Commun. Math. Phys. 169, 193–236 (1995)

Communications in Mathematical Physics

Construction of Field Algebras with Quantum Symmetry from Local Observables

Volker Schomerus

Harvard University, Cambridge, MA 02138, U.S.A.

Received: 18 January 1994

Abstract: It has been discussed earlier that (weak quasi-) quantum groups allow for a conventional interpretation as internal symmetries in local quantum theory. From general arguments and explicit examples their consistency with (braid-) statistics and locality was established. This work addresses the reconstruction of quantum symmetries and algebras of field operators. For every algebra $\mathcal{A}$ of observables satisfying certain standard assumptions, an appropriate quantum symmetry is found. Field operators are obtained which act on a positive definite Hilbert space of states and transform covariantly under the quantum symmetry. As a substitute for Bose/Fermi (anti-) commutation relations, these fields are demonstrated to obey a local braid relation.

Contents

1. Introduction

In classical mechanics, symmetries are described by *groups* of transformations acting on a phase space. In view of the predominant role group symmetries played in classical mechanics, it was natural to introduce them also into quantum theory. In fact, this was done soon after the discovery of quantum mechanics and group symmetries turned out to be an important tool to obtain predictions of quantum theory [81].

Since the late seventies much work was done to investigate quantum mechanical models on a two or three dimensional space-time. Solving models of $1 + 1$ dimensional quantum field theory, Sklyanin, Takhtadzhyan and Faddeev revealed a new algebraic structure that was called quantum algebra [20, 21]. In the more axiomatic treatment of Drinfel'd and Jimbo [18, 46] it appeared as a special class of Hopf algebras [1] and consequently as a generalization of groups. The new name *quantum group* was used henceforth. To cut a long story short we just mention that signs of quantum group symmetries have been found in many quantum mechanical models such as integrable spin chains (e.g. [69, 75]), rational conformal quantum field theories (e.g. [3, 68, 54, 44]), and massive integrable models (e.g. [73]). The process of generalization continued. In 1989 Drinfel'd introduced quasi quantum groups to be able to "twist" quantum groups. Non-trivial examples of quasi quantum groups were constructed and shown to be related to orbifold models [14]. For reasons that remain to be discussed later, all models which exhibit generalized *quantum symmetries* are defined on a low dimensional space time.

To begin with, let us list the most important algebraic structures which are common to all known examples of quantum symmetries. The basic structure is an associative *- algebra $\mathcal{G}^*$ (generalization of the group algebra). Representations of $\mathcal{G}^*$ give rise to an action of the quantum symmetry on vector spaces. Unitarity of the representation τ on a Hilbert space V means that $\tau(\xi)^* = \tau(\xi^*)$ for all $\xi \in \mathcal{G}^*$. Among the representations of a quantum symmetry, one can always find a "trivial" one-dimensional representation ε. Moreover, tensor products of representations can be formed. The tensor product of two representations τ, τ' on V, V' is a representation $\tau \boxtimes \tau'$ on $V \otimes V'$. It is furnished by a homomorphism $\Delta: \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ (a "co-product" of $\mathcal{G}^*$) according to the formula

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')\Delta(\xi) .$$

This tensor product need not be associative and commutative and it may involve truncation, i.e. the representation $\tau \boxtimes \tau'$ possibly vanishes on a non-trivial subspace of $V \otimes V'$. Finally, a notion of "contragredient" representations is furnished by a suitable anti-automorphism $\mathcal{S}: \mathcal{G}^* \mapsto \mathcal{G}^*$ (an "antipode" of $\mathcal{G}^*$). We use the name "bi-*-algebra with antipode" for such an algebraic structure. The precise definition is given below in Definition 3. The special case where $\Delta(e) = e \otimes e$ and the co-product is co-associative[1] is called a "Hopf-*-algebra."

We adopt the framework of second quantized quantum mechanics, so that there is a Hilbert space $\mathcal{H}$ of physical states which is generated from a unique ground state $|0\rangle$ by application of a set of field operators $\Psi^I_i(x, t)$. Superscripts $I, J, K, \ldots$ distinguish between field multiplets while subscripts label members of the

[1] co-associativity is the property $(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)$ of the co-product.

multiplets. A quantum mechanical system is said to possess a quantum symmetry $\mathcal{G}^*$, if the Hilbert space $\mathcal{H}$ carries an unitary representation $\mathcal{U}$ of $\mathcal{G}^*$, such that the ground state $|0\rangle$ is invariant and field operators $\Psi_i^I(x, t)$ transform covariantly. Invariance of the ground state means that $|0\rangle$ transforms according to the trivial one-dimensional representation ε of $\mathcal{G}^*$. The formulation of the transformation law of field operators involves the tensor product of representations of $\mathcal{G}^*$. More precisely, a field multiplet $\Psi_i^I(x, t)$ is said to transform covariantly according to the finite dimensional representation τ^I of $\mathcal{G}^*$, if

$$\mathcal{U}(\xi)\Psi_i^I(x, t) = \Psi_j^I(x, t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi)$$

holds for all $\xi \in \mathcal{G}^*$ [12]. The symmetry transformations considered here do not act on the space time argument of the field operators, in other words: they are internal symmetries. We give a more comprehensive explanation of the notion of quantum symmetry in the next section.

In second quantized quantum theory, Bose and Fermi statistics are implemented through local commutation or anticommutation relations of field operators which create particles,

$$\Psi_i^I(x, t)\Psi_j^J(y, t) = \pm\Psi_j^J(y, t)\Psi_i^I(x, t) \quad \text{for } x \neq y\,. \tag{1.1}$$

Consistency of a symmetry with Bose/Fermi statistics requires that this relation should be preserved by a symmetry transformation. This is indeed true for internal symmetry groups.

In two and less space dimensions, Bose/Fermi statistics is not the most general possibility, but braid group statistics can also occur. Fröhlich proposed that local (anti-) commutation relations of fields should be replaced by *local braid relations*,

$$\Psi_i^I(x, t)\Psi_j^J(y, t) = \omega^{IJ}\Psi_l^J(y, t)\Psi_k^I(x, t)\mathcal{R}_{kl,ij}^{IJ>} \tag{1.2}$$

if $x > y$ for some ordering of space coordinates and ω^{IJ} are phase factors. Originally, $\mathcal{R}_{kl,ij}^{IJ>}$ were proposed to be complex numbers.

The main question is, whether such local braid relations can be consistent with the transformation law under some non-trivial quantum symmetry. The answer is affirmative. It will be seen, however, that braid matrices $\hat{\mathcal{R}}$ with entries in $\mathcal{G}^*$ should be admitted.

In general, local braid relations of this more general type can be consistent with non-trivial quantum symmetries, provided the tensor product of representations is associative and commutative at least up to equivalence,

$$(\tau^I \boxtimes \tau^J) \cong (\tau^J \boxtimes \tau^I)\,, \tag{1.3}$$

$$(\tau^I \boxtimes \tau^J) \boxtimes \tau^K \cong \tau^I \boxtimes (\tau^J \boxtimes \tau^K)\,. \tag{1.4}$$

The precise formulation of this requirement will involve an element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ and a "re-associator" $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ with certain properties. They will furnish invertible intertwiners $R^{IJ} = (\tau^I \otimes \tau^J)(R)$ and $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$ between the representations on the right and left-hand side of Eq. (1.3, 1.4). When $\varphi = \sum_\sigma \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ one introduces $\varphi_{213} = \sum_\sigma \varphi_\sigma^2 \otimes \varphi_\sigma^1 \otimes \varphi_\sigma^3$. In this notation, consistent braid matrices are given by

$$\mathcal{R}_{kl,ij}^{IJ>} = (\tau_{ki}^I \otimes \tau_{lj}^J \otimes \mathcal{U})(\varphi_{213}(R \otimes e)\varphi^{-1}) \in \mathcal{U}(g^*)\,. \tag{1.5}$$

Bi-*-algebras with antipode $\mathcal{S}$, re-associator φ and R-element R are called *weak quasi quantum groups*. They were introduced in [56] as a generalization of Drinfeld's quasi quantum groups [19]. A precise definition will be given in Sect. 4. Consistency of weak quasi quantum group symmetries with local braid relations is discussed in more detail in Sect. 5.1.

It is shown in [57] that the chiral Ising model – i.e. the conformal quantum field theory in which observables are generated by a Virasoro algebra with central charge $c = \frac{1}{2}$ – provides an example, with the truncated quantum group algebra $U_q^T(sl_2)$, $q = \pm i$, as a symmetry[2]. To the best of our knowledge, this was the first time that the consistency of non-abelian local braid relations (1.2) has been demonstrated through the construction of a model. Originally it had been proposed that minimal conformal models have quantum groups as symmetries [3, 69], but this identification is not quite satisfactory, because the local braid relations, which should come with the symmetry, are not satisfied [54].

The main result of this work states that the picture we obtained from studying this example is generic. The notion of fusion rules of superselection sectors will be explained in detail in Sect. 3.1. Basically, superselection sectors are inequivalent *-representations of the algebra of observables in quantum mechanics. Under standard assumptions, tensor products of such representations can be defined and decomposed into irreducibles, with multiplicities N_K^{IJ} ("fusion rules"). The algebra of observables $\mathcal{A}$ has subalgebras $\mathcal{A}(\mathcal{O})$ which correspond to measurements in the bounded space time domain $\mathcal{O}$. It suffices to consider double cones $\mathcal{O}$, i.e. non-vanishing intersections of the interior of forward and backward light cones. These algebras form a net, i.e. there is an inclusion preserving map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ is an element in the set $\mathcal{K}$ of open double cones in spacetime.

Theorem 1 (*QFT Reconstruction theorem*). *Let $\mathcal{A}$ be a net of local observables which satisfies the standard assumptions (Haag–Kastler axioms, Haag-duality, locally generated sectors, finite statistics). Suppose that there is a bi-*-algebra with antipode such that the multiplicities in the Clebsch–Gordon decomposition agree with the fusion rules N_K^{IJ} defined by the superselection structure of $\mathcal{A}$. Then one can construct a complete local field system $(\mathcal{F}, \mathcal{G}^*, \mathcal{H}, \pi)$ with quantum symmetry. In detail this means the following:*

1. *There exists a net $\mathcal{F}$ of local quantum fields. The algebras $\mathcal{F}(\mathcal{O})$ which $\mathcal{F}$ assigns to every open double cone $\mathcal{O}$ in spacetime come equipped with a conjugation. The product in $\mathcal{F}(\mathcal{O})$ is denoted by $\cdot$.* The product is not necessarily associative, but it enjoys quasiassociativity properties as stated in Theorem *16 below.*
2. *$\mathcal{G}^*$ is a weak quasi quantum group with re-associator φ and R-matrix R. Elements $\xi \in \mathcal{G}^*$ act on $\mathcal{F}(\mathcal{O})$ as generalized derivations. The algebra $\mathcal{A}(\mathcal{O})$ of observables in $\mathcal{O}$ is the fixed-point algebra with respect to this action.*
3. *Quantum fields localized in relatively spacelike regions of spacetime satisfy local braid relations. If $\psi_\alpha \in \mathcal{F}(\mathcal{O}_1)$, $\psi'_\beta \in \mathcal{F}(\mathcal{O}_2)$ transform according to irreducible representations $\tau \cong \tau^J$ and $\tau' \cong \tau^K$ of $\mathcal{G}^*$, they read*

$$\psi_\alpha \cdot \psi'_\beta = \omega^{JK} \psi'_\gamma \cdot \psi_\delta (\tau_{\delta\alpha} \otimes \tau'_{\gamma\beta})(R) \tag{1.6}$$

[2] $U_q^T(sl_2)$ is a family of weak quasi quantum groups which is obtained from $U_q(sl_2)$ by a process of truncation [56].

whenever $\mathcal{O}_1 > \mathcal{O}_2$. *Here* ω^{IJ} *are certain phase factors depending on the equivalence classes of* τ, τ'. *Fields are local relative to the observables, i.e.* $\mathcal{F}(\mathcal{O}_1)$ *and* $\mathcal{A}(\mathcal{O}_2)$ *commute elementwise whenever* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are spacelike separated.*

4. *There is a Hilbert space* $\mathcal{H}$ *which carries a unitary representation* $\mathcal{U}$ *of* $\mathcal{G}^*$ *and a faithful representation* π *of* $\mathcal{F}$. *When* π *is restricted to* $\mathcal{A}$, *it determines a reducible representation of* $\mathcal{A}$ *on* $\mathcal{H}$. $\mathcal{H}$ *decomposes into a direct sum of "superselection sectors"* $\mathcal{H}^J$ *(irreducible representations of* $\mathcal{A}$*) each with some finite nonzero multiplicity* δ_J,

$$\mathcal{H} = \sum_J \mathcal{H}^J \otimes V^J, \quad \dim V^J = \delta^J .$$

V^J *are representation spaces for* $\mathcal{G}^*$. *States within the "vacuum sector"* $\mathcal{H}^0$ *are* $\mathcal{G}^*$*-invariant.* $\mathcal{H}^0$ *appears with a multiplicity* $\delta_0 = 1$ *and is cyclic for* $\pi(\mathcal{F}(\mathcal{O}))$.

5. *Suppose that* ψ_α, ψ_β *transform according to representations* τ, τ' *of* $\mathcal{G}^*$ *and denote their "representation operators"* $\pi(\psi_\alpha)$, $\pi(\psi'_\beta)$ *("field operators") as* Ψ_α, Ψ'_β. *Then the product* $\psi_\alpha \cdot \psi'_\beta$ *is represented by*

$$\pi(\psi_\alpha \cdot \psi'_\beta) = \sum \Psi_\gamma \Psi'_\delta (\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \mathcal{U})(\varphi) . \tag{1.7}$$

It follows in particular that field operators obey local braid relations of the form (1.2) *with a (not necessarily numerical)* $\mathcal{R}$*-matrix.*

The braid relations hold for all field operators, including composites, i.e. the braid relations pass to products and conjugates. The field operators satisfy quantum symmetric operator product expansions, see Theorem 18.

Theorem 2 (*Existence of the quantum symmetry*). *An appropriate bi-*-algebra* $\mathcal{G}^*$ *with antipode as required by the hypothesis of the preceding theorem exists, if the theory defined by* $\mathcal{A}$ *is rational, i.e. has only a finite number of superselection sectors.*

The above theorems and their proofs are expressed in the language of algebraic quantum field theory [43]. Some basic ideas from the algebraic theory of superselection sectors are reviewed in Sect. 3. Covariant field operators will be constructed in Sect. 3.3. Local braid relations are established in Sect. 5.2. To discuss the locality properties of field operators some mathematical results on symmetry algebras with re-associator φ and R-element R are needed. In order not to clutter the presentation in Sect. 5, This mathematical background will be anticipated in Sect. 4. Readers who do not want to enter the discussion of algebraic quantum field theory but have some basic knowledge in two-dimensional conformal quantum field theory may skip Sect. 3.1, 3.2. Besides of providing a more rigorous mathematical basis, arguments from algebraic theory of superselection sectors are used to demonstrate the universality of "vertex operators" and their properties. Indeed all related features familiar from two-dimensional conformal theories extend to a wide class of two- and three-dimensional models (without conformal symmetry).

Internal symmetries recently led to a classification of quantum field theories with permutation group statistics [17]. Doplicher and Roberts proved that a unique compact symmetry group G can be associated with every higher dimensional quantum field theory. Moreover, the quantum field on which the symmetry transformations act, commute or anticommute for spacelike separations. Theorem 1 generalizes the construction of Doplicher and Roberts to quantum field theories with braid group statistics. Some remarks on uniqueness, which fails to hold in the setting of Theorem 1, are included in the last section. For related work see Fröhlich and Kerler [50, 38], Häring [46], Majid [63], Rehren [70, 71] and Todorov et al. [45, 44].

2. The Notion of Quantum Symmetry

Let us begin with a short review on group symmetries in quantum mechanics. Consider some quantum mechanical system $(\mathscr{H}, \{\Psi\}, |0\rangle, H)$ within a second quantized formalism. The Hilbert space $\mathscr{H}$ of physical states should contain a unique ground state $|0\rangle$ with respect to the Hamiltonian H. We assume that $\mathscr{H}$ is generated from $|0\rangle$ by multiplets of field operators $\Psi^I_i(x, t)$ (I labels multiplets, i labels fields in the multiplet I) which create particles or excitations.

A compact group G is called (internal) symmetry of this system if there is a unitary representation $\mathscr{U}$: $G \to \mathscr{B}(\mathscr{H})$ such that the ground state $|0\rangle$ and the Hamiltonian H are invariant and field operators Ψ^I_i transform covariantly according to the representation τ^I of G. To state these requirements in mathematical terms, let us recall two notions from the representation theory of groups. Every group has a trivial representation ε_G: $G \to \mathbf{C}$ defined by $\varepsilon_G(\xi) = 1$ for all $\xi \in G$. The tensor product $\boxtimes$ of representations τ, τ' is given by

$$(\tau \boxtimes \tau')_{kl,ij}(\xi) = \tau_{ki}(\xi)\tau'_{lj}(\xi) \quad \text{for all} \quad \xi \in G\ . \tag{2.1}$$

If we set $\xi^* = \xi^{-1}$, unitarity of $\mathscr{U}$ asserts $\mathscr{U}(\xi)^* = \mathscr{U}(\xi^{-1}) = \mathscr{U}(\xi^*)$. Invariance of the ground state $|0\rangle$ can be expressed as

$$\mathscr{U}(\xi)|0\rangle = |0\rangle = |0\rangle\varepsilon_G(\xi) \quad \text{for all} \quad \xi \in G\ . \tag{2.2}$$

We say that $\Psi^I_i(x, t)$ transforms covariantly according to the representation τ^I of G, if for all $\xi \in G$,

$$\mathscr{U}(\xi)\Psi^I_i(x, t) = \Psi^I_j(x, t)\tau^I_{ji}(\xi)\mathscr{U}(\xi) = \Psi^I_j(x, t)(\tau^I \boxtimes \mathscr{U})_{ji}(\xi)\ . \tag{2.3}$$

Since we concentrate on internal symmetries, there is no action on the space-time variable of the field. For this reason we will often neglect to write arguments (x, t) explicitly. Adjoint field operators Ψ^{I*}_i transform covariantly according to the "conjugate" representation $\bar{\tau}^I(\xi) \equiv ({}^t\tau^I(\xi))^*$, for all $\xi \in G$ (here and in the following ^{t} denotes the transpose of a matrix and $*$ refers to a scalar product on the representation space of τ^I).

$$\mathscr{U}(\xi)\Psi^{I*}_i = \Psi^{I*}_j(\bar{\tau}^I \boxtimes \mathscr{U})_{ji}(\xi)\ . \tag{2.4}$$

In conclusion, the formulation of symmetry in quantum theory involves a conjugation $*$ to express unitarity, a trivial representation ε to state invariance and a tensor product $\boxtimes$ of representations to write down the covariance law. The mathematical structure behind these notions is known as bi-*-algebra with antipode.

Definition 3 (*Bi-*-algebra with antipode*). *A (not necessarily co-associative) bi-*-algebra* $(\mathscr{G}^*, \Delta, \varepsilon, *)$ *is furnished by an associative *-algebra* $\mathscr{G}^*$*, with unit e and* Δ: $\mathscr{G}^* \to \mathscr{G}^* \otimes \mathscr{G}^*$ *(co-product),* ε: $\mathscr{G}^* \to \mathbf{C}$ *(co-unit) which are *-homomorphisms.* Δ *is not required to be unit-preserving, i.e.* $\Delta(e) \neq e \otimes e$ *is permitted. The co-product* Δ *and the co-unit* ε *must satisfy*

$$(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta\ . \tag{2.5}$$

The bi--algebra has an antipode if there exists a* **C**-*linear antiautomorphism* $\mathcal{S}$: $\mathcal{G}^* \mapsto \mathcal{G}^*$ *(antipode) such that*

$$\sum_\sigma \mathcal{S}(\xi_\sigma^1)\alpha\xi_\sigma^2 = \alpha\varepsilon(\xi), \quad \sum_\sigma \xi_\sigma^1 \beta\mathcal{S}(\xi_\sigma^2) = \beta\varepsilon(\xi) \tag{2.6}$$

hold for two fixed (independent of ξ*) elements* $\alpha, \beta \in \mathcal{G}^*$ *and*

$$\mathcal{S}(\xi)^* = \mathcal{S}^{-1}(\xi^*), \quad \beta = \mathcal{S}(\alpha^*) . \tag{2.7}$$

If $\Delta(e) = e \otimes e$ *and if the co-product is co-associative, viz.* $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, *then* $\mathcal{G}^*$ *is called a "Hopf-*-algebra." The bi-*-algebra is called semisimple if* $\mathcal{G}^*$ *is a sum of full matrix algebras.*

Here ξ_σ^i are defined by the expression $\Delta(\xi) = \sum_\sigma \xi_\sigma^1 \otimes \xi_\sigma^2$ of the co-product Δ. Consistency of the antipode with the *-homomorphism involves a definition of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$, which is not unique (cf. [56]) since there are two possibilities to define a *-operation on $\mathcal{G}^{*\otimes_{N+1}}$,

$$\text{(I)} \quad (\xi \otimes \eta)^* = \xi^* \otimes \eta^* , \tag{2.8}$$

$$\text{(II)} \quad (\xi \otimes \eta)^* = \eta^* \otimes \xi^* \tag{2.9}$$

for all $\xi \in \mathcal{G}^*, \eta \in \mathcal{G}^{*\otimes_N}$. Throughout this paper we will only consider conjugations of type (I). Everything below can be worked out for conjugations of type (II) (cp. [61] for formulae). Note that $\Delta(e) = e \otimes e$ is not assumed. For us, the homomorphism property of Δ means $\Delta(\eta\xi) = \Delta(\eta)\Delta(\xi)$ and implies only that $\Delta(e)$ is a projector, i.e. $\Delta(e)\Delta(e) = \Delta(e)$. When we want to stress that Δ is not unit preserving, $(\mathcal{G}^*, \Delta, \varepsilon, *)$ will be called "weak" bi-*-algebra. One should also note the freedom in the choice of α, β in the definition of the antipode. For a given antipode $\mathcal{S}$, relations (2.6, 2.7) are invariant under the transformation $\alpha \mapsto \zeta\alpha$ with an arbitrary element ζ in the center of $\mathcal{G}^*$.

The co-product Δ determines a tensor product $\tau \boxtimes \tau'$ for representations τ, τ' of $\mathcal{G}^*$,

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)) \quad \text{for all } \xi \in \mathcal{G}^* . \tag{2.10}$$

With respect to this tensor product of representations, the co-unit ε furnishes a trivial one-dimensional representation. Triviality refers to the property $\varepsilon \boxtimes \tau = \tau = \tau \boxtimes \varepsilon$ for all representations τ of $\mathcal{G}^*$, which follows from (2.5). If $\Delta(e) \neq e \otimes e$ the tensor product of representations is truncated. This means that it is zero on a non-trivial subspace of the tensor product of representation spaces.

A "contragredient" representation $\tilde{\tau}$ of the representation τ of $\mathcal{G}^*$ can be defined with the help of the antipode $\mathcal{S}$,

$$\tilde{\tau}(\xi) \equiv {}^t\tau(\mathcal{S}^{-1}(\xi)) \quad \text{for all} \quad \xi \in \mathcal{G}^* . \tag{2.11}$$

Relations (2.6) assert that the tensor products $\tilde{\tau} \boxtimes \tau$ and $\tau \boxtimes \tilde{\tau}$ contain the trivial representation ε as a subrepresentation. It is this feature which motivates the name "contragredient" representation. We introduce the symbol $\bar{\tau}$ to denote the representation of $\mathcal{G}^*$ obtained as

$$\bar{\tau}(\xi) \equiv ({}^t\tau(\mathcal{S}^{-1}(\xi^*)))^* \quad \text{for all} \quad \xi \in \mathcal{G}^* . \tag{2.12}$$

Since $\bar{\tau}$ will turn out to determine the transformation law of adjoints, we refer to $\bar{\tau}$ as "conjugate" representation. If the representation τ is unitary, i.e. $\tau(\xi^*) = (\tau(\xi))^*$ for all $\xi \in \mathcal{G}^*$, the definition (2.12) assumes the simpler form $\bar{\tau}(\xi) = {}^t\tau(\mathcal{S}(\xi))$.

Let us explain how to abstract a bi-*-algebra with antipode from the representation theory of the compact group G. In this case, $\mathcal{G}^*$ should denote the group algebra of the compact gauge group G, i.e. a space of "linear combinations" of elements in G. All (anti-) homomorphisms of the group G can be uniquely extended to algebra-homomorphisms of the group algebra $\mathcal{G}^*$. Consequently it suffices to fix $\Delta_G, \varepsilon_G, *, \mathcal{S}_G$ on elements ξ in the group G. $\varepsilon_G, *$ have been defined above and comparison of (2.10) with (2.1) yields

$$\Delta_G(\xi) = \xi \otimes \xi \quad \text{for all} \quad \xi \in G . \tag{2.13}$$

Similarly we obtain the expression $\mathcal{S}_G(\xi) = \xi^{-1}$ for the antipode. Since $\Delta_G(e) = e \otimes e$, the co-product Δ_G is unit preserving. Assuming that the action of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$ is specified by $(\xi \otimes \eta)^* = \xi^* \otimes \eta^*$, $(\mathcal{G}^*, \Delta_G, \varepsilon_G, *, \mathcal{S}_G)$ is easily shown to satisfy all assumptions listed above. In this sense, group algebras are only special examples of bi-*-algebras with antipode.

Definition 4 (*Quantum symmetry*) [56]. *A *-algebra $\mathcal{G}^*$ with co-product Δ and co-unit ε is called* **quantum symmetry** *of the system $(\mathcal{H}, \{\Psi\}, |0\rangle, H)$, if there exists a representation*

$$\mathcal{U}\colon \mathcal{G}^* \mapsto \mathcal{B}(\mathcal{H}) \quad \textit{such that}$$

(i) *$\mathcal{U}$ is unitary in the sense that $\mathcal{U}(\xi^*) = \mathcal{U}(\xi)^*$ for all $\xi \in \mathcal{G}^*$.*
(ii) *The Hamiltonian H and the ground state $|0\rangle$ are invariant, i.e.*

$$[H, \mathcal{U}(\xi)] = 0, \quad \mathcal{U}(\xi)|0\rangle = |0\rangle\varepsilon(\xi) \quad \textit{for all} \quad \xi \in \mathcal{G}^* . \tag{2.14}$$

iii) *The field operators $\Psi^I_i(x,t)$ transform covariantly with respect to the representation τ^I of $\mathcal{G}^*$, i.e.*

$$\begin{aligned}\mathcal{U}(\xi)\Psi^I_i(x,t) &= \Psi^I_j(x,t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi) \\ &= \sum_p \Psi^I_j(x,t)\tau^I_{ji}(\xi^1_p)\mathcal{U}(\xi^2_p), \quad \textit{if } \Delta(\xi) = \sum_p \xi^1_p \otimes \xi^2_p .\end{aligned} \tag{2.15}$$

The covariant transformation law (2.15) tells us how to shift representation operators $\mathcal{U}(\xi)$ through fields from left to right. Together with the invariance of the ground state $|0\rangle$ it determines the transformation properties of states. We demonstrate this for the 1-excitation states,

$$\mathcal{U}(\xi)\Psi^I_i|0\rangle = \sum \Psi^I_j\tau^I_{ji}(\xi^1_p)\mathcal{U}(\xi^2_p)|0\rangle = \sum \Psi^I_j\tau^I_{ji}(\xi^1_p)\varepsilon(\xi^2_p)|0\rangle = \Psi^I_j|0\rangle\tau^I_{ji}(\xi) .$$

The transformation law of higher excitations can be found along the same lines. As a result one finds that they transform according to some tensor product of representations τ^I,

$$\mathcal{U}(\xi)\Psi^{I_1}_{i_1}\dots\Psi^{I_n}_{i_n}|0\rangle = \Psi^{I_1}_{j_1}\dots\Psi^{I_n}_{j_n}|0\rangle(\tau^{I_1}\boxtimes(\tau^{I_2}\boxtimes(\dots\boxtimes\tau^{I_n})))_{j_1\dots j_n, i_1\dots i_n}(\xi) .$$

The brackets in the tensor product of representations on the right-hand side are necessary since the tensor product (2.10) need not be associative. We will come back to this point later.

We would like to derive a transformation law for adjoint fields Ψ^{I*}_i as was done for group symmetries. However, the existence of an antipode $\mathcal{S}$ does not suffice for this purpose, even though it furnished an appropriate notion of contragredient and conjugate representations. Let us assume in addition that there exists an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ such that

$$\varphi(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\varphi \quad \text{for all } \xi \in \mathcal{G}^* . \tag{2.16}$$

We will give a detailed discussion of such elements φ below. They are called re-associators. Here we use the existence of $\varphi = \sum \varphi^1_\sigma \otimes \varphi^2_\sigma \otimes \varphi^3_\sigma$ to infer transformation properties of a covariant adjoint of field operators. The covariant adjoint can be regarded as an analog of the Lorentz covariant adjoint $\bar{\Psi} = \Psi^* \gamma_0$ of a Dirac spinor field.

Definition 5 (*Covariant adjoint*).

$$\bar{\Psi}^I_i \equiv (\Psi^I_j(\tau^I_{ji} \otimes \mathcal{U})(w))^* = \tau^I_{ij}(w^{1*}_\sigma)\mathcal{U}(w^{2*}_\sigma)\Psi^{I*}_j , \tag{2.17}$$

$$\text{with} \quad w = \sum w^1_\sigma \otimes w^2_\sigma \equiv \sum \varphi^2_\sigma \mathcal{S}^{-1}(\varphi^1_\sigma \beta) \otimes \varphi^3_\sigma .$$

The second equality in the definition (2.17) of the covariant adjoint holds only if τ^I is unitary.

Proposition 6. *The covariant adjoint transforms covariantly according to the representation $\bar{\tau}^I$ of $\mathcal{G}^*$,*

$$\mathcal{U}(\xi)\bar{\Psi}^I_i = \bar{\Psi}^I_j(\bar{\tau}^I \otimes \mathcal{U})_{ji}(\xi) . \tag{2.18}$$

Proof. From relations (2.6) and (2.16) one obtains

$$w(e \otimes \xi) = \sum \Delta(\xi^2_\sigma)w(\xi^1_\sigma \otimes e)$$

for all $\xi \in \mathcal{G}^*$. It follows that

$$\Psi^I_j(\tau^I_{ji} \otimes \mathcal{U})(w)\mathcal{U}(\xi) = (\tilde{\tau}^I_{ik} \otimes \mathcal{U})(\Delta(\xi))\Psi^I_k(\tau^I_{kj} \otimes \mathcal{U})(w) .$$

Taking the adjoint of this equation one derives the formula (2.18).

One should notice that the properties of the antipode $\mathcal{S}$ (2.6) and the discussion of covariance for adjoint fields differ from the earlier work ([56] f.). The original treatment turned out to provide an unnecessary restriction within the class of interesting quantum symmetries. In addition many crucial statements about covariant adjoints given below do not hold within the setting of [56]. It was suggested by Vecsernyes [79] to use Drinfeld's definition (2.6), and this is indeed appropriate as the results reported here confirm.

3. Algebraic Methods for Field Construction

According to the laws of local relativistic quantum mechanics, observables are selfadjoint operators acting in a Hilbert space $\mathcal{H}$ of physical states. The Hilbert space of physical states $\mathcal{H}$ may decompose into orthogonal subspaces $\mathcal{H}^J$, called *superselection sectors*, such that observables A do not make transitions between different subspaces $\mathcal{H}^J$ [80]. Different sectors $\mathcal{H}^J$ carry inequivalent irreducible positive energy representations of the algebra $\mathcal{A}$ of observables A, possibly with some multiplicity [42]. Among the sectors is the unique vacuum sector $\mathcal{H}^0$ which contains the vacuum $|0\rangle$ and appears with multiplicity 1.

When several superselection sectors exist, it is of interest to construct *additional field operators* which make transitions between superselection sectors so that the whole Hilbert space of physical states is generated from the vacuum by application of field operators. These fields operators should commute with all observables when their space-time arguments are spacelike localized.

By definition, observables have to commute with the generators of an internal gauge symmetry. A field operator which transforms according to some non-trivial representation of an internal gauge symmetry is necessarily non-observable, i.e. it maps states in different superselection sectors into each other. This implies that states in different sectors possibly transform according to inequivalent representations of the internal symmetry.

According to the algebraic theory of superselection sectors [15, 16], non-observable fields are constructed by adjoining localized endomorphisms ρ to the algebra $\mathscr{A}$ of observables. These ideas will be explained in some detail below. For a comprehensive introduction we recommend the recent book of Haag [43] and lectures by Roberts in [49].

3.1. Observables and Superselection Sectors in Local Quantum Field Theory. In this section, M_d is a d-dimensional space-time manifold with a global causal structure. $\mathscr{K}$ will denote the set of all double cones $\mathcal{O}$ (non-void intersections of forward and backward light cones) in M_d. To be specific let us choose M_d to be the d-dimensional Minkowski space. We will consider another possibility at the end of this subsection. Two subsets $\mathscr{S}_1, \mathscr{S}_2$ of M_d are called spacelike separated, if any two points $x_i \in \mathscr{S}_i$ are relatively spacelike. The causal complement $\mathscr{S}'$ of a $\mathscr{S} \subset M_d$ is the set of all points spacelike separated from $\mathscr{S}$. If $d = 2$ and $\mathcal{O}$ a double cone, $\mathcal{O}'$ decomposes into two disconnected components $\mathcal{O}_{\lessgtr}$. We use the notation $\mathcal{O}_1 \lessgtr \mathcal{O}$ whenever $\mathcal{O}_1 \subset \mathcal{O}_{\lessgtr}$. In the following, G is a group of global transformations of M_d which respect the causal structure. It is supposed to contain spacetime translations. Elements $g \in G$ transform regions $\mathcal{O} \in \mathscr{K}$ to $g\mathcal{O}$.

Consider a set of observable local quantum fields $\phi_b(x)$ on the space-time M_d. One might for example think of an energy-momentum tensor $T(x)$ or a family of real currents $J_a(x)$. They are described by a commutator which vanishes for relatively spacelike arguments. According to the Wightman theory [40, 77], fields are operator valued distributions. So they should be evaluated on real test functions f: $M_d \to \mathbf{R}$ to obtain (unbounded) operators on the Hilbert space of physical states,

$$\phi_b(f) = \int_{M_d} dx \phi_b(x) f(x) .$$

Regarded as operators on the vacuum sector $\mathscr{H}^0$, these smeared out fields "generate" the algebra of observables. We define an algebra $\mathscr{A}(\mathcal{O})$ of observables localized in $\mathcal{O}$ to be the von Neumann (weakly closed *-) algebra generated by all bounded functions of the operators $\phi_a(f)$ with $\mathrm{supp}(f) \in \mathcal{O}$. Properties of the set of observable fields $\phi_b(x)$ can be translated into properties of the family $\mathscr{A}(\mathcal{O})$ they generate. We would like to mention that in this step one has to be aware of certain subleties which might be overlooked at a first glance. In general, domain problems of the unbounded operators $\phi_a(f)$ spoil local commutativity $[\Phi(f_1), \Phi(f_2)] = 0$ of bounded functions Φ in $\phi(f_i)$ for spacelike separated support of f_i. For a review on the status of these problems see [85] and references therein.

At least in many applications to concrete models, we are led (cf. [43] for a thorough discussion) to consider a family of von Neumann algebras

$\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}} \subset \mathscr{B}(\mathscr{H}^0)$ in a Hilbert space $\mathscr{H}^0$ which satisfies the following basic (Haag–Kastler) axioms [42].

1. It is isotonic, i.e. $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathscr{A}(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O})$.
2. Einstein causality (or locality) is satisfied, which means that $\mathscr{A}(\mathcal{O}_1) \subset \mathscr{A}(\mathcal{O}_2)'$ if $\mathcal{O}_1 \subset \mathcal{O}'_2$. Here $\mathscr{A}(\mathcal{O}_2)'$ is the set of all bounded operators in $\mathscr{H}^0$ which commute with $\mathscr{A}(\mathcal{O}_2)$.
3. There is a strongly continuous unitary representation U_0 of G in $\mathscr{H}^0$ which implements automorphisms α_g: $\mathscr{A}(\mathcal{O}) \to \mathscr{A}(g(\mathcal{O}))$ defined by

$$\alpha_g(A) = U_0(g)AU_0(g^{-1}) \quad \text{for every} \quad g \in G\,. \tag{3.1}$$

   The generators of the translation subgroup should have their spectrum in the closed forward light cone. We will refer to these properties as covariance and spectrum condition respectively.

4. There is an (up to a phase) unique vector $|0\rangle \in \mathscr{H}^0$ which is invariant under the action of G, i.e. $U_0(g)|0\rangle = |0\rangle$. $|0\rangle$ is cyclic for each $\mathscr{A}(\mathcal{O})$.

Algebraic quantum field theory starts from this algebraic structure. Specific properties of underlying Wightman fields (which often exist in the applications but possibly not in general) are not needed in the analysis. So we might take the family $\mathscr{A}$ of local observables instead of a set of Wightman fields to define (the observable content of) the model.

All the properties of the family $\mathscr{A}$ stated above, reflect deep physical principles (Einstein causality, covariance, etc.). Later we will often need another assumption on the structure of $\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}}$, which cannot be justified on the same footing. The family $\mathscr{A}$ is said to satisfy *Haag duality* if

$$\mathscr{A}(\mathcal{O}) = \mathscr{A}(\mathcal{O}')' \quad \text{for all} \quad \mathcal{O} \in \mathscr{K}\,. \tag{3.2}$$

Here $\mathscr{A}(\mathcal{O}')$ is defined to be the C^*-algebra generated by the algebras $\mathscr{A}(\mathcal{O}_1)$ for $\mathcal{O}_1 \subset \mathcal{O}'$, $\mathcal{O}_1 \in \mathscr{K}$. Haag duality can be regarded as a strong version of Einstein causality, which asserts that $\mathscr{A}(\mathcal{O}) \subset \mathscr{A}(\mathcal{O}')'$. Generally speaking, breakdown of this duality indicates spontaneous breakdown of symmetry [74]. Bisognano and Wichmann have shown that for families $\mathscr{A}$ generated by Wightman fields one can always pass to the bi-dual $\mathscr{B}(\mathcal{O}) := \mathscr{A}(\mathcal{O}')'$ which then satisfies duality [5, 6]. We shall make some more specific remarks on the validity of (3.2) in conformal quantum field theories below.

According to our introductory remarks, superselection sectors carry irreducible representations of the observables algebras $\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}}$. A representation of $\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}}$ is a family of representations $\pi^{\mathcal{O}}$, $\mathcal{O} \in \mathscr{K}$ of $\mathscr{A}(\mathcal{O})$ on some Hilbert space $\mathscr{H}_\pi$ together with a strongly continuous representation U_π of the group G such that

1) $\pi^{\mathcal{O}_1}|_{\mathscr{A}(\mathcal{O}_2)} = \pi^{\mathcal{O}_2} \quad \text{if} \quad \mathcal{O}_2 \subset \mathcal{O}_1\,.$
2) $Ad_{U_\pi(g)} \circ \pi^{\mathcal{O}} = \pi^{g\mathcal{O}} \circ \alpha_{g|\mathscr{A}(\mathcal{O})}\,,$

where $Ad_{U_\pi(g)}$ is the adjoint action. For the defining representation on the vacuum sector $\mathscr{H}^0$ we will use the symbol π_0. One often prefers to work with representations of one C^*-algebra instead of representations of the family $\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}}$. This is possible since every representation of $\mathscr{A}(\mathcal{O})_{\mathcal{O}\in\mathscr{K}}$ defines a representation of the C^*-inductive limit

$$\mathscr{A} = \overline{\bigcup_{\mathcal{O}\in\mathscr{K}} \mathscr{A}(\mathcal{O})}\,. \tag{3.3}$$

Here the bar denotes closure with respect to the operator norm. The *algebra $\mathscr{A}$ of quasi-local observables* contains all the local algebras $\mathscr{A}(\mathcal{O})$. To perform the inductive limit it is essential that the sets $\mathcal{O} \in \mathscr{K}$ in the Minkowski space form a directed set.

We will be concerned only with a small subset of representations which have been singled out because of their relevance for elementary particle physics [7, 8]. A representation π is said to be locally normal if

$$\pi|_{\mathscr{A}(\mathcal{O})} \cong \pi_0|_{\mathscr{A}(\mathcal{O})} \quad \text{for all} \quad \mathcal{O} \in \mathscr{K} . \tag{3.4}$$

This has a direct physical interpretation. Elements in $\mathscr{A}(\mathcal{O})$ describe measurements which can be performed in $\mathcal{O}$. Typically, different superselection sectors can be distinguished only by their global properties ("total charge"). In other words, representations π of the algebra $\mathscr{A}$ of quasi-local observables become equivalent when they are restricted to the local subalgebras $\mathscr{A}(\mathcal{O})$. Local normality (3.4) will be tacitly assumed throughout this text. By definition, the Hilbert space $\mathscr{H}_\pi$ carries a strongly continuous representation $\mathscr{U}_\pi$ of the space time translations. When the spectrum of the corresponding generators is contained in the closed forward light cone, π is a *positive energy representation.*

In [15] Doplicher, Haag and Roberts introduced yet another criterion which selects in general an interesting subset of positive energy representations. They called π *locally generated with respect to* π_0, if

$$\pi|_{\mathscr{A}(\mathcal{O}')} \cong \pi_0|_{\mathscr{A}(\mathcal{O}')} \, . \tag{3.5}$$

This criterion looks similar to local normality (3.4). However it is much stronger, since sectors which satisfy (3.5) cannot be distinguished as long as measurements inside a given region $\mathcal{O}$ are forbidden. One might take quantum electrodynamics as a counterexample for this situation, because by Gauss law the charge inside $\mathcal{O}$ can be calculated from he flux through the surface of $\mathcal{O}$. To measure the latter, one need not enter the region $\mathcal{O}$. Since criterion (3.5) is obviously too restrictive one had to look for generalizations. Under some additional assumptions, Buchholz and Fredenhagen have been able to establish a similar criterion (localization in spacelike cones ("strings")) which allows to consider all positive energy representations with an isolated mass shell in $d \geqq 3$-dimensional quantum field theories [10]. To exploit this stringlike localization, Haag duality for spacelike cones $\mathscr{C}$ instead of double cones $\mathcal{O}$ should be supposed.

Conformal quantum field theories live on the tube $\tilde{M}_d = \mathbf{S}^1 \times \mathbf{R}$ [52, 53]. This causes minor changes in the standard framework described so far. For our purposes it will suffice to give some details concerning two-dimensional conformal quantum field theories. We parametrize the space time manifold $\tilde{M}_2$ by (τ, σ), $\tau = -\infty \ldots \infty$, $\sigma = 0 \ldots 2\pi$. $\tilde{M}_2$ contains Minkowski spaces M_ζ as subspaces (see Fig. 1). Their positions are fixed by the unique point $\zeta \in \tilde{M}_2$ at spacelike infinity of M_2. The manifold $\tilde{M}_2$ inherits from M_2 a global causal structure – i.e. a notion of positive timelike, spacelike, and negative timelike – which is invariant under the action of a covering of a direct product of two Möbius groups on the light cone coordinates $\sigma_\pm = \frac{1}{2}(\tau \mp \sigma)$. As usual, the set $\mathscr{K}$ should contain all double cones $\mathcal{O}$ in $\tilde{M}_2$. Double cones $\mathcal{O} \in \mathscr{K}$ have the obvious property that their causal complement $\mathcal{O}'$ is again in $\mathscr{K}$. This implies the selection criterion (3.5) becomes equivalent to local normality (3.4). Consequently, *every* (*locally normal*) *representation in*

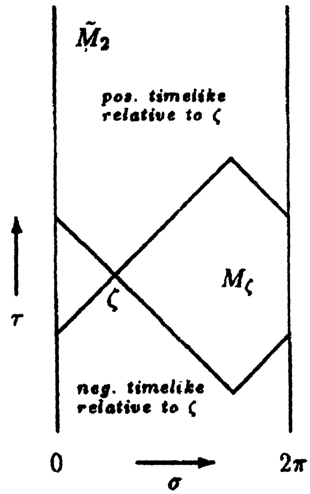

Fig. 1. Möbius invariant global causal structure on the tube $\tilde{M}_2 = \mathbf{R} = \mathbf{S}^1$. M_ζ is the Minkowski space with point ζ at "spacelike infinity." It consists of all points of $\tilde{M}_2$ which are relatively spacelike to ζ.

conformal quantum field theory is locally generated with respect to the vacuum representations π_0 [11].

In two-dimensional conformal quantum field theory observable fields often split into mutually commuting chiral components $\phi_\pm(\sigma_\pm)$ which depend only on one light cone coordinate. For such chiral observables, periodicity in the coordinate σ which parametrizes the space $\mathbf{S}^1$ implies periodicity in time τ. This means that $\phi_\pm$ can be regarded as a one-valued field on the circle $|z_\pm| = 1$, where $z_\pm = e^{i\sigma_\pm}$. We restrict attention to one chiral component of the observables and drop the suffix $+$ or $-$. Since spacelike separated double cones $\mathcal{O} \in \tilde{M}_2$ project onto disjoint intervals I on the circle $|z| = 1$, *chiral observables* $\phi(z_1)$ *and* $\phi(z_2)$ *commute when* $z_1 \neq z_2$, $(z_i \in \mathbf{S}^1)$. A set of chiral observables $\phi(z)$ generates a family of von Neumann algebras $\mathscr{A}(I)$ indexed by intervals $I \in \mathbf{S}^1$. To formulate Haag Kastler axioms for such families, double cones $\mathcal{O}$ should be replaced by intervals I. The complement $I' = \mathbf{S}^1 \backslash I$ substitutes for the causal complement $\mathcal{O}'$ and one of the Möbius groups $SL(2, \mathbf{R})/\mathbf{Z}_2$, acts as symmetry group G on the circle $\mathbf{S}^1$. In a positive energy representation π the generator L_0 of "rotations" has positive spectrum. Recently, Brunetti et al. and Fröhlich et al. proved Haag duality for chiral conformal quantum field theories [9, 37].

The construction of a C^∞-algebra of "quasi local observables" is not straightforward since intervals I in the circle $\mathbf{S}^1$ do not form a directed set. To define an inductive limit of local algebras, one has to remove a point $\zeta \in \mathbf{S}^1$ ("point at infinity")

$$\mathscr{A}_\zeta = \overline{\bigcup_{I \not\ni \zeta} \mathscr{A}(I)}\,. \tag{3.6}$$

Note that after the choice of $\zeta \in \mathbf{S}^1$, the complement of an interval $I \not\ni \zeta$ in $\mathbf{S}^1 \backslash \zeta$ decomposes into left and right components $I_\lessgtr$. Disjoint intervals $I_1, I_2 \subset \mathbf{S}^1 \backslash \zeta$ can be ordered like double cones in two-dimensional Minkowski space M_2.

Even though $\mathscr{A}_\zeta$ will suffice for all model independent studies, it is often inconvenient in the applications. It is an obvious disadvantage of $\mathscr{A}_\zeta$ that local algebras $\mathscr{A}(I)$ are not embedded into $\mathscr{A}_\zeta$ if $\zeta \in I$. This motivates to look for a C^∞-algebra $\mathscr{A}_{\text{univ}}$ such that

1. every local algebra $\mathscr{A}(I)$ can be embedded into $\mathscr{A}_{\text{univ}}$ by a unital map i^I such that
$$i^{I_1}|_{\mathscr{A}(I_2)} = i^{I_2} \quad \text{for all} \quad I_2 \subset I_1$$
and $\mathscr{A}_{\text{univ}}$ is generated by the algebras $i^I(\mathscr{A}(I))$, $I \subset \mathbf{S}^1$.
2. For every representation $\{\pi^I\}_{I \subset \mathbf{S}^1}$ of the family $\mathscr{A}(I)_{I \subset \mathbf{S}^1}$ there is a unique representation π of $\mathscr{A}_{\text{univ}}$ which satisfies $\pi \circ i^I = \pi^I$.

The "universal algebra" $\mathscr{A}_{\text{univ}}$ does is exist and is unique [27, 25], but its explicit construction is subtle. Unlike the algebras $\mathscr{A}$ resp. $\mathscr{A}_\zeta$ obtained from the inductive limits (3.3, resp. 3.6), the center of the universal algebra $\mathscr{A}_{\text{univ}}$ is in general non-trivial. This means that the vacuum representation π_0 of $\mathscr{A}_{\text{univ}}$ may not be faithful. Indeed it is possible that two charge operators localized in domains $I_1, I_2, I_1 \cup I_2 = \mathbf{S}^1$ add up to a global quantity which commutes with all elements in $\mathscr{A}_{\text{univ}}$. These charges may have different values in different superselection sectors. They must not be identified with multiples of the identity.

The setup for theories with charges localized along strings in 3-dimensional Minkowski space is very similar to the situation in chiral conformal quantum field theories (cf. [25] and references therein). Points ζ on the circle $\mathbf{S}^1$ are substituted by directions in the two-dimensional space and one uses spacelike cones $\mathscr{C}$ instead of intervals $I \subset \mathbf{S}^1$.

3.2. Localized Endomorphisms and Fusion Structure. A detailed analysis of the structure of superselection sectors was first performed by Doplicher, Haag and Roberts [15, 16]. It was restricted to sectors which are locally generated with respect to the vacuum sector and formulated for theories on the four-dimensional Minkowski space. The generalization to string-like localized sectors [10] gives essentially the same structure. If the dimension of the space time manifold is decreased, specific new features appear. In $d = 2$ (resp. $d = 3$) dimensional space-times, double cones (resp. spacelike cones) can be ordered and – as we will explain below – this gives rise to representations of the braid group. Such situations were considered more recently by Fredenhagen, Rehren and Schroer [24, 25]. Our short exposition will concentrate on representations localized in double cones $\mathcal{O}$ in Minkowski space. We included some remarks relevant to treat chiral conformal theories and string-like localized sectors in three-dimensional quantum field theory. For many further results and discussions the reader is referred to the original papers – especially to [24, 25] – and the reviews of Fredenhagen [26] or by Kastler, Mebkhout and Rehren (in [49]). Theories with charges localized in spacelike cones in M_3 have been considered explicitly by Frölich et al. [34, 35, 36]. Applications to models in two-dimensional conformal quantum field theory can be found in [11, 54, 55, 39].

From now on, families $\mathscr{A}(\mathcal{O})_{\mathcal{O} \in \mathscr{K}}$ are always supposed to satisfy Haag–Kastler axioms and Haag duality (3.2). Except from some remarks, representations are assumed to be localized on double cones (3.5) $\mathcal{O}$ in Minkowski space M_d.

Equivalence classes of locally generated positive energy representations form a set denoted by $\mathscr{R}ep$.

The notion of *localized endomorphisms* will provide the key to all further analysis. By definition, an endomorphism ρ of the C^* algebra $\mathscr{A}$ is a linear map ρ: $\mathscr{A} \to \mathscr{A}$ with the properties

$$\rho(AB) = \rho(A)\rho(B) ,$$

$$\rho(A^*) = \rho(A)^* ,$$

$$\rho(\mathbf{1}) = \mathbf{1} .$$

It is called an *automorphism* if it has an inverse. Endomorphisms of $\mathscr{A}$ fall into equivalence classes $[\rho]$ with respect to inner automorphisms, i.e. conjugation by unitaries $U \in \mathscr{A}$. ρ is said to be *localized* in $\mathcal{O} \in \mathscr{K}$ if

$$\rho(A) = A \quad \text{for all} \quad A \in \mathscr{A}(\mathcal{O}_1) \subset \mathscr{A},\, \mathcal{O}_1 \subset \mathcal{O}' .$$

An endomorphism ρ localized in $\mathcal{O}$ is called *transportable* whenever equivalent morphisms $\sigma \in [\rho]$ localized in the transformed region $g\mathcal{O}$ exist for all $g \in G$. Transportable endomorphisms localized in spacelike separated regions commute [15].

Endomorphisms of $\mathscr{A}$ can be used to obtain positive energy representations $\pi_0 \circ \rho$ of $\mathscr{A}$ on the Hilbert space $\mathscr{H}^0$. For every locally generated positive energy representation π of $\mathscr{A}$ there is a localized transportable endomorphism ρ such that

$$\pi \cong \pi_0 \circ \rho . \tag{3.7}$$

This can be seen as follows. According to the criterion (3.5), $\pi|_{\mathscr{A}(\mathcal{O}')}$ is unitary equivalent to $\pi_0|_{\mathscr{A}(\mathcal{O}')}$, i.e. for each double cone $\mathcal{O}$ there is a unitary V: $\mathscr{H}_\pi \to \mathscr{H}^0$ such that

$$V\pi(A) = \pi_0(A)V \quad \text{for all} \quad A \in \mathscr{A}(\mathcal{O}') .$$

By Haag duality, the map

$$\rho(A) = V\pi(A)V^*$$

defines an endomorphism of $\mathscr{A}$ localized in $\mathcal{O}$. It is transportable and has the desired property (3.7). Two localized endomorphisms ρ_i, $i = 1, 2$ are equivalent if and only if the representations $\pi_0 \circ \rho_i$ are equivalent. We conclude that elements of $\mathscr{R}ep$, i.e. equivalence classes of locally generated positive energy representations of $\mathscr{A}$, correspond one by one to equivalence classes $[\rho]$ of localized transportable endomorphisms ρ. This observation will be used to identify both objects.

Let us pause for a moment to comment on (chiral) conformal quantum field theory. In this case superselection sectors carry irreducible positive energy representations of the C^*-algebra $\mathscr{A}_\zeta$ introduced in the last section. By the results in [11, 9, 37], every positive energy representation is obtained as a composition $\pi_0 \circ \rho$ of the vacuum representation π_0 with a endomorphism ρ of $\mathscr{A}_\zeta$. We can assume ρ to be localized and transportable in an appropriate sense. The general considerations below can be established on $\mathscr{A}_\zeta$ without modifications. In practice it will be more convenient to work with endomorphisms ρ of the universal algebra $\mathscr{A}_{\text{univ}}$. Fredenhagen has shown [27] that the localized and locally transportable endomorphisms of $\mathscr{A}_{\text{univ}}$ exist for all elements of $\mathscr{R}ep$. They restrict to localized transportable endomorphisms of $\mathscr{A}_\zeta \subset \mathscr{A}_{\mathrm{univ}}$ when ζ lies outside the domain of localization. Basically, these remarks apply also to stringlike localized sectors in $2+1$ dimensional quantum field theory.

All this is much more than a technicality. Endomorphisms of $\mathscr{A}$ can be composed and thus lead to a proper definition of a product of sectors. Given two representations $\pi_i = \pi_0 \circ \rho_i$, $i = 1, 2$, on the Hilbert space $\mathscr{H}^0$, their product $\pi_1 \times \pi_2$ is defined by

$$\pi_1 \times \pi_2 = \pi_0 \circ \rho_1 \circ \rho_2 \ .$$

The equivalence class of $\pi_1 \times \pi_2$ is an element of $\mathscr{R}ep$ if equivalence classes of π_i, $i = 1, 2$, are. In other words a product in $\mathscr{R}ep$ is defined by $[\rho_1] \times [\rho_2] \equiv [\rho_1 \circ \rho_2]$. These assertions do not depend on the type of localization.

Two representations $\pi_0 \circ \rho_i$, $i = 1, 2$, in $\mathscr{H}^0$ have equivalent subrepresentations if an isometry $U \in \mathscr{B}(\mathscr{H}^0)$ intertwines between them, i.e. $U\pi_0(\rho_1(A)) = \pi_0(\rho_2(A))U$ for all local observables $A \in \mathscr{A}$. When restricted with the "source" projection U^*U of U, the representation $\pi_0 \circ \rho_1$ is equivalent to the restriction of $\pi_0 \circ \rho_2$ to the range of U. By Haag duality, such intertwining operators U can be obtained as an image of a local intertwiner $T \in \mathscr{A}$,

$$T\rho_1(A) = \rho_2(A)T \quad \text{for all} \quad A \in \mathscr{A} \ .$$

Local observables $T \in \mathscr{A}$ which satisfy this equation span a complex linear space $\mathscr{T}(\rho_1, \rho_2)$. If $S, T \in \mathscr{T}(\rho_1, \rho_2)$ then TS^* is a local observable in $\mathscr{T}(\rho_2, \rho_2)$. Schurs lemma asserts, that irreducibility of ρ implies $\mathscr{T}(\rho, \rho) \cong \mathbf{C}$. In conclusion, $\langle T, S \rangle \equiv TS^*$ defines a scalar product on $\mathscr{T}(\rho_1, \rho_2)$ if ρ_2 is irreducible. Therefore $\mathscr{T}(\rho_1, \rho_2)$ is a Hilbert space in this case.

The product of sectors $[\rho_i]$, $i = 1, 2$ is commutative in the sense that $[\rho_1 \circ \rho_2] = [\rho_2 \circ \rho_1]$. To see this we pick two endomorphisms σ_i from the equivalence classes $[\rho_i]$ which are localized in spacelike separated double cones. As we remarked above, their action on the observables commutes and so the assertion follows. For every pair of localized and transportable endomorphisms ρ_i, $i = 1, 2$ there is a unitary local intertwiner $\varepsilon(\rho_1, \rho_2) \in \mathscr{T}(\rho_1 \circ \rho_2, \rho_2 \circ \rho_1)$, the *statistics operator*. The collection of statistics operators in uniquely determined by the following equations (a detailed proof can be found in the contribution of Mebkhout et al. in [49]),

$$\varepsilon(\rho_1, \rho_2)\rho_1(T_2)T_1 = T_2\sigma_2(T_1)\varepsilon(\sigma_1, \sigma_2) \quad \text{for all} \quad T_i \in \mathscr{T}(\sigma_i, \rho_i) \ ,$$

$$\varepsilon(\rho_1 \circ \rho_2, \sigma) = \varepsilon(\rho_1, \sigma)\rho_1(\varepsilon(\rho_2, \sigma)), \quad \varepsilon(\sigma, \rho_1 \circ \rho_2) = \rho_1(\varepsilon(\sigma, \rho_2))\varepsilon(\sigma, \rho_1) \ ,$$

$$\varepsilon(\rho_1, \rho_2) = 1 \quad \text{whenever} \quad \rho_1 > \rho_2 \ . \tag{3.8}$$

In the last row, $\rho_1 > \rho_2$ refers to the order of localization regions, provided they can be ordered. For localization in double cones contained in a two-dimensional Minkowski space this is the case and $\rho_1 > \rho_2$ means that ρ_1 is localized on a domain $\mathcal{O}_1$ left from the localization region $\mathcal{O}_2$ of ρ_2, i.e. $\mathcal{O}_1 > \mathcal{O}_2$. Trivialization for $\rho_1 < \rho_2$ would give rise to the opposite statistics operator $\varepsilon(\rho_2, \rho_1)^*$. In higher dimensional quantum field theories there is no invariant distinction between left and right so that trivialization is possible for all pairs of spacelike separated endomorphisms. For two-dimensional light cone theories [stringlike localized sectors in theories on M_3], the notion $\lessgtr$ refers to the point ζ at infinity [the direction in the two-dimensional space] which we have to single out to define the inductive limit in (3.6).

It follows from these relations that the vacuum sector carries a representation of the colored braid group. In particular, $\sigma_i = \rho^{i-1}(\varepsilon(\rho, \rho))$ satisfy Artin relations ([4]),

$$\sigma_i \sigma_k = \sigma_k \sigma_i \quad \text{if } |k-i| \geqq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \,. \tag{3.9}$$

$$\sigma_i \sigma_i^{-1} = \mathbf{1} = \sigma_i^{-1} \sigma_i \,, \tag{3.10}$$

i.e. the elements σ_i and σ_i^{-1} $(i = 1 \ldots n-1)$ generate the braid group B_n. In theories without an invariant distinction between left and right, $\sigma_i = \sigma_i^{-1}$ so that we obtain a representation of the permutation group. Accordingly the statistics operators give rise to an intrinsic notion of statistics of a superselection sector.

At this stage we would like to restrict to sectors of *finite statistics*. An irreducible transportable endomorphism has finite statistics whenever it possesses a left inverse[3] Φ with $\Phi(\varepsilon(\rho, \rho)) \neq 0$. It has been shown by Longo [51] that this is equivalent to a finite Jones index [47] of the inclusion $\rho(\mathscr{A}(\mathscr{O})) \subset \mathscr{A}(\mathscr{O})$. We do not plan to justify this restriction (see however [23, 10]) but list the consequences required later. Whenever two sectors of finite statistics are composed, their product decomposes into a finite direct sum of sectors which have finite statistics again. This means that the subset $\mathscr{R}ep' \subset \mathscr{R}ep$ of finite direct sums of sectors with finite statistics is closed under products. Moreover, every sector with finite statistics has a unique conjugate, i.e. for every irreducible endomorphism ρ of $\mathscr{A}$ there exists a unique irreducible endomorphism $\bar{\rho}$ such that $\bar{\rho} \circ \rho$ contains the vacuum sector. One can show that $\dim \mathscr{T}(\bar{\rho} \circ \rho, id) = 1$.

Equivalence classes of irreducible representations in $\mathscr{R}ep'$ will be labelled by elements $I, J, K, \ldots$ of an index set $\mathscr{I}_{\mathscr{A}}$. We reserve $0 \in \mathscr{I}_{\mathscr{A}}$ for the vacuum sector. Let us fix a set of representative endomorphisms ρ_I, one for each superselection sector in $\mathscr{R}ep'$. According to the general remarks on sectors with finite statistics, the product of two endomorphisms $\rho_I \circ \rho_J$ is equivalent to a finite direct sum of irreducibles ρ_K. As we saw above, the space $\mathscr{T}(\rho_I \circ \rho_J | \rho_K) \equiv \mathscr{T}(IJ|K)$ defines the fusion rules N^{IJ}_K which appear in the decomposition

$$[\rho_I \circ \rho_J] = \bigoplus_K [\rho_K] N^{IJ}_K \,.$$

It follows from the associativity of the composition of endomorphisms and the commutativity we found above that the fusion rules are associative and commutative in the sense

$$N^{IJ}_K = N^{JI}_K, \quad \text{and} \tag{3.11}$$

$$\sum_M N^{IM}_L N^{JK}_M = \sum_M N^{MK}_L N^{IJ}_M \,. \tag{3.12}$$

The endomorphism $\rho_{\bar{J}}$ should be the unique conjugate $\bar{\rho}_J$ of ρ_J. Since the vacuum sector $[\rho_0]$ appears with multiplicity 1 in the decomposition of $[\rho_{\bar{J}} \circ \rho_J]$ we have

$$N^{I\bar{J}}_0 = \delta_{I,J} \,.$$

[3] By definition, a left inverse is a positive linear map $\Phi\colon \mathscr{A} \mapsto \mathscr{A}$ such that $\Phi(\rho(A)B\rho(C)) = A\Phi(B)C$ for all $A, B, C \in \mathscr{A}$ and $\Phi(\mathbf{1}) = \mathbf{1}$.

In the Hilbert space $\mathscr{T}(IJ|K)$ we choose an orthonormal basis $T_a({}_K{}^J{}_I) \in \mathscr{A}$. In detail this means that the operators $T_a({}_K{}^J{}_I)$, $a = 1 \ldots N_K^{IJ}$, satisfy

$$T_a({}_K{}^J{}_I)\rho_I \circ \rho_J(A) = \rho_K(A) T_a({}_K{}^J{}_I) \quad \text{for all} \quad A \in \mathscr{A}\ , \tag{3.13}$$

$$T_a({}_K{}^J{}_I) T_b({}_L{}^J{}_I)^* = \delta_{a,b}\delta_{L,K}, \quad \sum_{K,a} T_a({}_K{}^J{}_I)^* T_a({}_K{}^J{}_I) = 1\ . \tag{3.14}$$

The *fusion- and braiding- matrices* are complex valued matrices defined by

$$T_a({}_L{}^K{}_M) T_b({}_M{}^J{}_I) = \sum_N T_c({}_L{}^N{}_I)\rho_I(T_d({}_N{}^K{}_J)) F_{MN}\left[\begin{smallmatrix} K & J \\ L & I \end{smallmatrix}\right]^{cd}_{ab}\ , \tag{3.15}$$

$$T_a({}_L{}^J{}_M) T_b({}_M{}^K{}_I)\rho_I(\varepsilon(\rho_J, \rho_K)) = \sum_N T_c({}_L{}^K{}_N) T_d({}_N{}^J{}_I) B_{MN}\left[\begin{smallmatrix} J & K \\ L & I \end{smallmatrix}\right]^{cd}_{ab}\ . \tag{3.16}$$

As a special case of Eq. (3.16) for $I = 0$ we introduce the matrix $\Omega({}_L{}^J{}_M)$,

$$T_a({}_L{}^J{}_K)\varepsilon(\rho_J, \rho_K) \equiv T_b({}_L{}^K{}_J)\Omega^b_a({}_L{}^J{}_K)\ . \tag{3.17}$$

By Schurs' lemma, the coefficients F, B, Ω are certain complex matrices. They are determined by the model and depend only on the equivalence classes $[\rho_I]$ but not on ρ_I itself. The braiding matrices B, Ω are often denoted by $\Omega(+)$, $B(+)$ to emphasize their dependence on the choice of trivialization of the statistics operators $\varepsilon(\rho_J, \rho_K)$ (3.8). The corresponding matrices for the opposite statistics operators $\varepsilon(\rho_K, \rho_J)^*$ are called $B(-)$, $\Omega(-)$. We restrict attention to one trivialization and neglect to write $(\pm)$. A short calculation reveals the following proposition.

Proposition 7 (*Polynomial equations*) [24]. *The fusion- and braiding matrices F, Ω defined by* (3.16, 3.15) *solve the polynomial equations*

$$\sum_N F_{NP}\left[\begin{smallmatrix} K & J \\ L & I \end{smallmatrix}\right](\Omega({}_L{}^N{}_K) \otimes 1) F_{MN}\left[\begin{smallmatrix} J & I \\ L & K \end{smallmatrix}\right] = (1 \otimes \Omega({}_P{}^J{}_K)) F_{MP}\left[\begin{smallmatrix} J & K \\ L & I \end{smallmatrix}\right](1 \otimes \Omega({}_M{}^I{}_K))\ , \tag{3.18}$$

$$\sum_Q F_{QS}\left[\begin{smallmatrix} L & K \\ R & J \end{smallmatrix}\right]_{23} F_{NR}\left[\begin{smallmatrix} L & Q \\ P & I \end{smallmatrix}\right]_{12} F_{MQ}\left[\begin{smallmatrix} K & J \\ N & I \end{smallmatrix}\right]_{23} = P_{23} F_{MR}\left[\begin{smallmatrix} S & J \\ P & I \end{smallmatrix}\right]_{13} F_{NS}\left[\begin{smallmatrix} L & K \\ P & M \end{smallmatrix}\right]_{12}\ , \tag{3.19}$$

$$\sum_P F_{PQ}\left[\begin{smallmatrix} I & J \\ K & L \end{smallmatrix}\right] F_{PR}\left[\begin{smallmatrix} I & J \\ K & L \end{smallmatrix}\right]^* = \mathbf{1}\delta_{Q,R}, \quad \sum_Q F_{PQ}\left[\begin{smallmatrix} I & J \\ K & L \end{smallmatrix}\right]^* F_{RQ}\left[\begin{smallmatrix} I & J \\ K & L \end{smallmatrix}\right] = \mathbf{1}\delta_{P,R}\ , \tag{3.20}$$

$$\Omega({}_L{}^I{}_J)\Omega({}_L{}^I{}_J)^* = \mathbf{1}, \quad \Omega({}_L{}^I{}_J)^*\Omega({}_L{}^I{}_J) = \mathbf{1}\ , \tag{3.21}$$

$$F_{IJ}\left[\begin{smallmatrix} J & 0 \\ K & I \end{smallmatrix}\right] = \mathbf{1}\ . \tag{3.22}$$

The braiding-matrix B can be calculated from F, Ω.

$$B_{M,N}\left[\begin{smallmatrix} J & K \\ L & I \end{smallmatrix}\right] = \sum_P F_{NP}\left[\begin{smallmatrix} K & J \\ L & I \end{smallmatrix}\right]^*(1 \otimes \Omega({}_P{}^J{}_K)) F_{MP}\left[\begin{smallmatrix} J & K \\ L & I \end{smallmatrix}\right]\ . \tag{3.23}$$

We used an obvious matrix notation and $\mathbf{1}$ *denotes an appropriate unit matrix. F_{12} is defined on threefold tensor products $u \otimes v \otimes w$ by $F_{12}(u \otimes v \otimes w) = F(u \otimes v) \otimes w$, etc. P_{23} acts as permutation of the second and third component.*

The first two relations (3.18, 3.19) are the famous Moore–Seiberg "hexagon" and "pentagon" identities known from conformal quantum field theory [66, 67]. In our context they reflect deep properties of the fusion structure of superselection sectors. In particular, conformal symmetry was not assumed.

Proof. We do not want to prove all the relations but just demonstrate the type of calculations to be done at the example of Eq. (3.23). The product of operators which appears on the left-hand side of Eq. (3.16) can be manipulated in two different ways. One is just the step from the left to the right-hand side of (3.16). For the other we apply the definition (3.15) of the fusion matrix, the endomorphism property of ρ_I, the definition (3.17) and relation (3.15) in this order,

$$T_a({}_L{}^J{}_M)T_b(({}_M{}^K{}_I)\rho_I(\varepsilon(\rho_J,\rho_K)) = \sum_P T_c({}_L{}^P{}_I)\rho_I(T_d({}_P{}^J{}_K))\rho_I(\varepsilon(\rho_J,\rho_K))F_{MP}\begin{bmatrix} J & K \\ L & I\end{bmatrix}^{cd}_{ab}$$

$$\sum_P T_c({}_L{}^P{}_I)\rho_I(T_e({}_P{}^K{}_J))\Omega({}_P{}^J{}_K)^e_d F_{MP}\begin{bmatrix} J & K \\ L & I\end{bmatrix}^{cd}_{ab}$$

$$= \sum_{NP} T_f({}_L{}^K{}_N)T_g({}_N{}^J{}_I)\left(F_{NP}\begin{bmatrix} K & J \\ L & I\end{bmatrix}^*\right)^{jg}_{ce}\Omega({}_P{}^J{}_K)^e_d F_{MP}\begin{bmatrix} J & K \\ L & I\end{bmatrix}^{cd}_{ab} .$$

We used the first relation in (3.20) for the last equality. If we compare the result with the right-hand side of (3.16) we find that the same operator has been expressed by two linear combinations of the same basis elements. Consequently, the coefficients have to agree and this gives Eq. (3.23).

3.3. Covariant Field Operators. Now we are prepared to construct field operators Ψ which make transitions between different superselection sectors. We want them to transform non-trivially under the action of elements ξ from an "appropriate" symmetry algebra $\mathscr{G}^*$. A simple assumption on the structure of the representation theory of $\mathscr{G}^*$ will turn out to ensure the existence of such covariant field operators. They will be constructed as a sum of "vertex operators."

$\mathscr{G}^*$ is assumed to be semisimple bi-*-algebra with antipode. Equivalence classes $[\tau]$ of finite dimensional irreducible representations τ of $\mathscr{G}^*$ are labelled by elements of an index set $\mathscr{I}_{\mathscr{G}^*}$. Because of semisimplicity[4], tensor products of two finite dimensional irreducible representations τ, τ' on vector spaces, V, V' can be decomposed into irreducibles τ^α on V^α. The corresponding "Clebsch Gordon" intertwiners $C(r \boxtimes \tau'|\tau^\alpha)$: $V \otimes V' \mapsto V^\alpha$ form complex vector spaces $\mathscr{C}(\tau \boxtimes \tau'|\tau^\alpha)$,

$$C(\tau \boxtimes \tau'|\tau^\alpha)(\tau \boxtimes \tau')(\xi) = \tau^\alpha(\xi)C(\tau \boxtimes \tau'|\tau^\alpha) .$$

Recall that a representation τ of $\mathscr{G}^*$ on a Hilbert space V is unitary, if $\tau(\xi)^* = \tau(\xi^*)$ for all $\xi \in \mathscr{G}^*$

Assumption. *Let $I_{\mathscr{A}}$ denote the set of superselection sectors with finite statistics as in the preceding subsection. We assume that there is a bijection θ: $\mathscr{I}_{\mathscr{A}} \mapsto \mathscr{I}_{\mathscr{G}^*}$ and a set of unitary representatives $\tau^{\theta(I)}$ from the equivalence classes of irreducible representations of $\mathscr{G}^*$ such that*

$$\dim(\mathscr{C}(\tau^{\theta(I)} \boxtimes \tau^{\theta(J)}|\tau^{\theta(K)})) = N^{IJ}_K . \tag{3.24}$$

In other words: There is a unique equivalence class of irreducible representations of $\mathscr{G}^*$ associated with every superselection sector and fusion rules of the sectors are in agreement with the selection rules of the prospective symmetry. By uniqueness of the conjugate, θ will automatically map the vacuum sector to the

[4] Note that every finite dimensional representation of a semisimple algebra is a direct sum of irreducible representations.

equivalence class of the one-dimensional trivial representation ε of $\mathscr{G}^*$. Since ε is unitary we can always choose

$$\tau^{\theta(0)} = \varepsilon \, .$$

We will not distinguish between I and $\theta(I)$ in the following.

Given an algebra of observables $\mathscr{A}$ one may wonder whether a bi-*-algebra with antipode satisfying the above assumption does exist. We postpone this discussion until the next section. It will turn out that *suitable bi-*-algebras with antipode do always exist*, if $\mathscr{I}_{\mathscr{A}^*}$ is finite (cp. Corollary 14 below).

Given the required bi-*-algebra $\mathscr{G}^*$, with antipode we can start to build covariant field operators. They will act on a Hilbert space $\mathscr{H}$ of physical states which is a direct sum of irreducible representation spaces $\mathscr{H}^J$ for the algebra of observables $\mathscr{A}$, each with multiplicity δ_J determined by the dimension of the representation τ^J of $\mathscr{G}^*$, i.e. $\delta_J \equiv \dim(\tau^J)$,

$$\mathscr{H} = \bigoplus_{J \in \mathscr{I}} \bigoplus_{1}^{\delta_J} \mathscr{H}_i^J \, . \tag{3.25}$$

$\mathscr{H}^0$ carries the vacuum representation π_0 of $\mathscr{A}$ and it occurs with multiplicity one (since $\dim(\tau^0) = \dim(\varepsilon) = 1$).

Next we define a representation π of the observables algebra $\mathscr{A}$ on all of $\mathscr{H}$ by its restrictions to the subspaces $\mathscr{H}_m^J$,

$$\pi(A) = \pi_J(A) \quad \text{on} \quad \mathscr{H}_m^J (m = 1 \ldots \delta_J) \, . \tag{3.26}$$

According to our general discussion, the representation π_J on $\mathscr{H}_m^J$ is equivalent to $\pi_0 \circ \rho_J$ which is realized on the vacuum Hilbert space $\mathscr{H}^0$. This equivalence can be expressed by isometries i_{Jm}^*: $\mathscr{H}^0 \mapsto \mathscr{H}_m^J$ with the intertwining property

$$\pi_J(A) i_{Jm}^* = i_{Jm}^* \pi_0(\rho_J(A)) \, . \tag{3.27}$$

To specify the action of elements $\xi \in \mathscr{G}^*$ on $\mathscr{H}$, we choose an orthonormal basis e_m^J in the finite dimensional representation space V^J. When the corresponding matrix elements of $\tau^J(\xi)$ are denoted by $\tau_{km}^J(\xi)$, an unitary representation $\mathscr{U}$ of $\mathscr{G}^*$ on $\mathscr{H}$ is obtained according to

$$\mathscr{U}(\xi) i_{Jm}^* |\psi\rangle = i_{Jk}^* |\psi\rangle \tau_{km}^J(\xi) \tag{3.28}$$

for arbitrary $|\psi\rangle \in \mathscr{H}^0$. The symmetry acts as as a *gauge symmetry* (*of first kind*), i.e. all observables are invariant

$$[\mathscr{U}(\xi), \pi(A)] = 0 \quad \text{for all} \quad A \in \mathscr{A}, \xi \in \mathscr{G}^* \, . \tag{3.29}$$

Our main task is to construct field operators $\Psi_m^J(\rho_J)$ which make transitions between the sectors $\mathscr{H}_i^I$ with different I. They will have the following properties:

1. Intertwining property for representations of $\mathscr{A}$,

$$\pi(A) \Psi_m^J(\rho_J) = \Psi_m^J(\rho_J) \pi(\rho_J(A)) \, . \tag{3.30}$$

As a consequence, $\Psi_m^J(\rho_J)$ commutes with observables localized in the causal complement $\mathscr{O}'$ of the localization region $\mathscr{O}$ of the endomorphisms ρ_J. This property reflects locality of the field operators with respect to observables.

2. Field operators $\Psi^J_m(\rho_J)$ transform covariantly according to the representation τ^J of $\mathcal{G}^*$,

$$\mathcal{U}(\xi)\Psi^J_m(\rho_J) = \Psi^J_k(\rho_J)(\tau^J_{km} \otimes \mathcal{U})(\Delta(\xi)) \,. \tag{3.31}$$

The field operators $\Psi^J_m(\rho_J)$ are determined by these properties up to a phase factor. They will be build up from the following "vertex operators":

$$_k\Psi_l(_K{}^J{}_L)_a(\rho_J) : \mathcal{H}^J_l \mapsto \mathcal{H}^K_k \quad \text{for } a = 1 \ldots N^{JL}_K \,, \tag{3.32}$$

$$_k\Psi_l(_K{}^J{}_L)_a(\rho_J) = i^*_{Kk}\pi_0(T_a(_K{}^J{}_L))i_{Ll} \,. \tag{3.33}$$

Combining the intertwining relations (3.27) and (3.13) we find that these operators satisfy the intertwining property 1 for representations of $\mathcal{A}$. We extend the vertex operators $_k\Psi_l(_K{}^J{}_L)_a$ to all of $\mathcal{H}$ such that they vanish on states in $\mathcal{H}^I_i$ for all $i \neq l$, $I \neq L$. The extended operators are denoted by the same symbol. Their intertwining properties with observables is not affected by this extension.

To obtain a covariant field operator $\Psi^J_l(\rho_J)$ on the whole Hilbert space $\mathcal{H}$, we fix an orthonormal basis $C^a(JL|K)$, $a = 1 \ldots N^{JL}_K$, in the Hilbert spaces $\mathcal{C}(JL|K) \equiv \mathcal{C}(\tau^J \boxtimes \tau^L|\tau^K)$ of Clebsch Gordon intertwiners. By assumption, the fusion rules and tensor product decomposition match so that the index a assumes the same values as for the vertex operators. With this knowledge we can form the following linear combination of vertex operators:

$$\Psi^J_m(\rho_J) = \sum_{K,k,L,l} {}_k\Psi_l(_K{}^J{}_L)_a(\rho_J)\begin{bmatrix} J & L & K \\ m & l & k \end{bmatrix}^a \,. \tag{3.34}$$

The complex coefficients $[\,\cdot\,\vdots\,\cdot\,]^a$ are matrix elements of the Clebsch Gordon map $C^a(JL|K)$ in the basis $e^J_j \otimes e^L_l$, resp. e^K_k. The field operators $\Psi^J_m(\rho_J)$ meet all the requirements stated before.

Theorem 8. *Let $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}} \subset \mathcal{B}(\mathcal{H}^0)$ be a family of local observable algebras with properties as before. Suppose that $\mathcal{G}^*$ is a bi-*-algebra which satisfies assumption* (3.24). *Then there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, a unitary representation $\mathcal{U}$ of $\mathcal{G}^*$ on $\mathcal{H}$ and a family $\mathcal{B}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}} \subset \mathcal{B}(\mathcal{H})$ of *-algebras such that*

1. *The vacuum representation π_0 is a subrepresentation of π on $\mathcal{H}^0 \subset \mathcal{H}$ and appears with multiplicity* 1. *States in $\mathcal{H}^0$ are invariant with respect to the action of $\mathcal{G}^*$. In particular,*

$$\mathcal{U}(\xi)|0\rangle = |0\rangle\varepsilon(\xi) \quad \textit{for all} \quad \xi \in \mathcal{G}^* \,.$$

The Hilbert space $\mathcal{H}$ is generated from $|0\rangle$ by algebras $\mathcal{B}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$.

2. *The algebras $\mathcal{B}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ are generated by elements of $\mathcal{A}(\mathcal{O})$ and operators $\mathcal{U}(\xi)$ together with operators $\Psi^J_m(\rho_J)$, where ρ_J are endomorphisms of $\mathcal{A}$ localized in $\mathcal{O}$. The field operators $\Psi^J_m(\rho_J)$ are local relative to observables and transform covariantly according to the representation τ^J of $\mathcal{G}^*$. Explicitly this means*

$$\Psi^J_m(\rho_J)\pi(A) = \pi(A)\Psi^J_m(\rho_J) \quad \textit{for all} \quad A \in A(\mathcal{O}') \,, \tag{3.35}$$

$$\mathcal{U}(\xi)\Psi^J_m(\rho_J) = \Psi^J_k(\rho_J)(\tau^J_{km} \otimes \mathcal{U})(\Delta(\xi)) \,. \tag{3.36}$$

3. *Each equivalence class of irreducible representations in $\mathcal{R}ep'$ (i.e. locally generated sector with finite statistics) is realized as a subrepresentation of π.*

For these results the existence of an antipode $\mathscr{S}$ was superfluous. The antipode is used to define a covariant adjoint of field operators (Proposition 5). Recall that a controlled transformation behaviour of adjoint fields requires in addition an element φ satisfying (2.16). It will be argued later that the assumptions of the preceding theorem suffice to construct such intertwiners φ without any further input.

We note that operators $\pi(A)$ and $\mathscr{U}(\xi)$ with $A \in \mathscr{A}(\mathcal{O})$ and ζ in the center of $\mathscr{G}^*$ generate an invariant subalgebra of $\mathscr{B}(\mathcal{O})$ (i.e. elements commute with $\mathscr{U}(\xi)$ for all $\xi \in \mathscr{G}^*$). This implies that not all invariants in the algebra $\mathscr{B}(\mathcal{O})$ are observables of the model. A detailed discussion is given below.

For chiral conformal quantum field theories, field operators can be obtained in the same way. It was explained above that the corresponding endomorphisms ρ_J act on an algebra $\mathscr{A}_\zeta$ which depends on the choice of the "point at infinity" $\zeta \in \mathbf{S}^1$. This dependence on ζ does also show up in the field operators $\Psi^J_m(\rho_J)$ and the algebras of fields $\mathscr{B}^\zeta(I)$. If ζ is changed, field operators are multiplied with a unitary element from the center of the universal algebra $\mathscr{A}_{\text{univ}}$. This means that fields do not "live" on the circle $\mathbf{S}^1$ but on a covering thereof. The same behaviour is found for field operators which create charges localized along strings in a three-dimensional Minkowski space. For an enlightening discussion of these points the reader is referred to [25].

The field operators $\Psi^J_m(\rho_J)$ are localized in the localization domain of ρ_J. One may construct operators $\Psi^J_m(x, t)$ associated with a point (x, t) by taking appropriate limits. If ρ^α_J is a sequence of endomorphisms from the equivalence class of ρ_J such that the localization region shrinks to a point (x, t) in the limit $\alpha \to \infty$, $\Psi^J_m(x, t)$ is obtained formally as

$$\Psi^J_m(x, t) = \lim_{\alpha \to \infty} \mathscr{N}_{\rho^\alpha_J} \Psi^J_m(\rho^\alpha_J) ,$$

where $\mathscr{N}_{\rho^\alpha_J}$ are suitable normalization factors. This procedure has been successfully applied to charged fields of the $U(1)$-current algebra on the circle $\mathbf{S}^1$ [11] and there is much hope to develop a general technique for chiral conformal quantum field theories [48, 30].

4. Weak Quasi-Quantum Groups

The formulation of quantum symmetry in Sect. 2 involved only a bi-*-algebra structure. One cannot expect that every bi-*-algebra is actually realized as a quantum symmetry of a quantum mechanical system. It this point it should suffice to remark that only group symmetries seem to be realized in higher dimensional quantum field theory. We will elucidate the reasons later.

To describe distinguished algebraic structures within the class of bi-*-algebras we introduce and discuss some relevant notions. Weak quasi-quantum groups will be defined. They were introduced in [56] as a generalization of Drinfeld's quasi-quantum groups [19]. References to our physical framework have been avoided to emphasize the purely mathematical nature of the arguments. Our presentation is restricted to those parts of the theory which are needed later.

4.1. Fundamental Definitions and Results. In a bi-algebra, tensor products of representations are defined with the help of the coproduct Δ: $\mathscr{G}^* \mapsto \mathscr{G}^* \otimes \mathscr{G}^*$.

Properties of the tensor product (2.10) of representations can be traced back to properties of the co-product. As an example consider the group algebra associated with a compact group G. In this case, the tensor product (2.1) of representations is well known to be associative and commutative. This corresponds to a co-commutative and co-associative co-product Δ_G. A co-product Δ is called co-associative, if

$$(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\,, \tag{4.1}$$

and co-commutativity means that

$$\Delta(\xi) = \Delta'(\xi)\,. \tag{4.2}$$

Given the expansion $\Delta(\xi) = \sum \xi^1_\sigma \otimes \xi^2_\sigma$, Δ' is defined by $\Delta'(\xi) = \sum \xi^2_\sigma \otimes \xi^1_\sigma$. For Δ_G both properties can be verified from the explicit action (2.13) of Δ_G on elements in G. In the following we introduce a special class of bi-algebras for which tensor products of representations are at least commutative and associative *up to equivalence.*

Definition 9 (Quasi-co-associativity) [19, 56]. *The co-product Δ of a bi-*-algebra with antipode $(\mathcal{G}^*, \Delta, \varepsilon, *, \mathcal{S})$ is called* **quasi-co-associative**, *if an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ exists, such that*

1. *φ has a quasi-inverse $\varphi^{-1} \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ such that*
$$\varphi\varphi^{-1} = (id \otimes \Delta)\Delta(e), \quad \varphi^{-1}\varphi = (\Delta \otimes id)\Delta(e)\,. \tag{4.3}$$
2. *φ satisfies the "intertwining" relation*
$$\varphi(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\varphi \quad \text{for all} \quad \xi \in \mathcal{G}^*\,. \tag{4.4}$$
3. *The following* **pentagon equation** *holds:*
$$(id \otimes id \otimes \Delta)(\varphi)(\Delta \otimes id \otimes id)(\varphi) = (e \otimes \varphi)(id \otimes \Delta \otimes id)(\varphi)(\varphi \otimes e)\,. \tag{4.5}$$
4. *φ and the co-unit ε satisfy $(id \otimes \varepsilon \otimes id)(\varphi) = \Delta(e)$.*
5. *There exist elements α, β which satisfy relations (2.6 f.) together with the "normalization"*
$$\sum \mathcal{S}(\varphi^1_\sigma)\alpha\varphi^2_\sigma\beta\mathcal{S}(\varphi^3_\sigma) = e = \sum \phi^1_\sigma\alpha\mathcal{S}(\phi^2_\sigma)\beta\phi^3_\sigma\,, \tag{4.6}$$
where *ϕ^i_σ are defined through the expansion $\varphi^{-1} = \sum \phi^1_\sigma \otimes \phi^2_\sigma \otimes \phi^3_\sigma$*
6. *φ is unitary, i.e. $\varphi^* = \varphi^{-1}$.*

An element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ with these properties is called a **re-associator**.

Drinfel'd introduced the notion of quasi-co-associativity in [19] for the case without truncation, viz. $\Delta(e) = e \otimes e$. Without truncation, φ^{-1} is a true inverse of φ. Let us discuss the meaning of this definition in terms of representation theory. Consider tensor products of three representations τ^i, $i = 1, 2, 3$. Due to the freedom in placing the brackets, there exist two different ways to perform threefold tensor products, $(\tau^1 \boxtimes \tau^2) \boxtimes \tau^3$ *and* $\tau^1 \boxtimes (\tau^2 \boxtimes \tau^3)$. They are constructed from the two different combinations of the co-product, $(\Delta \otimes id)\Delta$ and $(id \otimes \Delta)\Delta$. If Δ is quasi-co-associative in the sense of the above definition, the two threefold tensor products are unitary equivalent. Indeed it follows from Eq. (4.3, 4.4) and the last item that φ furnishes an unitary intertwiner $(\tau^1 \otimes \tau^2 \otimes \tau^3)(\varphi)$. Definition 9.3 expresses equality of two intertwiners between fourfold tensor products of representations. The name derives from the fact that the equation describes commutativity of a pentagon shaped diagram, in which the edges are indexed with the five factors of the equation. The relation in 4 is consistent since $(\tau^1 \boxtimes \varepsilon) \boxtimes \tau^2 = \tau^1 \boxtimes \tau^2 = \tau^1 \boxtimes (\varepsilon \boxtimes \tau^2)$ by triviality of ε. Corresponding equations for the other components of φ follow with the help of the pentagon equation. In particular we will need the relation

$$(id \otimes id \otimes \varepsilon)(\varphi) = \Delta(e) . \tag{4.7}$$

The definition above has another highly nontrivial consequence. Given a co-product Δ and an antipode $\mathscr{S}$ the combination $(\mathscr{S} \otimes \mathscr{S})\Delta'(\mathscr{S}^{-1}(\xi))$ defines again a homomorphism $\mathscr{G}^* \mapsto \mathscr{G}^* \otimes \mathscr{G}^*$. The latter turns out to be equivalent to the co-product Δ, if Δ is quasi-co-associative [19]. To make this statement more precise we introduce the following notation:

$$\gamma = \sum \mathscr{S}(U_\sigma)\alpha V_\sigma \otimes \mathscr{S}(T_\alpha)\alpha W_\sigma ,$$

$$\text{with} \quad \sum T_\sigma \otimes U_\sigma \otimes V_\sigma \otimes W_\sigma = (\varphi \otimes e)(\Delta \otimes id \otimes id)(\varphi^{-1}) ,$$

$$f = \sum (\mathscr{S} \otimes \mathscr{S})(\Delta'(\phi^1_\sigma))\gamma\Delta(\phi^2_\sigma \beta \mathscr{S}(\phi^3_\sigma)) ,$$

$$\text{with} \quad \phi = \varphi^{-1} = \sum \phi^1_\sigma \otimes \phi^2_\sigma \otimes \phi^3_\sigma . \tag{4.8}$$

Drinfel'd proved in [19] that the element f satisfies

$$f\Delta(\xi)f^{-1} = (\mathscr{S} \otimes \mathscr{S})\Delta'(\mathscr{S}^{-1}(\xi)) \quad \text{for all} \quad \xi \in \mathscr{G}^* ,$$

$$\gamma = f\Delta(\alpha) . \tag{4.9}$$

The first equation asserts that f furnishes an intertwiner between the coproduct Δ and the combination of Δ and $\mathscr{S}$ on the right-hand side. In terms of representation theory, it can be restated as

$$(\bar\tau^I \boxtimes \bar\tau^J)(\xi)(\bar\tau^I \boxtimes \bar\tau^J)(f^*) = (\bar\tau^I \otimes \bar\tau^J)(f^*)(\tau^J \boxtimes \tau^I)^-(\xi) . \tag{4.10}$$

Here $(\tau^J \boxtimes \tau^I)^-$ is the conjugate of $\tau^J \boxtimes \tau^I$.

Definition 10 (*Quasi-triangularity*) [19, 56]. *A bi-*-algebra with antipode* ($\mathscr{G}^*$, Δ, ε, $*$, $\mathscr{S}$) *with quasi-co-associative co-product* Δ *and re-associator* φ *is called* **quasi-triangular**, *if there exists* $R \in \mathscr{G}^* \otimes \mathscr{G}^*$ *such that*

1. *R has a quasi-inverse $R^{-1} \in \mathscr{G}^* \otimes \mathscr{G}^*$ such that*

$$RR^{-1} = \Delta'(e), \quad R^{-1}R = \Delta(e) . \tag{4.11}$$

2. *R satisfies the "intertwinning relation"*

$$R\Delta(\xi) = \Delta'(\xi)R \quad \textit{for all} \quad \xi \in \mathscr{G}^* . \tag{4.12}$$

3. *R is unitary in the sense that $R^* = R^{-1}$.*
4. *The following hexagon equations are fulfilled:*

$$(id \otimes \Delta)(R) = \varphi^{-1}_{231} R_{13} \varphi_{213} R_{12} \varphi^{-1} ,$$

$$(\Delta \otimes id)(R) = \varphi_{312} R_{13} \varphi^{-1}_{132} R_{23} \varphi . \tag{4.13}$$

We used the standard notation. If $R = \sum r_a^1 \otimes r_a^2$, *then* $R_{13} = \sum r_a^1 \otimes e \otimes r_a^2$, *etc. Given the expansion* $\varphi = \sum \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ *and any permutation s of* 123 *we set* $\varphi_{s(1)s(2)s(3)} = \sum_\sigma \varphi_\sigma^{s^{-1}(1)} \otimes \varphi_\sigma^{s^{-1}(2)} \otimes \varphi_\sigma^{s^{-1}(3)}$.

The discussion of this definition parallels the one given for Definition 9. Quasi-triangularity implies that the two representations $\tau^1 \boxtimes \tau^2$ and $\tau^2 \boxtimes \tau^1$ are equivalent. The interwiner is furnished by $(\tau^1 \boxtimes \tau^2)$ (R). Equations $(\varepsilon \otimes id)R = e$, $(id \otimes \varepsilon)R = e$ follow with the help of the hexagon equation. The same holds true for all relations involving the action of the antipode S on components of R.

After these definitions we are prepared to explain the title of this section. A bi-*-algebra with antipode $\mathscr{S}$, re-associator φ and R-element R is called weak quasitriangular quasi-Hopf-*-algebra or simply *weak quasi-quantum group* [56]. They are generalizations of Drinfeld's quasi-quantum groups [19] in which truncation is not allowed, i.e. $\Delta(e) = e \otimes e$. For a quantum group [18, 82, 83], the re-associator φ is trivial, i.e. $\varphi = e \otimes e \otimes e$. In this framework group algebras appear as special examples of quantum groups when $R = e \otimes e$.

Let us discuss some properties of bi-*-algebras with antipode $\mathscr{S}$, re-associator φ and R-element R. The relations stated in Definitions 9, 10 imply validity of quasi-Yang Baxter equations,

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi = \varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}\ . \tag{4.14}$$

and this guarantees that R together with φ determines a representation of the braid group [59]. To state this result we introduce some notations. Write

$$e^n = e \otimes \cdots \otimes e \quad (n \text{ factors})\,, \tag{4.15}$$

and similarly for id^n. In addition we abbreviate $\mathscr{G}^{*\otimes n} = \mathscr{G}^* \otimes \cdots \otimes \mathscr{G}^*$ (n factors), and

$$\Delta^n = (id^{n-1} \otimes \Delta) \cdots (id \otimes \Delta)\Delta \quad \text{for } n \geqq 2\,, \tag{4.16}$$

$$\Delta^1 = \Delta, \quad \Delta^0 = id, \quad \Delta^{-1} = \varepsilon\,. \tag{4.17}$$

Furthermore we introduce the following permutation maps $\mathbf{P}_k^n : \mathscr{G}^{*\otimes n} \mapsto \mathscr{G}^{*\otimes n}$ defined by

$$\mathbf{P}_k^n(\xi_n \otimes \cdots \xi_{k+1} \otimes \xi_k \cdots \otimes \xi_1) = (\xi_n \otimes \cdots \xi_k \otimes \xi_{k+1} \cdots \otimes \xi_1)\,. \tag{4.18}$$

Theorem 11 (*Artin relations*) [58, 59]. *Let* $R^+ = R$ *and* $R^- = R'^{-1}$ *where* $'$ *interchanges factors in* $\mathscr{G}^* \otimes \mathscr{G}^*$. *For* $k = 1, \ldots, n-1$, *define maps* $\sigma_k^{n\pm} : \mathscr{G}^{*\otimes n} \mapsto \mathscr{G}^{*\otimes n}$ *by*

$$\sigma_k^{n\pm} = \Delta^{n-1}(e)\mathbf{P}_k^n(id^{n-k+1} \otimes \Delta^{k-2})(e^{n-k-1} \otimes \varphi_{213}(R^\pm \otimes e)\varphi^{-1})\,. \tag{4.19}$$

Then $\sigma_k^{n\pm}$ *obey Artin relations* (3.14) *and* σ_k^{n-} *is the quasi-inverse of* σ_k^{n+}, *i.e.*

$$\sigma_k^{n+}\sigma_k^{n-} = \Delta^{n-1}(e)\,.$$

The proof can be found in [59]. For the special case in which $\varphi = e \otimes e \otimes e$ and $\Delta(e) = e \otimes e$, the explicit relation to representations of the braid group prompted the discovery of quantum groups and can be used to construct them from solutions of the Yang Baxter equations (see e.g. [62]).

Let τ^I denote a complete set of representatives from the equivalence classes of irreducible representations of $\mathscr{G}^*$. τ^I are assumed to be finite dimensional and unitary representations on the Hilbert spaces V^I, i.e. $\tau^I(\xi^*) = (\tau^I(\xi))^*$. For semisimple $\mathcal{G}^*$, representations $\tau^I \boxtimes \tau^J$ can be decomposed into a direct sum of representations τ^K and this decomposition determines a Hilbert space $\mathcal{C}(IJ, K)$ of Clebsch Gordon intertwiners $C(IJ\,|\,K): V^I \otimes V^J \to V^K$ as in Sect. 3.3.

When a re-associator φ and an element R exist, the tensor product of representations is associative and commutative up to equivalence so that the dimensions $v_K^{IJ} = dim\mathcal{C}(IJ\,|\,K)$ furnish a solution of Eqs. (3.12). To describe the action of φ and R on the Clebsch Gordon intertwiners, we fix a set of complex phases $\omega^{IJ} = \overline{\omega^{IJ}}$ such that

$$\omega^{IJ}\omega^{IK} = \omega^{IL}, \quad \omega^{JI}\omega^{KI} = \omega^{LI} \tag{4.20}$$

whenever τ^L is a subrepresentation of $\tau^J \boxtimes \tau^K$. Note that one can always choose $\omega^{IJ} = 1$ for all I, J. Due to the intertwining properties (4.4, 4.12) of φ, R and Schur's lemma, φ, R determine a set of complex matrices Φ, ω defined by

$$C(IP\,|\,L)C(JK\,|\,P)_{23}\varphi^{IJK} = \sum_Q \Phi_{PQ}\begin{bmatrix} I & J \\ L & K \end{bmatrix}^* C(QK\,|\,L)C(IJ\,|\,Q)_{12}\,, \tag{4.21}$$

$$C(IJ\,|\,K)\hat{R}^{+IJ} = \omega^{IJ}\omega\begin{pmatrix} & J & \\ K & & I \end{pmatrix}C(JI\,|\,K)\,. \tag{4.22}$$

Here $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$, $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$ with $P: V^I \otimes V^J \mapsto V^J \otimes V^I$ the permutation map. The properties of R, φ give rise to relations among the matrices Φ, ω. A short calculation shows that they satisfy the polynomial equations (in Proposition 7) when Φ substitutes for F and the matrix ω appears in place of Ω.

Conversely we can start from a bi-*-algebra $\mathcal{G}^*$ with antipode with an associative and commutative set of dimensions $v_K^{IJ} = dim\mathcal{C}(IJ\,|\,K)$. Note that the dimensions of the matrices, F, Ω are the only parameters in the polynomial equations. In Proposition 7 these dimensions were given by the fusion rules N_K^{IJ}. However, it is realized immediately that every solution of (3.12) – in particular v_K^{IJ} – determines a set of polynomial equations. Our aim is to construct elements φ and R from a known solution of these polynomial equations. We will succeed for semisimple algebras $\mathcal{G}^*$.

Theorem 12 (*Reconstruction theorem*). *Let $\mathcal{G}^*$ be a semisimple bi-*-algebra with antipode. Suppose that the multiplicities v_K^{IJ} which appear in the Clebsch Gordon decomposition*

$$\tau^I \boxtimes \tau^J \cong \bigoplus v_K^{IJ}\tau^K$$

are commutative and associative in the sense of Eq. (3.12) *and that a solution of Eq.* (4.20) *has been fixed. Then every solution of the polynomial equations* (*Proposition* 7) *associated with v_K^{IJ} determines a pair of elements φ, R with properties as in Definition* 9, 10. *Their action on Clebsch Gordon maps is given by Eq.* (4.21, 4.22).

Proof. The proof of this theorem consists of two parts. First one has to show that for a given set of φ^{IJK}, $\hat{R}^{+IJ}$ defined by (4.21, 4.22) one can always find elements $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ and $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)\varphi$, $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$. We do this for R. Let M_K be the full matrix algebra that consists of $(dim(\tau^K) \times dim(\tau^K))$-matrices. Since $\mathcal{G}^*$ is semisimple, $\tau^K(\mathcal{G}^*) = M_K$ for all irreducible representations τ^K. By definition $P\hat{R}^{+IJ} \in M_J \otimes M_I$, so that $P\hat{R}^{+IJ}$ is a sum of tensor products of matrices, $P\hat{R}^{+IJ} = \sum_\sigma m_\sigma^1 \otimes m_\sigma^2, m_\sigma^1 \in M_J, m_\sigma^2 \in M_I$. We can find elements $s^i_\sigma \in \mathscr{G}^*$, $i = 1, 2$, such that $\tau^I(s^1_\sigma) = m^1_\sigma$ and $\tau^I(s^2_\sigma) = m^2_\sigma$. Take $S^{JI} \in \mathscr{G}^*$ to be the element $P^J \otimes P^I \sum s^1_\sigma \otimes s^2_\sigma$ (P^I is the minimal central projection corresponding to the irreducible representation τ^I) and repeat this construction for every pair (I, J) of representations. Finally, $R = \sum_{IJ} S^{IJ}$ satisfies $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$. The second part of the proof is to show that the elements φ, R satisfy all the relations in Definition (9, 10). This is an immediate consequence of the polynomial equations (Proposition 7). Proving the normalizations (4.6) involves an appropriate choice of α, β by exploiting the transformations $\alpha \mapsto \zeta\alpha$ described after relations (2.7). The quasi-inverses act according to

$$C(QK|L)C(IJ|Q)_{12}(\varphi^{-1})^{IJK} = \sum_P \Phi_{PQ}\left[\begin{smallmatrix} I & J \\ L & K \end{smallmatrix}\right] C(IP|L)C(JK|P)_{23}\,, \tag{4.23}$$

$$C(JI|K)\hat{R}^{-IJ} = \omega^{JI}\omega({}_K{}^J{}_I)C(IJ|K)\,, \tag{4.24}$$

with $(\varphi^{-1})^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi^{-1})$, $\hat{R}^{-IJ} = (\tau^J \otimes \tau^I)(R^{-1})P$ and $P: V^I \otimes V^J \mapsto V^J \otimes V^I$ the permutation map.

The idea to reconstruct elements φ and R from a solution of the polynomial equations appeared first in [57]. There it was done in a concrete example. In the language of categories, a similar observation was formulated by Majid [63] and extended to cases with truncation by Kerler in [50].

4.2. Construction of Weak Quasi-Quantum Groups. Looking for examples of weak quasi-quantum groups, the last theorem in the preceding subsection gives a simple strategy. Since solutions of the polynomial equations are known for many fusion rules v^{IJ}_K, we obtain a weak quasi-quantum group whenever we are able to find a suitable bi-*-algebra with antipode (in the sense of Theorem 12). The construction of bi-*-algebras with antipode with prescribed multiplicities v^{IJ}_K of the Clebsch Gordon decomposition is actually a simple algebraic problem. It can be solved if $\delta_I\delta_J \geqq \sum v^{IJ}_K \delta_K$ has a positive integer solution.

As usual, the multiplicities v^{IJ}_K are assumed to be associative and commutative in the sense of Eq. (3.12). We suppose that there is a unique label 0 with the property $v^{I0}_J = \delta_{I,J}$ and that – with respect to 0 – every label J has a conjugate $\bar{J}$. The latter is distinguished by the property $v^{IJ}_0 = \delta_{I\bar{J}}$.

In order to start our construction we choose a set of finite dimensions $\delta_J \geqq 1$ such that $\delta_0 = 1$ and $\delta_J = \delta_{\bar{J}}$. V^J should denote a δ_J-dimensional Hilbert space and e_J the unit operator on V^J. V^0 is spanned by one normalized vector $|e_0\rangle$ with dual $\langle e_0|$.

For certain choices of the dimensions δ_J we can hope to find a family of linear maps $C^a(IJ|K): V^I \otimes V^J \mapsto V^K$ ($a = 1 \cdots N^{IJ}_K$) which satisfies the following equations:

$$C^a(IJ|K)C^b(IJ|L)^* = \delta_{K,L}\delta_{a,b}e_K\,, \tag{4.25}$$

$$C(I0|I) = e_I = C(0I|I)\,, \tag{4.26}$$

$$C(\bar{I}I|0)_{23}C(I\bar{I}|0)^*_{12} = \kappa_I^{-1}e_I\,. \tag{4.27}$$

Here, the constants κ_I are supposed to be real numbers. We introduce the Hilbert space $V = \bigoplus_J V^J$ and extend e_J and $C^a(IJ|K)$ to linear maps on V, resp. $V \otimes V$ by $e_I V^J = e_I \delta_{I,J}$ etc..

Proposition 13. *Suppose that Eqs. (4.25, 4.26) can be solved by a family $C^a(IJ\,|\,K)$. Define a *-algebra $\mathscr{G}^*$ which consists of maps $\xi : V \mapsto V$ as follows:*

$$\mathscr{G}^* = \{\xi : V \mapsto V \,|\, \xi : V^I \mapsto V^I \text{ for all } I \in \mathscr{I}\}\ .$$

*The *-operation on $\mathscr{G}^*$ is given by the usual adjoint of maps $\xi : V \mapsto V$. There exist a co-product Δ, a co-unit ε and an antipode $\mathscr{S}$ which enjoy the usual properties. Explicitly they act on elements $\xi \in \mathscr{G}^*$ as*

$$\Delta(\xi) = \sum C^a(IJ\,|\,K)^* \xi C^a(IJ\,|\,K)\ , \tag{4.28}$$

$$\varepsilon(\xi) = \langle e_0 \,|\, \xi \,|\, e_0 \rangle\ , \tag{4.29}$$

$$\mathscr{S}(\xi) = \sum_I \kappa_I C(\bar{I}I\,|\,0)_{23}(e_I \otimes \xi \otimes e_I) C(I\bar{I}\,|\,0)^*_{12}\ . \tag{4.30}$$

Proof. Most of the properties are obvious. We shall give only some of the calculations and omit the others. Orthonormality (4.25) of $C^a(IJ\,|\,K)$ is needed for the co-product $\Delta : \mathscr{G}^* \mapsto \mathscr{G}^*$ to become a homomorphism,

$$\begin{aligned} \Delta(\xi)\Delta(\eta) &= \sum C^a(IJ\,|\,K)^* \xi C^a(IJ\,|\,K) \sum C^b(LM\,|\,N)^* \eta C^b(LM\,|\,N) \\ &= \sum C^a(IJ\,|\,K)^* \xi \delta_{a,b} \delta_{K,N} e_K \eta C^b(IJ\,|\,N) = \Delta(\xi\eta)\ . \end{aligned}$$

$\Delta(\xi)^* = \Delta(\xi^*)$ and the properties of ε are trivial. The normalization (4.26) is used to obtain

$$(\varepsilon \otimes id)\Delta(\xi) = \sum C(0J\,|\,J)^* \xi C(0J\,|\,J) = \xi\ . \tag{4.31}$$

To prove that $\mathscr{S} : \mathscr{G}^* \mapsto \mathscr{G}^*$ is an anti-homomorphism we note that the definition of $\mathscr{S}$ is equivalent to

$$C(I\bar{I}\,|\,0)(e \otimes \mathscr{S}(\xi)) = C(I\bar{I}\,|\,0)(\xi \otimes e)\ . \tag{4.32}$$

Consequently, the action of $\mathscr{S}$ on the product $\xi\eta$ can be evaluated according to

$$\begin{aligned} \mathscr{S}(\xi\eta) &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{23}(e_I \otimes \xi\eta \otimes e_I) C(I\bar{I}\,|\,0)^*_{12} \\ &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{23}(e_I \otimes \eta \otimes \mathscr{S}(\xi)) C(I\bar{I}\,|\,0)^*_{12} \\ &= \mathscr{S}(\eta)\mathscr{S}(\xi)\ . \end{aligned}$$

The behaviour of $\mathscr{S}$ with respect to the *-operation is checked as folows:

$$\begin{aligned} \mathscr{S}(\xi)^* &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{12}(e_I \otimes \eta^* \otimes e_I) C(I\bar{I}\,|\,0)^*_{23} \\ &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{12}(\mathscr{S}^{-1}(\xi^*) \otimes e_I \otimes e_I) C(I\bar{I}\,|\,0)^*_{23} \\ &= \mathscr{S}^{-1}(\xi^*) \sum_I \kappa_I \kappa_I^{-1} = \mathscr{S}^{-1}(\xi^*)\ . \end{aligned}$$

It remains to prove compatibility with the co-product. To do this we start again from Eq. (4.32):

$$\begin{aligned} \mathscr{S}(\xi^1_\sigma)\xi^2_\sigma &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{23} C(I\bar{I}\,|\,0)^*_{12} \mathscr{S}(\xi^1_\sigma)\xi^2_\sigma \\ &= \sum_I \kappa_I C(\bar{I}I\,|\,0)_{23}(e_I \otimes \Delta(\xi)) C(I\bar{I}\,|\,0)^*_{12} \end{aligned}$$

$$= \sum_I \kappa_I \varepsilon(\xi) C(\bar{I}I\,|\,0)_{23} C(I\bar{I}\,|\,0)^*_{12}$$

$$= \varepsilon(\xi) \,. \tag{4.33}$$

A similar calculation gives $\xi^1_\sigma \mathscr{S}(\xi^2_\sigma) = \varepsilon(\xi)$. We see that in these examples $\alpha = \beta = e$. Trivality of α, β is related to the condition (4.27). If κ_I is allowed to become a more general invertible map $V^I \mapsto V^I$, one has to encounter nontrivial α, β.

Irreducible unitary representations τ^J of $\mathscr{G}^*$ are obtained by restriction to V^J,

$$\tau^J(\xi) \equiv \xi\,|_{V^J} \,.$$

The tensor product $\tau^I \boxtimes \tau^J$ acts on $V^I \otimes V^J$ by

$$(\tau^I \boxtimes \tau^J)(\xi) = \sum_{K,a} C^a(IJ\,|\,K)^* \tau^K(\xi) C^a(IJ\,|\,K),$$

so that $C^a(IJ\,|\,K)$ have been identified as Clebsch Gordon intertwiners. By construction, $dim(\mathscr{C}(IJ\,|\,K)) = v^{IJ}_K$. We conclude that our initial problem is solved by the bi-*-algebra with antipode $(\mathscr{G}^*, \Delta, \varepsilon, *, \mathscr{S})$ in Proposition 13, provided it exists.

The above proposition reduces the problem of finding an appropriate bi-*-algebra with antipode to the solution of equations (4.25 f.). Of course such solutions will only exist for special choices of the dimensions δ_J. A necessary condition on δ_J can easily be derived from the orthonormality of $C^a(IJ\,|\,K)$ (4.25). It means that the $C^a(IJ\,|\,K)$ map vectors in the $\delta_I\delta_J$-dimensional Hilbert space $V^I \otimes V^J$ onto an orthogonal sum of δ_K-dimensional spaces which occur with a multiplicity N^{IJ}_K depending on K. The total dimension $\sum_K N^{IJ}_K \delta_K$ of the image cannot exceed the dimension $\delta_I\delta_J$ of $V^I \otimes V^J$,

$$\delta_I \delta_J \geqq \sum_K N^{IJ}_K \delta_K \,. \tag{4.34}$$

The first part in the proof of the following corollary shows that condition (4.34) is also sufficient.

Corollary 14. *Suppose that the number of labels I is finite and that the "fusion rules" v^{IJ}_K satisfy the standard assumptions. Then there is a semisimple bi-*-algebra $\mathscr{G}^*$ with antipode and irreducible unitary representations τ^I such that*

$$dim(\mathscr{C}(\tau^I\tau^J\,|\,\tau^K)) = v^{IJ}_K \,. \tag{4.35}$$

Proof. Let us first suppose that a solution of (4.34) exists. One starts by constructing maps $C(I\bar{I}\,|\,0)$: $V^I \otimes V^I \mapsto V^0$ with the property that $C(I\bar{I}\,|\,0)_{23} C(\bar{I}I\,|\,0)^*_{12}$ is proportional to e_I. They will satisfy relations (4.25, 4.27) after an appropriate normalization. A complete set of solutions of (4.25, 4.26, 4.27) can then be obtained by the usual orthonormalization procedure. Relation (4.34) guarantees that sufficiently many linear independent maps exist.

For a finite number of equivalence classes of irreducible representations, a solution of (4.34) is furnished by $\delta_0 = 1$ and $\delta_K = \delta = \max_{IJ}(\sum_K N^{IJ}_K)$, $J \neq 0$. This proves the corollary.

In the context of this subsection, Corollary 14 asserts the existence of (nontrivial) weak quasi-quantum groups. It should be remarked that it also answers the question which was raised by the assumption (3.24) at the beginning of Sect. 3.3., namely the question whether a bi-*-algebra $\mathscr{G}^*$ with antipode suitable to construct covariant field operators $\Psi_i^J(\rho_J)$ does exist. As is seen from Corollary 14, one can find appropriate bi-*-algebra with antipode for all rational models.

Example. Let us describe at least one explicit solution of Eq. (4.25 f) for the fusion rules N_K^{IJ} of the chiral critical Ising model. They can be found in Sect. 7. Since the dimensions $\delta_0 = 1$, $\delta_{1/2} = 2$, $\delta_1 = 1$ satisfy the condition (4.34), we know that a bi-*-algebra with antipode exists for these dimensions. A list of corresponding Clebsch Gordon maps which satisfy Eqs. (4.25, 4.26, 4.27) is given by (with **1** the two-dimensional unit matrix)

$$C(00\,|\,0) = 1, \qquad C(\tfrac{1}{2}\tfrac{1}{2}\,|\,0) = \tfrac{1}{\sqrt{2}}(1\;0\;0\;1)\,,$$
$$C(0\tfrac{1}{2}\,|\,\tfrac{1}{2}) = \mathbf{1}, \qquad C(11\,|\,0) = 1\,,$$
$$C(01\,|\,1) = 1, \qquad C(\tfrac{1}{2}\tfrac{1}{2}\,|\,1) = \tfrac{1}{\sqrt{2}}(1\;0\;0\;-1)\,,$$
$$C(\tfrac{1}{2}0\,|\,\tfrac{1}{2}) = \mathbf{1}, \qquad C(10\,|\,1) = 1\,,$$
$$C(\tfrac{1}{2}1\,|\,\tfrac{1}{2}) = \mathrm{diag}(1,\,-1), \qquad C(1\tfrac{1}{2}\,|\,\tfrac{1}{2}) = \mathrm{diag}(1,\,-1)\,.$$

Here $(1\;0\;0\;0) = (1\;0) \otimes (1\;0)$, $(0\;1\;0\;0) = (1\;0) \otimes (0\;1)$, etc. A solution of the polynomial equations associated with the Ising fusion rules N_K^{IJ} is well known and determines a re-associator φ and a R-matrix R according to Eqs. (4.21, 4.22). So we obtain the (smallest non-trivial) example of a weak quasi-quantum group.

5. Quantum Symmetry, Statistics and Locality

In quantum field theory, permutation group statistics is implemented through quadratic relations among the field operators, viz. canonical (anti-)commutation relations for Bosons (Fermions). The spin statistics theorem states that Fermions have spin $s = \frac{1}{2}, \frac{3}{2}, \ldots$, whereas Bosons have integer spin. More general values for the spin (remember that the spin labels representations of the rotation group, e.g. SO(2) in 2 space dimensions) are possible in low dimensional quantum field theory. They are associated with braid group statistics. It has been proposed to implement braid group statistics through local braid relations [32],

$$\Psi_i^I(x,t)\,\Psi_j^J(y,t) = \omega^{IJ}\Psi_l^J(y,t)\,\Psi_k^I(x,t)\mathscr{R}^{IJ>}_{kl,\,ij} \quad \text{for} \quad x > y\,. \tag{5.1}$$

Here, ω^{IJ} are complex phase factors. As in Sect. 2, the meaning of $x > y$ depends on the dimension of the space-time on which the theory lives. In contrast with [32] we do not restrit the $\mathscr{R}$-matrix to have **C**-number entries, but the matrix elements may take values in $\mathscr{U}(\mathscr{G}^*)$ instead. For the order $x < y$ a similar relation follows with $\mathscr{R}^{JI<}$ and $\omega^{JI} = \overline{\omega^{IJ}}$. Denoting with $\hat{\mathscr{R}}$ the matrix obtained from $\mathscr{R}$ by interchange of the first indices k, l, the two matrices $\mathscr{R}^{IJ>}$ and $\mathscr{R}^{JI<}$ obey

$$\hat{\mathscr{R}}^{IJ<} = (\hat{\mathscr{R}}^{JI>})^{-1}\,.$$

Note that for $\mathscr{R}^{IJ>}_{kl,\,ij} = 1$ and $\omega^{IJ} = \pm$, we recover Bose/Fermi-commutation relations as a special case of Eq. (5.1).

Consistently of local braid relations with the transformation law (2.15) distinguishes weak quasi-quantum groups from arbitrary bi-*-algebra symmetries [56]. We will review these arguments here. In the second subsection we determine a re-associator φ and a R-element R for the quantum symmetry of the algebra of fields $\mathscr{B}$ constructed in Sect. 3.3. Local braid relations of the field operators (3.34) and their covariant adjoints will be established with $\mathscr{R}^{IJ\gtrless}_{kl,ki}$ furnished by the elements φ, R.

5.1. Weak Quasi-Quantum Group Symmetry and Local Braid Relations. In a quantum theory with quantum symmetry, the matrix-elements $\mathscr{R}^{IJ\gtrless}_{kl,ij}$ are not free to take any values. Consistency of (5.1) with the transformation law of field operators and associativity of the product of operators constrain $\mathscr{R}$. If the quantum symmetry is quasi-co-associative and quasi-triangular with re-associator φ and R-element R, a solution of these constraints is given by

$$\mathscr{R}^{IJ\gtrless}_{kl,ij} = (\tau^I \otimes \tau^J \otimes \mathscr{U})_{kl,ij}(\varphi_{213}(R \otimes e)\varphi^{-1})$$

$$= \sum \tau^I_{ki}(\varphi^2_\sigma r^1_a \phi^1_\tau)\tau^J_{lj}(\varphi^1_\sigma r^2_a \phi^2_\tau)\mathscr{U}(\varphi^3_\sigma \phi^3_\tau) , \tag{5.2}$$

where we use the same notations as in the preceding section and $\phi = \varphi^{-1}$. From the second line we read off that the matrix elements $\mathscr{R}^{IJ\gtrless}_{kl,ij}$ are not numbers in general but operators in the Hilbert space. The expression is a linear combination of representation operators $\mathscr{U}(\varphi^3_\sigma \phi^3_\tau)$ and is therefore equal to a representation operator $U(\eta)$ of some element $\eta \in \mathscr{G}^*$. A numerical matrix $\mathscr{R}$ is obtained if and only if the re-associator φ is trivial, i.e. if $\varphi = e \otimes e \otimes e$.

To motivate Eq. (5.2) we demonstrate that *local braid relations* (5.1) *with* $\mathscr{R}^{IJ\gtrless}$ *given by* (5.2) *are consistent with the transformation law of fields*, i.e. that both sides of the equation transform in the same way. The products of covariant fields which appear in (5.1) are in general not covariant, if the co-product Δ is not co-associative. Suppose now that there exists a re-associator φ satisfying (4.4). Then one can construct a "covariant product" $\times$ of field operators [56, 58].

Definition 15 (*Covariant product of field operators*). *The covariant product of two multiplets* Ψ^I, Ψ^J *is the multiplet defined by*

$$(\Psi^I \times \Psi^J)_{ij} \equiv \sum \Psi^I_m \Psi^J_n \tau^I_{mi}(\varphi^1_\sigma)\tau^J_{nj}(\phi^2_\sigma)\mathscr{U}(\varphi^3_\sigma) . \tag{5.3}$$

By (4.4), $\Psi^I \times \Psi^J$ *transforms covariantly according to the tensor product representation* $\tau^I \boxtimes \tau^J$. Ordinary products of field operators can be recovered from covariant ones,

$$\Psi^I_i \Psi^J_j = \sum (\Psi^I \times \Psi^J)_{mn} \tau^I_{mi}(\phi^1_\sigma)\tau^J_{nj}(\phi^2_\sigma)\mathscr{U}(\phi^3_\sigma) . \tag{5.4}$$

Using this covariant product, local braid relations (5.1, 5.2) may be rewritten as

$$(\Psi^I \times \Psi^J)_{ij} = \omega^{IJ}(\Psi^J \times \Psi^I)_{lk}(\tau^I_{ki} \otimes \tau^J_{lj})(R) . \tag{5.5}$$

The arguments of the field operators, which we neglect to write, satisfy conditions as in Eq. (5.1). Because of the covariance properties of field operators, both sides transform in the same if the element $R \in \mathscr{G}^* \otimes \mathscr{G}^*$ obeys intertwining property (4.12).

To prepare for our second consistency check one should notice that associativity of the product of field operators is equivalent to the following:

Proposition 16 (*Quasi-associativity of the covariant product*). *The covariant product is quasi-associative in the sense that products* $\Psi^{J_n}_{j_n} \times \cdots \times \Psi^{J_1}_{j_1}$ *with arbitrary positions* *of brackets can be written as a complex linear combination of products* $\Psi^{J_n}_{k_n} \times \cdots \times \Psi^{J_1}_{k_1}$ *with any other specification of brackets. In particular one has*

$$((\Psi^I \times \Psi^J) \times \Psi^K)_{ijk} = (\Psi^I \times (\Psi^J \times \Psi^K))_{i'j'k'} (\tau^I \otimes \tau^J \otimes \tau^K)_{i'j'k', ijk}(\varphi) \, . \tag{5.6}$$

Proof. This proposition is a consequence of the pentagon equation (4.5) for φ.

Consistency requires that the matrix $\mathscr{R}^{IJ>}$ satisfies constraints which come from the possibility of interchanging triples $\Psi^I_i(x,t)\Psi^J_j(y,t)\Psi^K_k(z,t)$, $x > y > z$, of fields in two different ways, leading to the same result where fields in the multiplet Ψ^K appear in the leftmost position followed by Ψ^J and finally Ψ^I to the right. The constraints are exploited most easily if one starts from the threefold $\times$-product $\Psi^I \times (\Psi^J \times \Psi^K)$. By an alternating application of Eq. (5.6) and Eq. (5.5) we end up with $(\Psi^K \times \Psi^J) \times \Psi^I$. There are two ways to perform these manipulations which give the same result, provided that R, φ satisfy the quasi-Yang Baxter equations (4.14).

In our present context Eqs. (4.13) imply validity of local braid relations for composite operators, i.e. when the product $\Psi^{I_1} \times \Psi^{I_2}$ is inserted in place of Ψ^I. More precisely suppose that local braid relations (5.5) hold for the products $\Psi^{I_i}(x_i,t) \times \Psi^J(y,t)$, $i = 1,2$ and $x_i > y$. Then by (4.13) we have the following *local braid relations for composites*:

$$((\Psi^{I_1} \times \Psi^{I_2}) \times \Psi^J)_{i_1 i_2 j} = \omega^{I_1 J}\omega^{I_2 J}(\Psi^J \times (\Psi^{I_1} \times \Psi^{I_2}))_{l k_1 k_2}((\tau^{I_1} \boxtimes \tau^{I_2})_{k_1 k_2 i_1 i_2} \otimes \tau^J_{lj})(R) \, .$$

Similar considerations for the field operator Ψ^J lead to

$$(\Psi^I \times (\Psi^{J_1} \times \Psi^{J_2}))_{i j_1 j_2} = \omega^{I J_1}\omega^{I J_2}((\Psi^{J_1} \times \Psi^{J_2}) \times \Psi^I)_{l_1 l_2 k}(\tau^I_{ki} \otimes (\tau^{J_1} \boxtimes \tau^{J_2})_{l_1 l_2 j_1 j_2})(R) \, .$$

For the proof one uses the quasi-associativity (5.6) of the $\times$ product and local braid relations (5.5) as in the preceding paragraph.

When we defined the covariant conjugation[5] in Sect. 2, we had to make the assumption (2.16) about the existence of an appropriate intertwiner φ. If φ is the re-associator of a weak quasi-quantum group, the covariant conjugation has certain distinguished properties. We state them here without proofs. For details the reader should consult Appendix A. Recall that the covariant adjoint of a multiplet Ψ^I_i transforming covariantly according to some representation τ^I of $\mathscr{G}^*$ was defined by (rel. (2.17))

$$\overline{\Psi}{}^I_i = (\Psi^I_j(\tau^I_{ji} \otimes id)(w))^* \, . \tag{5.7}$$

$\overline{\Psi}{}^I_i$ transforms covariantly according to the representation $\bar\tau^I$ so that its covariant adjoint is again well defined,

$$\overline{\overline{\Psi}}{}^I_i = (\overline{\Psi}{}^I_j(\bar\tau^I_{ji} \otimes id)(w))^* = \Psi^I_i \, . \tag{5.8}$$

The last equality follows with the help of the pentagon equation (4.5) for φ. It states that the covariant conjugation is involutive. Being kind of a substitute for the ordinary adjoint $*$ (for $\varphi = e \otimes e \otimes e$ the covariant adjoint is given by $\overline{\Psi}{}^I = \Psi^{I*}$), we expect the covariant conjugation to be consistent with the product of fields. The precise statement is

$$\overline{(\Psi^I \times \Psi^J)}_{ij} = (\overline{\Psi}{}^J \times \overline{\Psi}{}^I)_{kl}(\bar\tau^J_{kj} \otimes \bar\tau^I_{li})(f^*) \, . \tag{5.9}$$

[5]It is now regarded as a complex anti-linear map which sends every covariant multiplet of fields to its covariant adjoint

The term involving f^* should not surprise. In view of relation (4.10) it can be understood from consistency with the transformation properties of relation (5.9). As a last property of covariant conjugation we want to mention that the adjoints $\bar{\Psi}^I$, $\bar{\Psi}^J$ of covariant fields obey local braid relations

$$\bar{\Psi}^I_i \Psi^J_j = \omega^{IJ} \bar{\Psi}^J_l \bar{\Psi}^I_k (\bar{\tau}^I_{ki} \otimes \bar{\tau}^J_{lj} \otimes \mathscr{U})(\mathscr{R}) , \tag{5.10}$$

if Ψ^I, Ψ^J satisfy (5.1, 5.2).

The local braid relations for composites together with relations (5.10, 5.9) reveal a remarkable *stability of local braid relations with respect to covariant multiplication and conjugation.* This has a major conceptual significance. Supppose we were able to construct a set of "fundamental" covariant fields Ψ^I_i for a given model and we checked that they and their covariant adjoints obey local braid relations. Then we can apply covariant multiplication and conjugation to obtain arbitrary composites. No matter how fancy they are, they will always obey local braid relations. Since $(\varepsilon \otimes id)R = (id \otimes \varepsilon)R = e$, it follows in particular that invariant composites are local observables, i.e. they commute with all fields in the theory which are localized at spacelike distance (provided that $\omega^{0I} = \omega^{I0} = 1$ which holds at least for every solution of (4.20)).

We mentioned that local braid relations are expected to give rise to representations of the braid group B_n on the space of n-particle excitations. When $\mathscr{R}^{IJ>}$ is given by Eq. (5.2), this is indeed the case. Consider states which are created from the ground state $|0\rangle$ by application of n-fold product of field operators $\Psi^{I_k}_{i_k}(x_k, t)$. Suppose that $x_1 < x_2 < \cdots < x_n$, for instance. Operators ζ_k, $(k = 1 \cdots n-1)$ should act on such states by interchange of field operators $\Psi^{I_k}_{i_k}$ and $\Psi^{I_{k+1}}_{i_{k+1}}$,

$$\zeta^n_k \Psi^{I_1}_{i_1} \cdots \Psi^{I_k}_{i_k} \Psi^{I_{k+1}}_{i_{k+1}} \cdots \Psi^{I_n}_{i_n} |0\rangle = \Psi^{I_1}_{i_1} \cdots \Psi^{I_{k+1}}_{i_{k+1}} \Psi^{I_k}_{i_k} \cdots \Psi^{I_n}_{i_n} |0\rangle . \tag{5.11}$$

Using (5.1) and (5.2) one verifies by a short calculation that the action of ζ^n_k is obtained from the maps σ^n_k: $\mathscr{G}^{* \otimes n} \mapsto \mathscr{G}^{* \otimes n}$ as introduced in Theorem 11. Since the latter obey Artin relations (3.9) and the additional factors ω^{IJ} cancel in the calculations, Eq. (5.11) defines a representation of the braid group as it was announced.

5.2. Quantum Symmetry and Local Braid Relations in $\mathscr{B}$. When the algebra of field operators $\mathscr{B}$ with quantum symmetry $\mathscr{G}^*$ was discussed in Sect. 3.3, we did not consider the locality properties of field operators. We will demonstrate now that the field operators $\psi^J_m(\rho_J)$ constructed in Sect. 3.3 satisfy local braid relations among themselves and with their covariant adjoints. We need no extra conditions on the quantum symmetry $\mathscr{G}^*$ than those stated there, namely that the fusion rules of the bi-*-algebra $\mathscr{G}^*$ with antipode. A re-associator φ and a R-element R are constructed from the fusion- and braiding matrix of the quantum field theory. In agreement with the general discussion, they determine the matrix $\mathscr{R}^{IJ}_{kl,ij}$ according to formula (5.2).

Our first aim is to construct appropriate elements φ, R. We use the same notations as above. In Sect. 3.3 we supposed that the dimensions $dim\mathscr{C}(IJ|K)$ coincide with the fusion rules N^{IJ}_K defined by the superselection structure. The latter are commutative and associative in the sense of Eq. (3.12) so that we are now in the position to apply Theorem 12. It asserts that every solution of the polynomial equations (Proposition 7) associated with the fusion rules N^{IJ}_K yields elements φ, R which satisfy the relations in Definitions 9, 10. In the present situation the desired solution is furnished by the fusion- and braiding-matrices, F, Ω of the quantum field theory (Proposition 7). Formulae (4.21) for φ and (4.22) for R take the following explicit form:

$$C(IP\,|\,L)C(JK\,|\,P)_{23}\varphi^{IJK} = \sum_Q F_{PQ}\begin{bmatrix} I & J \\ L & K \end{bmatrix}^* C(QK\,|\,L)C(IJ\,|\,Q)_{12}\,, \tag{5.12}$$

$$C(IJ\,|\,K)\hat{R}^{+IJ} = \omega^{IJ}\Omega({}_K{}^J{}_I)C(JI\,|\,K)\,, \tag{5.13}$$

where $\omega^{IJ} = \overline{\omega^{JI}}$ denotes an arbitrary solution of (4.20) by complex phase factors. The bi-*-algebra $(\mathcal{G}^*, \Delta, \varepsilon, *, \mathcal{S})$ with antipode $\mathcal{S}$, re-associator φ and R-element R is dual to the quantum field theory in the sense of Fröhlich [33].

Let us now turn to the discussion of local braid relations among the field operators $\Psi^J_m(\rho_J)$. The statistics operator $\varepsilon(\rho_J, \rho_K)$ was introduced in (3.8) as an element of the observables algebra $\mathcal{A}$. Thus, the action $\pi(\varepsilon(\rho_J, \rho_K))$ on the Hilbert space $\mathcal{H}$ is well defined. Local braid relations will follow from

$$\Psi^J_j(\rho_J)\Psi^K_k(\rho_K)\pi(\varepsilon(\rho_J, \rho_K)) = \omega^{JK}\Psi^K_m(\rho_K)\Psi^J_n(\rho_J)\mathcal{R}^{JK>}_{nm,jk}\,, \tag{5.14}$$

with $\mathcal{R}$ given by (5.2). To prove this identity we insert Definition (3.34) for the field operators $\Psi^I_m(\rho_I)$ into the left-hand side and use the interwining relation (3.27) and Definition (3.23) of the braiding-matrix B,

$$\Psi^J_j(\rho_J)\Psi^K_k(\rho_K)\pi(\varepsilon(\rho_J, \rho_K)) = \sum i^*_{Ll} T_c({}_L{}^K{}_N) T_d({}_N{}^J{}_I) i_{Ii} B_{MN}\begin{bmatrix} J & K \\ L & I \end{bmatrix}^{cd}_{ab} \begin{bmatrix} J & M & L \\ j & m & l \end{bmatrix}^a \begin{bmatrix} K & I & M \\ k & i & m \end{bmatrix}^b .$$

With the help of (3.23) the braiding matrix B can be expressed in terms of the matrices F, Ω so that it is possible to apply Definition (5.12, 5.13) of φ and $\mathcal{R}$. The result is

$$\Psi^J_j(\rho_J)\Psi^K_k(\rho_K)\pi(\varepsilon(\rho_J, \rho_K)) = \omega^{JK}\sum i^*_{Ll} T_c({}_L{}^K{}_N) T_d({}_N{}^J{}_I) i_{Ii'} \begin{bmatrix} K & N & L \\ m & v & l \end{bmatrix}^c \begin{bmatrix} J & I & N \\ n & i & v \end{bmatrix}^d \tau^I_{i'i}(\mathcal{R}^{IJ>}_{nm,jk})\,.$$

This equals the right-hand side of Eq. (5.14). Since the statistics operators were normalized to 1 for $\rho_J > \rho_K$ (cf. (3.8)), local braid relations follow from Eq. (5.14).

Local braid relations betwen field operators Ψ^I and their covariant adjoints are obtained in two steps. First it is checked by a calculation similar to the previous one that

$$\Psi^J_j(\rho_J)\pi(\varepsilon(\rho_K, \rho_J))(\mathcal{R}^{KJ>})^{-1}_{mn,kj}\Psi^{K*}_k(\rho_K) = \overline{\omega^{KJ}}\,\Psi^{K*}_m(\rho_K)\Psi^J_n(\rho_J)\,. \tag{5.15}$$

Then the lemma in Appendix A establishes the braid relation we are looking for,

$$\Psi^K_k(\rho_K)\bar{\Psi}^J_j(\rho_J) = \omega^{KJ}\bar{\Psi}^J_n(\rho_J)\Psi^K_m(\rho_K)(\tau^K_{mk} \otimes \bar{\tau}^J_{nl} \otimes \mathcal{U})(\varphi_{213}R_{12}\varphi^{-1})\,. \tag{5.16}$$

Here $\rho_K > \rho_J$ has been assumed. Braid relations among conjugate fields follow from the braid relations (5.14) and the result (5.10).

Theorem 17 (*Local braid relations in* $\mathcal{B}$). *Suppose that the fusion rules* N^{IJ}_K *defined by the superselection structure of* $\mathcal{A}$ *coincide with the multiplicities in the Clebsch Gordon decomposition of an* (*otherwise ordinary*) *semisimple bi-*-algebra with antipode* $\mathcal{G}^*$ (*in the sense of* (3.24)). *Then the algebra of field operators* $\mathcal{B} = \mathcal{B}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ *with quantum symmetry* $\mathcal{G}^*$ *constructed in Sect.* 3.3 *has the following properties*:

1. *The field operators* (3.34) *which generate the algebras* $\mathcal{B}(\mathcal{O})$ *satisfy local braid relations*

$$\Psi^I_i(\rho_I)\Psi^J_j(\rho_J) = \omega^{IJ}\Psi^J_l(\rho_J)\Psi^I_k(\rho_I)\mathcal{R}^{IJ>}_{kl,ij} \quad \textit{for} \quad \rho_I > \rho_J\,. \tag{5.17}$$

Analogous relations (5.16) *hold between the field operators and their covariant conjugates.*

2. *The operators* $\mathscr{R}^{IJ>}_{lk,ij}$ *are furnished by elements* $\varphi \in \mathscr{G}^* \otimes \mathscr{G}^* \otimes \mathscr{G}^*$ *and* $R \in \mathscr{G}^* \otimes \mathscr{G}^*$ *which satisfy all axioms in Definition* 9, 10,

$$\mathscr{R}^{IJ>}_{lk,ij} = (\tau^I \otimes \tau^J \otimes \mathscr{U})_{lk,ij}(\varphi_{213} R_{12} \varphi^{-1}) . \tag{5.18}$$

3. *The quantum symmetry* $\mathscr{G}^*$ *is a weak quasi quantum group with re-associator* φ *and R-element R so that braid relations are "stable" under covariant multiplication* (5.3) *and conjugation* (2.17) (*in the sense discussed above*).

Even though this theorem was formulated for quantum field theories on the Minkowski space for sectors which satisfy criterion (3.5), analogous results hold for stringlike localized sectors or quantum fields in chiral conformal quantum field theory.

Local braid relations do not exploit all the nice properties of the field operators $\Psi^J_j(\rho_J)$. It was remarked in [57] that they also satisfy operator product expansions.

Theorem 18 ($\mathscr{G}^*$-*covariant operator product expansions*). *The field operators* (3.34) *satisfy*

$$(\Psi^J(\rho_J) \times \Psi^K(\rho_K))_{jk} = \sum \Psi^M_m(\rho_M) T_b({}_M{}^K{}_J) \begin{bmatrix} J & K & M \\ j & k & m \end{bmatrix}^b . \tag{5.19}$$

This holds for any choice of the morphism ρ_M *within the class* $[\rho_M]$.

Proof. We use the definition (3.34) of the field operators and definition (5.12) of φ to obtain the first equality in

$$\begin{aligned}(\Psi^J(\rho_J) \times \Psi^K(\rho_K))_{jk} &= \sum i^*_{Ll} T_c({}_L{}^J{}_N) T_d({}_N{}^K{}_I) i_{Ii} \left(F_{NM} \begin{bmatrix} J & K \\ L & I \end{bmatrix}^*\right)^{cd}_{ab} \begin{bmatrix} M & I & L \\ m & i & l \end{bmatrix}^a \begin{bmatrix} J & K & M \\ j & k & m \end{bmatrix}^b \\ &= \sum i^*_{Ll} T_a({}_L{}^M{}_I) \rho_I(T_b({}_M{}^K{}_J)) i_{Ii} \begin{bmatrix} M & I & L \\ m & i & l \end{bmatrix}^a \begin{bmatrix} J & K & M \\ j & k & m \end{bmatrix}^b \\ &= \sum \Psi^M_m(\rho_M) T_b({}_M{}^K{}_J) \begin{bmatrix} J & K & M \\ j & k & m \end{bmatrix}^b .\end{aligned}$$

Application of relation (3.15) leads to the second line where we finally insert the definition (3.34) again to obtain the result.

Operator product expansions on the vacuum can be used to convert expectation values of products of field operators into vacuum expectation values of observables. In contrast with [55], the relations (18) hold on the whole Hilbert space $\mathscr{H}$.

In the case of quantum field theories with permutation group statistics, Doplicher and Roberts have established the existence of a group algebra $\mathscr{G}^*$ which satisfies the assumption of Theorem 17. Moreover, they found that the elements R, φ determined by (5.12, 5.13) become trivial, when the phases ω^{IJ} are fixed to be -1 if I, J label sectors with para-Fermi statistics and $+1$ otherwise. Triviality of R, φ means $R = e \otimes e$ and $\varphi = e \otimes e \otimes e$ so that we recover ordinary Bose-/Fermi (anti-) commutation relations from Eq. (5.27) and the covariant conjugation coincides with taking adjoints.

6. The Field Algebra $\mathscr{F}$

Field operators Ψ^J_m and the representation operators $\mathscr{U}(\xi)$ generate the associative $*$-algebras $\mathscr{B}(\mathscr{O}) \subset B(\mathscr{H})$. In quantum field theories with permutation group statistics, one is used to work with field algebras $\mathscr{F} = \mathscr{F}(\mathscr{O})_{\mathscr{O} \in \mathscr{K}}$ which are generated by fields localized in $\mathcal{O}$. The (group) symmetry acts on the algebras $\mathcal{F}(\mathcal{O})$ such that invariants under this action are local, i.e. commute with all spacelike separated fields of the theory. In general, invariant elements within the algebra $\mathcal{B}(\mathcal{O})$ will not have this property, even if they can be expressed purely in terms of field operators localized in $\mathcal{O}$ without any factors $\mathcal{U}(\xi)$. When φ is nontrivial there is no way to conclude that such invariant operators are local with respect to the other fields. Instead these invariants are products of local fields and operators $\mathcal{U}(\zeta)$ with ζ in the center of $\mathcal{G}^*$ [6]. In other words, the associative algebra $\mathcal{B}(\mathcal{O})$ is too large. It is the factor $\mathcal{U}(\zeta)$ which destroys locality (unless $\zeta = 1$). Nevertheless there is a good substitute for the family $\mathcal{F}(\mathcal{O})$ even in the case of braid group statistics. Its definition involves the covariant product $\times$ which was introduced in Definition 15.

Definition 19 (*Field algebra*). *A net of (not necessarily associative) algebras $\mathcal{F}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ with conjugation $^-$ is called* **field algebra** *$\mathcal{F}$ of the model, if there is a linear injective map $\pi:\mathcal{F}\mapsto\mathcal{B}(\mathcal{H})$ and a set of generators ψ^J_j such that $\pi(\psi^I_i) = \Psi^I_i$, $\pi(\bar\psi^I_i) = \bar\Psi^I_i$ and*

$$\pi(\psi^{I_1}_{i_1}\cdot(\cdots\cdot(\psi^{I_{n-1}}_{i_{n-1}}\cdot\psi^{I_n}_{i_n})\cdots)) = (\Psi^{I_1}\times(\cdots\times(\Psi^{I_{n-1}}\times\Psi^{I_n})\cdots))_{i_1\cdots i_{n-1}i_n}\,. \tag{6.1}$$

We say that π is a **representation of the field algebra** *$\mathcal{F}$.*

This definition means that the field operators Ψ^J_m faithfully represent the elements ψ^J_m of the field algebra as linear operators on $\mathcal{H}$. We will see below that the product $\cdot$ in $\mathcal{F}(\mathcal{O})$ inherits its properties from the covariant product $\times$. In particular it is non-associative in general. The representation operators Ψ^J_m can be multiplied with the ordinary associative operator product. The non-associative $\cdot$ product in $\mathcal{F}(\mathcal{O})$ and the associative operator product are related indirectly via the covariant product. Actually, this situation is quite familiar from the representation theory of Lie algebras. The non-associative Lie bracket serves as a perfect analogue of the $\cdot$ product on $\mathcal{F}$. Within a representation of a Lie algebra, representation operators can be multiplied using the associative product of linear maps. The commutator of linear maps expresses the non-associative Lie product in terms of the operator product.

Given the set of representation operators Ψ^J_m and an appropriate element φ to define their covariant products, the field algebra $\mathcal{F}$ is unique and can be constructed explicitly. In a first step we define a family of *field sets* $F(\mathcal{O})\subset\mathcal{B}(\mathcal{H})$ to be the linear span of components of covariant multiplets obtained as a covariant product of the fundamental field operators. In other words, elements in $F(\mathcal{O})$ are linear combinations of

$$\Psi^{I_1,\cdots I_{n-1},I_n}_{i_1,\cdots i_{n-1},i_n} \equiv (\Psi^{I_1}\times(\cdots\times(\Psi^{I_{n-1}}\times\Psi^{I_n})\cdots))_{i_1\cdots i_{n-1}i_n}\,, \tag{6.2}$$

where all field operators are localized in $\mathcal{O}$. Obviously, the operator product of two elements in $F(\mathcal{O})$ will not be in $F(\mathcal{O})$ in general. To obtain a product $\cdot$ on $F(\mathcal{O})$ we make use of the covariant product (5.3). The definition

$$\Psi^{I_1,\cdots I_n}_{i_1,\cdots i_n}\cdot\Psi^{J_1,\cdots J_m}_{j_1,\cdots j_m} = \Psi^{I_1,\cdots I_n,J_1,\cdots J_m}_{i_1,\cdots i_n,j_1,\cdots j_m} \tag{6.3}$$

extends by linearity to $F(\mathcal{O})$ and furnishes a product on the field sets $F(\mathcal{O})$. It follows from relation (5.9) that the covariant conjugation maps the field sets into itself and hence restricts to a conjugation of the field algebra. This concludes the construction of $\mathcal{F}$.

[6] I thank Karl-Henning Rehren for this remark

Next we introduce an action of the quantum symmetry $\mathscr{G}^*$ on $\mathscr{F}(\mathcal{O})$, i.e. we define a linear map $\xi : \mathscr{F}(\mathcal{O}) \mapsto \mathscr{F}(\mathcal{O})$ for every $\xi \in \mathscr{G}^*$ (in an abuse of notation we denote elements in $\mathscr{G}^*$ and the associated linear maps by the same letter ξ),

$$\begin{aligned} &\xi(\psi^{I_1}_{i_1} \cdot (\cdots \cdot (\psi^{I_{n-1}}_{i_{n-1}} \cdot \psi^{I_n}_{i_n}) \cdots)) \\ &\equiv (\psi^{I_1}_{k_1} \cdot (\cdots \cdot (\psi^{I_{n-1}}_{k_{n-1}} \cdot \psi^{I_n}_{k_n}) \cdots))(\tau^{I_1}_{k_1 i_1} \boxtimes (\cdots \boxtimes (\tau^{I_{n-1}}_{k_{n-1} i_{n-1}} \boxtimes \tau^{I_n}_{k_n i_n}) \cdots))(\xi) \,. \end{aligned} \tag{6.4}$$

A tuple (ψ_α) of elements $\psi_\sigma \in \mathscr{F}(\mathcal{O})$ is said to transform covariantly according to the representation τ of $\mathscr{G}^*$ if

$$\xi(\psi_\alpha) = \psi_\beta \tau_{\beta\alpha}(\xi) \,, \tag{6.5}$$

for all $\xi \in \mathscr{G}^*$. $\phi \in \mathscr{F}$ is invariant if it transforms according to the trivial representation ε of $\mathscr{G}^*$, i.e. if $\xi(\phi) = \phi\varepsilon(\xi)$.

The following theorem collects all the important properties of the field algebra $\mathscr{F} = \mathscr{F}(\mathcal{O})_{\mathcal{O} \in \mathscr{K}}$.

Theorem 20 (Properties of the field algebra $\mathscr{F}$).

1. *The algebras $\mathscr{F}(\mathcal{O})$ are quasi-associative in the sense that the product $\psi^{J_n}_{j_n} \cdots \psi^{J_1}_{j_1}$ with arbitrary specification of the position of brackets can be written as a complex linear combination of products $\psi^{J_n}_{k_n} \cdots \psi^{J_1}_{k_1}$ with any other specification of brackets. Re-association is performed with the help of the formulae*

$$((\psi_\alpha \cdot \psi'_\beta) \cdot \psi''_\gamma) = (\psi_\delta \cdot (\psi'_\varepsilon \cdot \psi''_\kappa))(\tau_{\alpha\delta} \otimes \tau'_{\beta\varepsilon} \otimes \tau''_{\gamma\kappa})(\varphi) \,, \tag{6.6}$$

$$(\psi_\delta \cdot (\psi'_\varepsilon \cdot \psi''_\kappa)) = ((\psi_\alpha \cdot \psi'_\beta) \cdot \psi''_\gamma)(\tau_{\delta\alpha} \otimes \tau'_{\varepsilon\beta} \otimes \tau''_{\kappa\gamma})(\varphi^{-1}) \,. \tag{6.7}$$

   They are valid if (ψ_α), (ψ'_β), (ψ''_β) transform according to representations τ, τ', τ'' of $\mathscr{G}^*$.

2. *Quantum fields obey local braid relations, i.e. if $\psi_\alpha \in \mathscr{F}(\mathcal{O}_1)$, $\psi'_\beta \in \mathscr{F}(\mathcal{O}_2)$ transform according to irreducible representations $\tau \cong \tau^J$ and $\tau' \cong \tau^K$ of $\mathscr{G}^*$,*

$$\psi_\alpha \cdot \psi'_\beta = \omega^{JK} \psi'_\gamma \cdot \psi_\delta (\tau_{\delta\alpha} \otimes \tau'_{\gamma\beta})(R) \tag{6.8}$$

   whenever $\mathcal{O}_1 > \mathcal{O}_2$.

3. *$\xi \in \mathscr{G}^*$ acts on the algebras $\mathscr{F}(\mathcal{O})$ as a generalized derivation, i.e.*

$$\xi(\psi \cdot \psi') = \sum_\sigma \xi^1_\sigma(\psi) \cdot \xi^2_\sigma(\psi') \tag{6.9}$$

   for arbitrary $\psi, \psi' \in \mathscr{F}$.

4. *The conjugation $\bar{}$ on $\mathscr{F}(\mathcal{O})$ is involute, $\bar{\bar{\psi}} = \psi$, and satisfies*

$$\overline{(\psi \cdot \psi')}_{\alpha\beta} = (\bar{\psi}' \cdot \bar{\psi})_{\delta\gamma} (\bar{\tau}'_{\delta\beta} \otimes \bar{\tau}_{\gamma\alpha})(f^*) \,, \tag{6.10}$$

   where f denotes the element (4.8) and ψ, ψ' are again assumed to transform covariantly according to the representations τ, τ'.

5. *The algebras $\mathscr{A}(\mathcal{O})$ consist of all invariants in $\mathscr{F}(\mathcal{O})$, i.e.*

$$\mathscr{A}(\mathcal{O}) \cong \{\psi \in \mathscr{F}(\mathcal{O}) \,|\, \xi(\psi) = \psi\varepsilon(\xi)\} \,. \tag{6.11}$$

   On invariants $\phi \in \mathscr{A}(\mathcal{O})$ the covariant conjugation restricts to the adjoint $*$, $\bar{\phi} = \phi^*$.

Observe that the version of local braid relations we used in this theorem implies local braid relations for arbitrary pairs of generating fields ψ^I_j. If the quantum fields ψ_α transforms according to a reducible representation τ, Clebsch Gordon interwiners corresponding to the irreducible subrepresentations of τ furnish a linear decomposition of the multiplet (ψ_α). The relations 2 apply – by construction – to the individual summands in this linear combination so that local braid relations for arbitrary quantum fields ψ_α can be worked out.

Let us just remark at the end that operator product expansions of the form (18) give rise to operator product expansion in the field algebra,

$$\psi^J_m(\rho_J)\cdot\psi^K_n(\rho_K) = \sum \psi^M_m(\rho_M)\, T_b({}_M{}^K{}_J)\left[{}^J_j\,{}^K_k\,{}^M_m\right]^b \,. \tag{6.12}$$

7. Discussion and Outlook

For the construction of the field algebra $\mathscr{F}$ with weak quasi-quantum group symmetry, every bi-*-algebra with antipode $\mathscr{G}^*$ can be admitted, provided the multiplicities in the Clebsch Gordon decomposition coincide with the fusion rules of quantum field theory. This selection criterion for $\mathscr{G}^*$ as well as the explicit expressions for the re-associator and the R-matrix were completely determined by the observables $\mathscr{A}$ of the model.

In [54] we used algebraic methods to construct a field algebra with quantum symmetry for the critical chiral Ising model. Instead of C^*-algebras we used the Virasoro algebra $\mathrm{Vir}_{c=1/2}$ as a chiral Lie algebra $\mathrm{Lie}\mathscr{A}$ of observables. This Lie algebra admits three inequivalent unitary irreducible positive energy representations π_J in the Hilbert spaces $\mathscr{H}^J$, $J = 0, \frac{1}{2}, 1$. To construct suitable endomorphisms, we had to enlarge $\mathrm{Vir}_{c=1/2}$ to a Lie algebra $\mathrm{Lie}\bar{\mathscr{A}}$. All unitary irreducible positive energy representations π_J of the Virasoro algebra $\mathrm{Vir}_{c=1/2}$ on $\mathscr{H}^J$ extend to representations of $\mathrm{Lie}\bar{\mathscr{A}}$ in the same Hilbert space $\mathscr{H}^J$. A complete set of endomorphisms ρ_J of $\mathrm{Lie}\bar{\mathscr{A}}$ to reach all the sectors was found explicitly. The fusion rules were calculated from the explicit expressions.

$$\begin{aligned}
[\rho_{1/2}\circ\rho_{1/2}] &= [\rho_0] + [\rho_1]\,,\\
[\rho_{1/2}\circ\rho_1] &= [\rho_1\circ\rho_{1/2}] = [\rho_{1/2}]\,,\\
[\rho_1\circ\rho_1] &= [\rho_0]\,,
\end{aligned} \tag{7.1}$$

and $[\rho_0\circ\rho_J] = [\rho_J\circ\rho_0] = [\rho_J]$ for all $J = 1, \frac{1}{2}, 1$. The matrices N^{IJ}_K can be read off.

It was suggested in [56, 57] to regard a "truncated" version of the quantum group algebra $U_q(sl_2)$ as quantum symmetry of the chiral Ising model. As we have seen above, the critical Ising model (as any other rational model) admits infinitely many quantum symmetries. It is important to remark that they are all truncated (in particular they cannot be group symmetries). Without truncation the dimensions δ_J of irreducible representations τ^J have to satisfy

$$\delta_I\delta_J = \sum_K N^{IJ}_K \delta_K \,. \tag{7.2}$$

When N^{IJ}_K are the fusion rules of a quantum field theory with permutation group statistics, an integer solution of this equation is furnished by the "statistics dimensions" d_J of the superselection sectors [16]. The set of statistics dimensions d_J gives at the same time the dimensions of representations of the symmetry groups constructed by Doplicher and Roberts. On the simple level of Eq. (7.2) differences with the situation in low dimensional quantum field theory show up already. As one can easily check for the fusion rules of the chiral critical Ising model, positive integer solutions of (7.2) do not exist. Whenever this happens it excludes the possibility of non-truncated quantum symmetries. If one admits for truncation, the condition (7.2) on the dimensions δ_J of irreducible representations becomes an inequality, which has an infinite number of solutions so that there is a priori an enormous freedom in the construction of quantum symmetries (and consequently in the construction of field algebras).

If we fix a particular integer solution of the inequality $\delta_I \delta_J \geqq \sum N_K^{IJ} \delta_K$, the remaining freedom is reduced to a "twist" in the sense of [19]. As H. Rehren remarked in [72], this corresponds to the possibility of Klein transformations of field operators.

It should be mentioned at the end that soliton sectors in massive two-dimensional quantum field theory do not fit into the present theory of superselection structure. The usual analysis applies only to a special class of models which was discussed by Fröhlich [31]. Inspired by the properties of classical multi-soliton solutions; Fredenhagen proposed that in generic situations, soliton sectors can only be composed if they "fit together" [27, 28]. The structure of possible "quantum symmetries" is not known for these more general cases, but will probably be quite different from the quantum symmetries treated here.

Validity of the analysis in this paper is also restricted to finite statistics and does not extend to quantum field theories with infinite statistics. Models of such theories exist [29].

8. Appendix A: Properties of Covariant Conjugation

This appendix is devoted to the properties of the covariant conjugation which was introduced in Sect. 2. A list of properties is given in Sect. 5.1. Their proofs will be sketched below.

As we remarked before, the covariance law can be used to shift operators $\mathscr{U}(\xi)$ from the left to right of the fields Ψ_i^I. Yet we did not discuss how to achieve the reverse, i.e. move elements from right to left. It is precisely the existence of an interwiner φ satisfying (4.4) which allows to do this. Let us introduce the element $w = \sum \varphi_\sigma^2 \mathscr{S}^{-1}(\varphi_\sigma^1 \beta) \otimes \varphi_\sigma^3$ as before and define a new multiplet ${}_i\Psi^I$ according to

$$ {}_i\Psi^I(x,t) \equiv \Psi_j^I(x,t)(\tau_{ji}^I \otimes \mathscr{U})(w) \,. \tag{8.3} $$

This new tuple has a "good" transformation behaviour, namely

$$ {}_i\Psi^I \mathscr{U}(\xi) = (\tilde{\tau}_{ij}^I \boxtimes \mathscr{U})(\xi)\, {}_j\Psi^I \,, \tag{8.4} $$

where $\tilde{\tau}(\xi) \equiv {}^t\tau^I(\mathscr{S}^{-1}(\xi))$ for all $\xi \in \mathscr{G}^*$ and all representations τ of $\mathscr{G}^*$. We will often refer to transformation laws of the form (8.4) as "left covariance" to distinguish them from the (right-) covariance (2.15). Relation (8.4) is actually a simple consequence of the intertwining property (4.4) and the relations (2.6).

If the element φ is the re-associator of a weak quasi-quantum group (i.e. φ satisfies all the relations in Sect. 4.1) the map (8.3) from (right-) covariant to left-covariant multiplets has a number of useful properties. They are summarized in the following proposition.

Proposition 21 (*Properties of Rel.* (8.3)). *Assuming that the quantum symmetry* $\mathcal{G}^*$ *is a weak quasi-quantum group one can show*

1. *The map* (8.3) *from right- to left-covariant elements has an inverse,*
$$\Psi^I_i(x,t) = (\tilde{\tau}^I_{ji} \otimes \mathcal{U})(v)\,{}_i\Psi^I(x,t)\,. \tag{8.5}$$
Here $v \in \mathcal{G}^* \otimes \mathcal{G}^*$ *is defined by* $v = \sum \mathcal{S}(\phi^1_\sigma)\alpha\phi^2_\sigma \otimes \phi^3_\sigma$ *with* ϕ^i_σ *given through the expansion of* $\phi = \varphi^{-1}$.
2. *The passage from right- to left covariant elements is consistent with local braid relations in the following sense. Suppose that* Ψ^I_i, Ψ^J_j *satisfy local braid relations* (5.5) *Then*
$${}_i\Psi^I\,{}_j\Psi^J = \omega^{IJ}(\tilde{\tau}^I_{ik} \otimes \tilde{\tau}^J_{jl} \otimes \mathcal{U})(\varphi_{213}R_{12}\varphi^{-1})\,{}_l\Psi^J\,{}_k\Psi^J\,. \tag{8.6}$$
3. *On covariant products,* (8.3) *acts according to*
$$(\Psi^J \times \Psi^I)_{mn}((\tau^J \boxtimes \tau^I)_{mnji} \otimes \mathcal{U})(w) = (\tilde{\tau}^I_{ik} \otimes \tilde{\tau}^J_{jl} \otimes \mathcal{U})(f_{12}\varphi^{-1})\,{}_l\Psi^J\,{}_k\Psi^I\,, \tag{8.7}$$
where $f \in \mathcal{G}^*$ *is the element* (4.8).

Proof. A detailed prof of this proposition is beyond the scope of this text. We start with the first item and make only some remarks on the other two. The relation (8.5) is a straightforwad application of the pentagon equation (4.5). The latter can be restated as

$$(id \otimes id \otimes \Delta)(\varphi^{-1})(e \otimes \varphi) = (\Delta \otimes id \otimes id)(\varphi)(\varphi^{-1} \otimes e)(id \otimes \Delta \otimes id)(\varphi^{-1})\,,$$

Using the properties (2.6) of the antipode one derives

$$(id \otimes \Delta)(v)\varphi = \sum (\mathcal{S}(\phi^1_\tau \phi^1_\sigma)\alpha\phi^2_\tau \otimes \phi^3_\tau \otimes \phi^3_\sigma)(\Delta(\phi^2_\sigma) \otimes e)\,,$$

and then after a similar step (with $v = \sum v^1_\sigma \otimes v^2_\sigma$)

$$\sum \Delta(v^2_\sigma)w(\mathcal{S}^{-1}(v^1_\sigma) \otimes e) = \sum (\phi^3_\sigma \mathcal{S}^{-1}(\alpha\phi^2_\sigma\beta)\phi^1_\sigma \otimes e)\Delta(e) = \Delta(e)\,,$$

where the last equality follows from relation (4.6). This equation can be exploited in the following calculation:

$$\begin{aligned}(\tilde{\tau}^I_{ji} \otimes \mathcal{U})(v)\,{}_i\Psi^I &= \Psi^I_k(\tilde{\tau}^I_{ji} \otimes \tau^I_{kl} \otimes \mathcal{U})((id \otimes \Delta)(v))(\tau^I_{li} \otimes \mathcal{U})(w)\\ &= \Psi^I_k(\tau^I_{kj} \otimes \mathcal{U})(\Delta)(e)) = \Psi^I_j\,.\end{aligned}$$

We prove the second item by iteration of arguments leading to the following lemma.

Lemma. *Suppose that* Ψ^I_i, Ψ^J_j *satisfy local braid relations* (5.5). *Then*

$$\omega^{IJ}\,{}_j\Psi^J \Psi^I_i = \Psi^I_k(\tau^I_{ki} \otimes \tilde{\tau}^J_{jl} \otimes \mathcal{U})(\mathcal{R})\,{}_i\Psi^J\,, \tag{8.8}$$

with $\mathcal{R} = \varphi_{213}R_{12}\varphi^{-1}$.

Proof. From the defining relations of a weak quasi-quantum group one can derive the following *generalized quasi-triangularity*

$$\mathcal{R}^{-1}_{134}[(id \otimes id \otimes \Delta)(\varphi)]_{2314}(id \otimes \Delta \otimes id)(\mathcal{R}) = (id \otimes id \otimes \Delta)(\mathcal{R})(e \otimes \varphi)\,. \tag{8.9}$$

The meaning of the lower indices has been explained in Definition 10. This equation is a good starting point to obtain

$$(\varrho^1_\sigma \otimes \Delta(\varrho^3_\sigma))(e \otimes w)(e \otimes \mathcal{S}^{-1}(\varrho^2_\sigma) \otimes e) = \mathcal{R}^{-1}[(id \otimes \Delta)(w)]_{213} , \qquad (8.10)$$

where ϱ^i_σ are the components of $\mathcal{R}$. As a consequence

$$\begin{aligned} \Psi^I_k(\tau^I_{ki} \otimes \tilde{\tau}^J_{jl} \otimes \mathcal{U})(\mathcal{R})_i\Psi^J &= \Psi^I_k \Psi^J_l(\tau^I_{ki} \otimes \tau^J_{lj} \otimes \mathcal{U})(\mathcal{R}^{-1}[(id \otimes \Delta)(w)]_{213}) \\ &= \omega^{IJ} \Psi^J_n \Psi^I_m(\tau^I_{mi} \otimes \tau^J_{nj} \otimes \mathcal{U})([(id \otimes \Delta)(w)]_{213}) \\ &= \omega^{IJ} \Psi^I_n(\tau^J_{nj} \otimes \mathcal{U})(w) \Psi^I_i \\ &= \omega^{IJ}{}_j\Psi^J \Psi^I_i . \end{aligned}$$

This is the proposed equation.

Proof of Proposition 21 (*continued*) Things become really cumbersome when we come to the third item. Here is a rough sketch of the proof. One has to insert the definition of left covariant multiplets into the left-hand side of the equation. Next the covariance law is applied to move all factors which involve elements in the symmetry algebra $\mathcal{G}^*$ to the right. Once this is done, the resulting expression is rewritten with the help of the pentagon equation (4.5) and the properties (2.6) of the antipode. One has to use (4.5) three times until everything simplifies as a consequence of the formulae (4.9f.) for the element f. This completes the proof of the proposition.

The covariant conjugation (2.17) introduced in Sect. 2 amounts to the prescription

$$\bar{\Psi}^I_i \equiv ({}_i\Psi^I)^* . \qquad (8.11)$$

The transformation law (2.18) of $\bar{\Psi}^I$ is a direct consequence of relation (8.4) above. All properties of the covariant conjugation listed in Sect. 5.1 follow from Proposition 21 (notice that $w^* = v$).

Acknowledgements. It is a pleasure to thank Gerhard Mack for many helpful and encouraging conversations and for providing the final stimulus to finish this work. I also profited a lot from discussions with Henning Rehren, Klaus Fredenhagen and Kornel Szlachanyi. The warm hospitality at the Erwin Schrödinger Institute (where many of the discussions took place) and in particular of Harald Grosse is gratefully acknowledged. I am indebted to Arthur Jaffe for the opportunity to continue my work in his group. This project was partially supported by the NSF grant No. PHY-91-20626.

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Communicated by A. Jaffe