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Elliptic Quantum Many-Body Problem and Double Affine Knizhnik–Zamolodohikov Equation

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Abstract: The elliptic-matrix quantum Olshanetsky–Perelomov problem is introduced for arbitrary root systems by means of an elliptic version of the Dunkl operators. Its equivalence with the double affine generalization of the Knizhnik–Zamolodchikov equation (in the induced representations) is established.

0. Introduction

We generalize the affine Knizhnik–Zamolodchikov equation from [Ch1,2,3] replacing the corresponding root systems by their affine counterparts. To explain the construction in the case of the root system of $\mathbf{gl}_n$, let us first introduce the **affine Weyl group** $\mathbf{S}_n^a$. It is the semi-direct product of the symmetric group $\mathbf{S}_n$ and the lattice $A = \oplus_{i=1}^{n-1}\mathbf{Z}\varepsilon_{ii+1}$, where the first acts on the second permuting $\{\varepsilon_i, \varepsilon_{ij} = \varepsilon_i - \varepsilon_j\}$ naturally. This group is generated by the adjacent transpositions

$$s_i = (ii+1),\ 1 \leqq i < n, \text{ and } s_0 = s_{n1}^{[1]}, \text{ where } s_{ij}^{[k]} = (ij)(k\varepsilon_{ij}) \in \mathbf{S}_n^a\,.$$

Setting

$$s_{ij}^{[k]}(b) = b - (\varepsilon_{ij}, b)(\varepsilon_{ij} + kc),\ s_{ij}^{[k]}(c) = c,\ \ b \in B = \bigoplus_{i=1}^{n} \mathbf{Z}\varepsilon_i\,, \tag{0.1}$$

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we obtain an action of $\mathbf{S}_n^a$ in $\tilde{B} = B \oplus \mathbf{Z}c$. In particular,

$$s_0(b) = b + (b, \varepsilon_{1n})(c - \varepsilon_{1n}),\ a(\varepsilon_i) = \varepsilon_i - (a, \varepsilon_i)c,\ a \in A, 1 \leqq i \leqq n\,.$$

Put

$$x_{\tilde{b}} = kx_c + \sum_{i=1}^{n} k_i x_i,\ z_{\tilde{b}} = k\xi + \sum_{i=1}^{n} k_i z_i \text{ for } \tilde{b} = kc + \sum_{i=1}^{n} k_i \varepsilon_i\,.$$

The **double affine degenerate (graded) algebra** $\mathfrak{H}'$ is generated by the group algebra $\mathbf{C}[\mathbf{S}_n^a]$, pairwise commutative elements $\{x_b,\ b \in B\}$, and central x_c, satisfying the relations (depending on $\eta \in \mathbf{C}$):

$$s_i x_b = x_{s_i(b)} s_i + \eta(\varepsilon_{ii+1}, b),\ 1 \leqq i < n,\ s_0 x_b = x_{s_0(b)} s_0 + \eta(\varepsilon_{n1}, b)\,. \tag{0.2}$$

This algebra is a double affine generalization of that considered by Drinfeld and Lusztig and a degeneration of the double affine Hecke algebra from [Ch8] (for $\mathfrak{gl}_n$).

Let us fix $\mu \in \mathbf{C}$ and set

$$\mathrm{ct}_{ij}^{[k]} = \mathrm{ct}(z_{ij} + k\xi) \text{ for } \mathrm{ct}(t) = (\exp(t) - 1)^{-1},\ z_{ij} = z_i - z_j\,.$$

We introduce the differential operators of the first order:

$$\begin{aligned}\mathscr{D}_{\varepsilon_i} = \mathscr{D}_i &= \partial/\partial z_i - \eta \sum_{n \geqq j > i} \mathrm{ct}_{ij}^{[0]}(s_{ij}^{[0]} - \mu) + \eta \sum_{1 \leqq j < i} \mathrm{ct}_{ji}^{[0]}(s_{ji}^{[0]} - \mu) \\ &\quad - \eta \sum_{j \neq i} \sum_{k > 0} \left(\mathrm{ct}_{ij}^{[k]}(s_{ij}^{[k]} - \mu) - \mathrm{ct}_{ji}^{[k]}(s_{ji}^{[k]} - \mu)\right) + \mu\eta(n/2 - i + 1)\,, \\ \mathscr{D}_c &= \partial/\partial\zeta + \eta\mu n,\ 1 \leqq i, j \leqq n\,.\end{aligned} \tag{0.3}$$

We consider the sums formally as infinite linear combinations of the elements $\tilde{w} \in \mathbf{S}_n^a$ with the coefficients depending on $\{z, \xi\}$ and one more complex variable ζ. Assuming that $\Re(\xi) > 0$, we can introduce a norm in this space to make all series convergent.

The family of operators $\{\mathscr{D}_i' = \mathscr{D}_i - x_i,\ \mathscr{D}_c' = \mathscr{D}_c - x_c\}$ is commutative and $\mathbf{S}_n^a$-invariant with respect to the following simultaneous action of this group on the coefficients (that are from $\mathfrak{H}'$) and the arguments $\{z_{\tilde{b}}, \zeta\}$:

$$\tilde{w}(\hat{h}) = \tilde{w}\hat{h}\tilde{w}^{-1},\ \hat{h} \in \mathfrak{H}',\ \tilde{w}(z_{\tilde{b}}) = z_{\{\tilde{w}(\tilde{b})\}},\ \tilde{b} \in \tilde{B}\,,$$

$$s_i(\zeta) = \zeta \text{ for } 1 \leqq i < n,\ s_0(\zeta) = \zeta - \xi + z_{1n}\,. \tag{0.4}$$

The invariance means that $\tilde{w}(\mathscr{D}_{\tilde{u}}') = \mathscr{D}_{\tilde{w}(\tilde{u})}'$, where $\mathscr{D}_{\alpha\tilde{u}+\beta\tilde{v}}' = \alpha\mathscr{D}_{\tilde{u}}' + \beta\mathscr{D}_{\tilde{v}}'$ for $\alpha, \beta \in \mathbf{Z}, \tilde{u}, \tilde{v} \in \tilde{B}$, and $\tilde{w} \in \mathbf{S}_n^a$. Actually this family is invariant even with respect to the action of the bigger group generated by W and B (instead of A). It leads to a natural extension of the above $\mathfrak{H}'$. The introduction of ∂/∂_ζ and the precise choice of constants in (0.3) is necessary to ensure the B-invariance. As to W-invariance, the central extension is not necessary.

The **double affine KZ** is the system $\{\mathscr{D}_u'\Phi = 0,\ u \in B\}$ for a function $\Phi(z)$ with the values in $\mathfrak{H}'$ or its representations. Here ξ is considered as a parameter.

Let us factorize $\mathfrak{H}'$ by the ideal (x_c). The symmetric polynomials in $x_1, \ldots, x_n$ belong to the center of the resulting algebra $\mathfrak{H}_0'$. Given a character of the algebra of symmetric x-polynomials and a finite dimensional $\mathbf{C}[\mathbf{S}_n^a]$-modules V, the corresponding induced $\mathfrak{H}_0'$-module is finite dimensional as well. When considered in this representation, the series in (0.3) become convergent (at least for rather big $\Re(\xi)$) and turn into functions of elliptic type. The corresponding double KZ is equivalent to a V-valued version of the **elliptic quantum many-body problem** from [OP]. It also generalizes the spin QMB introduced in [Ch5] unifying the Calogero–Sutherland and the Haldane–Shastry models.

To introduce the elliptic QMB let us start with the same formulas (0.3) assuming now that $s_{ij}^{[k]}$ act on the arguments $\{z_b, \zeta\}$ as in (0.4). We will write $\sigma(\tilde{w})$ and $\sigma_{ij}^{[k]}$instead of $\tilde{w}$ and $s_{ij}^{[k]}$ to emphasize this. The corresponding **elliptic Dunkl operators** (which are scalar but not pure differential anymore) will be denoted by $\{\Delta_i, \Delta_c, \Delta_{\tilde{b}}\}$ (instead of $\{\mathscr{D}\}$). The map

$$\tilde{w} \to \sigma(\tilde{w}),\ x_{\tilde{b}} \to \Delta_{\tilde{b}},\ \tilde{w} \in \mathbf{S}_n^a,\ \tilde{b} \in \tilde{B}\,,$$

gives a homomorphism from the algebra $\mathfrak{H}'$ into the algebra of operators acting on the space of (scalar) functions of $\{z, \zeta\}$. Imposing the relation $\Delta_c = 0$ we obtain an embedding of $\mathfrak{H}'_0$. This theorem plays the key role in the paper.

Second, given an arbitrary symmetric polynomial $p = p(x_1, \ldots, x_n)$, we use (0.4) to represent

$$p(\Delta_1, \ldots, \Delta_n) = \sum_{\tilde{w} \in \mathbf{S}_n^a} D_{\tilde{w}} \sigma(\tilde{w}), \text{ where } D_{\tilde{w}} \text{ are differential}\,.$$

Then we replace every $\sigma(\tilde{w})$ by the image of $\tilde{w}^{-1}$ in $\mathrm{Aut}_{\mathbf{C}}\, V$ setting $\partial/\partial\zeta = -\eta\mu n$ afterwards.

The resulting operators $\{L_p\}$ are $\mathbf{S}_n^a$-invariant and pairwise commutative. We emphasize that $\partial/\partial\zeta$ is not present in the final answer but appears in the intermediate calculations when we place $\sigma(\tilde{w})$ on the right (the action of $\mathbf{S}_n^a$ on $b \in B$ involves c).

If V is one-dimensional, $\{L_p\}$ coincide with the OP operators for $\mu = 0$. When $\mu = \pm 1$ (with one-dimensional V of the same "sign") they are conjugated to these operators (by proper remarkable scalar functions).

The element $p_2 = \sum_{i=1}^n x_i^2$ leads (up to a constant) to the **Schrödinger operator**

$$H = \sum_{i=1}^{n} \partial^2/\partial z_i^2 + \text{const} \sum_{i<j} \wp(z_i - z_j) \tag{0.5}$$

in terms of the Weierstrass elliptic function with the periods $\{(2\pi\imath), \xi\}$.

In this paper we consider arbitrary root systems and any initial representations V of the corresponding affine Weil groups. We note that the commutative families of scalar H-operators for the A, B, D types (with certain uniqueness theorems) were obtained recently by direct methods (due to Heckman–Opdam) in [OOS].

It is worth mentioning that for $\mu = 1$ (and certain special η) the operators L_p are expected to be the radial parts of Laplace operators for Kac–Moody symmetric spaces at the critical level $c + n = 0$. The latter condition gives the existence of the "big" center of the corresponding universal enveloping algebra (which is necessary to start the Harish–Chandra, Helgason theory of radial part). It is directly connected with the substitution $\partial/\partial\zeta = -\eta\mu n$.

Something can be done when $\partial/\partial\zeta = \eta\mu\nu$ for arbitrary $\nu \in \mathbf{C}$ (which corresponds presumably to the "affine" harmonic analysis at arbitrary level). Let us introduce one more operator

$$\Delta_d = \partial/\partial\zeta - \eta \sum_{i<j} \sum_{k>0} k \left(\mathrm{ct}_{ij}^{[k]}(\sigma_{ij}^{[k]} - \mu) + \mathrm{ct}_{ji}^{[k]}(\sigma_{ji}^{[k]} - \mu) \right)\,. \tag{0.6}$$

It does not commute with $\{\Delta_i\}$, but the operators $\Delta_{\hat{b}} = \Delta_b + k\Delta_c + l\Delta_d$ for $\hat{b} = b + kc + ld$ still satisfy the cross-relations:

$$\sigma_i \Delta_{\hat{b}} = \Delta_{s_i(\hat{b})}\sigma_i + \eta(\varepsilon_{ii+1}, \hat{b}),\ 0 \leqq i < n,\ \varepsilon_{01} = c - \varepsilon_{1n}\ ,$$
$$s_i(c) = c,\ s_j(d) = d \text{ for } 1 \leqq j < n,\ s_0(d) = d - \varepsilon_{01}\ , \tag{0.7}$$

where the form (,) is extended to $\mathbf{R}^{n+2}$ in the following way:

$$(c,c) = (c,\varepsilon_i) = 0 = (d,\varepsilon_i) = (d,d),\ 1 \leqq i \leqq n,\ (c,d) = 1\ .$$

It gives that the operator $2\Delta_d\Delta_c + \sum_{i=1}^n \Delta_i^2$ is $\mathbf{S}_n^a$-invariant. Its reduction in the above sense is also invariant and is conjugated to the **parabolic operator** $\hat{H} = H + 2\eta\mu(n+\nu)\partial/\partial\xi$ in the setup of (0.5).

There is another way. We can exclude $\partial/\partial\zeta$ from the construction (put $\nu = 0$) considering the operators

$$\tilde{\Delta}_k = \exp(2\pi i\Delta_k/(\eta\mu n))$$

instead of Δ_k. The corresponding $\{\tilde{L}_p\}$ will be pairwise commutative and $\mathbf{S}_n^a$-invariant. It resembles the construction of the center of Kac–Moody algebras (after a proper completion) due to Kac.

When $\mu = 1$ and V is the corresponding one-dimensional representation, $\hat{H}$ was introduced in [EK]. Presumably this operator and $\{L_p\}$ and the double affine KZ are related to the elliptic r-matrix KZ from [Ch1] (with the additional equation from [E]) and the so-called Bernard KZ equation (see [FW,EK]).

In conclusion we would like to note that all above constructions have difference counterparts (based on the non-degenerate double affine Hecke algebras from [Ch8]) and hopefully ensure a basis for the elliptic Macdonald theory (see e.g. [M,R] and [O,Ch6]). The latter is related to the Macdonald theory at roots of unity. The connections with q-deformations of the "double loop algebras" and the so-called elliptic algebras are also expected (in the case of A_n).

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1. Double Hecke Algebras

We follow [Ch 3] (see also [Ch 5,6,7]). Reduced root systems only will be discussed here. All the definitions and statements can be extended to the general case. Minor changes in formulas are necessary for divisible roots.

Given a Euclidean form (,) on $\mathbf{R}^n$ and a root system $R = \{\alpha\} \subset \mathbf{R}^n$ of type $A_n, B_n, \ldots, G_2$, let s_α be the orthogonal reflections in the hyperplanes $(\alpha, u) = 0,\ u \in \mathbf{R}^n$. Further, $\{\alpha_1, \ldots \alpha_n\}$ are the simple roots relative to some fixed Weyl chamber, R_+ the set of all positive (written $\alpha > 0$) roots, W the Weyl group generated by s_α(or by $s_i \overset{\text{def}}{=} s_{\alpha_i}, 1 \leqq i \leqq n$), $\mathbf{C}[W] = \oplus_w \mathbf{C}w$ the group algebra of $W \ni w$.

We introduce $a_i = \alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, the dual fundamental weights $b_1, \ldots, b_n$ satisfying the relations $(b_i, \alpha_j) = \delta_i^j$ for the Kronecker delta, and the lattices

$$A = \bigoplus_{i=1}^n \mathbf{Z}a_i \subset B = \bigoplus_{i=1}^n \mathbf{Z}b_i\ .$$

Let us fix a W-invariant set $\eta = \{\eta_\alpha \in \mathbf{C}, \alpha \in R\}$. The W-invariance (${}^w\eta_\alpha = \eta_{w(\alpha)}, w \in W$) gives that $\eta_\alpha = \eta'$ or η'' respectively for the short and long roots (R is supposed to be reduced). We put $\eta_i = \eta_{\alpha_i}$ and define the η-generalization of the ρ and the Coxeter number h:

$$2\rho_\eta = \sum_{\alpha>0} \eta_\alpha \alpha = \sum_{i=1}^{n} \eta_i (\alpha_i, \alpha_i) b_i \in \mathbf{R}^n ,$$

$$h_\eta = \eta_\theta + (\rho_\eta, \theta) \text{ for the maximal root } \theta \in R_+ . \tag{1.1}$$

We will use the same notations for other W-invariant sets instead of η.

The following affine completion is common in the theory of the Kac–Moody algebras (see e.g. [Ka, Ch6]). Let us extend the above pairing to $\mathbf{R}^{n+1} = \mathbf{R}^n \oplus \mathbf{R}c$ setting $(c, c) = 0 = (c, u)$.

The vectors (affine roots) $\tilde{\alpha} = \alpha + kc$ for $\alpha \in R, k \in \mathbf{Z}$, form the **affine root system** $R^a \supset R$. We add $\alpha_0 \stackrel{\text{def}}{=} c - \theta$ to the set of simple roots and put $\eta_{\tilde{\alpha}} = \eta_\alpha$, $\eta_0 = \eta_\theta = \eta''$. The corresponding set R^a_+ of positive roots coincides with $R_+ \cup \{\alpha + kc, \alpha \in R, k > 0\}$. Let $\tilde{B} \stackrel{\text{def}}{=} B \oplus \mathbf{Z}c$. Given $\tilde{\alpha} = \alpha + kc \in R^a$, $a \in A$, $\tilde{u} = u + \kappa c \in \mathbf{R}^{n+1}$,

$$s_{\tilde{\alpha}}(\tilde{u}) = \tilde{u} - (u, \alpha^\vee)(\alpha + kc), \quad a'(\tilde{u}) = \tilde{u} - (u, a)c . \tag{1.2}$$

The **affine Weyl group** W^a is generated by all $s_{\tilde{\alpha}}$. One can take the simple reflections $s_j = s_{\alpha j}, 0 \leqq j \leqq n$, as its generators. This group is the semi-direct product $W \ltimes A'$ of its subgroups W and $A' = \{a', a \in A\}$, where

$$a' = s_\alpha s_{\{\alpha + c\}} = s_{\{-\alpha + c\}} s_\alpha \text{ for } a = \alpha^\vee, \alpha \in R .$$

Definition 1.1 The degenerate (graded) double affine Hecke algebra $\mathfrak{H}'$ *is algebraically generated by the group algebra* $\mathbf{C}[W^a]$ *and the pairwise commutative*

$$x_{\tilde{u}} \stackrel{\text{def}}{=} \sum_{i=1}^{n} (u, \alpha_i) x_i + \kappa x_c \textit{ for } \tilde{u} = u + \kappa c \in \mathbf{R}^{n+1} , \tag{1.3}$$

satisfying the following relations:

$$s_i x_{\tilde{u}} - x_{\{s_i(\tilde{u})\}} s_i = \eta_i (u, \alpha_i), \;\; 0 \leqq i \leqq n . \tag{1.4}$$

The **restricted algebra** $\mathfrak{H}'_0$ *is the factor-algebra* $\mathfrak{H}'/(x_c)$ (*the quotient by the central ideal* (x_c)). □

Without $i = 0$ we arrive at the defining relations

$$s_i x_i - (x_i - x_{a_i}) s_i = \eta_i, \;\; s_i x_j = x_j s_i, \text{ where } 1 \leqq i \neq j \leqq n, \; a_i = \alpha_i^\vee ,$$

of the graded affine Hecke algebra from [L] (see also [Ch 3,5]). We mention that $\mathfrak{H}'$ is a degeneration of the double affine Hecke algebras introduced in (Ch6,7].

Let $\mathbf{C}[x] \stackrel{\text{def}}{=} \mathbf{C}[x_1, \ldots, x_n, x_c]$ be the algebra of polynomials in terms of $\{x_{\tilde{u}}\}$. We denote the subalgebra of W-**invariant polynomials** (with respect to the action of W on $\{\tilde{u}\}$) by $\mathbf{C}[x]^W$. Later the same notations will be used for other letters instead of x.

Theorem 1.2 *An arbitrary element $\hat{h} \in \mathfrak{H}'$ can be uniquely represented in the (left) form $\hat{h} = \sum_{\tilde{w} \in W^a} f_{\tilde{w}} \tilde{w}$ and the (right) form $\hat{h} = \sum_{\tilde{w} \in W^a} \tilde{w} g_{\tilde{w}}$, where $f_{\tilde{w}}, g_{\tilde{w}} \in \mathbf{C}[x]$. The center of $\mathfrak{H}_0$ contains $\mathbf{C}[x]^W$.*

Proof. The first statement results from Theorem 2.3, [Ch7] established in the non-degenerate case (see also [Ch6]). Following [Ch3] one can check that the center of $\mathfrak{H}_0$ contains $\mathbf{C}[x]^W$. □

Induced representations. Let V be a $\mathbf{C}[W^a]$-module, $V^0 = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ its dual with the natural action $(\tilde{w}(l(v)) = l(\tilde{w}^{-1}v),\ l \in \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})), \tau$ and τ^0 the corresponding homomorphisms from $\mathbf{C}[W^a]$ to $\mathrm{End}_{\mathbf{C}} V$ and $\mathrm{End}_{\mathbf{C}} V^0$. We will use the diagonal action:

$$\delta(\tilde{w})(v \otimes x_{\tilde{u}}) = \tau^0(\tilde{w})(v) \otimes \tilde{w}(x_{\tilde{u}}), \tilde{w}(x_{\tilde{u}}) = x_{\{\tilde{w}(\tilde{u})\}}\ ,$$

$$\text{for } v \otimes x_{\tilde{u}} \in \mathscr{V} \stackrel{\text{def}}{=} V^0 \otimes_{\mathbf{C}} \mathbf{C}[x],\ \tilde{w} \in W^a,\ \tilde{u} \in \mathbf{R}^n\ . \tag{1.5}$$

The next proposition holds good for the entire $\mathfrak{H}'$. However the latter has the trivial center $= \mathbf{C}x_c$ (we need a "big" center to construct finite dimensional representations). Till the end of the section, $x_c = 0$ and $x_{\tilde{u}}$ are identified with the corresponding x_u.

Proposition 1.3 *The universal (free) $\mathfrak{H}_0$-module generated by the $\mathbf{C}[W^a]$-module V^0 is isomorphic to $\mathscr{V}$ with the natural action of $\mathbf{C}[x]$ by multiplications and the following action of s_i :*

$$\hat{s}_i = \delta(s_i) + \eta_i x_{a_i}^{-1}(1 - s_i),\ 0 \leqq i \leqq n,\ a_0 = c - \theta^{\vee}\ , \tag{1.6}$$

where $x_{a_i}^{-1}(1 - s_i)(f) = x_{a_i}^{-1}(f - s_i(f))$ for $f \in \mathscr{V}$ (s_i acts only on x).

Proof follows [Ch3,5]. □

We fix a set $\lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbf{C}$ and consider the quotient $\mathscr{V}_\lambda$ of $\mathscr{V}$ by the (central) relations $p(x_1, \ldots, x_n) = p(\lambda_1, \ldots, \lambda_n)$ for all $p \in \mathbf{C}[x]^W$.

Finally, we introduce:

$$V(\lambda) \stackrel{\text{def}}{=} (\mathscr{V}_\lambda)^0 = \mathrm{Hom}_{\mathbf{C}}(\mathscr{V}_\lambda, \mathbf{C}),\ \hat{h}(l(u)) = l(\hat{h}^0(u)),\ u \in \mathscr{V}_\lambda, l \in (\mathscr{V}_\lambda)^0\ ,$$

$$s_i^0 = s_i,\ x_i^0 = x_i,\ (\hat{h}_1\hat{h}_2)^0 = \hat{h}_2^0\hat{h}_1^0,\ \hat{h}_{1,2} \in \mathfrak{H}_0\ . \tag{1.7}$$

The anti-involution $\hat{h} \to \hat{h}^0$ is well-defined because relations (1.4) are self-dual.

The above construction gives two canonical W^a-homomorphisms:

$$id : V^0 \to \mathscr{V} \to \mathscr{V}_\lambda,\ tr : V(\lambda) \to V\ .$$

Proposition 1.4. *If a $\mathfrak{H}'$-submodule $\mathscr{U} \subset V(\lambda)$ is non-zero then its image $tr(\mathscr{U})$ is non-zero too.*

Proof. It is clear, since $\mathscr{V}_\lambda$ is generated by V^0 as an $\mathfrak{H}'$-module. □

If V is finite-dimensional then $\dim_{\mathbf{C}} V(\lambda) = |W| \dim_{\mathbf{C}} V$, where $|W|$ is the number of elements of W. The main examples will be for one-dimensional representations of W^a which are described by W-invariant sets $\varepsilon \subset \{\pm 1\}$:

$$\tau_\varepsilon(s_i) = \varepsilon_i, \ \ \tau_\varepsilon(a') = 1, \ \ 0 \leqq i \leqq n, \ a \in A\,. \tag{1.8}$$

Let us denote the corresponding V, $V(\lambda)$ by $\mathbf{C}_\varepsilon$, $\mathbf{C}_\varepsilon(\lambda)$ for the latter reference.

2. Affine r-Matrices

Following [Ch 1,3,5] we introduce abstract classical r-matrices with the values in an arbitrary $\mathbf{C}$-algebra $\mathscr{F}$ and show how to extend non-affine r-matrices to affine ones. The notations are from Sect. 1. Let us denote $\mathbf{R}\tilde{\alpha} + \mathbf{R}\tilde{\beta} \subset \mathbf{R}^n$ by $\mathbf{R}\langle\tilde{\alpha},\tilde{\beta}\rangle$ for $\tilde{\alpha},\tilde{\beta} \in R^a$.

Definition 2.1. *a) A set* $r = \{r_{\tilde{\alpha}} \in \mathscr{F}, \tilde{\alpha} \in R^a_+\}$ *is an* **affine r-matrix** *if*

$$[r_{\tilde{\alpha}}, r_{\tilde{\beta}}] = 0\,, \tag{2.1}$$

$$[r_{\tilde{\alpha}}, r_{\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + [r_{\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\beta}}] = 0\,, \tag{2.2}$$

$$[r_{\tilde{\alpha}}, r_{\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + [r_{\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\alpha}+2\tilde{\beta}}] + [r_{\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\beta}}] + [r_{\tilde{\alpha}+2\tilde{\beta}}, r_{\tilde{\beta}}] = 0, \ \ [r_{\tilde{\alpha}}, r_{\tilde{\alpha}+2\tilde{\beta}}] = 0\,, \tag{2.3}$$

$$\begin{aligned}
&[r_{\tilde{\alpha}}, r_{3\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{2\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{3\tilde{\alpha}+2\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}}, r_{\tilde{\beta}}] + \\
&[r_{3\tilde{\alpha}+\tilde{\beta}}, r_{2\tilde{\alpha}+\tilde{\beta}}] + [r_{2\tilde{\alpha}+\tilde{\beta}}, r_{3\tilde{\alpha}+2\tilde{\beta}}] + [r_{2\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\alpha}+\tilde{\beta}}] + [r_{3\tilde{\alpha}+2\tilde{\beta}}, r_{\tilde{\alpha}+\tilde{\beta}}] + [r_{\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\beta}}] = 0\,, \\
&[r_{3\tilde{\alpha}+\tilde{\beta}}, r_{3\tilde{\alpha}+2\tilde{\beta}}] + [r_{3\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\beta}}] + [r_{3\tilde{\alpha}+2\tilde{\beta}}, r_{\tilde{\beta}}] = 0 = [r_{3\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\alpha}+\tilde{\beta}}] = [r_{2\tilde{\alpha}+\tilde{\beta}}, r_{\tilde{\beta}}],
\end{aligned} \tag{2.4}$$

under the assumption that $\tilde{\alpha},\tilde{\beta} \in R^a_+$ *and*

$$\mathbf{R}\langle\tilde{\alpha},\tilde{\beta}\rangle \cap R^a = \{\pm\tilde{\gamma}\}, \tilde{\gamma} \text{ runs over all the indices} \tag{2.5}$$

in the corresponding identities.

b) A **closed** ***r*-matrix** *(or a closure of the above* r*) is a set* $\{r_{\tilde{\alpha}} \in \mathscr{F}, \tilde{\alpha} \in R^a\}$ *(extending* r *and) satisfying relations* (2.1)–(2.4) *for arbitrary (positive, negative)* $\tilde{\alpha},\tilde{\beta} \in R^a$ *such that the corresponding condition* (2.5) *is fulfilled. If the indices are from* R_+ *(or* R*) we call* r **non-affine**. □

We note that (2.5) for identity (2.1) means that

$$(\tilde{\alpha},\tilde{\beta}) = 0 \text{ and } \mathbf{R}\langle\tilde{\alpha},\tilde{\beta}\rangle \cap R^a = \{\pm\tilde{\alpha}, \pm\tilde{\beta}\}\,. \tag{2.6}$$

It is equivalent to the existence of $\tilde{w} \in W^a$ such that $\tilde{\alpha} = \tilde{w}(\alpha_i), \tilde{\beta} = \tilde{w}(\alpha_j)$ for simple $\alpha_i \mp \alpha_j (0 \leqq i,j \leqq n)$ disconnected (not neighbouring) in the affine Dynkin diagram of R^a. In the most interesting examples, (2.1) holds true for arbitrary orthogonal roots.

The corresponding assumptions for (2.2)–(2.4) give that $\tilde{\alpha},\tilde{\beta}$ are simple roots of a two-dimensional root subsystem in R^a of type A_2, B_2, G_2. Here $\tilde{\alpha},\tilde{\beta}$ stay for α_1, α_2 in the notations from the figure of the systems of rank 2 from [B]. One can represent them as follows : $\tilde\alpha = \tilde w(\alpha_i), \tilde\beta = \tilde w(\alpha_j)$ for a proper $\tilde w$ from W^a and joined (neighbouring) α_i, α_j.

Given an arbitrary r, we always have the following closures (the standard one and the extension by zero):

$$r_{-\tilde\alpha} = -r_{\tilde\alpha},\ \ r_{-\tilde\alpha} = 0,\ \ \tilde\alpha \in R^a_+ \,. \tag{2.7}$$

If there exists an action of $W^a \ni \tilde w$ on $\mathscr{F}$ such that

$$\tilde w(r_{\tilde\alpha}) = r_{\tilde w(\tilde\alpha)} \text{ for } \tilde\alpha,\ \tilde w(\tilde\alpha) \in R^a_+ \,,$$

then the extension of r satisfying these relations for all $\tilde w$ is well-defined and closed (the invariant closure).

Theorem 2.2 *Let us assume that r is a closed non-affine r-matrix and the group $A \ni a$ (see* (1.2)) *acts on the algebra $\mathscr{F} \ni f$ (written $f \to a(f)$) obeying the following condition:*

$$a(r_\alpha) = r_\alpha \text{ whenever } (a,\alpha) = 0, a \in A, \alpha \in R \,. \tag{2.8}$$

Then the elements

$$r_{\tilde\alpha} \overset{\text{def}}{=} a(r_\alpha) \text{ for } a \text{ such that } \tilde\alpha = a'(\alpha) = \alpha - (a,\alpha)c \tag{2.9}$$

are well-defined (*do not depend on the choice of the element a satisfying* (2.9) *for a given $\tilde\alpha \in R^a$*) *and form a closed* (*affine*) *r-matrix.*

Proof is the same as that of Theorem 2.3 from [Ch4] in the case of quantum R-matrices. □

Theorem 2.3 *a*) *Given an affine r-matrix, let us suppose that the algebra $\mathscr{F}$ is supplied* (*as a* **C**-*linear space*) *with a norm $||f||$ and the following series are absolutely convergent:*

$$\tilde r_\alpha \overset{\text{def}}{=} r_\alpha + \sum_{k>0}(r_{kc+\alpha} - r_{kc-\alpha}), \alpha \in R_+ \,,$$
$$y_u \overset{\text{def}}{=} \sum_{\tilde\alpha\in R^a_+} (u,\tilde\alpha)r_{\tilde\alpha} = \sum_{\alpha\in R_+} (u,\alpha)\tilde r_\alpha,\ \ u \in \mathbf{R}^n \,. \tag{2.10}$$

If any pairwise products of these series are also absolutely convergent, then $\tilde r$ is a non-affine r-matrix and $[y_u, y_v] = 0$ for any $u, v \in \mathbf{R}^n$.

b) *Let the group W^a act in $\mathscr{F}$ by continuous automorphisms relative to the norm and r be W^a-invariant:*

$$\tilde w(r_{\tilde\alpha}) = r_{\tilde w(\tilde\alpha)} \text{ for all } \tilde w \in W^a, \tilde\alpha \in R^a \,, \tag{2.11}$$

for a proper closure of r. Then $\tilde r$ is W-invariant and

$$s_i(y_u) - y_{s_i(u)} = (u,\alpha_i)(r_{\alpha_i} + s_i(r_{\alpha_i})),\ \ 0 \leqq i \leqq n,\ \ u \in \mathbf{R}^n \,. \tag{2.12}$$

Proof. The commutativity in the non-affine case is established in [Ch 3], Proposition 3.2. As to (2.12), see [Ch 3], Corollary 3.6 and the end of Sect. 1 from [Ch 5]. The considerations in the affine case are the same. We calculate separately the sums of the pairwise commutators for any subspaces $\mathbf{R}\langle\tilde\alpha,\tilde\beta\rangle \cap R^a$. □

Let us fix one more W^a-invariant set $\mu = \{\mu_{\tilde{\alpha}}, \tilde{\alpha} \in R^a\}$. Here are two examples of the above construction.

Theorem 2.4. *a) Using the variables* $\{x\}$ *from* (1.3),(1.5), *let* $\mathscr{F}$ *be the algebra* $\mathscr{F}^\flat$ *generated by* $\mathbf{C}[W^a]$ *and*

$$\mathbf{C}\{x\} = \mathbf{C}[\mathrm{ct}(x_{\alpha^\vee} + kx_c), \tilde{\alpha} = \alpha + kc \in R^a_+]$$

with the cross-relations $\tilde{w}x_u = x_{\tilde{w}(u)}\tilde{w}$, *where* $\mathrm{ct}(t) = (\exp(t) - 1)^{-1}$. *Then*

$$r^\flat_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}\mathrm{ct}(x_a + kx_c)(\mu_{\tilde{\alpha}} - s_{\tilde{\alpha}}), \tilde{\alpha} = \alpha + kc \in R^a, a = \alpha^\vee , \tag{2.13}$$

is a W^a*-invariant closed r-matrix and* $s_i r^\flat_{\alpha_i} + r^\flat_{\alpha_i} s_i = \eta_i(s_i - \mu)$ *for* $0 \leqq i \leqq n$.

b) Now $\mathscr{F} = \mathscr{F}^\#$ *is the algebra generated by* $\mathscr{F}^\flat$ *and* $\mathbf{C}\{z\} = \mathbf{C}[\mathrm{ct}(z_{\tilde{\alpha}})]$, *where*

$$z_{u+\kappa c} = \sum_{i>0}(u, b_i)z_i + \kappa\xi, u \in \mathbf{R}^n, \quad \textit{for complex} \ \tilde{z} = \{z_1, \ldots, z_n, z_c = \xi\}$$

commuting with $\mathbf{C}[W^a]$. *The following functions of* $\tilde{z}$

$$r^\#_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}\mathrm{ct}(z_\alpha + k\xi)(s_{\tilde{\alpha}} - \mu_{\tilde{\alpha}}) + r^\flat_{\tilde{\alpha}}, \ \ \tilde{\alpha} = \alpha + kc \in R^a , \tag{2.14}$$

also form an r-matrix which is invariant relative to the diagonal (simultaneous) action δ *of* W^a *that is the product of the action of* W^a *on* $\{x\}$ *and the analogous action* σ *on* $\{z\} : \sigma(\tilde{w})(z_u + \kappa\xi) = z_{\tilde{w}(u)} + \kappa\xi$. *Moreover* $\delta(s_i)(r^\#_{\alpha_i}) + r^\#_{\alpha_i} = 0$ *for* $0 \leqq i \leqq n$.

Proof. The theorem for $\mu = 1$ is a straightforward affine extension of Corollary 3.6 from [Ch 3] (see also the end of Sect. 2, [Ch 5]). These r-matrices are quasiclassical limits of the quantum R-matrices from [Ch 4], Propositions 3.5, 3.8 ((1.6) is a rational counterpart of one of them). Calculating the corresponding commutators (2.1–4) we obtain a set of relations that are the coefficients of $s_{\tilde{\alpha}}$ and $s_{\tilde{\alpha}}s_{\tilde{\beta}}$ (the latter never coincide with the first). If $s_{\tilde{\alpha}}s_{\tilde{\beta}} = 1$ then $\tilde{\alpha} = \tilde{\beta}$ and the corresponding commutator equals zero. Hence if the r-matrix relations are checked for one nonzero μ they are valid for all of them. □

We regard $\{x_i, x_c, z_i, \xi\}$ as the coordinates of the space $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$, where W^a acts on the first component in the obvious way. The following proposition introduces a completion of the semi-direct product of $\mathbf{C}[W^a]$ and a proper "functional" extension of the algebra $\mathbf{C}\{x, z\} = \mathbf{C}[\mathrm{ct}(x_{\alpha^\vee} + kx_c), \mathrm{ct}(z_{\tilde{\alpha}} + k\xi)]$. This definition allows us to apply Theorem 2.3. We will use it permanently in the next sections as well. The discussion will be continued in the next paper.

Proposition 2.5 *Let* $\text{æ} \geqq \varepsilon > 0$, $M > 1$, $m \in \mathbf{Z}_+$,

$$\Xi\text{æ}(M) = \{(x, x_c, z, \xi) \ \textit{such that} \ \Re(\xi), \Re(x_c) \geqq \text{æ} ,$$

$$|\mathrm{ct}(x_a + kx_c)|, \ \ |\mathrm{ct}(z_\alpha + k\xi)| < M > \exp|x_a|, \exp|z_\alpha|\}$$

for all $k \in \mathbf{Z}$, $\alpha \in R$, $a = \alpha^\vee$. *A formal series* $f = \sum_{\tilde w} f_{\tilde w}(x, x_c, z, \xi)\tilde w$ *for scalar* $f_{\tilde w}$, $\tilde w \in W^a$ *is called m-convergent if the following norm is absolutely convergent for any* $\ae$, M, ε:

$$\|f\| \overset{\text{def}}{=} \sum_{\tilde w} \sup\{|f_{\tilde w}| \text{ in } \Xi_{\ae}(M)\}\|\tilde w\|\,,$$
$$\text{where } \|\tilde w\| = \exp\left((\ae - \varepsilon) l(\tilde w)(2h-2)^{-1}4^{1-m}\right), \tag{2.15}$$

$l(\tilde w)$ *is the length of* $\tilde w \in W^a$ *with respect to the generators* $\{s_i, 0 \leqq i \leqq n\}$, h *the Coxeter number,* $|\ |$ *the absolute value. Then products of any* m *series from* (2.10) *are m-convergent (for both* $r^\flat$ *and* $r^\#$*). The diagonal action of* W^a *is continuous.*

Proof. Let us start with $\mu = 0$. Without $r^\flat$, (2.15) follows from the estimate

$$l(s_{\alpha+kc}) \leqq kl(a') + \text{const} \leqq k(2h-2) + \text{const, for } k \geqq 0,\ \ a = \alpha^\vee\,, \tag{2.16}$$

(see e.g. [Ch4], Proposition 1.6 and [Ch7], (1.15)). Here the factor 4^{1-m} does not appear. Given $\alpha(1), \ldots, \alpha(m) \in R_+$, let us consider the product $\tilde r^\flat_{\alpha(1)} \ldots \tilde r^\flat_{\alpha(m)}$ that is the sum of

$$\Pi_k = r^\flat_{\tilde\alpha(1)} \ldots r^\flat_{\tilde\alpha(m)}, k = \{k(1), \ldots, k(m)\} \subset \mathbf{Z}_+, \tilde\alpha(i) = k(i)\ c \pm \alpha(i) \in R^a_+\,.$$

We should fix $C > 0$ and calculate the number of the terms such that $\|\Pi_k\| > C$. Now $\{s_{\tilde\alpha}\}$ from $\{r^\flat\}$ act on the arguments $\{x_i\}$ moving them from $\Xi_{\ae}(M)$:

$$\Pi_k = (-1)^m \prod_i \eta_{\tilde\alpha(i)}(\exp x_{\tilde\alpha^i} - 1)^{-1}\tilde w, \text{ where } \tilde w = \prod_i s_{\tilde\alpha(i)}\,,$$

$$\tilde\alpha^1 = \tilde\alpha(1),\ \ \tilde\alpha^2 = s_{\tilde\alpha(1)}(\tilde\alpha(2)), \ldots, \tilde\alpha^m = (s_{\tilde\alpha(1)} \ldots s_{\tilde\alpha(m-1)})(\tilde\alpha(m))\,. \tag{2.17}$$

Lemma 2.6. *Let* $\tilde\alpha^i = \alpha^i + k^i c$, $\alpha^i \in R$, $\ \ k_\pm = \max\{0, \pm k^i, 1 \leqq i \leqq m\}$. *Then*

$$c_m k_+ \geqq k_- \text{ for } c_m = (v+1)^{m-1} - 1\,, \tag{2.18}$$

where v *is* 1 *for* A, D, E, 3 *for* G_2, *and* 2 *for the other root systems. The number of the terms* Π_k *with given* k_+ *is less than* $(c_m+1)^m(k_+)^m$. *The length of the corresponding elements* $\tilde w$ *is not more than* $(2h-2)(c_m+1)k_+$.

Proof. We argue by induction on m. The inequality for $k_\pm$ is clear for $m = 1$ since $k^1 = k(1)$ is always non-negative. Supposing that (2.18) is valid for m, let us add one more factor $\tilde r^\flat_{\tilde\alpha(0)}$ on the left and denote the new pair of extreme values of $\{\pm k^i, 0 \leqq i \leqq n\}$ by $k'_\pm$. Then

$$k_+ - vk^0 \leqq k'_+ \geqq k^0, \quad k'_- \leqq k_- + vk^0\,,$$

$$c_m(1+v)k'_+ \geqq c_m(k'_+ + vk^0) \geqq k_- \geqq k'_- - vk^0 \geqq k'_- - vk'_+\,.$$

Hence $(c_m(1+v)+v)k'_+ \geqq k'_-$, which provides the necessary estimate. As to the length, $l(\tilde w) = l(\tilde w^{-1})$, $\tilde w^{-1} = s_{\tilde\alpha^m} \cdots s_{\tilde\alpha^1}$, and we can use (2.16). □

The lemma gives that $\|\Pi_k\| <$ const $\exp(-k_+\varepsilon)$ for a rather big k_+. Here const $= (M\max|\eta|)^m$. Finally, the sum of the norms of the terms Π_k with given k_+ can be estimated as $C(k_+)^m \exp(-k_+\varepsilon)$ for a constant C. It gives the convergence.

If we have a "mixed" product (2.17) where some of x are replaced by z, then the reasoning is quite similar. We apply again the induction taking into consideration mostly the first term (with $k(1) = k^1$). The changes of the arguments of the others can be controlled in the same way. When $\mu \neq 0$, we can use the estimate without μ for smaller m. □

Corollary 2.7. *Let us denote the operators y from* (2.10) *considered for* r^b *by* $\{y_u^b, u \in \mathbf{R}^n\}$ *and introduce*

$$x_u^b = y_u^b + (\rho_{\eta\mu}, u),\ x_c^b = h'_{\eta\mu} \overset{\text{def}}{=} h_{\eta\mu}(\theta, \theta)/2,\ x_{u+\kappa c}^b = x_u^b + \kappa x_c^b\,. \tag{2.19}$$

Then the group algebra $\mathbf{C}[W^a]$ *and* $\{x_{\tilde{\alpha}}^b\}$ *satisfy relations from Definition* 1.1 *and form a representation of* $\mathfrak{H}'$ (*which is faithful in* $\mathfrak{H}'/(x_c - h'_{\eta\mu})$). □

3. Dunkl operators and KZ

Let us extend $\mathbf{C}$-linearly the standard pairing (,) to $\mathbf{C}^n$ and then to $\mathbf{C}^{n+2} = \mathbf{C}^n \oplus \mathbf{C}_c \oplus \mathbf{C}_d$ setting $(c, d) = 1$, $(c, c) = (c, u) = 0 = (d, u) = (d, d)$ for $u \in \mathbf{C}^n$ (see e.g. [Ka], Chapter 6). Given $\tilde{\alpha} = \alpha + kc \in R^a$, $\alpha \in A$, the formulas

$$s_{\tilde{\alpha}}(\hat{u}) = \hat{u} - \{(u, \alpha) + vk\}\alpha^\vee - \{vk^2(\alpha^\vee, \alpha^\vee)/2 + (u, \alpha^\vee)k\}c\,,$$

$$a'(\hat{u}) = \hat{u} + va - \{v(a, a)/2 + (u, a)\}c, \quad \hat{u} = u + \kappa c + vd \in \mathbf{C}^{n+2}\,,$$

$$z_{\hat{u}} = \sum_{i=1}^{n} (u, b_i) z_i + \kappa\xi + v\zeta, \quad \sigma(\tilde{w})(z_{\hat{u}}) \overset{\text{def}}{=} z_{\{\tilde{w}(\hat{u})\}}, \quad \sigma_{\tilde{\alpha}} = \sigma(s_{\tilde{\alpha}}) \tag{3.1}$$

define an action of $\tilde{w} \in W^a$ on $\hat{u} \in \mathbf{C}^{n+2}$ and $W_\sigma^a \overset{\text{def}}{=} \sigma(W^a)$ on $z_{\hat{u}}$.

The linear functions $z_i = z_{\alpha i}$, $1 \leqq i \leqq n$, ξ, ζ will be regarded as coordinates of $\mathbf{C}^{n+2}$. For instance, $\partial z_{\tilde{\alpha}}/\partial z_i$ is the multiplicity of α_i in $\tilde{\alpha} = \alpha + kc \in R^a$, $\partial z_{\tilde{\alpha}}/\partial \xi = k$, $\partial z_{\tilde{\alpha}}/\partial \zeta = 0$. We will also use the derivatives

$$\partial_{\tilde{u}} = \partial_u + \kappa \partial/\partial\zeta, \quad \partial_u(z_{\hat{v}}) = (v, u), \quad \tilde{u} = u + \kappa c \in \mathbf{C}^{n+1}, \quad \hat{v} \in \mathbf{C}^{n+2}\,,$$

with the following evident properties:

$$\partial_{r\tilde{u}+t\tilde{v}} = r\partial_{\tilde{u}} + t\partial_{\tilde{v}}, \quad \sigma(\tilde{w})(\partial_{\tilde{u}}) = \partial_{\tilde{w}(\tilde{u})}, \quad r, t \in \mathbf{C}, \quad \tilde{w} \in W^a\,,$$

$$\partial_{b_i} = \partial/\partial_{z_i}, \quad 1 \leqq i \leqq n, \quad \partial_c = \partial/\partial\zeta\,. \tag{3.2}$$

We extend $(\rho_{\eta\mu}, \cdot\,)$ to a linear function on $\tilde{u} = u + \kappa c \in \mathbf{C}^{n+1}$ by the formulas (see (1.1))

$$\tilde{\rho}_{\eta\mu}(\tilde{u}) = (\rho_{\eta\mu} + h_{\eta\mu}(\theta, \theta)d/2,\ \tilde{u}) = (\rho_{\eta\mu}, u) + \kappa h'_{\eta\mu}, \quad h'_{\eta\mu} = h_{\eta\mu}(\theta, \theta)/2\,, \tag{3.3}$$

to ensure the relations $\tilde{\rho}_{\eta\mu}(\alpha_i) = \eta_i\mu_i(\alpha_i, \alpha_i)/2$ for all $0 \leqq i \leqq n$.

Following Theorem 2.4, let us introduce the algebra $\mathscr{F}_\sigma^a$ generated by $\mathbf{C}[W_\sigma^a]$ and $\mathbf{C}\{z\} = \mathbf{C}[\mathrm{ct}(z_{\tilde{\alpha}}), \tilde{\alpha} \in R_+^a]$. We will need another W^a (without σ) commuting with z and the corresponding algebra $\mathscr{F}^a$ generated by W^a instead of W_σ^a. The definition of the sequence of norms (in terms of m, M) from Proposition 2.5 remains the same (but there is no dependence on $\{x\}$!).

We can write $\mathscr{F}^a = \mathbf{C}\{z\} \otimes_{\mathbf{C}} \mathbf{C}[W^a]$. As to $\mathscr{F}^a_\sigma$, it is the semi-direct (smash) tensor product where the second algebra acts naturally on the first.

The algebra of differential operators in $\partial_1, \dots, \partial_n$, ∂_c with the coefficients in $\mathscr{F}^a_\sigma$ will be denoted by $\mathscr{F}^a_\sigma[\partial]$. We will also use $\mathscr{F}^a[\partial]$ (the derivatives are always with respect to z, ζ).

Theorem 3.1. *The following family of differential-difference* **Dunkl operators** *defined for $\tilde{u} = u + \kappa c \in \mathbf{R}^{n+1}$:*

$$\Delta_{\tilde{u}} \stackrel{\text{def}}{=} \partial_u + \kappa\partial/\partial\zeta - \sum_{\tilde{\alpha}>0} \eta_{\tilde{\alpha}}(u, \alpha)\mathrm{ct}(z_{\tilde{\alpha}})(\sigma_{\tilde{\alpha}} - \mu_{\tilde{\alpha}}) + \tilde{\rho}_{\eta\mu}(\tilde{u}), \tag{3.4}$$

is commutative. Moreover, $\{\sigma_i = \sigma(s_i)\}$ for $0 \leqq i \leqq n$ and $\{\Delta_{\tilde{u}}\}$ satisfy relations (1.4), *and the map*

$$\Sigma : s_{\tilde{\alpha}} \mapsto \sigma_{\tilde{\alpha}}, \quad x_{\tilde{u}} \mapsto \Delta_{\tilde{u}} \tag{3.5}$$

gives an injective homomorphism from $\mathfrak{H}'$ into the algebra of convergent series from $\mathscr{F}^a_\sigma[\partial]$. The convergence of differential operators is coefficient-wise with respect to the norms for sufficiently big m, M. If $\Delta_c = \partial/\partial\zeta + h'_{\eta\mu}$ is replaced by zero, then Σ maps via $\mathfrak{H}'_0$.

Proof. Without $\{\partial_{\tilde{u}}\}$, the commutativity follows from Corollary 2.7 (x are to be replaced by z). The contribution of the derivatives to the commutators of $\Delta_{\tilde{u}}$ is trivial since $[\partial_{\tilde{u}}, r^b_{\tilde{\alpha}}] = 0$ if $(\tilde{u}, \tilde{\alpha}) = 0$ and

$$[\partial_{\tilde{u}}, (\tilde{v}, \tilde{\alpha})r^b_{\tilde{\alpha}}] - [\partial_{\tilde{v}}, (\tilde{u}, \tilde{\alpha})r^b_{\tilde{\alpha}}] = [\partial_{(\tilde{v},\tilde{\alpha})\tilde{u}-(\tilde{u},\tilde{\alpha})\tilde{v}}, r^b_{\tilde{\alpha}}] = 0 \text{ for all } \tilde{u}, \tilde{v}\,.$$

Here (see (2.13))

$$r^b_{\tilde{\alpha}} = \eta_{\tilde{\alpha}}\mathrm{ct}(z_a + k\xi)(\mu_{\tilde{\alpha}} - s_{\tilde{\alpha}}),\ \tilde{\alpha} = \alpha + kc \in R^a,\ a = \alpha^\vee\,.$$

The other properties of Σ follow from the same Corollary 2.7. □

The theorem is valid even when the map σ satisfies the following weaker properties:

$$\sigma_{\tilde{\alpha}} z_{\tilde{u}} = z_{\tilde{u}'}\sigma_{\tilde{\alpha}},\ \sigma_{\tilde{\alpha}}\partial_{\tilde{u}} = \partial_{\tilde{u}'}\sigma_{\tilde{\alpha}},\ \text{for } \tilde{u}' = s_{\tilde{\alpha}}(\tilde{u}),\ \tilde{u} \in \mathbf{C}^{n+1}\,, \tag{3.6}$$

$$\sigma_{\tilde{\alpha}_1}\sigma_{\tilde{\alpha}_2} = \sigma_{\tilde{\beta}_1}\sigma_{\tilde{\beta}_2} \text{ if } s_{\tilde{\alpha}_1}s_{\tilde{\alpha}_2} = s_{\tilde{\beta}_1}s_{\tilde{\beta}_2},\ \tilde{\alpha}, \tilde{\beta} \in R^a\,. \tag{3.7}$$

Indeed, the necessary relations are written in terms of commutators (cf. [Ch 5], Sect. 2).

Definition 3.2. *Let us take $\mathbf{C}[W^a]$ which commutes with z, ξ, ζ (we omit σ to differ it from $\mathbf{C}[W^a_\sigma]$). Given $\Delta \in \mathscr{F}^a_\sigma[\partial]$, we represent it in the form*

$$\Delta = \sum_{\tilde{w}\in W^a} D_{\tilde{w}}\sigma(\tilde{w}), \quad \textit{where } D_{\tilde{w}} \textit{ are differential,} \tag{3.8}$$

and introduce the operator from $\mathscr{F}^a[\partial]$

$$\mathrm{Red}(\Delta) \stackrel{\text{def}}{=} \sum_{\tilde{w}\in W^a} D_{\tilde{w}}\tilde{w}^{-1} \tag{3.9}$$

with the coefficients in the completion of the group algebra $\mathscr{F}^a = \mathbf{C}\{z\}\otimes_{\mathbf{C}} \mathbf{C}[W^a]$. Replacing $\partial_c = \partial/\partial\zeta$ by $-h'_{\eta\mu}$ in $\mathrm{Red}(\Delta)$ we obtain $\mathrm{Red}_0(\Delta) \in \mathscr{F}^a[\partial_1, \dots, \partial_n]$. Both operations are continuous. □

Theorem 3.3. *Given arbitrary* Δ *and* W^a_σ*-invariant* Δ' *from* $\mathscr{F}^a_\sigma[\partial]$,

$$\mathrm{Red}(\Delta\Delta') = \mathrm{Red}(\Delta)\mathrm{Red}(\Delta')\,.$$

If $p \in \mathbf{C}[x_1,\ldots,x_n]^W$, *then the (differential)* **OP operators**

$$L_p \stackrel{\mathrm{def}}{=} \mathrm{Red}_0(p(\Delta_{b_1},\ldots,\Delta_{b_n}) \in \mathscr{F}^a[\partial_1,\ldots,\partial_n]$$

are pairwise commutative and W^a_δ*-invariant with respect to the diagonal action* $\delta(\tilde{w}) = \sigma(\tilde{w}) \otimes \tilde{w}$, *where* $\tilde{w}$ *act in* $\mathbf{C}[W^a]$ *by conjugations* (cf. (1.5)).

Proof. We completely follow [Ch 5], Theorem 2.4. □

It is worth mentioning that the operators $p(\Delta_{b_1},\ldots,\Delta_{b_n})$ for $p \in \mathbf{C}[x_1,\ldots,x_n]^W$ are not W^a-invariant. Therefore Red destroys their commutativity and has to be replaced by Red_0 that can be done for a special value of $\partial_c = \partial/\partial_\zeta$ only.

We can exclude $\partial/\partial\zeta$ from the construction considering the operators

$$\tilde{\Delta}_{b_k} = \exp(2\pi i \Delta_{b_k}/h'_{\eta\mu})$$

instead of Δ_{b_k}. The corresponding $p(\Delta_{b_1},\ldots,\tilde{\Delta}_{b_n})$ will be W^a-invariant. Hence we can use Red (and do not need the central element c at all). However $\{\tilde{\Delta}_{b_k}\}$ are rather complicated to deal with. They are similar to the difference elliptic Dunkl operators which will be discussed in the next paper.

Theorem 3.4. *Let us introduce the* **KZ operators** *that are differential operators of the first order with convergent coefficients from* $\mathbf{C}\{z\} \otimes_{\mathbf{C}} \mathfrak{H}'$:

$$\mathscr{D}_{\tilde{u}} = \partial_{\tilde{u}} - \sum_{\tilde{\alpha}>0} \eta_{\tilde{\alpha}}(\tilde{u}, \tilde{\alpha})\mathrm{ct}(z_{\tilde{\alpha}})(s_{\tilde{\alpha}} - \mu_{\tilde{\alpha}}) + \tilde{\rho}_{\mu\eta}(\tilde{u}) - x_{\tilde{u}},\ \tilde{u} \in \mathbf{C}^{n+1}\,. \tag{3.10}$$

They are pairwise commutative and satisfy the following invariance property with respect to the above diagonal action δ *extended to* $\mathfrak{H}' \supset \mathbf{C}[W^a]$:

$$\delta(\tilde{w})(\mathscr{D}_{\tilde{u}}) = \mathscr{D}_{\tilde{w}(\tilde{u})},\ \tilde{w} \in W^a,\ \tilde{u} \in \mathbf{C}^{n+1}\,. \tag{3.11}$$

Proof. First of all, the contribution of the derivations is zero (see the proof of Theorem 3.1). Then the commutators $[\mathscr{D}_{\tilde{u}}, \mathscr{D}_{\tilde{v}}]$ and the differences $\delta(\tilde{w})(\mathscr{D}_{\tilde{u}}) - \mathscr{D}_{\tilde{w}(\tilde{u})}$ for all $\tilde{u}, \tilde{v}, \tilde{w}$ belongs to $\mathbf{C}[W^a]$. We have to check that they vanish. Theorem 2.4 gives that they really equal zero in the representation of $\mathfrak{H}'$ from Corollary 2.7. But the latter is faithful when restricted to $\mathbf{C}[W^a]$. □

The isomorphism. We will show that KZ considered in certain induced representations of $\mathfrak{H}'$ is equivalent to the proper eigenvalue problem for the above Dunkl operators. It generalizes the constructions from [Ma] and [Ch 5]. Let us start with the following general remark. If $\tilde{w} \in W^a$ are bounded operators in a certain algebra with a norm $\mathscr{N}$, then the series for $\mathscr{D}$ and Δ and the products of any m among them are convergent for rather big $\mathfrak{R}(\xi)$. Indeed, (2.15) leads to the estimate

$$\exp(\text{æ}(2h-2)^{-1}4^{1-m}) > \max\{\mathscr{N}(s_i),\ 0 \leqq i \leqq n\}\,.$$

The norm always exists if W^a and $\{x\}$ act in finite dimensional representations.

Thus the **KZ equation**, which is the system

$$\mathscr{D}_u\varphi(z_1,\ldots,z_n)=0, \quad u\in\mathbf{C}^n\,, \tag{3.12}$$

is well-defined when the values of φ are taken in any finite dimensional representations of $\mathfrak{H}'$. The **extended KZ** is obtained for $\tilde{u}$ instead of u:

$$\mathscr{D}_u\phi(z_1,\ldots,z_n,\zeta)=0,\ \partial\phi/\partial\zeta+h'_{\eta\mu}=0,\ h'_{\eta\mu}=h_{\eta\mu}(\theta,\theta)/2\,. \tag{3.12a}$$

If φ satisfies (3.12) then $\phi=\varphi\exp(-h'_{\eta\mu}\zeta)$ is a solution of (3.12a). But this trivial extension is important for the main theorem below.

We may use standard results about the solutions of differential equations (assuming that $\mathfrak{R}(\xi)$ is rather big). Here and further ξ is considered as a parameter ($\partial/\partial\xi$ does not appear in $\mathscr{D}$, Δ).

Following Sect. 1, let V be a finite dimensional $\mathbf{C}[W^a]$-module, τ the corresponding homomorphisms from $\mathbf{C}[W^a]$ to $\mathrm{End}_{\mathbf{C}}V$. We fix a set $\lambda=\{\lambda_1,\ldots,\lambda_n\}\subset\mathbf{C}$ and consider the $\mathfrak{H}'_0$-module $V(\lambda)$ introduced in (1.17) with the $\mathbf{C}[W^a]$-homomorphism $tr: V(\lambda)\mapsto V$. The homomorphism $\mathfrak{H}'_0\mapsto\mathrm{End}_{\mathbf{C}}V(\lambda)$ will be denoted by $\hat{\tau}$.

Main Theorem 3.5. *Let $\mathscr{K}$ be the space of solutions $\varphi(z)$ of* (3.12) *in $V(\lambda)$ defined in a neighbourhood of a given point (its dimension coincides with* $\dim_{\mathbf{C}}V(\lambda)=|W|\dim_{\mathbf{C}}V$). *Then the map tr: $\varphi\mapsto\psi=tr(\varphi)$ is an isomorphism onto the space $\mathscr{M}$ of solutions of the* **quantum many-body problem**

$$L_p\psi(z)=p(\lambda_1,\ldots,\lambda_n)\psi(z),\ p(x_1,\ldots,x_n)\in\mathbf{C}[x]^W\,, \tag{3.13}$$

for the operators $\{L\}$ introduced in Theorem 3.3.

Proof. The statement is a direct generalization of Theorem 4.6 from [Ch 5]. We will remind the main type steps of the proof (adapted to the affine case).

In the set up of Theorem 2.5, let us pick a set $Z\subset\mathbf{C}^{n+1}$ obtained from $\Xi_{\mathfrak{a}}(M)$ by certain cutoffs and obeying the following conditions. It is connected and simply connected. The image of the intersection $\bigcap_{\tilde{w}}\tilde{w}(Z)$ in the quotient $\Xi_{\mathfrak{a}}(M)/W^a$ is connected. Assuming that $\mathfrak{R}(\xi)$ is rather big, we can fix an invertible analytical solution $\Phi(z,\zeta)$ of (3.12a) for $z\in Z$ and arbitrary ζ with the values in $\mathrm{End}_{\mathbf{C}}V(\lambda)$.

The functions $\sigma(\tilde{w})\Phi$, $\tilde{w}\in W^a$, are well-defined in open subsets of Z, we may introduce the "monodromy matrices" T:

$$\hat{\tau}(\tilde{w})\Phi(z,\zeta)=\sigma(\tilde{w}^{-1})(\Phi(z,\zeta))T_{\tilde{w}}(z,\zeta),\quad \tilde{w}\in W^a\,, \tag{3.14}$$

which are well-defined for almost all $z\in Z$ and locally constant (use the invariance of $\mathscr{D}_{\tilde{u}}$). They satisfy the one-cocycle relation

$$T_{\tilde{w}_1\tilde{w}_2}=\sigma(\tilde{w}_2^{-1})(T_{\tilde{w}_1})T_{\tilde{w}_2},\quad \tilde{w}_1,\tilde{w}_2\in W^a\,,$$

which results in the following action $\bar{\sigma}$ of W^a:

$$\bar{\sigma}(\tilde{w})(F(z,\zeta))\stackrel{\mathrm{def}}{=}\sigma(\tilde{w})(F(z,\zeta))T_{\tilde{w}^{-1}}(z,\zeta),\quad \tilde{w}\in W^a\,, \tag{3.15}$$

on $\mathrm{End}_{\mathbf{C}}V(\lambda)$-valued functions F defined for almost all $z\in Z$.

Substituting $\bar{\sigma}_\alpha=\bar{\sigma}(s_\alpha)$ for $\hat{\tau}(s_\alpha)$ for $\hat{\tau}(s_\alpha)$ we rewrite KZ for Φ as the system

$$\bar{\Delta}_{\tilde{u}}(\Phi)=\hat{\tau}(x_{\tilde{u}})\Phi,\ \tilde{u}\in\mathbf{C}^{n+1}\,, \tag{3.16}$$

where the operators $\bar{\Delta}_{\tilde{u}}$ are introduced by formulas (3.4) with σ replaced by $\bar{\sigma}$. The latter obeys relations (3.6), (3.7), which ensure the validity of Theorems 3.1,3.3. The operators $\bar{L}_p$ constructed for $\bar{\sigma}$ (by replacing $\bar{\sigma}(\tilde{w})$ on the right with $\tilde{w}^{-1}$) coincide with L_p for σ. Hence,

$$p(\bar{\Delta}_{\tilde{u}})(\Phi) = (\hat{\tau}(p(x_{\tilde{u}}))(\Phi) = p(\lambda)\varphi\,,$$
$$L_p(\Phi) = p(\lambda)\Phi \text{ for } p(x_1,\ldots,x_n) \in \mathbf{C}[x]^W\,. \tag{3.17}$$

The last formula contains no $\{x\}$ and therefore commutes with tr. More precisely, given $e \in V(\lambda)$,

$$tr(L_p)(\phi_e)) = p(\lambda)tr(\Phi e)\,.$$

Since an arbitrary solution $\varphi \in \mathscr{K}$ can be represented in the form Φe for a proper e, the image of $\mathscr{K}$ belongs to $\mathscr{M}$. The dimension of the latter is not more than $\dim_{\mathbf{C}}\mathscr{K}$. However tr has no kernel due to Proposition 1.4 (as it was checked in [Ch5]). □

It is worth mentioning that one can introduce the monodromy of KZ more traditionally. It is necessary to fix a point z^0 and to replace Φ in right-hand side of (3.14) by its analytical continuation along a certain path from z^0 to $\tilde{w}(z^0)$ (see [Ch 2, 5]). This approach gives a representation of the "elliptic" braid group which is directly connected with the induced representations of the double affine Hecke algebras from [Ch 7, 8].

4. Examples

We will calculate the first (quadratic) L-operators for the simplest $\mu \subset \{\pm 1\}$ and $\mu = 0$, and discuss their basic properties. More complete analysis will be continued in the next paper(s).

The following elliptic functions ς, ϑ "almost" coincide (but do not coincide) with the classical ζ, ϑ_1. To avoid confusions we changed a little the standard notations. Let

$$\varsigma(t) = \sum_{k=0}^{\infty} \mathrm{ct}(k\xi + t) - \sum_{k=1}^{\infty} \mathrm{ct}(k\xi - t)\,,$$
$$\vartheta(t) = (\exp(t/2) - \exp(-t/2)) \prod_{k=1}^{\infty} (1 - \exp(-k\xi + t))(1 - \exp(-k\xi - t))\,,$$
$$\varrho(t) = \sum_{k=1}^{\infty} k(\mathrm{ct}(k\xi + t) + \mathrm{ct}(k\xi - t))\,. \tag{4.1}$$

Here $t, \xi \in \mathbf{C}$, $\Re(\xi) > 0$. All these functions are $2\pi\imath\mathbf{Z}$-invariant. One has the following relations (which can be deduced from the corresponding properties of ζ and ϑ_1 or proved directly):

$$\varsigma(t + m\xi) = \varsigma(t) + m, \quad \vartheta(t + m\xi) = -\exp(t + \xi/2)\vartheta(t)\,,$$
$$\varsigma(t) + \varsigma(-t) = -1,\ \vartheta(-t) = -\vartheta(t),\ \varrho(-t) = \varrho(t)\,,$$
$$\partial(\log\vartheta(t))/\partial t = \tilde{\varsigma}(t) \stackrel{\mathrm{def}}{=} \varsigma(t) + 1/2,\ \partial(\log\vartheta(t))/\partial\xi = \varrho(t)\,,$$
$$\varrho(t - \xi) = \varrho(t) + \varsigma(t),\ \varsigma' \stackrel{\mathrm{def}}{=} \partial\varsigma/\partial t = \varpi - \tilde{\varsigma}(t)^2 - 2\varrho(t)\,. \tag{4.2}$$

As to the latter (up to a constant ϖ), check that the difference of the two functions has no poles and is periodic with respect to the shifts by ξ (everything is periodic relative to $2\pi\imath\mathbf{Z}$).

Let us take $\mu = \pm 1$ and the corresponding one-dimensional $V = \mathbf{C}_\mu$ (see (1.8)). Our first aim is to determine $L_2 = L_{p_2}$ (Theorem 3.3) for

$$p_2(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i x_{\alpha_i},\ x_{\alpha_i} = \sum_j (\alpha_j,\, \alpha_i) x_j\,.$$

The calculations are rather simple because $\mathrm{Red}_0(\sigma_{\tilde\alpha} - \mu_{\tilde\alpha}) = 0$:

$$\mathrm{Red}_0(\Delta_{b_i}\Delta_{\alpha_i}) = \partial_i\partial_{\alpha_i} + (\rho_{\eta\mu},\, \alpha_i)\partial_i + (\rho_{\eta\mu},\, b_i)\partial_{\alpha_i} +$$

$$(\rho_{\eta\mu},\, \alpha_i)(\rho_{\eta\mu},\, b_i) + \ \mathrm{Red}_0\{-\sum_{\tilde\alpha>0} \eta_{\tilde\alpha}(b_i,\, \tilde\alpha)\mathrm{ct}(z_{\tilde\alpha})(\sigma_{\tilde\alpha} - \mu_{\tilde\alpha})\partial_{\alpha_i}\}\,, \tag{4.3}$$

where the last term equals

$$+\sum_{\alpha+kc>0} \eta_\alpha\mu_\alpha(b_i,\, \alpha)(\alpha_i,\, \alpha^\vee)(\mathrm{ct}(z_\alpha + k\xi))(\partial_{\alpha_i} - kh'_{\eta\mu})\,.$$

Here we applied (3.1), (3.2) and replaced $\partial/\partial\zeta$ by $-h'_{\eta\mu}$. To sum up the terms (4.3) with respect to i, we use the definition of $\rho_{\eta\mu}$ and the relations

$$b = \sum_{i=1}^{n}(b,\, b_i)\alpha_i = \sum_{i=1}^{n}(b,\, \alpha_i)b_i\,.$$

Finally, $L_2 = \ \mathrm{Red}_0(\Sigma_{i=1}^n \Delta_{b_i}\Delta_{\alpha_i})$

$$= \sum_{i=1}^{n} \partial_i\partial_{\alpha_i} + 2\partial_{\rho_{\eta\mu}} + (\rho_{\eta\mu},\, \rho_{\eta\mu}) + 2\sum_{\alpha\in R_+} \eta_\alpha\mu_\alpha(\varsigma(z_\alpha)\partial_\alpha - h'_{\eta\mu}\varrho(z_\alpha))$$

$$= \sum_{i=1}^{n} \partial_i\partial_{\alpha_i} + (\rho_{\eta\mu},\, \rho_{\eta\mu}) + 2\sum_{\alpha\in R_+} \eta_\alpha\mu_\alpha(\tilde\varsigma(z_\alpha)\partial_\alpha - h'_{\eta\mu}\varrho(z_\alpha))\,. \tag{4.4}$$

The next calculation will be a reduction of L_2 to the Schrödinger operator (without linear differentiations). We will introduce the following elliptic generalization of the "standard product" playing the main role in the Macdonald theory, Heckman–Opdam theory, and the theory of integral solutions of KZ:

$$\omega(z) = \omega(-z) \stackrel{\mathrm{def}}{=} \prod_{\alpha\in R} \vartheta(z_\alpha)^{\eta_\alpha\mu_\alpha/2}\,.$$

Actually we will need in this paper only the formulas (see (4.2):

$$\partial_u(\omega) = \omega \sum_{\alpha\in R_+} \eta_\alpha\mu_\alpha(u,\, \alpha)\tilde\varsigma(z_\alpha), \quad \text{for } u \in \mathbf{C}^n\,,$$

$$\partial\omega/\partial\xi = \omega \sum_{\alpha\in R_+} \eta_\alpha\mu_\alpha\varrho(z_\alpha)\,. \tag{4.5}$$

The first gives that $H_2 \stackrel{\text{def}}{=} \omega L_2 \omega^{-1}$ is free of linear differential operators. More precisely, $H_2 = \sum_{i=1}^{n} \partial_i \partial_{\alpha_i} + (\rho_{\eta\mu}, \rho_{\eta\mu}) - U(z)$, where

$$\begin{aligned} U(z) &= 2 \sum_{\alpha \in R_+} \eta_\alpha \mu_\alpha (\tilde{\varsigma}(z_\alpha) \partial_\alpha(\omega)\omega^{-1} + h'_{\eta\mu} \varrho(z_\alpha)) \\ &\quad + \sum_{i=1}^{n} ((\partial_{\alpha_i}(\omega)\omega^{-1})(\partial_i(\omega)\omega^{-1}) - \partial_i \{\partial_{\alpha_i}(\omega)\omega^{-1}\}) \\ &= \sum_{\alpha>0} \eta_\alpha \mu_\alpha ((\alpha, \alpha)\varsigma'(z_\alpha) + 2h'_{\eta\mu} \varrho(z_\alpha)) \\ &\quad + \sum_{\alpha, \beta>0} \eta_\alpha \mu_\alpha \eta_\beta \mu_\beta (\alpha, \beta) \tilde{\varsigma}(z_\alpha) \tilde{\varsigma}(z_\beta) . \end{aligned} \tag{4.6}$$

Lemma 4.1.

$$\sum_{\alpha, \beta>0} \eta_\alpha \eta_\beta (\alpha, \beta) \tilde{\varsigma}(z_\alpha) \tilde{\varsigma}(z_\beta) = h'_\eta \sum_{\alpha>0} \eta_\alpha \tilde{\varsigma}^2(z_\alpha) + C(\eta) . \tag{4.7}$$

Proof. Let us fix $b \in B$ and replace z_u by $z_{b'(u)} = z_{u-(b,u)c} = z_u - (b,u)\xi$ for $u = \alpha, \beta$ in (4.7). The change of the left-hand side is

$$\begin{aligned} &\sum_{\alpha, \beta>0} \eta_\alpha \eta_\beta (\alpha, \beta)((b, \alpha)\tilde{\varsigma}(z_\alpha) + (b, \beta)\tilde{\varsigma}(z_\beta) + (b, \alpha)(b, \beta)) \\ &\quad = h'_\eta \sum_{\alpha>0} 2\eta_\alpha (b, \alpha)\tilde{\varsigma}(z_\alpha) + (h'_\eta)^2 (b, b) . \end{aligned} \tag{4.8}$$

Here we used the main property of h'_η :

$$\sum_{\alpha>0} \eta_\alpha (u, \alpha)(v, \alpha) = h'_\eta (u, v) \text{ for } u, v \in \mathbf{C}^n .$$

The same holds for the right-hand side. Hence, their difference is B-periodic and has no singularities. The latter can be checked directly or deduced from (4.8) with t^{-1} instead of $\tilde{\varsigma}(t)$ (use the r-matrix relations). Thus the difference is a constant C depending on η. □

Finally, applying the lemma and replacing $2\varrho(z_\alpha) + \tilde{\varsigma}(z_\alpha)^2$ by $\varpi - \varsigma'(z_\alpha)$ (see (4.2)), we arrived at the formula for U and the following

Theorem 4.2. a) *If $\mu \subset \{\pm 1\}$ and $V = \mathbf{C}_\mu$ is the corresponding one-dimensional representation of W^a, then the reduction procedure for $p_2 = \sum_i x_i x_{\alpha_i}$ gives the operator L_2 conjugated (by ω) with*

$$\begin{aligned} H_2 &= \sum_{i=1}^{n} \partial_i \partial_{\alpha_i} + \sum_{\alpha>0} \eta_\alpha \mu_\alpha \{h'_{\eta\mu} - (\alpha, \alpha)\}\varsigma'(z_\alpha) \\ &\quad + (\rho_{\eta\mu}, \rho_{\eta\mu}) - \varpi h'_{\eta\mu} \sum_{\alpha>0} \eta_\alpha \mu_\alpha - C(\eta\mu) . \end{aligned} \tag{4.9}$$

b) *The operator H_2 can be included into the family of pairwise commutative differential operators $H_p \stackrel{\text{def}}{=} \omega L_p \omega^{-1}$, $p \in \mathbf{C}[x]^W$, which are W-invariant. Their coefficients are B-periodic with respect to the action $z_u \to z_u - (u, b)\xi$, $b \in B$. They are self-adjoint relative to the complex involution taking z_u to $-z_u$ and leaving ∂_u invariant.*

c) *Operators* $\{L_p\}$ *are* W*-invariant as well. Moreover, they are* B*-invariant for the action:*

$$z_u \to z_u - (u, b)\xi, \quad \partial_u \to \partial_u + (u, b)h_{\eta\mu}, \; b \in B, \; u \in \mathbf{C}^n ,$$

and formally self-adjoint with respect to the following pairing:

$$\langle f(z), g(z)\rangle = \int \omega^2 f(z) g(-z) dz_1 \dots dz_n .$$

Proof. The previous calculation gives a). Begining with c), the invariance relative to W^a (generated by W and A) is due to Theorem 3.3. It can be naturally extended to the bigger group with B instead of A. We will not discuss this extension in this paper. The self-adjointness results from the same property of $\Delta_{\tilde{u}}$, which can be checked directly using the definition of ω. It gives the analogous properties of H. For instance, let us check the periodicity:

$$\tilde{\omega}^{-1}\partial_u\tilde{\omega} = \partial_u + \sum_{\alpha>0} \eta_\alpha\mu_\alpha(b, \alpha)(\alpha, u) = \partial_u + h'_{\eta\mu}(u, b) ,$$

$$\text{where } \tilde{\omega} = \omega(z_\alpha \to z_\alpha - (b, \alpha)\xi) = \omega \exp(-\sum_{\alpha\in R} \eta_\alpha\mu_\alpha(b, \alpha)\xi/2) . \quad \square$$

Without going into detail we mention that one can generalize the construction of the shift operators from [Op, He] to the elliptic case. It is connected with Theorem 3.5 for $\mathbf{C}_\mu$ (see [FV]). The most interesting applications of these operators are expected when $\mu = 1$ because in this case the operators L_p preserve certain subspaces of W-invariant elliptic functions.

To define these spaces let us fix $m \in \mathbf{Z}_+$ and introduce the set

$$\tilde{P}^+_m \stackrel{\text{def}}{=} \{\tilde{\beta} = k_1\omega_1 + \dots + k_n\omega_n + kc, \; (\tilde{\beta}, \theta) \leqq m'\}, \; m' \stackrel{\text{def}}{=} m(\theta, \theta)/2 ,$$

$$\text{where } \omega_i = (\alpha_i, \alpha_i)b_i/2, \quad k_1, \dots, k_n \in \mathbf{Z}_+, \; k \in \mathbf{Z}. \tag{4.10}$$

The linear space generated by the orbit sums

$$\Upsilon_{\tilde{\beta}_+} = \sum_{\hat{\beta}\in W^a(\tilde{\beta}_+ + m'd)} \exp(z_{\hat{\beta}}) \text{ for } \tilde{\beta}_+ \in \tilde{P}^+_m \tag{4.11}$$

over the algebra of formal series $\sum_{l<l_0} c_l \exp(l\xi)$, $c_l \in \mathbf{C}$, (convergent for $\Re(\xi) > 0$) will be denoted by $\mathscr{L}_m$. This construction is due to Looijenga and closely related to the characters of Kac–Moody algebras. The operators $\{L_p\}$ for $\mu = 1$ leave $\mathscr{L}_m$ invariant if $m' = -h'_\eta$. Moreover they preserve subspaces $\mathscr{L}_m(\tilde{\beta}_+)$ for $\tilde{\beta}_+ \in \tilde{P}^+_m$ generated by

$$\Upsilon_{\tilde{\gamma}_+} \text{ such that } \tilde{\gamma}_+ = \tilde{\beta}_+ - \sum_{i=0}^{n} k_i\alpha_i \in \tilde{P}^+_m, \; \{k_i\} \subset \mathbf{Z}_+ .$$

It results directly from the corresponding properties of the elliptic Dunkl operators and allow us to introduce the elliptic Jacobi–Jack–Macdonald polynomials $J_{\tilde{\beta}_+}$ as eigenfunctions of $\{L_p\}$ in $\mathscr{L}_m(\tilde{\beta}_+)$ with leading terms $\Upsilon_{\tilde{\beta}_+}$. A further discussion will be continued in the next papers.

Parabolic operator. A demerit of the above constructions is the constraint $\partial/\partial\zeta + h'_{\eta\mu} = 0$ corresponding to the condition $x_c = 0$ in the Hecke algebra $\mathfrak{H}'$. We will show that something can be done even without this restriction.

$$\text{Let } \Delta_{\hat{u}} = \Delta_{\tilde{u}} + v\Delta_d \text{ for } \hat{u} = \tilde{u} + vd \in \mathbf{C}^{n+2},$$

$$\Delta_d = \partial/\partial\xi - \sum_{\alpha\in R}\sum_{k\in\mathbf{Z}_+} \eta_\alpha k \mathrm{ct}(z_\alpha + k\xi)(\sigma_{\alpha+k\xi} - \mu_\alpha). \tag{4.12}$$

The operators $\Delta_{\hat{u}}$ are not pairwise commutative but still satisfy the following cross-relations (see (1.4)):

$$\sigma_i\Delta_{\hat{u}} - \Delta_{\{s_i(\hat{u})\}}\sigma_i = \eta_i(\hat{u}, \alpha_i), \quad 0 \leqq i \leqq n, \ \hat{u} \in \mathbf{C}^{n+2}, \tag{4.13}$$

relative to the action from (3.1). It gives (together with the previous considerations) the following theorem.

Theorem 4.3. *The operator* $\mathcal{M} = 2\Delta_d\Delta_c + \sum_{i=0}^n \Delta_{b_i}\Delta_{\alpha_i}$ *and its reduction* $M = \mathrm{Red}(\mathcal{M})$ *are* W^a*-invariant. If* $V = \mathbf{C}_\mu$, $\mu = 1$, $\partial/\partial\zeta = m'$, $m \in \mathbf{Z}_+$, *then*

$$M = 2(m' + h'_\eta)\partial/\partial\xi + \sum_{i=1}^n \partial_i\partial_{\alpha_i} + (\rho_\eta, \rho_\eta) + 2\sum_{\alpha\in R_+} \eta_\alpha(\zeta(z_\alpha)\partial_\alpha + m'\varrho(z_\alpha)),$$

$$N \stackrel{\mathrm{def}}{=} \omega M\omega^{-1} = 2(m' + h'_\eta)\partial/\partial\zeta + H_2(\textit{see } (4.9)). \tag{4.14}$$

The operator M *preserves the spaces* $\mathcal{L}_m$ *and* $\mathcal{L}_m(\tilde{\beta}_+)$ *for arbitrary* $\tilde{\beta}_+ \in \tilde{P}^+_m$.
□

The operator N was introduced by Etingof and Kirillov in [EK] for $\mathfrak{sl}_n$ together with its certain eigenfunctions (the generalized characters that are the traces of proper vertex operators of $\hat{\mathfrak{sl}}_n$). In a recent work, they extended the definition of N to arbitrary root systems and proved directly the properties mentioned in the theorem. To be more precise, their formulas are different but with certain minor changes seem to be equivalent to (4.14) (e.g. they use more special parameters). If it is so, then our approach (based on the Dunkl operators) gives another proof of their result. The construction of the generalized characters is still known for $\mathfrak{sl}_n$ only.

Matrix Schrödinger operator. The next application (which is a straightforward extension of Corollary 2.8 from [Ch 5]) will be for arbitrary representations and $\mu = 0$. Let us calculate $L_2^0 = L_2^{\mu=0}$ for $p = p_2$ (see above). Applying Red and imposing the condition $\partial/\partial\zeta = 0$, one has:

$$L_2^0 = \sum_{i=1}^n \partial_{b_i}\partial_{\alpha_i} - \sum_{\tilde{\alpha}=\alpha+kc>0} \eta_\alpha(\alpha, \alpha)\mathrm{ct}'(z_{\tilde{\alpha}})s_{\tilde{\alpha}}$$

$$+ \sum_{\tilde{\alpha}, \tilde{\beta}>0} \eta_\alpha\eta_\beta(\alpha, \beta)\mathrm{ct}(z_{\tilde{\alpha}})\mathrm{ct}(\sigma_{\tilde{\alpha}}(z_{\tilde{\beta}}))s_{\tilde{\beta}}s_{\tilde{\alpha}}, \tag{4.15}$$

where $\mathrm{ct}'(t) = \partial\mathrm{ct}(t)/\partial t = -(\exp(t/2) - \exp(-t/2))^{-2}$. Following [Ch 5], Lemma 2.7, we check that the contribution of the terms with $\tilde{\alpha} \neq \tilde{\beta}$ in the last sum equals zero. Hence we arrived at the follwing theorem:

Theorem 4.4. *The differential* $\mathbf{C}[W^a]$*-valued operators*

$$L_2^0 = \sum_{i=1}^{n} \partial_{\beta_i}\partial_{\alpha_i} + \sum_{\tilde{\alpha}>0} (\alpha, \alpha)\eta_\alpha \mathrm{ct}'(z_{\tilde{\alpha}})(\eta_\alpha - s_{\tilde{\alpha}}) \tag{4.16}$$

and $L_p^{\mu=0}$ *defined for* $p \in \mathbf{C}[x]^W$ *are pairwise commutative. Moreover they are* W^a*-invariant with respect to the* δ*-action on* z *and on* $\mathbf{C}[W^a]$ *(by conjugations). When considered in finite dimensional representations of the latter, the coefficients are convergent matrix-valued functions for sufficiently big* $\Re(\xi)$. □

We can obtain the scalar OP operators (for arbitrary root systems) from this construction as well. Let $\{s_{\tilde{\alpha}}\}$ be taken in one-dimensional representations $\mathbf{C}_\varepsilon$ (see (1.8)). Then

$$L_2^0 = \sum_{i=1}^{n} \partial_{\beta_i}\partial_{\alpha_i} + \sum_{\alpha>0} (\alpha, \alpha)\eta_\alpha(\eta_\alpha - \varepsilon_\alpha)\varsigma'(z_\alpha)\,. \tag{4.17}$$

The corresponding L_p are W-invariant and their coefficients are elliptic $= B$-periodic (cf. Theorem 4.2). Generally speaking, the coefficients are "matrix" elliptic functions with the values in the endomorphisms of vector bundles over elliptic curves. Ignoring the differential operations in (4.16) and substituting "good" z, we obtain "periodic" generalizations of Haldane–Shastry hamiltonians. Presumably the points of finite order of the corresponding elliptic curve and the critical points of the scalar hamiltonians ((4.17) without the differentiations and after a proper normalization) lead to integrable models (see [BGHP, F, P]).

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