Commun. Math. Phys. 168, 563–570 (1995)

Communications in
Mathematical
Physics

Anderson Localization for the Almost Mathieu Equation: II. Point Spectrum for $\lambda > 2$

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Received: 23 August 1993

Abstract: We prove that for any $\lambda > 2$ and a.e. ω, θ the pure point spectrum of the almost Mathieu operator $(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda\cos(2\pi(\theta + n\omega))\Psi_n$ contains the essential closure $\hat{\sigma}$ of the spectrum. Corresponding eigenfunctions decay exponentially. The singular continuous component, if it exists, is concentrated on a set of zero measure which is nowhere dense in $\hat{\sigma}$.

1. Introduction

This paper is another attack on the almost-Mathieu operator on $\ell^2(Z)$:

$$(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda\cos(2\pi(\theta + n\omega))\Psi_n .$$

This simple-looking operator has been studied extensively for many years. We refer the reader to [1, 2] for a still incomplete list of references. The critical (and physical) value of the coupling constant λ is $\lambda = 2$ (we assume without loss of generality that $\lambda \geqq 0$); it is believed that at $\lambda = 2$ there occurs a transition from pure absolutely continuous to pure point spectrum. The ω here is supposed to be "irrational enough," since for rational ω the potential is periodic and the spectrum is absolutely continuous for all λ, and for Liouville ω (abnormally well approximated by rationals) and $\lambda > 2$ the spectrum of $H(\theta)$ is purely singular continuous [3, 4]. Up to recently the only rigorous reason for this belief was that for $\lambda > 2$ and irrational ω the Lyapunov exponents are positive, which proves the absence of the absolutely continuous part of the spectrum [5, 6]. By Aubry duality there is no pure point spectrum for $\lambda < 2$ [7]. The latest development for any $\lambda < 2$ is the proof of existence of absolutely continuous spectrum that was given by Last [8] for a.e. ω, θ and by Gesztesy and Simon [13] for all ω, θ. Last [8] also proved that for a.e. ω the absolutely continuous spectrum, σ_{ac}, coincides with the spectrum, σ, up to a set of zero Lebesgue measure.

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In the present paper we study λ's above the critical value. Localization, i.e., pure point spectrum with exponentially decaying eigenfunctions, was proved by Sinai [9] and Fröhlich, Spencer and Wittwer [10] in the perturbative regime: λ "big enough." In [11] we developed a nonperturbative way of proving localization, which worked for $\lambda \geqq 15$. The method also allowed us to prove localization in the middle of the spectrum for $\lambda \geqq 5.4$. These restrictions on λ were caused exclusively by the roughness of the a priori estimates (from above) on the growth of the formal solution Ψ_E, i.e., on the Lyapunov exponents. In this paper we show that the results of [8, 13] together with Aubry duality and certain regularity of the Lyapunov exponent [12] give a much nicer estimate for "most of" the Lyapunov exponents. We combine this soft argument with the method of [11] to prove the existence of "a lot of" pure point spectrum for a.e. ω. The result holds for any $\lambda > 2$ but we have to pay for that by not being able to rule out the possibility of some singular continuous spectrum.

We already denoted the spectrum of $H(\theta)$ by σ, and now denote the singular continuous part of the spectrum by σ_{sc}, and the pure point part of the spectrum by σ_{pp}. It is well known that for irrational ω the sets σ, σ_{sc} and σ_{pp} (understood as the closure of the set of eigenvalues) do not depend on the phase θ for a.e. θ. Although for $\lambda < 2$ the results on the absolutely continuous spectrum probably hold for every θ (see [13]), the pure point spectrum for $\lambda > 2$ should be an essentially "a.e." result, since for generic θ the spectrum is purely singular continuous in this case [14].

We set

$$\Theta = \left\{\theta\text{: for every } s > 1 \text{ the relation } \left|\sin 2\pi\left(\theta + \frac{k}{2}\omega\right)\right| < k^{-s} \text{ holds for infinitely many } k\text{'s}\right\} .$$

We will assume that the phase θ does not belong to this set of zero measure. As we already mentioned, for $\lambda > 2$ the arithmetic nature of ω plays a major role. Let $\frac{p_n}{q_n}$ be the n^{th} continuous fractions approximant of ω. Throughout the rest of the paper we assume that ω is Diophantine, i.e., an irrational such that for some $r > 1$ and $C > 0$ we have $|q_n\omega - p_n| > Cq_n^{-r}$. In Theorem 1 we will also use another, rather technical, restriction on ω:

$$\frac{C}{q_n^r} < |q_n\omega - p_n| = \frac{o(1)}{q_n} \text{ as } n \to \infty . \tag{1}$$

The set of ω's described in (1) has full Lebesgue measure [15].

For $\lambda > 2$ the spectrum σ is a set of positive Lebesgue measure [13]: $|\sigma| \geqq 2\lambda - 4$. We set

$$\hat{\sigma} = \{E \in \sigma : \text{for any } \varepsilon > 0\ |(E - \varepsilon, E + \varepsilon) \cap \sigma| > 0\} .$$

We will prove

Theorem 1. *Suppose ω satisfies* (1). *Then for any $2 < \lambda < 15$ we have*

1. $|\sigma_{\mathrm{pp}}| = |\sigma|$,

2. *σ_{sc}, if it exists, has measure zero and is nowhere dense in $\hat{\sigma}$ (in the relative topology).*

3. *For $\theta \notin \Theta$ the set of eigenvalues of the operator $H(\theta)$ is dense in $\hat{\sigma}$ and the corresponding eigenfunctions are exponentially decaying.*

The zero measure set of Diophantine ω's not satisfying (1) happens to include the golden mean – the most popular object for numerical studies. Without the technical assumption (1) we prove a slightly weaker result:

Theorem 1′. *For any Diophantine* ω *and any* $2 < \lambda < 15$ *we have*

1. $|\sigma_{pp}| \geqq 2\lambda - 4$.
2. *There exists a closed set* $A \subset \sigma_{pp}$, $|A| \geqq 2\lambda - 4$, *such that* σ_{sc} *is nowhere dense in* A.
3. *For* $\theta \notin \Theta$ *the set of eigenvalues of the operator* $H(\theta)$ *is dense in* A *and the corresponding eigenfunctions are exponentially decaying.*

Remarks.

1. As can be seen from the proof, in order to prove complete localization in $\hat{\sigma}$ for $\lambda > 2$ it suffices to prove the continuity of the Lyapunov exponent $\gamma(E)$. For λ large enough the continuity of $\gamma(E)$ follows from the proof in [9]. This continuity would also be enough to rule out the singular continuous spectrum for $\lambda < 2$ [16].
2. An estimate on the rate of the exponential decay of the eigenfunctions can be easily obtained from the proof and is given by $|\Psi(x)| \leqq \text{const}\left(\frac{\lambda}{2}\right)^{(-\frac{1}{4}+\varepsilon)|x|}$ for any $\varepsilon > 0$. This decay is slower than what is suggested by the Lyapunov exponent.
3. Theorems 1 and 1′ can be proved with θ satisfying a weaker condition. Namely, let us fix $1 < \mu < \left(\frac{\lambda}{2}\right)^{\frac{1}{16}}$ and put
$$\Theta_\mu = \{\theta : \left|\sin 2\pi\left(\theta + \frac{k}{2}\omega\right)\right| < \mu^{-k} \text{ holds for infinitely many } k\text{'s}\}.$$
Then for $\theta \notin \Theta_\mu$ the same result holds, but the estimate on the decay of the eigenfunctions will be: $|\Psi(x)| \leqq \text{const}\left(\left(\frac{\lambda}{2}\right)^{-\frac{1}{4}}\mu^4\right)^{|x|}$. It is quite clear that the rate of decay should depend on μ. This result can be compared with the fact that for $\theta \in \Theta_\mu$, μ sufficiently large, the operator $H(\theta)$ has no pure point component in the spectrum [14].
4. The Diophantine property can be made weaker (see the comment in [11]) but not too weak since for Liouville ω the spectrum of $H(\theta)$ is purely singular continuous. The upper bound in (1) appears here only because of the same bound in [8] and is presumably an artifact of Last's (and, of course, of the present) proof.
5. All the results certainly hold for $\lambda \geqq 15$ as well [11].

In Sect. 2 we describe our method for proving localization which is a certain modification of the method of [17] (see also [10, 18]). In Sect. 3 we formulate our main technical result, Theorem 2, which is very similar to Theorem 2 in [11], and present the soft arguments which prove Theorems 1 and 1′ from Theorem 2. In Sect. 4 we prove Theorem 2.

2. The General setup

We start with some definitions

Definition. *A formal solution* $\Psi_E(x)$ *of the equation* $H(\theta)\Psi_E = E\Psi_E$ *will be called a* **generalized eigenfunction** *if* $|\Psi_E(x)| \leqq C(1+|x|)$ *for some* $C = C(\Psi_E) < \infty$. The energy E *for which such a solution exists will be called a* **generalized eigenvalue**.

It is well known that to prove pure point spectrum one only needs to prove that generalized eigenfunctions belong to ℓ^2 (see [17]). We denote the Green's function $(H-E)^{-1}$ of the operator $H(\theta)$ restricted to the interval $[x_1, x_2]$ with zero boundary conditions at $x_1 - 1$ and $x_2 + 1$ by $G_{[x_1,x_2]}(E)$. Let us fix a number $m < 1$.

Definition. *A point* $y \in Z$ *will be called* (m, k)*-regular if there exists an interval* $[x_1, x_2]$ *containing* y *such that*

$$|G_{[x_1,x_2]}(y, x_i)| < m^k, \text{ and } \operatorname{dist}(y, x_i) \leqq k;\ i = 1, 2\ .$$

Otherwise y *will be called* (m, k)*-singular.*

Let E be a generalized eigenvalue of H_θ, $\Psi(x)$ the corresponding generalized eigenfunction.

Lemma 1. *For every* $x \in Z$ *such that* $\Psi(x) \neq 0$ *there exists* $k_0 = k_0(x, m, \theta, E) < \infty$ *such that for* $k > k_0$ *the point* x *is* (m, k)*-singular.*

Lemma 1 is the same kind of statement as Lemma 3.1 in [10] and so is the proof. □

Suppose one can prove that (m, k)-singular points are "far apart" which we formulate as the following

Quasilemma. *Suppose the points* x_1, x_2 *are* (m, k)*-singular,* k *is large enough and* $\operatorname{dist}(x_1, x_2) > \frac{k}{2}$, *then* $\operatorname{dist}(x_1, x_2) > k$.

Then the rest of the proof can be organized as follows. Assume without loss of generality that $\Psi(0) \neq 0$. Let $|x|$ be bigger than $k_0(0, m, \theta, E)$ and sufficiently large so that we can apply the Quasilemma with $k = |x|$. Suppose x is $(m, |x|)$-singular. Since 0 is $(m, |x|)$-singular, the Quasilemma asserts that $\operatorname{dist}(0, x) = |x| > |x|$. The contradiction implies that x is $(m, |x|)$-regular. Thus we have that there exists an interval $[x_1, x_2]$ containing x such that

$$|x_i - x| \leqq |x|,\ |G_{[x_1,x_2]}(x, x_i)| \leqq m^{|x|},\ i = 1, 2\ .$$

We now can use the formula

$$\Psi(x) = G_{[x_1,x_2]}(x, x_1)\Psi(x_1 - 1) + G_{[x_1,x_2]}(x, x_2)\Psi(x_2 + 1)$$

to obtain the estimate:

$$|\Psi(x)| \leqq 2C(1 + 2|x|)m^{|x|}\ .$$

This argument shows that the localization follows immediately if we make a *l*emma out of the Quasilemma, i.e., provide it with hypotheses and, of course, with a proof.

3. Proof of Theorem 1

We define

$$B(\theta, E, \lambda) = \begin{pmatrix} E - \lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}, \; B_k(\theta, E, \lambda) = B(\theta + k\omega, E, \lambda) \, ,$$

$$M_k(\theta, E, \lambda) = B_k(\theta, E, \lambda) \ldots B_0(\theta, E, \lambda) \, .$$

The Lyapunov exponent $\gamma(E, \lambda)$ is given by $\gamma(E, \lambda) = \inf_k \int_0^1 |k|^{-1} \ln \| M_k(\theta, E, \lambda) \| d\theta$. Recall that for $\lambda > 2$ we have $\gamma(E, \lambda) \geqq \ln(\lambda/2) > 0$. Theorem 1 will follow from

Theorem 2. *Let $\theta \notin \Theta$, ω be as in Theorem 1 and suppose E, λ are such that $\frac{\ln\frac{\lambda}{2}}{\gamma(E,\lambda)} > \frac{3}{4}$ and E is a generalized eigenvalue of $H(\theta)$. Then the corresponding generalized eigenfunction $\Psi(x)$ is exponentially decaying.*

Proof of Theorem 1. Take any $\lambda > 2$. The Aubry duality and the Thouless formula yield famous relations [19, 6, 7]

$$\sigma(H(\theta, \lambda)) = \frac{\lambda}{2} \sigma\left(H\left(\theta, \frac{4}{\lambda}\right)\right),$$

$$\gamma(E, \lambda) = \ln\frac{\lambda}{2} + \gamma\left(\frac{2E}{\lambda}, \frac{4}{\lambda}\right). \tag{2}$$

Since $\frac{4}{\lambda} < 2$, we can use the result of Last [8] that for a.e. $E \in \sigma\left(H\left(\theta, \frac{4}{\lambda}\right)\right)$ the Lyapunov exponent $\gamma\left(E, \frac{4}{\lambda}\right)$ is equal to 0.

Craig and Simon [12] have proven that $\gamma(E)$ is continuous at points E where $\gamma(E) = 0$. We set

$$G = \left\{E : \gamma\left(E, \frac{4}{\lambda}\right) < \frac{1}{3} \ln \frac{\lambda}{2}\right\}.$$

The set $\frac{\lambda}{2} G \cap \sigma$ contains an open (in $\hat{\sigma}$) dense (in $\hat{\sigma}$) set of measure $|\sigma|$, thus the set $\sigma \backslash \frac{\lambda}{2} G$ is a nowhere dense in $\hat{\sigma}$ set of zero measure. For $E \in \frac{\lambda}{2} G$ relation (2) implies that $\gamma(E, \lambda) < \frac{4}{3} \ln \frac{\lambda}{2}$ and we can apply Theorem 2 to obtain that for $\theta \notin \Theta$ (thus, for a full measure set) $\sigma_c \cap \frac{\lambda}{2} G = \emptyset$ which gives statements 1 and 2 of the theorem. To obtain the third statement we recall that the spectrum, σ, does not depend on θ for *all* θ (see, e.g., [1]). Thus for $\theta \notin \Theta$ the set of the eigenvalues is dense in $\frac{\lambda}{2} G \cap \sigma$ and, consequently, in $\hat{\sigma}$. $\square$

Proof of Theorem 1′. Gesztesy and Simon [13] have proven that for any irrational ω and any λ the inequality $|\sigma_{ac}| \geqq 4 - 2\lambda$ holds. This plus the Ishii–Pastur–Kotani theorem (see [1]) and the same argument as above implies the result. $\square$

4. Proof of Theorem 2

As we learned from Sect. 2 it suffices to prove the Quasilemma under the conditions of Theorem 2.

Following [11], let us put

$$P_k(\theta, E) = \det[(H(\theta) - E)|_{[0,k-1]}] .$$

$P_k(\theta, E)$ is an even function of the argument $\theta + \frac{k-1}{2}\omega$ and can be written as a polynomial of the degree k in $\cos\left(2\pi\left(\theta + \frac{k-1}{2}\omega\right)\right)$:

$$P_k(\theta, E) = \sum_{j=0}^{k} b_j(E)\cos^j\left(2\pi\left(\theta + \frac{k-1}{2}\omega\right)\right) .$$

To simplify the notation we will sometimes omit the dependence on E.

It is easy to see that $b_k = 2\lambda^k$. We now fix $E \in R$; $1 < m_1 < \frac{\lambda}{2}$. Given $k > 0$ we set

$$A_k = \{x\colon |P_k(\theta + x\omega)| > m_1^k\} .$$

For any $x_1, x_2 = x_1 + k - 1$, $x_1 \leqq y \leqq x_2$ we have

$$|G_{[x_1,x_2]}(x_1, y) = \left|\frac{P_{x_2-y}(\theta + (y+1)\omega)}{P_k(\theta + x_1\omega)}\right| ,$$

$$|G_{[x_1,x_2]}(y, x_2) = \left|\frac{P_{y-x_1}(\theta + x_1\omega)}{P_k(\theta + x_1\omega)}\right| . \tag{3}$$

Since $P_k(\theta, E)$ is one of the entries of the matrix $M_k(\theta, E)$ we have the evident upper bound $|P_k(\theta, E)| \leqq \|M_k(\theta, E)\|$. It is proved in [12] that for *all* θ, E, λ the inequality

$$\limsup |k|^{-1} \ln \|M_k(\theta, E, \lambda)\| \leqq \gamma(E, \lambda)$$

holds. Thus for any $\varepsilon > 1$ there exists $k(\varepsilon, E)$ such that for $k > k(\varepsilon, E)$ we have

$$\ln|P_k(\theta, E)| < k(\varepsilon\gamma(E, \lambda)) . \tag{4}$$

It follows from (3), (4) that for $x_1 \in A_k$, $x_2 = x_1 + k - 1$, $k > k(\varepsilon, \theta, E)$, $\frac{1}{m_1} < m < 1$ and $y \in [x_1, x_2]$ such that

$$k\left(1 - \frac{\ln(m_1 m)}{n\varepsilon\gamma(E, \lambda)}\right) < y - x_1 < k\left(\frac{\ln(m_1 m)}{\varepsilon\gamma(E, \lambda)}\right) ,$$

we have

$$|G_{[x_1,x_2]}(y, x_i)| < m^k, \ i = 1, 2 . \tag{5}$$

We set $c_{\lambda,\varepsilon} = \dfrac{\ln(m_1 m)}{\varepsilon\gamma(E, \lambda)}$.

Proposition 1. *Suppose $y \in Z$ is (m, k)-singular. Then for any such x that $k(1 - c_{\lambda,\varepsilon}) \leqq y - x \leqq kc_{\lambda,\varepsilon}$ we have that x does not belong to A_k.*

Proposition 1 follows immediately from (5). □

The rigorous statement of the Quasilemma can now be obtained from the following lemma:

Lemma 2 [11]. *Suppose ω is Diophantine, $E \in R$. For $\theta \notin \Theta$ there exists $k_1(\theta, E)$ such that for $k > k_1(\theta, E)$ if the points x_1, x_2 satisfy*

1) $x_i, x_i + 1, \ldots, x_i + \left[\frac{k+1}{2}\right] \notin A_k,\ i = 1, 2,$

2) $\operatorname{dist}(x_1, x_2) > \left[\frac{k+1}{2}\right],$

then

$$\operatorname{dist}(x_1, x_2) > \alpha^k$$

with $\alpha = \alpha(m_1, m, \lambda, E, s, r) > 1$.

Indeed, let us suppose that $\dfrac{\ln\frac{\lambda}{2}}{\gamma(E,\lambda)} > \dfrac{3}{4}$. Then three exist $1 < m_1 < \dfrac{\lambda}{2}$, $m < 1$ and $\varepsilon > 1$ such that $2c_{\lambda,\varepsilon} - 1 > \frac{1}{2}$. Let k be bigger than $\max[k(\varepsilon, E), k_1(\theta, E)]$. Suppose x_1', x_2' are (m, k)-singular and $\operatorname{dist}(x_1', x_2') > \left[\frac{k+1}{2}\right]$. Since $2c_{\lambda,\varepsilon} - 1 > \frac{1}{2}$, we obtain using Proposition 1 that the points $x_1 = x_1' - c_{\lambda,\varepsilon}k$ and $x_2 = x_2' - c_{\lambda,\varepsilon}k$ satisfy the conditions of Lemma 2. Applying Lemma 2 we get that $\operatorname{dist}(x_1', x_2') > \alpha^k > k$ for large k. After that the rest of the argument is the same as in the end of Sect. 2. □

5. *Acknowledgement.* It is a pleasure to thank A. Klein for many useful discussions.

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Communicated by T. Spencer