}(\lambda)$,

$$\overleftarrow{\exp}\left(\hbar \int_0^1 dt \frac{\lambda X_i^{(n)}}{(t-1)\lambda + z_1 - z_n}\right) = \left(\frac{z_n - z_1}{\lambda + z_n - z_1}\right)^{\hbar X_i^{(n)}}. \tag{6.9}$$

Thus, using (2.8) and performing the change of variables $x = \lambda(1-t)/(z_n - z_1)$ we get:

$$g_i^{(n)}(\lambda) = \overleftarrow{\exp}\left(\hbar \int_{\frac{\lambda}{z_n - z_1}}^{0} dx\, \tilde{\omega}(x)\right)\left(\frac{z_n - z_1}{\lambda + z_n - z_1}\right)^{\hbar X_i^{(n)}}, \tag{6.10}$$

with

$$\tilde{\omega}(x) = (1+x)^{-\hbar X_i^{(n)}}\left(\sum_{\substack{1 \leqq j \leqq i \\ i+1 \leqq k \leqq n}} \frac{t_{kj}(1-u_{kj})}{(x+u_{kj})(1+x)}\right)(1+x)^{\hbar X_i^{(n)}} \tag{6.11}$$

and $u_{kj} = (z_k - z_j)/(z_n - z_1)$. Since

$$\tilde{\omega}(x) = \sum_{p=0}^{+\infty} \frac{\hbar^p}{p!}\left(\sum_{\substack{1 \leqq j \leqq i \\ i+1 \leqq k \leqq n}} \mathrm{ad}^p(X_i^{(n)}) \cdot t_{kj}\right) \frac{(1-u_{kj})\log^p(1+x)}{(x+u_{kj})(x+1)}, \tag{6.12}$$

and the integral

$$\int_0^{+\infty} dx \frac{\log^p(1+x)}{(x+u_{kj})(1+x)} \tag{6.13}$$

converges, we deduce that

$$\overleftarrow{\exp}\left(\hbar \int_{+\infty}^{0} dx\, \tilde{\omega}(x)\right) \tag{6.14}$$

converges as a formal series in h, which means that all the integrals appearing in its expansion converge.

2. The asymptotic region (6.6) is defined by $u_{kj} = 1, 1 \leqq j \leqq i, i+1 \leqq k \leqq n$. In this region $\tilde{\omega}(x) = 0$, thus $D_i^{(n)}(z_n - z_1)^{-\hbar X_i^{(n)}} = 1$.

3. The formula (2.5) yields:

$$\frac{\partial}{\partial z_j} g_i^{(n)}(\lambda) = \hbar \int_0^1 dt\, h(1,t) \frac{\partial}{\partial z_j}\left(-\sum_{\substack{1 \leqq l \leqq i \\ i+1 \leqq k \leqq n}} \frac{\lambda t_{lk}}{\lambda(1-t) + z_k - z_l}\right) h(t,0), \tag{6.15}$$

where

$$h(x,y) = \overleftarrow{\exp}\left(\hbar \int_y^x du\, \omega(u)\right). \tag{6.16}$$

Using

$$\frac{\partial}{\partial z_j}\left(\sum_{\substack{1\leqq l\leqq i\\ i+1\leqq k\leqq n}} \lambda \frac{t_{lk}}{\lambda(1-t)+z_k-z_l}\right) = \frac{\partial}{\partial t}\left(\sum_{i+1\leqq k\leqq n} \frac{t_{jk}}{\lambda(1-t)+z_k-z_j}\right) \tag{6.17}$$

if $j \leqq i$, and integrating by parts over t, the r.h.s. of (6.15) becomes:

$$\begin{aligned}\hbar \sum_{i+1\leqq k\leqq n} \frac{t_{jk}}{z_j-z_k} g_i^{(n)}(\lambda) - \hbar g_i^{(n)}(\lambda) \sum_{i+1\leqq k\leqq n} \frac{t_{jk}}{\lambda+z_j-z_k} - \hbar \int_0^1 dt\ h(1,t)\\ \times \left[\sum_{i+1\leqq k\leqq n} \frac{t_{jk}}{\lambda(1-t)+z_k-z_j}, \sum_{\substack{1\leqq j'\leqq i\\ i+1\leqq k'\leqq n}} \frac{\lambda t_{j'k'}}{\lambda(1-t)+z_{k'}-z_{j'}}\right] h(t,0)\,.\end{aligned} \tag{6.18}$$

Using in the second term the classical Yang–Baxter equation:

$$\left[\frac{t_{jk}}{z_j-z_k}, \frac{t_{j'k}}{z_{j'}-z_k}\right] + \left[\frac{t_{jj'}}{z_j-z_{j'}}, \frac{t_{jk}}{z_j-z_k} + \frac{t_{j'k}}{z_{j'}-z_k}\right] = 0\,, \tag{6.19}$$

and again the derivation property (2.5) we get:

$$\begin{aligned}\frac{\partial}{\partial z_j} g_i^{(n)}(\lambda) = \hbar \left(\sum_{\substack{k=1\\ k\neq j}}^{n} \frac{t_{jk}}{z_j-z_k}\right) g_i^{(n)}(\lambda) - \hbar g_i^{(n)}(\lambda)\\ \times \left(\sum_{\substack{j'=1\\ j'\neq j}}^{i} \frac{t_{jj'}}{z_j-z_{j'}} + \sum_{k=i+1}^{n} \frac{t_{jk}}{z_j-z_k-\lambda}\right)\,.\end{aligned} \tag{6.20}$$

Now multiply on the right this equality by $\lambda^{\hbar X_i^{(n)}}$, and notice that $[X_i^{(n)}, t_{jj'}] = 0, 1 \leqq j, j' \leqq i$ and

$$\lim_{\lambda\to\infty} \lambda^{-\hbar X_i^{(n)}} \sum_{k=i+1}^{n} \frac{t_{jk}}{z_j-z_k-\lambda} \lambda^{\hbar X_i^{(n)}} = 0\,. \tag{6.21}$$

Here the limit means that each term in the h expansion tends to zero. This proves part 3 of the theorem for $1 \leqq j \leqq i$ (the proof for the case $i+1 \leqq j \leqq n$ is identical). □

Remark. The distancing operator is equivalently defined as being the solution of the differential equations (6.7) on V_n, satisfying the asymptotic conditions (6.6).

Let A be a standard ribbon graph such that its top (resp. bottom) endpoints are $(z_1,\ldots,z_i) \in \Delta_i$ (resp. $(z'_1,\ldots,z'_p) \in \Delta_p$), and let B be a standard ribbon graph such that its top (resp. bottom) endpoints are $(z_{i+1},\ldots,z_n) \in \Delta_{n-i}$ (resp. $(z'_{p+1},\ldots,z'_q) \in \Delta_{q-p}$). (Here "endpoint" really means the middle point of the intersection of an open ribbon with $\mathbb{R} \times \{0\} \times \{0,1\}$.) We put these graphs side by side such that the top (resp. bottom) endpoints $(z_1,\ldots,z_i,z_{i+1},\ldots,z_n)$ (resp. $(z'_1,\ldots,z'_p,z'_{p+1},\ldots,z'_q)$) of $A \square B$ lie in Δ_n (resp. Δ_q). This is always possible up to a global translation of B. The product $A \square B$ is not standard. Let $A \square_{\mathrm{st}} B$ be the same graph as $A \square B$ but with the standard arrangement of parentheses.

Theorem 5.

$$Z(A \square_{\rm st} B) = D_i^{(n)}(z_1,\ldots,z_n)(Z(A) \otimes Z(B))(D_p^{(q)}(z_1',\ldots,z_q'))^{-1} \,. \tag{6.22}$$

Proof. Let $(A \square_{\rm st} B)_\lambda$ be the ribbon graph whose top (resp. bottom) endpoints are

$$\begin{aligned}&(z_1 - \lambda/2,\ldots,z_i - \lambda/2,\, z_{i+1} + \lambda/2,\ldots,z_n + \lambda/2)\\ &(\text{resp. } (z_1' - \lambda/2,\ldots,z_p' - \lambda/2,\, z_{p+1}' + \lambda/2,\ldots,z_q' + \lambda/2))\,.\end{aligned} \tag{6.23}$$

Then $A \square_{\rm st} B$ and $\eta_i^{(n)}(z_1,\ldots,z_n)(A \square_{\rm st} B)_\lambda(\eta_p^{(q)}(z_1',\ldots,z_q'))^{-1}$ are isotopic ribbon graphs (see Fig. 16), where the path $\eta_i^{(n)}$ is defined in (6.2), and the same symbol is used here for the associated standard ribbon graph. We also set $(\eta_i^{(n)})^{-1}(t) = \eta_i^{(n)}(1-t), 0 \leqq t \leqq 1$.

Thus

$$Z(A \square_{\rm st} B) = g_i^{(n)}(\lambda) Z((A \square_{\rm st} B)_\lambda)\,(g_p^{(q)}(\lambda))^{-1} \tag{6.24}$$

$$= \left(g_i^{(n)}\lambda^{\hbar X_i^{(n)}}\right)\left(\lambda^{-\hbar X_i^{(n)}} Z((A \square_{\rm st} B)_\lambda))\lambda^{\hbar X_p^{(q)}}\right)\left(g_p^{(q)}\lambda^{\hbar X_p^{(q)}}\right)^{-1} . \tag{6.25}$$

From the definition of the regularized Kontsevich integral one sees that

$$Z((A \square_{\rm st} B)_\lambda) = Z(A) \otimes Z(B) + \sum_{n \geqq 1} \hbar^n f_{(n)}(\lambda)\,, \tag{6.26}$$

where $f_{(n)}(\lambda)$ is a finite sum of terms of the type:

$$D(A \square B, P_{AB}) \int_{t_{\min}}^{t_{\max}} dt \frac{\dot z_i - \dot z_j}{z_i - z_j + \lambda} f(t) \tag{6.27}$$

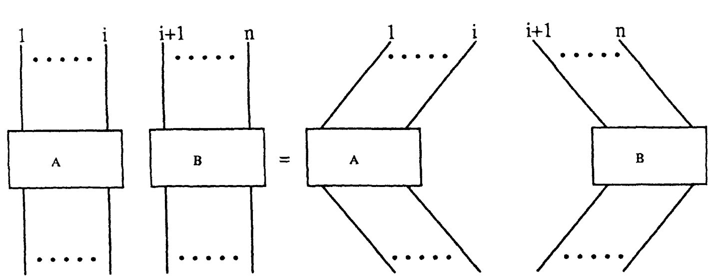

Fig. 16.

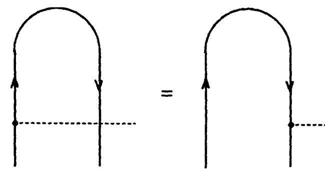

Fig. 17.

such that P_{AB} contains at least one pair (z_i, z_j) with $(z_i, t) \in A, (z_j, t) \in B$, and $\int_{t_{\min}}^{t_{\max}} f(t)dt$ is convergent. Hence

$$\lim_{\lambda \to \infty} f_{(n)}(\lambda) \log^p(\lambda) = 0 . \tag{6.28}$$

Moreover the following relation holds: $X_i^{(n)}(Z(A) \otimes Z(B)) = (Z(A) \otimes Z(B))X_p^{(q)}$. This relation follows from (3.2) and the fact that $D(t_{ik} + t_{i+1,k}) = 0$ (resp. $(t_{ik} + t_{i+1,k})D = 0$) if D is a diagram of degree zero in which the bottom (resp. top) end-points labeled i and $i+1$ belong to the same connected component (see Fig. 17).

Therefore

$$\lim_{\lambda \to \infty} \lambda^{-\hbar X_i^{(n)}} Z((A \square_{\mathrm{st}} B)_\lambda) \lambda^{\hbar X_p^{(q)}} = Z(A) \otimes Z(B) , \tag{6.29}$$

and the theorem is proved. $\square$

7. The Equivalence Theorem

The relation between the Drinfeld associator and the distancing operators involves two functions on V_{i+1} with values in $\mathscr{B}_{i+1}[[h]]$:

$$\tilde{g}_{i+1}(z_1, \ldots, z_i, z_{i+1}) = D_{i-1}^{(i+1)} D_{i-2}^{(i-1)} D_{i-3}^{(i-2)} \cdots D_1^{(2)} (z_{i+1} - z_i)^{\hbar t_{i,i+1}} , \tag{7.1}$$

$$g_{i+1}(z_1, \ldots, z_i, z_{i+1}) = D_i^{(i+1)} D_{i-1}^{(i)} D_{i-2}^{(i-1)} \cdots D_1^{(2)} . \tag{7.2}$$

Theorem 6.

$$(\tilde{g}_{i+1})^{-1} g_{i+1} = \Phi_{KZ}(hX_{i-1}^{(i)}, ht_{i,i+1}) . \tag{7.3}$$

Proof. The key point is the fact that $\tilde{g}_{i+1}$ and g_{i+1} are solutions of the Knizhnik–Zamolodchikov equations:

$$\frac{\partial g}{\partial z_j} = \hbar \left(\sum_{\substack{k=1 \\ k \neq j}}^{i+1} \frac{t_{jk}}{z_j - z_k} \right) g \tag{7.4}$$

with the following asymptotic behaviour:

$$\tilde{g}_{i+1} \sim (z_2 - z_1)^{\hbar t_{12}} \cdots (z_{i-1} - z_1)^{\hbar X_{i-2}^{(i-1)}} (z_{i+1} - z_1)^{\hbar X_{i-1}^{(i+1)}} (z_{i+1} - z_i)^{\hbar t_{i,i+1}}$$

$$\text{in the region} \quad \begin{cases} z_{i+1} - z_1 \gg z_{i-1} - z_1 \gg \cdots \gg z_2 - z_1 \\ z_{i+1} - z_1 \gg z_{i+1} - z_i \,; \end{cases} \tag{7.5}$$

$$g_{i+1} \sim (z_2 - z_1)^{\hbar t_{12}} \cdots (z_{i-1} - z_1)^{\hbar X_{i-2}^{(i-1)}} (z_i - z_1)^{\hbar X_{i-1}^{(i)}} (z_{i+1} - z_1)^{\hbar X_i^{(i+1)}}$$

$$\text{in the region} \quad z_{i+1} - z_1 \gg z_i - z_1 \gg z_{i-1} - z_1 \gg \cdots \gg z_2 - z_1 \,. \tag{7.6}$$

We recall that $X_i^{(n)} = \sum_{j=1}^{i} \sum_{k=i+1}^{n} t_{jk}$. Note that all the factors appearing in the asymptotic behaviour of g_{i+1} and $\tilde{g}_{i+1}$ mutually commute. The relations (7.4), (7.5), and (7.6) are direct consequences of the properties of the distancing operators established in Theorem 4.

Any solution g of the KZ equations (7.4) can be expressed in terms of the reduced variables $u_k = (z_k - z_1)/(z_{i+1} - z_1), 2 \leqq k \leqq i$:

$$g(z_1, \ldots, z_i, z_{i+1}) = G(u_2, \ldots, u_i)(z_{i+1} - z_1)^{\hbar X}, \, X = \sum_{1 \leqq j < k \leqq i+1} t_{jk} \,. \tag{7.7}$$

Observe that X is a central element in $\mathscr{B}_{i+1}$. The function $g(z_1, \ldots, z_{i+1})$ satisfies the KZ equations in the variables $z_j, j = 1, \ldots, i+1$ if and only if $G(u_2, \ldots, u_i)$ satisfies the equations

$$\frac{\partial G}{\partial u_j} = \hbar \left(\frac{t_{j1}}{u_j} + \frac{t_{j,i+1}}{u_j - 1} + \sum_{2 \leqq k \leqq i, \, k \neq j} \frac{t_{jk}}{u_j - u_k} \right) G \tag{7.8}$$

for $j = 2, \ldots, i$. Now if we set

$$U_{i+1} = u_2^{-\hbar t_{12}} \ldots u_{i-1}^{-\hbar X_{i-2}^{(i-1)}} G_{i+1} \,, \tag{7.9}$$

and the same for $\tilde{U}_{i+1}, \tilde{G}_{i+1}$, we obtain that U_{i+1} and $\tilde{U}_{i+1}$ are analytic functions in the domain $1 > u_i > u_{i-1} \geqq u_{i-2} \geqq \cdots \geqq u_2 \geqq 0$, and they obey the same linear differential equations. Moreover, at the point $u_{i-1} = u_{i-2} = \cdots = u_2 = 0$ it follows from (7.8) that they satisfy the equation

$$\frac{\partial U}{\partial u_i} = \hbar \left(\frac{X_{i-1}^{(i)}}{u_i} + \frac{t_{i,i+1}}{u_i - 1} \right) U \tag{7.10}$$

with the asymptotic behaviour:

$$U_{i+1} \sim (u_i)^{\hbar X_{i-1}^{(i)}} \qquad u_i \to 0 \,, \tag{7.11}$$

$$\tilde{U}_{i+1} \sim (1 - u_i)^{\hbar t_{i,i+1}} \quad u_i \to 1 \,. \tag{7.12}$$

Thus, at the point $u_{i-1} = u_{i-2} = \cdots = u_2 = 0$,

$$(\tilde{U}_{i+1})^{-1} U_{i+1} = \Phi_{KZ}(hX_{i-1}^{(i)}, ht_{i,i+1}) \,, \tag{7.13}$$

and by the uniqueness of the solution of differential equations this equality holds in the whole domain $1 > u_i > u_{i-1} \geqq u_{i-2} \geqq \cdots \geqq u_2 \geqq 0$. □

We have seen already in Sect. 3 that any closed ribbon graph is a composition of elementary standard graphs $X_{m,n}^{\pm}, \bigcap_{m,n}$ and $\bigcup_{m,n}$. Recall (see appendix) that $\bigcap_{m,n}'$

and $\bigcup'_{m,n}$ are diagrams in $\mathscr{D}_K$ of order zero with supports given by $\bigcap_{m,n}$, $\bigcup_{m,n}$. Put $\Phi_{KZ}^{(i)} = \Phi_{KZ}(hX_{i-1}^{(i)}, ht_{i,i+1})$.

Theorem 7. *The value of the regularized Kontsevich integral on the elementary graphs is*:

$$Z\left(X_{i,n}^{\pm}\right) = g_n(\Phi_{KZ}^{(i)})^{-1} \exp\left(\pm\frac{h}{2}t_{i,i+1}\right) D_{\sigma_{i,i+1}}\Phi_{KZ}^{(i)} g_n^{-1}, \tag{7.14}$$

$$Z\left(\bigcap_{i,n}\right) = g_{n-2}\left(\bigcap'_{i,n}\Phi_{KZ}^{(i)}\right) g_n^{-1}, \tag{7.15}$$

$$Z\left(\bigcup_{i,n}\right) = g_n\left((\Phi_{KZ}^{(i)})^{-1}\bigcup'_{i,n}\right) g_{n-2}^{-1}. \tag{7.16}$$

Comparing with the results obtained with the functor $F_U : \mathscr{R} \to \mathscr{D}_K[[h]]$ in Theorem 1, we see that for closed ribbon graphs L (framed links), $\hat{Z}(L) = F_U(L)$. This gives another proof that $\hat{Z}$ is an invariant of ribbon graphs.

Proof. Let $(z_1,\ldots,z_n) \in \Delta_n$ and consider a path in V_n:

$$X_{i,n}^{\pm}(t) = (z_1(t),\ldots,z_n(t)), \tag{7.17}$$

associated to the ribbon graph $X_{i,n}^{\pm}$. We choose the parametrization $z_j(t) = z_j$ if $j \neq i, i+1$ and

$$z_i(t) + z_{i+1}(t) = z_i + z_{i+1}, \tag{7.18}$$

$$z_i(t) - z_{i+1}(t) = e^{\pm i\pi t}(z_i - z_{i+1}). \tag{7.19}$$

In order to compute the corresponding Kontsevich integral, we move away to infinity all strands surrounding the crossing. More precisely, we move first the n^{th} strand on the right, then the $n-1^{\text{th}}$, and so on until the strand $i+2$. Then we are left with the diagram $X_{i,i+1}^{\pm}$. Therefore we write $X_{i,n}^{\pm} = X_{i,n-1}^{\pm}\square\mathrm{id}_1 = (X_{i,n-2}^{\pm}\square\mathrm{id}_1)\square\mathrm{id}_1 = \cdots = X_{i,i+1}^{\pm}\square_{\mathrm{st}}\mathrm{id}_{n-i-1}$, and apply the tensorization theorem (6.22) $n-i-1$ times to get:

$$Z(X_{i,n}^{\pm}) = (D_{n-1}^{(n)}\cdots D_{i+1}^{(i+2)})Z(X_{i,i+1}^{\pm})(D_{n-1}^{(n)}\cdots D_{i+1}^{(i+2)})^{-1}. \tag{7.20}$$

After that we send to infinity the distance between the strands $i-1$ and i, which translates into further conjugation of $Z(X_{i,i+1}^{\pm})$ by $D_{i-1}^{(i+1)}$. We then use the value of the Kontsevich integral on an isolated crossing: $Z(X_{1,2}^{\pm}) = \exp(\pm(h/2)t_{12})D_{\sigma_{1,2}}$, and the fact that $(z_{i+1}-z_i)^{\hbar t_{i,i+1}}D_{i-2}^{(i-1)}\cdots D_1^{(2)}$ commutes with $t_{i,i+1}$ to obtain:

$$Z(X_{i,i+1}^{\pm}) = \tilde{g}_{i+1}\exp\left(\pm\frac{h}{2}t_{i,i+1}\right)D_{\sigma_{i,i+1}}\tilde{g}_{i+1}^{-1}. \tag{7.21}$$

Now (7.14) follows from Theorem 6.

Next we prove (7.15). Proceeding as in the case of crossings, we successively move away from the maximum the strands $n, n-1,\ldots,i+2$, then we send to infinity the distance between the strands $i-1$ and i on the bottom of the maximum, noting that this last operation is not required on the top. We apply the tensorization theorem to $\bigcap_{i,n} = \mathrm{id}_{i-1}\square_{\mathrm{st}}\bigcap_{1,2}\square_{\mathrm{st}}\mathrm{id}_{n-i-1}$, and we get:

$$Z\left(\bigcap_{i,n}\right) = (D_{n-3}^{(n-2)} \cdots D_{i-1}^{(i)}) Z\left(\bigcap_{1,2}\right) (D_{n-1}^{(n)} \cdots D_{i+1}^{(i+2)} D_{i-1}^{(i+1)})^{-1}$$
$$= (D_{n-3}^{(n-2)} \cdots D_{i-1}^{(i)} D_{i-2}^{(i-1)} \cdots D_1^{(2)}) Z\left(\bigcap_{1,2}\right) (D_{n-1}^{(n)}$$
$$\cdots D_{i+1}^{(i+2)} D_{i-1}^{(i+1)} D_{i-2}^{(i-1)} \cdots D_1^{(2)})^{-1} . \tag{7.22}$$

Since the value of the Kontsevich integral on an isolated maximum is $Z(\bigcap_{1,2}) = \bigcap'_{1,2}(z_2 - z_1)^{\hbar t_{12}}$, we arrive at (7.15). The proof of (7.16) is similar. □

Appendix

We prove in this appendix that $Z(\bigcap_{m,n})$ and $Z(\bigcup_{mn})$ converge. In the following, we use the notation $Z(\gamma)$ instead of $Z(\hat{G}^{\gamma})$, for γ a path in V_m, which is not supposed to be closed. We start by computing $Z(\gamma)$ when γ is a path with 2 components of opposite orientations. We parametrize the components as follows:

$$z_i(t) = x_i + (y_i - x_i)t, \quad 0 \leqq t \leqq 1 , \tag{A.1}$$

for $i = 1,2$, with $x_i, y_i \in \mathbb{R}$, $x_1 < x_2$ and $y_1 < y_2$. Choose an $\varepsilon > 0$ such that $\varepsilon \leqq \min\{x_2 - x_1, y_2 - y_1\}$. Then Eq. (5.1) or (5.9) leads to

$$Z_\varepsilon(\gamma) = Z(\gamma) = \exp\left(-\overline{\Omega}_{12}\log\frac{y_1 - y_2}{x_1 - x_2}\right) = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)^{-\overline{\Omega}_{12}} . \tag{A.2}$$

Here we have used the notation $\overline{\Omega}_{ij} = \hbar\Omega_{ij}$. Suppose $x_2 - x_1 > y_2 - y_1$, take now $\varepsilon = y_2 - y_1$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^{\overline{\Omega}_{12}} Z_\varepsilon(\gamma) \tag{A.3}$$

exists, and the same is true in the other case $\varepsilon = x_2 - x_1 < y_2 - y_1$, namely

$$\lim_{\varepsilon\to 0} Z_\varepsilon(\gamma)\varepsilon^{-\overline{\Omega}_{12}} . \tag{A.4}$$

Consider now γ_{mn}, the same path with $m - 1$ (resp. $n - m - 1$) vertical lines on the left (right) and the two center lines as in γ. The parametrization we choose is $z_i = \text{const} \in \mathbb{R}$ for $i \neq m, m+1$ and z_i as in (A.1) for $i = m, m+1$. We will now prove that the limits

$$\lim_{\varepsilon\to 0} \varepsilon^{\overline{\Omega}_{m,m+1}} Z_\varepsilon(\gamma_{mn}) \tag{A.5}$$

and

$$\lim_{\varepsilon\to 0} Z_\varepsilon(\gamma_{mn})\varepsilon^{-\overline{\Omega}_{m,m+1}} \tag{A.6}$$

exist, where $\varepsilon = \min\{x_{m+1} - x_m, y_{m+1} - y_m\}$, and $\max\{x_{m+1} - x_m, y_{m+1} - y_m\}$ is fixed. The KZ connection can be written as

$$\omega = -\overline{\Omega}_{m,m+1}\frac{dz_m - dz_{m+1}}{z_m - z_{m+1}} + \omega' , \tag{A.7}$$

where

$$\omega' = \sum_{k \neq m,m+1} s_m s_k \left(\overline{\Omega}_{m,k} \frac{dz_m}{z_m - z_k} - \overline{\Omega}_{m+1,k} \frac{dz_{m+1}}{z_{m+1} - z_k} \right) , \tag{A.8}$$

and $s_i = \pm 1$ is the orientation of the i^{th} component. Using the factorization formula (2.7), we find

$$Z(\gamma_{mn}) = g(y_m - y_{m+1}) \overleftarrow{\exp} \int_0^1 \left(g(z_m - z_{m+1})^{-1} \omega' g(z_m - z_{m+1}) \right) , \tag{A.9}$$

where

$$g(z) = \left(\frac{z}{x_m - x_{m+1}} \right)^{-\overline{\Omega}_{m,m+1}} . \tag{A.10}$$

Therefore, the relation (3.2) implies that

$$g(y_m - y_{m+1})^{-1} Z(\gamma_{mn}) = \overleftarrow{\exp} \int_0^1 (\omega' + \omega'') , \tag{A.11}$$

where

$$\omega'' = \sum_{p=1}^{\infty} \frac{1}{p!} \log^p \left(\frac{z_m - z_{m+1}}{x_m - x_{m+1}} \right) \mathrm{ad}^p(\overline{\Omega}_{m,m+1}) \sum_{k \neq m,m+1} s_m s_k \overline{\Omega}_{mk}$$
$$\times \left(\frac{dz_m}{z_m - z_k} - \frac{dz_{m+1}}{z_{m+1} - z_k} \right) , \tag{A.12}$$

and to prove that (A.5) exists, it is enough to show that

$$I_p = \int_0^1 dt \, \log^p(z_{m+1} - z_m) \left(\frac{\dot{z}_m}{z_m - z_k} - \frac{\dot{z}_{m+1}}{z_{m+1} - z_k} \right) \tag{A.13}$$

converges when $y_m \to y_{m+1}$, for any positive integer p. Here we have adopted the notation $dz/dt = \dot{z}$. Writing

$$I_p = \int_0^1 dt \, \log^p(z_{m+1} - z_m) \left(\frac{\dot{z}_m (z_{m+1} - z_m)}{(z_m - z_k)(z_{m+1} - z_k)} - \frac{\dot{z}_{m+1} - \dot{z}_m}{z_{m+1} - z_k} \right) , \tag{A.14}$$

and noting that $(z_{m+1} - z_m) \log^p(z_{m+1} - z_m) \to 0$ as $z_{m+1} - z_m \to 0$, we see that the first term converges. As for the second term, we put

$$(\dot{z}_{m+1} - \dot{z}_m) \log^p(z_{m+1} - z_m) = \frac{d}{dt} F_p(z_{m+1} - z_m) , \tag{A.15}$$

where

$$F_p(z) = \int_{z_0}^{z} dx \log^p x , \tag{A.16}$$

and observe that the function $F_p(z)$ is defined at $z = 0$. Thus, integrating by parts proves that

$$\int_0^1 \frac{dt \, dF_p(z_{m+1} - z_m)/dt}{z_{m+1} - z_k} \tag{A.17}$$

converges. This concludes the proof that the limit (A.5) exists. Similarly, one can show the existence of (A.6).

Notice that since the KZ connection is flat, (A.5) and (A.6) do not depend on the details of the path γ_{mn}, but only on its endpoints. From the definition of the regularized integral (5.2), we see that for $\varepsilon = \min\{x_{m+1} - x_m, y_{m+1} - y_m\}$, keeping $\max\{x_{m+1} - x_m, y_{m+1} - y_m\}$ fixed,

$$Z\left(\bigcap_{m,n}\right) = \lim_{\varepsilon \to 0} \bigcap{}'_{m,n} \varepsilon^{\overline{\Omega}_{m,m+1}} Z_\varepsilon(\gamma_{mn}) = \lim_{\varepsilon \to 0} \left(\varepsilon^{\hbar \Theta_m} \cdot \bigcap{}'_{m,n}\right) Z_\varepsilon(\gamma_{mn}), \tag{A.18}$$

$$Z\left(\bigcup_{m,n}\right) = \lim_{\varepsilon \to 0} Z_\varepsilon(\gamma_{m,n}) \varepsilon^{-\overline{\Omega}_{m,m+1}} \bigcup{}'_{m,n} = \lim_{\varepsilon \to 0} Z_\varepsilon(\gamma_{m,n}) \left(-\varepsilon^{\hbar \Theta_m} \cdot \bigcup{}'_{m,n}\right), \tag{A.19}$$

where $\bigcap'_{mn}$ and $\bigcup'_{mn}$ are diagrams in $\mathcal{D}_K$ of order zero with supports given by $\bigcap_{mn}$, $\bigcup_{mn}$ such that the extremities of the line containing the maximum (resp. minimum) are y_m and y_{m+1}(x_m and x_{m+1}). $\square$

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