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Aspects of Fractional Superstrings

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Abstract: We investigate some issues relating to recently proposed fractional superstring theories with $D_{\text{critical}} < 10$. Using the factorization approach of Gepner and Qiu, we systematically rederive the partition functions of the $K = 4, 8$, and 16 theories and examine their spacetime supersymmetry. Generalized GSO projection operators for the $K = 4$ model are found. Uniqueness of the twist field, $\phi^{K/4}_{K/4}$, as source of spacetime fermions is demonstrated.

Section 1: Introduction

In the last few years, several generalizations of standard (supersymmetric) string theory have been proposed [20, 28, 25, 22]. One of them [16, 7, 10, 6, 9, 14, 15, 13] uses the (fractional spin) parafermions introduced from the perspective of 2-D conformal field theory (CFT) by Zamolodchikov and Fateev [31] in 1985 and further developed by Gepner and Qiu [18].[3] In a series of papers, possible new string theories with local parafermionic world sheet currents (of fractional conformal spin) giving critical dimensions $D = 6$, 4, 3, and 2 have been proposed [16, 7, 10, 6, 9].

At the heart of these new "fractional superstrings" are $\mathbb{Z}_K$ parafermion conformal field theories (PCFT's) with central charge $c = 2(K-1)/(K+2)$. (Equivalently, these are $SU(2)_K/U(1)$ conformal field theories.) The (integer) level-K PCFT

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[3] This is not to be confused with the original definition of "parafermions." The term "parafermion" was introduced by H.S. Green in 1953 [19]. Green's parafermions are defined as spin-1/2 particles that do not obey standard anticommutation rules, but instead follow more general trilinear relations [5, 27, 12, 3, 21].

contains a set of unitary primary fields ϕ^j_m, where $0 \leqq j, |m| \leqq K/2; j, m \in \mathbb{Z}/2$, and $j - m = 0 \pmod 1$, which have the identifications

$$\phi^j_m = \phi^j_{m+K} = \phi^{\frac{K}{2}-j}_{m-\frac{K}{2}}\,. \tag{1.1}$$

In the range $|m| \leqq j$, the conformal dimension is $h(\phi^j_m) = \dfrac{j(j+1)}{K+2} - \dfrac{m^2}{K}$. At a given level the fusion rules are

$$\phi^{j_1}_{m_2} \times \phi^{j_2}_{m_2} = \sum_{j=|j_1-j_2|}^{r} \phi^j_{m_1 m_2}\,, \tag{1.2}$$

where $r \equiv \min(j_1 + j_2, K - j_1 - j_2)$.[4] This CFT contains a subset of primary fields,

$$\{\phi_i \equiv \phi^0_m \equiv \phi^{K/2}_{-K/2+i};\quad 0 \leqq i \leqq K-1\} \tag{1.3}$$

($\phi^\dagger_i \equiv \phi_{K-i}$) which, under fusion, form a closed subalgebra possessing a $\mathbb{Z}_K$ Abelian symmetry:

$$\phi_i \times \phi_j = \phi_{i+j\,(\mathrm{mod}\,K)}\,. \tag{1.4}$$

The conformal dimensions, $h(\phi_i)$, of the fields in this subgroup have the form

$$h(\phi_i) = \frac{i(K-i)}{K}\,. \tag{1.5}$$

It has been proposed that string models based on tensor products of a level-K PCFT are generalizations of the Type II $D = 10$ superstring [16, 7, 10, 6, 9]. The standard $c = \frac{1}{2}$ fermionic superpartner, $\psi(z)$, of the holomorphic world sheet scalar, $X(z)$, is replaced by the "energy operation," $\varepsilon(z) \equiv \phi^1_0(z)$, of the $\mathbb{Z}_K$ PCFT. (Similar substitution occurs in the antiholomorphic sector.) Note that ε is not in the $\mathbb{Z}_K$ Abelian subgroup, and thus is not a $\mathbb{Z}_K$ parafermion, except for the degenerate $K = 2$ superstring case, where $\phi^1_0 \equiv \phi^0_1$. ε has conformal dimension (spin) $2/(K+2)$, which is "fractional" (*i.e.*, neither integer nor half-integer) for $K \neq 2$. This accounts for the name of these models. Each $\varepsilon - X$ pair has a total conformal anomaly (or central charge) $c = 3K/(K+2)$.

The naive generalization of the (holomorphic) supercurrent (SC) of the standard superstring, $J_{\mathrm{SC}}(z) = \psi(z) \cdot \partial_z X(z)$, (where ψ is a real world sheet fermion) to $J(z) = \phi^1_0(z) \cdot \partial_z X(z)$ proves to be inadequate [9]. Instead, the proposed "fractional supercurrent" (FSC) is

$$J_{\mathrm{FSC}}(z) = \phi^1_0(z) \cdot \partial_x X(z) + :\phi^1_0(z)\phi^1_0(z):\,. \tag{1.6}$$

$:\phi^1_0\phi^1_0:$ (which vanishes for $K = 2$) is the first descendent field of ϕ^1_0. $J_{\mathrm{FSC}}(z)$ is the generator of a local "fractional" world sheet supersymmetry between $\varepsilon(z)$ and $X(z)$, extending the Virasoro algebra of the stress-energy tensor $T(z)$. This local current

[4] For clarity, we use " $\times$ " to denote fusion of two fields or operators, in contrast to choosing "$\otimes$" (or no operator at all) to denote a tensor product.

of spin $h(J) = 1 + 2/(K+2)$ has fractional powers of $1/(z-w)$ in the OPE with itself, implying a non-local world sheet interaction and, hence, producing cuts on the world sheet. The corresponding chiral "fractional superconformal algebra" [9] is,

$$T(z)T(w) = \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \cdots, \tag{1.7a}$$

$$T(z)J_{\mathrm{FSC}}(w) = \frac{hJ_{\mathrm{FSC}}(w)}{(z-w)^2} + \frac{\partial J_{\mathrm{FSC}}(w)}{(z-w)} + \cdots, \tag{1.7b}$$

$$J_{\mathrm{FSC}}(z)J_{\mathrm{FSC}}(w) = \frac{1}{(z-w)^{2h}} + \frac{\frac{2h}{c}T(w)}{(z-w)^{2h-2}} + \frac{\lambda_K(c_0)J_{\mathrm{FSC}}(w)}{(z-w)^h}$$
$$+ \frac{\frac{1}{2}\lambda_K(c_0)\partial J_{\mathrm{FSC}}(w)}{(z-w)^{h-1}} + \cdots, \tag{1.7c}$$

where $c = Dc_0$. D is the critical dimension, $c_0 = 3K/(K+2)$ is the central charge for one dimension, and λ_K is a constant [8].

The relationship between critical dimension, D, and the level, K, of the parafermion CFT may be shown to be

$$D = 2 + \frac{16}{K}, \tag{1.8}$$

for $K = 2, 4, 8, 16$, and ∞. (The $K = 2$ theory is the standard Type II superstring theory with its partition function expressed in terms of string functions rather than theta-functions, which implies a set of identities between these two classes of functions.) In [16, 7, 10, 6, 9] the relationship (1.8) is derived by requiring a massless spin-1 particle in the open string spectrum, produced by $\phi_0^1(z)^\mu$ (where μ is the spacetime index) operating on the vacuum.

The purpose of this paper is to examine a number of issues relating to these models: In Sect. 2 we derive the partition functions of the $D = 6, 4$, and 3 theories (corresponding to $K = 4, 8$ and 16 respectively), using the factorization method of Gepner and Qiu [18], as well as demonstrating a new approach to obtaining the superstring partition function. In Sect. 3 we consider other necessary elements of string theory. In particular, we propose a generalization of the GSO projection that applies to the fractional superstring and we address the question of whether similar theories at different Kač–Moody levels can be constructed. Additionally, in this section, a comparison with the superstring is made and we attempt to elucidate its features in the current, more general context.

Section 2: Factorization of the Fractional Superstring Partition Functions

We now construct the level-K partition functions of these theories from the well understood characters of SU(2) primary fields. These closed string partition functions Z_K have the general form

$$Z_K = a_A|A_K|^2 + a_B|B_K|^2 + a_C|C_K|^2, \tag{2.1}$$

where a_A, a_B, and a_C are integer coefficients. The A_K term in Eq. (2.1) contains the massless graviton and gravitino. These $D < 10$ fractional superstrings have a new feature not present in the standard $D = 10$ superstrings (which correspond to $K = 2$). This is the existence of the massive B_K- and C_K-sectors. These additional sectors were originally derived by the authors of refs. [16, 7, 10] by applying S transformations to the A_K-sector and then demanding modular invariance of the theory. In this section, we will discuss (1) new aspects of the relationship between the B_K- and C_K-sectors and the A_K-sector, and (2) the presence of spacetime supersymmetry (SUSY) in all sectors. (Specifically, these type II models have $N = 2$ spacetime SUSY, with the holomorphic and antiholomorphic sectors each effectively contributing an $N = 1$ SUSY. Hence, heterotic fractional superstrings possess only $N = 1$ SUSY.)

We will demonstrate that spacetime SUSY results from the action of a twist current used in the derivation of these partition functions. Only by this twisting can cancellation between bosonic and fermionic terms occur at each mass level in the A_K- and B_K-sectors. The same twisting results in a "self-cancellation" of terms in the C_K-sector (which exists only in the four- and three-dimensional models). This self-cancellation may suggest an anyonic interpretation of the C_K-sector states. That uncompactified spacetime anyons can presumably exist only in three or less dimensions would seem to contradict our claim that the $K = 8$ ($D = 4$) model may contain spacetime anyons. We will argue shortly that one dimension of the $K = 8$ fractional string is probably compactified. Examination of the B_K-sector in the $D = 4$ model further suggests this. Anyonic interpretation of the C_K-sector fields was first proposed in ref. [16].

Before we systematically derive the fractional superstring partition functions (FSPF's) for each critical dimension, we will review the character, $Z(\phi^j_m(z))$, for the Verma module, $[\phi^j_m]$,[5] containing a single (holomorphic) parafermionic primary field $\phi^j_m(z)$ and its descendents. The form of the character is

$$Z(\phi^j_m(z)) = q^{-c/24} \operatorname{tr} q^{L_0} \tag{2.2a}$$

$$= \eta(\tau) c^{2j}_{2m}(\tau) , \tag{2.2b}$$

where $q = \mathrm{e}^{2\pi\tau}$ (with τ the one-loop modular parameter) and η is the Dedekind eta-function,

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{2.3}$$

with $q = e^{2\pi i\tau} \cdot c^{2j}_{2m}$ is a string function [24, 23] defined by

$$c^{2j}_{2m} = q^{h^j_m + \frac{1}{i(K+2)}} \frac{1}{\eta^3} \sum_{r,s=0}^{\infty} (-1)^{r+q} q^{r(r+1)/2 + s(s+1)/2 + rs(K+1)}$$

$$\times \left\{ q^{r(j+m)+s(j-m)} - q^{K+1-2j+r(K+1-j-m)+s(K+1-j+m)} \right\} \tag{2.4a}$$

$$= q^{h^j_m - \frac{c(SU(2)_K}{s(K+2)}} (1 + \cdots). \tag{2.4b}$$

[5] From here on, we do not distinguish between the primary field ϕ^J_m and its complete Verma module $[\phi^j_m]$. Thus, ϕ^{-j}_m can represent either, depending on the context.

In this notation $h^j_m \equiv h(\phi^j_m)$ and $c(SU(2)_K) = \dfrac{3K}{K+2}$. Also, as per standard convention, the level of the string function is suppressed. These string functions obey the same equivalences as their associated primary fields ϕ^j_m:

$$c^{2j}_{2m} = c^{2j}_{2m+2K} = c^{K-2j}_{2m-K/2} \,. \tag{2.5a}$$

Additionally

$$c^{2j}_{2m} = c^{2j}_{-2m} \,. \tag{2.5b}$$

Since the $K = 2$ theory is the standard Type II superstring theory,[6] expressing its partition function in terms of string functions rather than theta-functions can be accomplished simply using the following set of identities:

$$K=2: \begin{cases} 2\eta^2(c^1_1)^2 = \vartheta_2/\eta \,; \\ \eta^2(c^0_0 + c^2_0)^2 = \vartheta_3/\eta \,; \\ \eta^2(c^0_0 - c^2_0)^2 = \vartheta_4/\eta \,. \end{cases} \tag{2.6}$$

For each spacetime dimension in these theories, a term in the partition function of the form (2.2b) is tensored with the partition function $Z(X)$ for an uncompactified chiral boson $X(z)$. Since,

$$Z(X(z)) \propto \frac{1}{\eta(\tau)} \,, \tag{2.7}$$

the $\eta(\tau)$ factors cancel out in $Z(\phi^j_m(z)) \times Z(X(z))$. Similar cancellation of $\bar\eta(\bar\tau)$ occurs in the antiholomorphic sector. In the following partition functions, we generally suppress the trivial factor of $(\operatorname{Im}\tau)^{-s/K}$ contributed together by the $D-2$ holomorphic and anti-holomorphic world sheet boson partition functions.

2.1. Derivation of the Partition Functions. By the string function equivalences, the partition functions for the level-K fractional superstrings in refs. [16, 7, 10, 11] in critical spacetime dimensions $D = 2 + 16/\mathrm{K} = 10, 6, 4$, and 3 can be written (in light-cone gauge as:

$D = 10 \quad (K = 2)$: $\qquad Z_2 = |A_2|^2$, where

$$\begin{aligned} A_2 &= \frac{1}{2}\{(c^0_0 + c^2_0)^8 - (c^0_0 - c^2_0)^8\}_{\text{boson}} - 8(c^1_1)^8_{\text{fermion}} \\ &= 8\{(c^0_0)^7 c^2_0 + 7(c^0_0)^5(c^2_0)^3 + 7(c^0_0)^3(c^2_0)^5 + c^0_0(c^2_0)^7\}_{\text{boson}} \\ &\quad - 8(c^1_1)^8_{\text{fermion}} \,, \end{aligned} \tag{2.8}$$

$D = 6 \quad (K = 4)$: $\qquad Z_4 = |A_4|^2 + 3|B_4|^2$, where

$$\begin{aligned} A_4 &= 4\{(c^0_0 + c^4_0)^3(c^2_0) - (c^2_0)^4\} \\ &\quad + 4\{(c^0_2 + c^4_2)^3(c^2_2) - (c^2_2)^4\} \,, \end{aligned} \tag{2.9a}$$

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[6] The $K = 2$ parafermion model is a $c = \frac{1}{2}$ CFT that corresponds to a critical Ising (free fermion) model.

$$B_4 = 4\{(c_0^0 + c_0^4)(c_2^0 + c_2^4)^2(c_0^2) - (c_0^2)^2(c_2^2)^2\}$$
$$+ 4\{(c_2^0 + c_2^4)(c_0^0 + c_0^4)^2(c_2^2)^2 - (c_2^2)^2(c_0^2)^2\}\ , \tag{2.9b}$$

$D = 4 \quad (K = 8)$: $\quad Z_8 = |A_8|^2 + |B_8|^2 + 2|C_8|^2$, where

$$A_8 = 2\{(c_0^0 + c_0^8)(c_0^2 + c_0^6) - (c_0^4)^2\}$$
$$+ 2\{(c_4^0 + c_4^8)(c_4^2) - (c_4^6) - (c_4^4)^2\}\ , \tag{2.10a}$$

$$B_8 = 2\{(c_0^0 + c_0^8)(c_4^2 + c_4^6) - (c_0^4 c_4^4)\}$$
$$+ 2\{(c_4^0 + c_4^8)(c_0^2 + c_0^6) - (c_4^4 c_0^4)\}\ , \tag{2.10b}$$

$$C_8 = 2\{(c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2\}$$
$$+ 2\{(c_2^0 + c_2^8)(c_2^2 + c_2^6) - (c_2^4)^2\}\ , \tag{2.10c}$$

$D = 3 \quad (K = 16)$: $\quad Z_{16} = |A_{16}|^2 + |C_{16}|^2$, where

$$A_{16} = \{(c_0^2 + c_0^{14}) - c_0^8\}$$
$$+ \{(c_8^2 + c_8^{14}) - c_0^8\} \tag{2.11a}$$

$$C_{16} = \{(c_4^2 + c_4^{14}) - c_4^8\}$$
$$+ \{(c_4^2 + c_4^{14}) - c_4^8\}\ . \tag{2.11b}$$

The $D = 10$ partition function, written in string function format, was obtained by the authors of refs. [16, 10] as a check of their program, both by computer generation and by the $K = 2$ string functions/Jacobi ϑ-functions equivalences. In each model, the massless spin-2 particle and its supersymmetric partner arise from the A_K-sector. The B_K- and C_K-sectors were obtained by acting on the A_K-sector with the $SL(2, \mathbb{Z})$ modular group generators, $S: \tau \to -1/\tau$, and $T: \tau \to \tau + 1$. At each level of K, the contribution of each sector is separately zero. This is consistent with spacetime SUSY and suggests cancellation between bosonic and fermionic terms at each mass level. This leads to the following identities [16].

$$A_2 = A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0\ . \tag{2.12}$$

The factorization method of Gepner and Qiu [18] for string function partition functions allows us to rederive the above partition functions systematically. Gepner and Qiu have shown that we can express any general modular invariant parafermionic partition function,

$$Z = |\eta|^2 \sum N_{l,n,\bar{l},\bar{n}} c_n^l \bar{c}_n^{\bar{l}}\ , \tag{2.13a}$$

in the form

$$Z = |\eta|^2 \sum \frac{1}{2} L_{l,\bar{l}} M_{n,\bar{n}} c_n^l \bar{c}_{\bar{n}}^{\bar{l}}\ , \tag{2.13b}$$

(with $c^{l=2j}_{n=2m} = 0$ unless $l - n \in 2\mathbb{Z}$ since $\phi^j_m = 0$ for $j - m \notin \mathbb{Z}$). This factorization of $N_{l,n,\bar{l},\bar{n}}$ results from the transformation properties of c^l_n under the modular group generators S and T:

$$S : c^l_n \to \frac{1}{\sqrt{-i\tau K(K+2)}} \sum_{l'=0}^{K} \sum_{\substack{n'=-K+1 \\ l'-n' \in 2\mathbb{Z}}}^{K} \exp\left\{\frac{i\pi n n'}{K}\right\} \sin\left\{\frac{\pi(l+1)(l'+1)}{K+2}\right\} c^{l'}_{n'} , \tag{2.14a}$$

$$T : c^l_n \to \exp\left\{2\pi i \left\{\frac{l(l+2)}{4(K+2)} - \frac{n^2}{4K} - \frac{K}{8(K+2)}\right)\right\} c^l_n , \tag{2.14b}$$

which are an effect of the definition of string function c^l_n (at level-K) in terms of the $SU(2)_K$ affine characters χ_l and the Jacobi theta-function $\vartheta_{n,K}$:[7]

$$\chi_l(\tau) = \sum_{n=-K+1}^{K} c^l_n(\tau)\vartheta_{n,K}(\tau) . \tag{2.15}$$

Gepner and Qiu proved that as a result of the factorization,

$$N_{l,n,\bar{l},\bar{n}} = \frac{1}{2} L_{l,\bar{l}} M_{m,\bar{m}} , \tag{2.16}$$

we can construct all modular invariant partition functions (MIPF's) for parafermions from a product of modular invariant solutions for the $(l, \bar{l})$ and $(n, \bar{n})$ indices separately. That is, Eq. (2.13b) is modular invariant if and only if the $SU(2)$ affine partition function

$$W = \sum_{l,\bar{l}=0}^{K} L_{l,\bar{l}}\chi_l(\tau)\bar{\chi}_{\bar{l}}(\bar{\tau}) , \tag{2.17a}$$

and the $U(1)$ partition function

$$V = \frac{1}{|\eta(\tau)|^2} \sum_{n,\bar{n}=-K+1}^{K} M_{n,\bar{n}}\vartheta_{n,K}\bar{\vartheta}_{\bar{n},K} \tag{2.17b}$$

are simultaneously modular invariant; i.e. $N_{l,n,\bar{l},\bar{n}} = \frac{1}{2}L_{l,\bar{l}}M_{n,\bar{n}}$ belongs to a MIPF (2.13a) if and only if $L_{l,\bar{l}}$ and $M_{n,\bar{n}}$ correspond to MIPF's of the forms (2.17a) and (2.17b), respectively.

The proof of Gepner and Qiu can also be applied to show that for modular invariance of a d-dimensional (where $d = D - 2$) parafermion tensor product theory,

$$Z = |\eta|^2 \sum \mathbf{N}_{\vec{l},\vec{n},\vec{\bar{l}},\vec{\bar{n}}} c^{l_1}_{n_1} c^{l_2}_{n_2} \cdots c^{l_d}_{n_d} c^{\bar{l}_1}_{\bar{n}_1} c^{\bar{l}_2}_{\bar{n}_2} \cdots c^{\bar{l}_d}_{\bar{n}_d} , \tag{2.18a}$$

[7] The associated relationship between the level-K $SU(2)$ primary fields Φ^j and the parafermionic ϕ^j_m is

$$\Phi^j = \sum_{m=-j}^{j} \phi^j_m : \exp\left\{i\frac{m}{\sqrt{K}}\varphi\right\}: ,$$

where φ is the $U(1)$ boson field of the $SU(2)$ theory.

is necessary that there be affine $\times\, U(1)$ factorization,

$$Z = |\eta|^2 \sum \frac{1}{2} \mathbf{L}_{\vec{l},\vec{\bar{l}}} \mathbf{M}_{\vec{\bar{n}},\vec{n}} c^{l_1}_{n_1} c^{l_2}_{n_2} \cdots c^{l_d}_{n_d} c^{\bar{l}_1}_{\bar{n}_1} c^{\bar{l}_2}_{\bar{n}_2} \cdots c^{\bar{l}_d}_{\bar{n}_d} \,, \tag{2.18b}$$

with $\mathbf{L}$, and $\mathbf{M}$ corresponding to d-dimensional modular invariant generalizations of Eqs. (2.17a) and (2.17b).

Due to the nature of a $U(1)$ CFT, it is obvious that a tensor $\mathbf{M}$ corresponding to a MIPF for d-factors of $U(1)$ CFT's can be written as a tensor product of d-independent matrix $\mathbf{M}$ solutions to (2.17b) "twisted" by simple currents $\mathcal{T}$.[8]

In the following pages we demonstrate that the factorization approach to deriving the FSPF's, suggests much about the meaning of the different sectors in fractional superstrings, the related "projection" terms, the origin of spacetime supersymmetry, and the significance of a special $U(1)$ twist current.

2.2. Affine Factor and "W" Partition Function. In the A_K-sectors defined by Eqs. (2.9a, 2.10a, 2.11a), the terms inside the first (upper) set of brackets, carry "$n \equiv 2m = 0$" subscripts and can be shown to correspond to spacetime bosons; while the terms inside the second (lower) set carry "$n = K/2$" and correspond to spacetime fermions. Expressing the A_K-sectors in this form makes a one-to-one correspondence between bosonic and fermionic states in the A_K-sector manifest. If we remove the subscripts on the string functions in the bosonic and fermionic subsectors (which is parallel to replacing c^l_k with χ_l) we find the subsectors become equivalent. In fact, under this operation of removing the "n" subscripts and replacing each string function by its corresponding affine character (which we will denote by $\overset{\text{affine}}{\Longrightarrow}$), all sectors become equivalent up to an integer coefficient:

$D = 6 \quad (K = 4)$:

$$A_4, B_4 \overset{\text{affine}}{\Longrightarrow} A_4^{\text{aff}} \equiv (\chi_0 + \chi_K)^3 \chi_2 - (\chi_{K/2})^4 \,, \tag{2.19a}$$

$D = 4 \quad (K = 8)$:

$$A_8, B_8\, C_8 \overset{\text{affine}}{\Longrightarrow} A_8^{\text{aff}} \equiv (\chi_0 + \chi_K)(\chi_2 + \chi_{K-2}) - (\chi_{K/2})^2 \,, \tag{2.19b}$$

$D = 3 \quad (K = 16)$:

$$A_{16}, C_{16} \overset{\text{affine}}{\Longrightarrow} A_{16}^{\text{aff}} \equiv (\chi_2 + \chi_{K-2}) - \chi_{K/2} \,. \tag{2.19c}$$

We see that the B- and C-sectors are not arbitrary additions, necessitated only by modular invariance, but rather are naturally related to the physically motivated A-sectors: the corresponding affine partition function is the same for each sector.

[8] A simple current, $\mathcal{T}$, is a primary field of a CFT which, when fused with any other primary field, Φ_l, (including itself) in the CFT produces only a single primary field as a product state: $\mathcal{T} \times \Phi_l = \Phi_{l'}$.

Further, the affine characters, A_K^{aff} in Eqs. (2.19a, 2.19b, 2.19c) all have the general form

$$A_K^{\text{aff}} \equiv (\chi_0 + \chi_K)^{D-3}(\chi_2 + \chi_{K-2}) - (\chi_{K-2})^{D-2} \tag{2.20}$$

(with $(\chi_2 + \chi_{K-2})$ replaced by χ_2 for $K = 16$, since then $\chi_2 = \chi_{K-2}$). The corresponding partition function is

$$W_K = |A_K^{\text{aff}}|^2 . \tag{2.21}$$

This class of partition functions (2.21) is indeed modular invariant and possesses special qualities. (Note that the modular invariance of W requires A_K^{aff} to transform back into itself under S.) This is easiest to show for $K = 16$. The $SU(2)_{16}$ MIPF's for $D = 3$ are trivial to classify since at this level the A–D–E classification forms a complete basis set of modular invariants, even for MIPF's containing terms with negative coefficients. The only free parameters in $K = 16$ affine partition functions $Z(SU(2)_{16})$ are integers a, b, and c where

$$Z(SU(2)_{K=16}) = a(Z(\mathrm{A}_{17}) + bZ(\mathrm{D}_{10}) + cZ(\mathrm{E}_7) . \tag{2.22}$$

Demanding that neither a left- nor a right-moving tachyonic state be in the Hilbert space of states in the $K = 16$ fractional superstring when the intercept v, defined by

$$L_0|\text{physical}\rangle = v|\text{physical}\rangle , \tag{2.23}$$

is positive, removes these degrees of freedom and requires $a = -(b + c) = 0$, independent of the possible $U(1)$ partition functions. These specific values for a, b, and c give us (2.21) for this level:

$$W_{16} = Z(\mathrm{D}_{10}) - Z(\mathrm{E}_7) = |A_{16}^{\text{aff}}|^2 . \tag{2.24}$$

The corresponding partition functions for $K = 8$ and 4 can also be expressed as the difference of two known partition functions: W_8 is the difference between a $\mathrm{D}_6 \otimes \mathrm{D}_6$ MIPF and an exceptional MIPF derived using the conformal embedding

$$SU(2)^{(1)}_{K=8} \otimes SU(2)^{(1)}_{K=8} \otimes SO(8)^{(1)}_{K=4} \subset SO(32)^{(1)}_{K=1} \tag{2.25}$$

and the triality of $SO(8)$. [17, 29] W_4 is similarly the difference between $\mathrm{D}_4 \otimes \mathrm{D}_4 \otimes \mathrm{D}_4 \otimes \mathrm{D}_4$ MIPF and a simple current invariant of $\mathrm{D}_4 \otimes \mathrm{D}_4 \otimes \mathrm{D}_4 \otimes \mathrm{D}_4$. Although it is not possible to create the needed simple currents from $SU(2)_4$ fields, this may be realized using those of $SU(3)_1$. This is possible because the D_4 invariant of $SU(2)_4$ is equivalent to the diagonal invariant of $SU(3)_1$. When rewritten in $SU(3)_1$ language, the $[SU(2)_4]^4$ tensor product field $(\Phi^0 + \Phi^K)^3\Phi^2(\bar{\Phi}^{K/4})^4$ becomes the simple current, $(\Phi^0_{SU(3)})^3\bar{\Phi}^{\pm 3}_{SU(3)}(\bar{\Phi}^{\pm 3}_{SU(3)})^4$ of $[SU(3)_1]^4$, where $\Phi^0_{SU(3)}$ is the $SU(3)_1$ identity field, and $\Phi^{+3}_{SU(3)}$ and $\Phi^{-3}_{SU(3)}$ denote the 3 and $\bar{3}$ representation fields, respectively.

The parafermion partition function corresponding to Eqs. (2.20) and (2.21) is

$$Z_K(\text{affine factor}) = |(c_0^0 + c_0^K)^{D-3}(c_0^2 + c_0^{K-2}) - (c_0^{K/2})^{D-2}|^2 , \tag{2.26a}$$

representing the states created by the fields

$$\begin{aligned}&((\phi_0^0 + \phi_0^{K/2})^{D-3}(\phi_0^1 + \phi_0^{(K-2)/2}) - (\phi_0^{K/4})^{D-2})\\ &((\bar{\phi}_0^0 + \phi_0^{K/2})^{D-3}(\bar{\phi}_0^1 + \phi_0^{(K-2)/2}) - (\bar{\phi}_0^{K/4})^{D-2})\end{aligned} \tag{2.26b}$$

acting on the parafermion vacuum. In "factoring" the parafermion partition functions (2.9a, b), (2.10a–c), and (2.11a, b), into separate partition functions for the affine and $U(1)$ contributions we do not intend to imply that the string functions can actually be factored into $c_0^1 \times c_n^0 = c_n^l$, nor do we imply that Z_K (affine factor) above or $Z_K(U(1)$ factor) presented in the following subsection are modular invariant. Rather, we mean to use partition functions (2.26a) and (2.30b, 2.32b, 2.34b) only as an artificial construct for developing a deeper understanding of the function of the parafermion primary fields ϕ_0^j and ϕ_m^0 in these models. Though not true for the partition functions, factorization is, indeed, valid for the primary fields, ϕ_m^j: $\phi_0^j \otimes \phi_m^0 = \phi_m^j$ (for integer j, m).

2.3. $U(1)$ Factor and the "V" Partition Function. We now consider the $U(1)$ factor, $\mathbf{M}$, carrying the $(n, \bar{n})$-indices in the FSPF's. Since all A_K-, and B_K-, and C_K-sectors in the level-K fractional superstring partition function (and even the boson and fermion subsectors separately in A_K) contain the same affine factor, it is clearly the choice of the $U(1)$ factor which determines the spacetime supersymmetry of the fractional superstring theories. That is, spacetime spins of particles in the Hilbert space of states depend upon the $\mathbf{M}$'s that are allowed in tensored versions of Eq. (2.17b). In the case of matrix $\mathbf{M}$ rather than a more complicated tensor, invariance of (2.17b) under S requires that the components $M_{n\bar{n}}\phi_m^0$ be related by

$$M_{n',n'} = \frac{1}{2K} \sum_{n,\bar{n}} M_{n,\bar{n}} e^{i\pi nn'/K} e^{i\pi \bar{n}\bar{n}'/K} , \tag{2.27a}$$

and T invariance demands that

$$\frac{n^2 - \bar{n}^2}{4K} \in \mathbb{Z} , \quad \text{if } M_{n,\bar{n}} \neq 0 . \tag{2.27b}$$

At every level-K there is a unique modular invariant function corresponding to each factorization [18], $\alpha \times \beta = K$, where $\alpha, \beta \in \mathbb{Z}$. Denoting the matrix elements of $\mathbf{M}^{\alpha,\beta}$ by $M_{n,\bar{n}}^{\alpha,\beta}$, they are given by

$$M_{n,\bar{n}}^{\alpha,\beta} = \frac{1}{2} \sum_{\substack{x \in \mathbb{Z}_{2\beta} \\ y \in \mathbb{Z}_{2\alpha}}} \delta_{n,\alpha x + \beta y} \delta_{\bar{n}, \alpha x - \beta y} . \tag{2.28}$$

By (2.28), $M_{n,\bar{n}}^{\alpha,\beta} = M_{n,\bar{n}}^{\beta,\alpha}$. Hence, $\mathbf{M}^{\alpha,\beta}$ and $\mathbf{M}^{\beta,\alpha}$ result in equivalent FSPF's. To avoid this redundance, we demand that $\alpha \leqq \beta$.

Thus, for $K = 4$ the two distinct choices for the matrix $\mathbf{M}^{\alpha,\beta}$ are $\mathbf{M}^{1,4}$ and $\mathbf{M}^{2,2}$; for $K = 8$, we have $\mathbf{M}^{1,8}$ and $\mathbf{M}^{2,4}$; and for $K = 16$, the three alternatives are $\mathbf{M}^{1,16}$, $\mathbf{M}^{2,8}$, and $\mathbf{M}^{4,4}$. $\mathbf{M}^{1,K}$ represents the level-K diagonal, $n = \bar{n}$, partition function. $\mathbf{M}^{\alpha,\beta=\frac{K}{\alpha}}$ corresponds to the diagonal partition function twisted by a $\mathbb{Z}_\alpha$ symmetry. (Twisting by $\mathbb{Z}_\alpha$ and $\mathbb{Z}_{K/\alpha}$ produce isomorphic models.) Our investigations revealed that all possible simple tensor product combinations of these $\mathbf{M}^{\alpha,\beta}$ matrices are insufficient for producing fractional superstrings with spacetime SUSY (and, thus, no tachyons). We have found that twisting by a special simple $U(1)$ current (shown below) is required to achieve this. Of the potential choices of $\mathbf{M}$ from a $U(1)$ MIPF that could be combined with $\mathbf{L}$ from an affine MIPF, one can show (as indicated from string function identities) that the following are the only ones producing numerically zero FSPF's:

$D = 6 \quad (K = 4)$:

The $\mathbf{M} = \mathbf{M}^{2,2} \otimes \mathbf{M}^{2,2} \otimes \mathbf{M}^{2,2} \otimes \mathbf{M}^{2,2}$ model twisted by the simple $U(1)$ current[9]

$$\mathscr{T}_4 \equiv \phi^0_{K/4}\phi^0_{K/4}\phi^0_{K/4}\phi^0_{K/4}\phi^0_{K/4}\bar{\phi}^0_0\bar{\phi}^0_0\bar{\phi}^0_0\bar{\phi}^0_0 \tag{2.29}$$

has the following $U(1)$ partition function:

$$\begin{aligned} V_4 = {} & [(\vartheta_{0,4} + \vartheta_{4,4})^4(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4 + (\vartheta_{2,4} + \vartheta_{-2,4})^4(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^4 \\ & + (\vartheta_{0,4} + \vartheta_{4,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2 \\ & + (\vartheta_{2,4} + \vartheta_{-2,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4]_{\text{untwisted}} \\ & + [(\vartheta_{2,4} + \vartheta_{-2,4})^4(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4 + (\vartheta_{4,4} + \vartheta_{0,4})^4(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^4 \\ & + (\vartheta_{2,4} + \vartheta_{-2,4})^2(\bar{\vartheta}_{4,4} + \bar{\vartheta}_{0,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2 \\ & + (\vartheta_{4,4} + \vartheta_{0,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2(\bar{\vartheta}_{2,4} + \bar{\vartheta}_{-2,4})^2(\bar{\vartheta}_{0,4} + \bar{\vartheta}_{4,4})^4]_{\text{untwisted}} \,. \end{aligned} \tag{2.30a}$$

Writing this in parafermionic form, and then using string function identities, followed by regrouping according to A_4 and B_4 components, results in

$$Z_4(U(1)\,\text{factor}) = |((c^0_0)^4 + (c^0_2)^4|^2_{(A_4)} + |((c^0_0)^2(c^0_2) + (c^0_2)^2(c^0_0)^2|^2_{(B_4)} \,, \tag{2.30b}$$

which represents the tensor product primary fields

$$\begin{aligned} & \{[(\phi^0_0)^4 + (\phi^0_1)^4][(\bar{\phi}^0_0)^4 + (\bar{\phi}^0_1)^4] \\ & \quad + [(\phi^0_0)^2(\phi^0_1)^2 + (\bar{\phi}^0_0)^2(\phi^0_1)^2][(\bar{\phi}^0_0)^2(\bar{\phi}^0_1)^2 + (\bar{\phi}^0_1)^2(\phi^0_0)^2]\} \end{aligned} \tag{2.30c}$$

acting on the parafermion vacuum.

$D = 4 \quad (K = 8)$:

The $\mathbf{M} = \mathbf{M}^{2,4} \otimes \mathbf{M}^{2,4}$ model twisted by the simple $U(1)$ current

$$\mathscr{T}_8 \equiv \phi^0_{K/4}\phi^0_{K/4}\bar{\phi}^0_0\bar{\phi}^0_0 \tag{2.31}$$

corresponds to

$$\begin{aligned} V_8 = {} & [(\vartheta_{0,8} + \vartheta_{8,8})(\bar{\vartheta}_{0,8} + \bar{\vartheta}_{8,8}) + (\vartheta_{4,8} + \vartheta_{-4,8})^8(\bar{\vartheta}_{4,8} + \bar{\vartheta}_{-4,8})]^2_{\text{untwisted}} \\ & + [(\vartheta_{2,8} + \vartheta_{-6,8})^2(\bar{\vartheta}_{2,8} + \bar{\vartheta}_{-6,8}) + (\vartheta_{-2,8} + \vartheta_{6,8})(\bar{\vartheta}_{-2,8} + \bar{\vartheta}_{6,8})]_{\text{untwisted}} \\ & + [(\vartheta_{4,8} + \vartheta_{-4,8})^8(\bar{\vartheta}_{0,8} + \bar{\vartheta}_{8,8}) + (\vartheta_{0,8} + \vartheta_{8,8})(\bar{\vartheta}_{4,8} + \bar{\vartheta}_{-4,8})]^2_{\text{twisted}} \\ & + [(\vartheta_{6,8} + \vartheta_{-2,8})(\bar{\vartheta}_{2,8} + \bar{\vartheta}_{-6,8}) + (\vartheta_{2,8} + \vartheta_{-6,8})(\bar{\vartheta}_{-2,8} + \bar{\vartheta}_{6,8})]^2_{\text{untwisted}} \,. \end{aligned} \tag{2.32a}$$

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[9] Recall that the parafermion primary fields ϕ^0_m have simple fusion rules, $\phi^0_m \otimes \phi^0_{m'} = \phi^0_{m+m' \,(\text{mod}\, K)}$ (for $m, m' \in \mathbb{Z}$) and form a $\mathbb{Z}_K$ closed subalgebra. This fusion rule, likewise, holds for the $U(1)$ fields: $\exp\{i^m_K \varphi\}$:. The isomorphism makes it clear that any simple current, $\mathscr{T}_K$, in this subsection that contains only integer m can be expressed equivalently either in terms of these parafermion fields or in terms of $U(1)$ fields. In view of the following discussion, we express all of the simple twist currents, $\mathscr{T}_K$, in parafermion language.

Hence, we have

$$Z_8(U(1)\,\text{factor}) = |(c_0^0)^2 + (c_4^0)^2|^2_{(A_8)} + |(c_0^0)(c_4^0) + (c_4^0)(c_0^0)|^2_{(B_8)} + 4|(c_2^0)^2|^2_{(C_8)}\,, \tag{2.32b}$$

with the related tensor product primary fields

$$\{[(\phi_0^0)^2 + (\phi_2^0)^2][\bar\phi_0^0)^2 + (\bar\phi_1^0)^2] + (\phi_0^0\phi_2^0 + \phi_2^0\phi_0^0)(\bar\phi_0^0\bar\phi_2^0 + \bar\phi_2^0\bar\phi_0^0) + 4(\phi_1^0)^2(\bar\phi_1^0)\}\,. \tag{2.32c}$$

$D = 3 \quad (K = 16)$:

The $\mathbf{M} = \mathbf{M}^{4,4}$ model twisted by the simple $U(1)$ current

$$\mathscr{T}_{16} \equiv \phi_{K/4}^0\bar\phi_0^0 \tag{2.33}$$

produces,

$$V(K=16) = |(\vartheta_{0,16} + \vartheta_{16,16}) + (\vartheta_{8,16} + \vartheta_{-8,16})|^2_{\text{untwisted}} + |(\vartheta_{4,16} + \vartheta_{-4,16}) + (\vartheta_{12,16} + \vartheta_{-12,16})|^2_{\text{untwisted}}\,. \tag{2.34a}$$

Thus, the corresponding parafermion partition function is

$$Z_{16}(U(1)\,\text{factor}) = |c_0^0 + c_8^0|^2_{(A_{16})} + 4|c_4^0|^2_{(C_{16})}\,, \tag{2.34b}$$

and its primary fields are

$$\{(\phi_0^0 + \phi_4^0)(\bar\phi_0^0 + \bar\phi_4^0) + 4(\phi_2^0)(\bar\phi_2^0)\}\,. \tag{2.34c}$$

(In this case the twisting is trivial since $\mathscr{T}_{16}$ is already present in the initial untwisted model.)

We wish to point out that the partition function for the standard $D = 10$ superstring can also be factored into affine and $U(1)$ parts:

$D = 10 \quad K = 2$;

$$A_2 \overset{\text{affine}}{\Longrightarrow} \sum_{i\,\text{odd}=1}^{7} \binom{8}{i} (\chi_0)^i(\chi_K)^{8-i} - (\chi_{K/2})^8\,. \tag{2.35a}$$

The accompanying $U(1)$ factor is

$$V_2 = |(\vartheta_{0,2})^8 + (\vartheta_{1,2})^8 + (\vartheta_{-1,2})^8 + (\vartheta_{2,2})^8|^2 + 35|(\vartheta_{0,2} + \vartheta_{2,2})^4(\vartheta_{1,2} + \vartheta_{-1,2})^4|^2 + 35[(\vartheta_{1,2} + \vartheta_{-1,2})^4(\vartheta_{0,2} + \vartheta_{2,2})^4][(\bar\vartheta_{0,2} + \bar\vartheta_{2,2})^4(\bar\vartheta_{1,2} + \bar\vartheta_{-1,2})^4]\,. \tag{2.35b}$$

This originates from the

$$\mathbf{M} = \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1} \otimes \mathbf{M}^{2,1}$$

model being twisted by the simple $U(1)$ current

$$\mathscr{T}_2^{\text{theta}} \equiv (:\exp\{i\varphi/2\}:)^{10}\,. \tag{2.36}$$

The difference between the factorization for $K = 2$ and those for $K > 2$ is that here we cannot define an actual parafermion twist current $(\phi^0_{K/4})$ since $\phi^0_{K/4} = 0$ for $K = 2$. Relatedly, the effective $Z_2(U(1)$ factor) contributing to Eq. (2.8) reduces to just the first mod-squared term in Eq. (2.35b) since $c^l_n \equiv 0$ for $l - n \neq 0 \pmod 2$.

All of the above simple twist currents for $K > 2$ are of the general form

$$\mathscr{T}_K = (\phi^0_{K/4})^{D-2}(\bar{\phi}^0_0)^{D-2}\,. \tag{2.37}$$

(Note that $\bar{\mathscr{T}}_K \equiv (\phi^0_0)^{D-2}(\bar{\phi}^0_{K/4})^{D-2}$ is automatically generated as a twisted field also.) We believe this specific class of twist currents is the key to spacetime supersymmetry in the parafermion models. Without the twisting effects of $\mathscr{T}_K$, numerically zero modular invariant FSPF's in three, four, and six dimensions cannot be formed and thus spacetime SUSY would be impossible. This twisting also reveals much about the necessity of non-A_K-sectors. Terms from the twisted and untwisted sectors of these models become equally mixed in the $|A_K|^2$, $|B_K|^2$, and $|C_K|^2$ contribution to the level-K partition function. Further, this twisting keeps the string functions with $n \not\equiv 0,\, K/2 \pmod K$ from mixing with those possessing $n \equiv 0,\, K/2 \pmod K$. This is especially significant since we believe the former string functions in the C_K-sector likely correspond to spacetime fields of fractional spin-statistics (*i.e.*, anyons) and the latter in both A_K and B_K to spacetime bosons and fermions. If mixing were allowed, normal spacetime SUSY would be broken and replaced by a fractional supersymmetry, most-likely ruining Lorentz invariance for $D > 3$.

Since in the antiholomorphic sector $\mathscr{T}_K$ acts as the identity, we will focus on its effect in the holomorphic sector. In the A_K-sector the operator $(\phi^0_{K/4})^{D-2}$ transforms the bosonic (fermionic) nonprojection[8] fields into the fermionic (bosonic) projection fields and vice-versa.

For example, consider the effect of this twist current on the represented in

$$A_4 \equiv A_4^{\text{boson}} - A_4^{\text{fermion}}\,, \tag{2.38a}$$

where

$$A_4^{\text{boson}} = 4\{(c^0_0 + c^4_0)^3(c^2_0) - (c^2_0)^4\}\,, \tag{2.38b}$$

$$A_4^{\text{fermion}} = 4\{(c^2_2)^4 - (c^0_2 + c^4_2)^3(c^2_2)\}\,. \tag{2.38c}$$

Twisting by $(\phi^0_{K/2})^{D-2}$ transforms the related fields as

$$(\phi^0_0 + \phi^2_0)^3(\phi^1_0) \overset{(\phi^0_{K/4})^{D-2}}{\Leftrightarrow} (\phi^2_0 + \phi^0_1)^3(\phi^1_1)\,, \tag{2.39a}$$

$$(\phi^0_1)^4 \overset{(\phi^0_{K/4})^{D-2}}{\Leftrightarrow} (\phi^1_1)^4\,. \tag{2.39b}$$

[10] We use the same language as the authors of refs. [11]. Nonprojection refers to the bosonic and fermionic fields in the A_K^{boson} and A_K^{fermion} subsectors, respectively, corresponding to string functions with positive coefficients, whereas projection fields refer to those corresponding to string functions with negative signs. With this definition comes an overall minus sign coefficient on A_K^{fermion}, as shown in Eq. (2.39a). For example, in (2.39b), the bosonic non-projection fields are $(\phi^0_0 + \phi^2_0)^3(\phi^1_0)$ and the bosonic projection is $(\phi^1_0)^4$. Similarly, in (2.39c) the fermionic non-projection field is $(\phi^1_1)^4$ and the projections are $(\phi^0_1 + \phi^2_1)^3(\phi^1_1)$.

Although the full meaning of the projection fields is not yet understood, the authors of refs. [7] and [11] argue that the corresponding string functions should be interpreted as "internal" projections, *i.e.*, cancellations of degrees of freedom in the fractional superstring models. (See also [14, 15, and 13]). Relatedly, the authors show that when the A_K-sector is written as $A_K^{\text{boson}} - A_K^{\text{fermion}}$, as done above, the q-expansions of both A_K^{boson} and A_K^{fermion} are all positive. Including the fermionic projection terms results in the identity

$$\eta^{D-2} A_K^{\text{fermion}} = (D-2)\left(\frac{(\vartheta_2)^4}{16\eta^4}\right)^{\frac{D-2}{8}} . \tag{2.40a}$$

Equation (2.40a) is the standard theta-function expression for $D-2$ world sheet Ramond Majorana–Weyl fermions. Further,

$$\eta^{D-2} A_K^{\text{boson}} = (D-2)\left(\frac{(\vartheta_3)^4 - (\vartheta_4)^4}{16\eta^4}\right)^{\frac{D-2}{8}} . \tag{2.40b}$$

Now consider the B_K-sectors. For $K=4$ and 8 the operator $(\phi^0_{K/4})^{D-2}$ transforms the primary fields corresponding to the partition functions terms in the first set of brackets on the RHS of Eqs. (2.9b, 2.10b) into the fields represented by the partition functions terms in the second set. For example, in the $K=4$ ($D=6$) case,

$$(\phi^0_0 + \phi^2_0)(\phi^1_0)(\phi^0_1 + \phi^2_1)^2 \overset{(\phi^0_{K/4})^{D-2}}{\Leftrightarrow} (\phi^2_0 + \phi^0_1)(\phi^1_1)(\phi^2_0 + \phi^0_0)^2 , \tag{2.41a}$$

$$(\phi^0_1)^2(\phi^1_1)^2 \overset{(\phi^0_{K/4})^{D-2}}{\Leftrightarrow} (\phi^1_1)^2(\phi^1_0)^2 . \tag{2.41b}$$

Making an analogy with what occurs in the A_K-sector, we suggest that $(\phi^0_{K/4})^{D-2}$ transforms bosonic (fermionic) nonprojection fields into fermionic (bosonic) projection fields and vice-versa in the B_K-sector also. Thus, use of the twist current $\mathscr{T}_K$ allows for bosonic and fermionic interpretation of these fields[11]:

$$B_4 \equiv B_4^{\text{boson}} - B_4^{\text{fermion}} , \tag{2.42a}$$

where

$$B_4^{\text{boson}} = 4\{(c^0_0 + c^4_0)(c^2_0)(c^0_2 + c^4_2)^2 - (c^2_0)^2(c^2_2)^2\} , \tag{2.42b}$$

$$B_4^{\text{fermion}} = 4\{(c^2_2)^2(c^2_0)^2 - (c^0_2 + c^4_2)(c^2_2)(c^0_0 + c^4_0)^2\} . \tag{2.42c}$$

What appears as the projection term, $(c^2_0)^2(c^2_2)^2$, for the proposed bosonic part acts as the nonprojection term for the fermionic half when the subscripts are reversed. One interpretation is this implies a compactification of two transverse dimensions.[12] Let us choose the compactified directions to correspond to the last

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[11] Similar conclusions have been reached by K. Dienes and P. Argyres for different reasons. They have, in fact, found theta-function expressions for the B_K^{boson}- and B_K^{fermion}-subsectors [15].

[12] This was also suggested in ref. [7] working from a different approach.

2 strings functions in a term. Thus, the spin-statistics of the physical states of the $D = 6$ model as observed in four-dimensional uncompactified spacetime is determined by the (matching) n subscripts of the first two string functions (corresponding to the two uncompactified transverse dimensions) in each term of four string functions, $c_n^{l_1} c_n^{l_2} c_{n'}^{l_3} c_{n'}^{l_4}$. (Assuming instead that the first two string functions corresponded to the compactified dimensions, means interchanging the definitions of B_4^{boson} and B_4^{fermion} above.) The B_8 terms can be interpreted similarly when one dimension is compactified.

However, the C_K-sectors are harder to interpret. Under $(\phi_{K/4}^0)^{D-2}$ twisting, string functions with $K/4$ subscripts are invariant, transforming back into themselves. Thus, following the pattern of A_K and B_K we would end up writing, for example, C_{16} as

$$C_{16} = C_{16}^a - C_{16}^b \,, \tag{2.43a}$$

where,

$$C_{16}^a = (c_4^2 + c_4^{14}) - c_4^8 \,, \tag{2.43b}$$

$$C_{16}^b = c_4^8 - (c_4^2 + c_4^{14}) \,. \tag{2.43c}$$

The transformations of the corresponding primary fields are not quite as trivial, though. $(\phi_2^1 + \phi_2^7)$ is transformed into its conjugate field $(\phi_{-2}^7 + \phi_{-2}^1)$ and likewise ϕ_2^4 into ϕ_{-2}^4, suggesting that C_{16}^a and C_{16}^b are the partition functions for conjugate fields. Remember, however, that $C_{16} = 0$. Even though we may interpret this sector as containing two conjugate spacetime fields, this (trivially) means that the partition function for each is identically zero. We refer to this effect in the C_K-sector as "self-cancellation." One interpretation is that there are no states in the C_K sector of the Hilbert space that survive all of the internal projections. If this is correct, a question may arise as to the consistency of the $K = 8$ and 16 theories. Alternatively, perhaps anyon statistics allow two (interacting?) fields of either identical fractional spacetime spins $s_1 = s_2 = 2m/K$, or spacetime spins related by $s_1 = 2m/K = 1 - s_2$, where in both cases $0 < m < K/2 \pmod 1$, to somehow cancel each other's contribution to the partition function.

Using the $\phi_m^j \equiv \phi_{m+K}^j \equiv \phi_{m-\frac{K}{2}}^{\frac{K}{2}-j}$ equivalences at level-$K \in 4\mathbb{Z}$, a PCFT has $K/2$ distinct classes of integer m values. If one associates these classes with distinct spacetime spins (statistics) and assumes m and $-m$ are also in the same classes since $(\phi_m^0)^\dagger = \phi_{-m}^0$, then the number of spacetime spin classes reduces to $K/4 + 1$. Since $m = 0$ ($m = K/4$) is associated with spacetime bosons (fermions), we suggest that general m correspond to particles of spacetime spin $2m/K$, $2m/K + \mathbb{Z}^+$, or $\mathbb{Z}^+ - 2m/K$. If this is so, most likely $\text{spin}(m) \in \{2m/K, \mathbb{Z}^+ + 2m/K\}$ for $0 < m < K/4 \pmod{K/2}$ and $\text{spin}(m) \in \mathbb{Z}^+ - 2m/K$ for $-K/4 < m < 0 \pmod{K/2}$. This is one of the few spin assignment rules that maintains the equivalences of the fields ϕ_m^j under $(j, m) \to (k/2 - j, m - K/2) \to (j, m + K)$ transformations. According to this rule, the fields in the C_K-sectors have quarter spins (statistics), which agrees with prior claims [16, 7, 10].

Also, we do not believe products of primary fields in different m classes in the B_K-sectors correspond to definite spacetime spin states unless some dimensions are compactified. Otherwise by our interpretation of m values above, Lorentz invariance in uncompactified spacetime would be lost. In particular, Lorentz invariance requires that either all or none of the transverse modes in uncompactified spacetime be fermionic spinors. Further, B-sector particles cannot correspond to fractional spacetime spin particles for a consistent theory. Thus, the $D = 6(4)$ model must have two (one) of its dimensions compactified. (This implies that the $D = 6$, 4 partition functions are incomplete: momentum (and winding) factors for the two compactified dimensions would have to be added, while maintaining modular invariance.)

Note that the B_8-sector of the $D = 4$ model appears for more reasons than just modular invariance of the theory. By the above spacetime spin assignments, this model suggests massive spin-quarter states (anyons) in the C_K-sectors, which presumably cannot exist in $D > 3$ uncompactified dimensions. However, the B_K-sector, by forcing compactification to three dimensions where anyons are allowed, would save the model, making it self-consistent. Of course, anyons in the $K = 16$ theory with $D = 3$ are physically acceptable. (Indeed, no B_K-sector is needed and none exists, which would otherwise reduce the the theory to zero transverse dimensions.) Thus, $K = 8$ and $K = 16$ models are probably both allowed solutions for three uncompactified spacetime dimensional models. If this interpretation is correct, then it is the B_K-sector for $K = 8$ which makes that theory self-consistent.

An alternative, less restrictive, assignment of spacetime spin is possible. Another view is that the m quantum number is not fundamental for determining spacetime spin. Instead, the transformation of states under $(\phi^0_{K/4})^{D-2}$ can be considered to be what divides the set of states into spacetime bosonic and fermionic classes. With this interpretation, compactification in the B_K-sector is no more necessary than in the A_K-sector. Unfortunately, it is not *a priori* obvious, in this approach, which group of states is bosonic, and which fermionic. In the A_K-sector, the assignment can also be made phenomenologically. In the B_K-sector, we have no such guide. Of course, using the m quantum number to determine spacetime spin does not truly tell us which states have bosonic or fermionic statistics, since the result depends on the arbitrary choice of which of the two (one) transverse dimensions to compactify.

A final note of caution involves multiloop modular invariance. One-loop modular invariance amounts to invariance under S and T transformations. However modular invariance at higher orders requires an additional invariance under U transformations: Dehn twists mixing loops of of neighboring tori of $g > 1$ Riemann surfaces [26, 2, 4, 1]. We believe neither our new method of generating the one-loop partitions, nor the original method of Argyres et al. firmly prove the multiloop modular invariance that is required for a truly consistent theory.

Section 3: Beyond the Partition Function: Additional Comments

The previous discussion of the FSPF's in Sect. 2 does not fully demonstrate the consistency of the fractional superstrings, nor does it sufficiently compare them to the $K = 2$ superstring. In this section, we now comment further on these aspects of potential string theories: we consider the analog of the GSO projection, and then discuss the uniqueness of the "twist" field $\phi^{K/4}_{K/4}$ for producing spacetime fermions.

3.1. *Generalized Commutation Relations and the GSO Projection.* One of the major complications of generalizing from the $K = 2$ fermion case to $K > 2$ is that the parafermions (and bosonic field representations) do not have simple commutation relations [31]. What are the commutation relations for non-(half) integral spin particles? Naively, the first possible generalization of standard (anti-)commutation relations for two fields A and B with fractional spins seems to be:

$$AB - e^{[i4\pi \,\mathrm{spin}(A)\,\mathrm{spin}(B)]} BA = 0 \tag{3.1}$$

(which reduces to the expected result for bosons and fermions). This is too simple a generalization, however [30]. Fractional spin particles must be representations of the braid group. Zamolodchikov and Fateev [31] have shown that world sheet parafermions (of fractional spin) have complicated commutation relations that involve an infinite number of modes of a given field. For example:

$$\sum_{l=0}^{\infty} C^{(l)}_{(-1/3)} [A_{n+(1-q)/3-l} A^{\dagger}_{m-(1-q)/3+l} + A_{n+(2-q)/3-l} A^{\dagger}_{n-(2-q)/3+l}]$$
$$= -\frac{1}{2}\left(n - \frac{q}{3}\right)\left(n + 1 - \frac{q}{3}\right)\delta_{n+m,0} + \frac{8}{3c} L_{n+m} \tag{3.2a}$$

and

$$\sum_{l=0}^{\infty} C^{(l)}_{(-2/3)} [A_{n-q/3-l} A_{m+(2-q)/3+l} - A_{m-q/3-l} A_{n+(2-q)/3+l}]$$
$$= \frac{\lambda}{2}(n - m) A^{\dagger}_{(2-2q)/3+n+m}\,, \tag{3.2b}$$

where A is a parafermion field, and L_n are the generators of the Virasoro algebra. λ is a real coefficient, n is integer and $q = 0, 1, 2 \pmod 3$ is a $\mathbb{Z}_3$ charge of Zamolodchikov and Fateev that can be assigned to each primary field in the $K = 4$ model. The coefficients $C^{(l)}_{(\alpha)}$ are determined by the power expansion

$$(1 - x)^{\alpha} = \sum_{l=0}^{\infty} C^{(l)}_{(\alpha)} x^l\,. \tag{3.3}$$

As usual, $c = 2(K-1)/(K+2)$ is the central charge of the level-K PCFT.

These commutation relations are derived from the OPE of the related fields [31]. (Hence more terms in a given OPE should result in more complicated commutation relations.) Similar relations between the modes of two different primary fields should also be derivable from their OPE's. The significance of these commutation relations is that they severely reduce the number of distinct physical states in parafermionic models. There are several equivalent ways of creating a given physical state from the vacuum using different mode excitations from different parafermion primary fields in the same CFT. Thus, the actual Hilbert space of states for this $K = 4$ model will be much reduced compared to the space prior to moding out by these equivalences.[13] Although the fields in the parafermion CFT do not (anti-)commute, but instead have complicated commutation relations, some insight can be gained by comparing the $D = 6$, $K = 4$ FSC model to the standard $D = 10$ superstring. We can, in fact, draw parallels between ε and the standard fermionic superpartner, ψ, of an uncompactified boson X. In the free fermion approach, developed both by Kawai, Lewellen and Tye and by Antoniadis, Bachas and Kounnas, generalized GSO projections based on boundary conditions of the world sheet fermions are formed [26, 2, 4, 1]. Fermions with half-integer modes (NS-type) are responsible for $\mathbb{Z}_1$ (trivial) projections; fermions with integer modes (R-type) induce $\mathbb{Z}_2$ projections. In the non-Ramond sectors these $\mathbb{Z}_2$ projections remove complete states, while in the Ramond sector itself, remove half of the spin modes, giving chirality. Fermions with general complex boundary conditions,

$$\psi(\sigma = 2\pi) = -\, e^{i\pi\frac{a}{b}}\psi(\sigma = 0)\,, \tag{3.4}$$

form in the non-Ramond sector ${}_{2b}$ projections if a is odd and $\mathbb{Z}_b$ projections if a is even (with a and b coprime and chosen in the range $-1 \leqq a/b < 1$). For free-fermionic models, the GSO operator, coming from a sector where the set of world sheet fermions $\{\psi^i\}$ have boundary conditions

$$\psi^i(2\pi) = -\, e^{i\pi x^i}\psi^i(0)\,, \tag{3.5a}$$

and acting on a physical state $|\text{phys}\}_{\vec{y}}$ in a sector where the same fermions have boundary conditions

$$\psi^i(2\pi) = -\, e^{i\pi y^i}\psi^i(0)\,, \tag{3.5b}$$

takes the form,

$$\{e^{i\pi\vec{x}\cdot\vec{F}_{\vec{y}}} = \delta_{\vec{y}}C(\vec{y}\,|\,\vec{x})\}\,|\text{phys}\rangle \tag{3.6}$$

for states surviving the projection. Those states not satisfying the demands of he GSO operator for at least one sector $\vec{x}$ will not appear in the partition function of the corresponding model. In Eq. (3.6), $\vec{F}_{\vec{y}}$ is the (vector) fermion number operator for states in sector $\vec{y}$. $\delta_{\vec{y}}$ is -1 if either the left-moving or right-moving $\psi^{\text{spacetime}}$ are periodic and 1 otherwise. $C(\vec{y}\,|\,\vec{x})$ is a phase with value chosen from an allowed set of order $g_{\vec{y},\vec{x}} = GCD(N_{\vec{y}}, N_{\vec{x}})$, where $N_{\vec{y}}$ is the lowest positive integer such that $N_{\vec{y}} \times \vec{y} = \vec{0}$ (mod 2).

Now consider the ε fields in the $K = 4$ parafermion theory. The normal untwisted, (i.e., Neveu–Schwarz) modes of ε are $\varepsilon^{+}_{-\frac{1}{3}-n}$ and $\varepsilon^{-}_{-\frac{1}{3}-n_3}$, where $n \in \mathbb{Z}$. That is, untwisted $\varepsilon = \varepsilon^{+} + \varepsilon^{-}$ has the following normal-mode expansions:

N-S Sector:

$$\begin{aligned}\varepsilon^{+}(\sigma_1, \sigma_2) = \sum_{n=1}^{\infty} [&\varepsilon_{n-1/3}\exp\{-\,i(n - 1/3)(\sigma_1 + \sigma_2)\} \\ &+ \bar{\varepsilon}_{2/3-n}\exp\{-\,i(2/3 - n)(\sigma_1 + \sigma_2)\}]\,,\end{aligned} \tag{3.7a}$$

[13] These equivalence have subsequently been explicitly shown and the distinct low mass^2 fields determined in Argyres et al. [11].

$$\varepsilon^-(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty} [\varepsilon_{1/3-n}\exp\{-i(1/3-n)(\sigma_1+\sigma_2)\}$$
$$+ \bar{\varepsilon}_{n-2/3}\exp\{-i(n-2/3)(\sigma_1+\sigma_2)\}] \,, \tag{3.7b}$$

(where $\varepsilon_r^\dagger = \varepsilon_{-r}$ and $\bar{\varepsilon}_r^\dagger = \bar{\varepsilon}_{-r}$). The associated boundary conditions in this sector are

$$\varepsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi/3}\varepsilon^+(\sigma_1) \,, \tag{3.8a}$$

$$\varepsilon^-(\sigma_1 + 2\pi) = e^{+i2\pi/3}\varepsilon^-(\sigma_1) \,. \tag{3.8b}$$

Like the standard fermion, the ε operators at level-four can be in twisted sectors, where the normal-mode expansions have the following form:

General Twisted Sector:

$$\varepsilon^+(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty} [\varepsilon_{n-1/3-a/2b}\exp\{-i(n-1/3-a/2b)(\sigma_1+\sigma_2)\}$$
$$+ \bar{\varepsilon}_{2/3-n-a/2b}\exp\{-i(2/3-n-a/2b)(\sigma_1+\sigma_2)\}] \,, \tag{3.9a}$$

$$\varepsilon^-(\sigma_1,\sigma_2) = \sum_{n=1}^{\infty} [\varepsilon_{1/3-n+/2b}\exp\{-i(1/3-n+a/2b)(\sigma_1+\sigma_2)\}$$
$$+ \bar{\varepsilon}^{n-}{}_{2/3+a/2b}\exp\{-i(n-2/3+a/2b)(\sigma_1+\sigma_2)\}] \,. \tag{3.9b}$$

The associated boundary conditions are

$$\varepsilon^+(\sigma_1 + 2\pi) = e^{+i2\pi(1/3)}e^{i\pi a/b}\varepsilon^+(\sigma_1) \,, \tag{3.10a}$$

$$\varepsilon^-(\sigma_1 + 2\pi) = e^{-i2\pi(1/3)}e^{-i\pi a/b}\varepsilon^-(\sigma_1) \,. \tag{3.10b}$$

From the analogy of free-fermion models, we suggest that in $K = 4$ parafermion models the presence of a sector containing twisted ε fields with boundary conditions (3.10a) or (3.10b) will result in $\mathbb{Z}_b$ or $\mathbb{Z}_{2b}$ GSO projections, depending on whether a is even or odd respectively. (We assume a and b are relative primes and $-2/3 \leqq a/b < 4/3$.)

Zero modes correspond to $a/b = -2/3$. Thus, we conjecture that the presence of these (twisted) zero models ε_n, $n \in \mathbb{Z}$ in a model, result in a generalized $\mathbb{Z}_3$ GSO projection. Admittedly, this is suggested with the hindsight of having the partition function for this theory. Nevertheless, we mention this in attempting to give more physical meaning to the partition function. Likewise for $K = 8$ and 16, one might expect $\mathbb{Z}_5$ and $\mathbb{Z}_9$ projections, respectively. Such projections for $K = 8$ and 16 could be significantly altered though, by the effects of the non-Abelian braiding of the non-local interactions.

One other aspect to notice is that within the range $-2/3 \leqq a/b < 4/3$ there are actually two distinct N-S sectors, corresponding not just to $a/b = 0$, but also $a/b = 2/3$. This corresponding to $\mathbb{Z}_2$ symmetry $\varepsilon^3 \leftrightarrow \varepsilon^-$. Though this symmetry may be obvious, it could explain the origin of the additional $\mathbb{Z}_2$ projection we will shortly discuss.

For the $K = 4$FSC model, one expects a GSO projection to depend on a generalization of fermion number. However, the naive generalization to parafermion number, $F(\phi_0^1)$, is insufficient. We find that we must also consider the multiplicities of the other two "physically distinguished" fields, the twist field, ϕ_1^1 and the field ϕ_1^0, which raises the m quantum number.

In order to derive the MIPF we discovered that, indeed, a $\mathbb{Z}_3$ projection must be applied to both the left-moving modes (LM) and right-moving modes (RM) independently. Survival of a physical state, $|\text{phys}\rangle$, in the Hilbert space under this $\mathbb{Z}_3$ projection requires

$$\{e^{i\pi\frac{2}{3}\cdot[\vec{F}_{\mathrm{LM(RM)}}(\phi_0^1) + \vec{F}_{\mathrm{LM(RM)}}(\phi_1^1)]} = e^{i\pi\frac{2}{3}}\}\,|\text{phys}\rangle\,, \tag{3.11a}$$

or equivalently

$$\{Q_{3,\mathrm{LM(RM)}} \equiv \sum_i F_{i,\mathrm{LM(RM)}}(\phi_0^1) + \sum_i F_{i,\mathrm{LM(RM)}}(\phi_1^1) = 1 \pmod 3\}\,|\text{phys}\rangle\,, \tag{3.11b}$$

where $F_i(\phi_m^j)_{\mathrm{LM(RM)}}$ is the number operator for the field ϕ_m^j along the i direction for left-moving (right-moving) modes. One may note that this projection does not prevent mixing holomorphic A_4-sector and antiholomorphic B_4-sector terms. However, it need not, for this is separately prevented by the standard requirement that $M_{\mathrm{LM}}^2 = M_{\mathrm{RM}}^2$, i.e., $L_0 = \bar{L}_0$, which here results in the RM factors in the partition function being the complex conjugates of the LM, giving only mod-squared terms in the partition function.

Prior to projection by this extended GSO operator, we consider all physical states associated with the LM partition function terms in the expansion of $(c_0^0 + c_0^4 + c_0^2)^4$ or $(c_2^2 + c_2^4)^4$ to be in the A_4-sector. (The RM physical states in the A sector have parallel association with the complex conjugates of these partition function terms.) Similarly, we initially place in the B_4-sector all the LM physical states associated with the partition function terms in the expansion of $(c_2^2 + c_2^4)^2\,(c_0^0 + c_0^4 + c_0^2)^2$ or $(c_0^0 + c_0^4 + c_0^2)^2(c_2^2 + c_2^4)^2$. There is however, a third class of states; let us call this the "D_4" class. This latter class would be present in the original Hilbert space if not for an additional $\mathbb{Z}_2$ GSO projection. Left moving states in the D_4 class, would have partition functions that are terms in the expansion of $(c_0^0 + c_0^4 + c_0^2)^3(c_2^2 + c_2^4)^2$ or $(c_2^2 + c_2^4)^3(c_0^0 + c_0^4 + c_0^2)$. The thirty-two D_4 terms in the expansions (pairing c_0^0 and c_0^4, rather than expanding $(c_0^0 + c_0^4)^n$ as in Table 3.1) are likewise dividable into classes based on their associated $\mathbb{Z}_3$ charges, Q_3. Twelve have charge 0 (mod 3), twelve have charge 1 (mod 3) and eight have charge 2 (mod 3). Without the $\mathbb{Z}_2$ projection it is impossible to to keep just the correct terms in the A_4- and B_4-sectors, and also project away all of the D_4-sector terms. Simple variations of the projection (3.11a) cannot do the job. All D_4 terms can be eliminated, without further projections on the A_4 and B_4 terms, by an obvious $\mathbb{Z}_2$ projection,

$$\left\{\sum_i F_{i,\mathrm{LM(RM)}}(\phi_1^1) + \sum_i F_{i,\mathrm{LM(RM)}}(\phi_{\pm 1}^0) = 0 \bmod 2\right\}|\text{phys}\rangle\,. \tag{3.11c}$$

(Note that for $K = 2$, ϕ_1^1 is equivalent to the vacuum and ϕ_0^1 is indistinguishable from the usual fermion, ϕ_0^1. Thus for $K = 2$ there is no additional $\mathbb{Z}_2$ GSO projection.)

Consideration of these D class states reveals some physical meaning to our particular $\mathbb{Z}_3$ charge and the additional $\mathbb{Z}_2$ projection. First, in all sectors the charge Q_3 commutes with $(\phi_{K/4}^0)^{D-2}$, which transforms between non-projection and projections states of opposite space-time statistics in the A_4- and B_4-sectors. Second, the values of this charge are also associated with specific mass^2 (mod 1) levels. Third, only for the A_4- and B_4-sector states does mass^2 (mod 1) commute with the same twist operator $(\phi_{K/4}^0)^{D-2}$. Recall, in Sect. 2 we suggested that twisting by this latter field was the key to spacetime SUSY. Without any of our projections the mass^2 levels (mod 1) of states present would be mass$^2 = 0, \frac{1}{12}, \frac{2}{12}, \ldots, \frac{11}{12}$. When acting on D_4-sector fields, $(\phi^0_{K/4})^{D-2}$ transforms mass$^2 = i/12$ (mod 1) states into mass$^2 = (i+6)/12$ (mod 1) states. Thus, states in the D_4-sector paired by supersymmetry would be required to appear in different sectors (i.e., different mod-squared terms) of the partition function, in order to preserve T invariance. As a result, the paired contributions to the partition function cannot cancel, proving that D_4 terms cannot be part of any supersymmetric theory. Although mass^2 (mod 1) commutes with $(\phi^0_{K/4})^{D-2}$ in the $A_4(Q_3 = 0)$, $A_4(Q_3 = -1)$, $B_4(Q_3 = 0)$, $B_4(Q_3 = -1)$ subsectors, within these subsectors (1) there is either a single bosonic state or fermionic state of lowest mass without superpartner of equal mass, and/or (2) the lowest mass states are tachyonic. (See Table 3.1.) Thus, our specific GSO projections in terms of our $\mathbb{Z}_3$ charge projection and our $\mathbb{Z}_2$ projection equal to spacetimes SUSY.

Our assignments of states as spacetime bosons or fermions in the B_4-sector, uses an additional projection that we believe distinguishes between the two. Following the pattern in Eqs. (2.8b) with bosonic/fermionic assignment of related states defined in Eqs. (2.43a–c), we suggest that for these states the two primary fields, $\phi^{j_3}_{m_3}$ and $\phi^{j_4}_{m_4=m_3}$ (implicitly) assigned compactified spacetime indices must be

Table 3.1. Masses of $K = 4$ Highest Weight States (Represented by their associated characters)

A_4-Sector			Survives		B_4-Sector		
			Q_3	CSO			
Boson	Mass^2	Fermion			Boson	Mass^2	Fermion
$(c^4_0)^2(c^4_0)^2$	$3\frac{2}{3}$		0	No	$(c^4_0)^2(c^4_2)^2$	$3\frac{1}{6}$	$(c^4_2)^2(c^4_0)^2$
$(c^2_0)(c^4_0)(c^4_0)^2$	3		1	Yes	$(c^2_0)(c^4_0)(c^4_2)^2$	$2\frac{1}{2}$	$(c^2_2)(c^4_2)(c^4_0)^2$
$(c^0_0)(c^4_0)(c^4_0)^2$	$2\frac{2}{3}$	$(c^4_2)^2(c^4_2)^2$	0	No	$(c^0_0)(c^4_0)(c^4_2)^2$	2^{1}	$(c^4_2)^2(c^0_0)(c^4_0)$
$(c^4_0)^2(c^2_0)^2$	$2\frac{1}{3}$		-1	No	$(c^4_0)^2(c^2_2)^2$	$1\frac{5}{6}$	$(c^4_2)^2(c^2_0)^2$
$(c^2_0)^2(c^4_0)^2$					$(c^2_0)^2(c^4_2)^2$		$(c^2_2)^2(c^4_0)^2$
					$(c^2_0)(c^4_0)(c^2_2)(c^4_2)$		$(c^2_2)(c^4_2)(c^2_0)(c^4_0)$
$(c^2_0)(c^0_0)(c^4_0)^2$	2	$(c^2_2)(c^4_2)(c^4_2)^2$	1	Yes	$(c^2_0)(c^0_0)(c^4_2)^2$	$1\frac{1}{2}$	$(c^2_2)(c^4_2)(c^0_0)(c^4_0)$
$(c^2_0)(c^4_0)(c^2_0)^2$	$1\frac{2}{3}$		0	No	$(c^2_0)(c^4_0)(c^2_2)^2$	$1\frac{1}{6}$	$(c^2_2)(c^4_2)(c^2_0)^2$
$(c^0_0)^2(c^4_0)^2$					$(c^0_0)^2(c^4_2)^2$		
$(c^4_0)^2(c^0_0)^2$							$(c^4_2)^2(c^0_0)^2$
$(c^0_0)(c^4_0)(c^2_0)^2$	$1\frac{1}{3}$	$(c^2_2)^2(c^4_2)^2$	-1	No	$(c^0_0)(c^4_0)(c^2_2)^2$	$\frac{5}{6}$	
		$(c^4_2)^2(c^2_2)^2$			$(c^0_0)(c^2_0)(c^2_2)(c^4_2)$		$(c^2_2)(c^4_2)(c^0_0)(c^2_0)$
$(c^2_0)^2(c^2_0)^2$	1		1	Yes	$(c^2_0)^2(c^2_2)^2$	$\frac{1}{5^2}$	$(c^2_2)^2(c^2_0)^2$
$(c^2_0)(c^4_0)(c^0_0)^2$							$(c^2_2)(c^4_2)(c^0_0)^2$
$(c^2_0)(c^0_0)(c^2_2)^2$	$\frac{2}{3}$	$(c^2_2)(c^4_2)(c^2_2)^2$	0	No	$(c^2_0)(c^0_0)(c^2_2)^2$	$\frac{1}{6}$	
$(c^0_0)(c^4_0)(c^0_0)^2$							
$(c^0_0)^2(c^2_0)^2$	$\frac{1}{3}$		-1	No	$(c^0_0)^2(c^2_2)^2$	$-\frac{1}{6}$	
$(c^2_0)^2(c^0_0)^2$							$(c^2_2)^2(c^0_0)^2$
$(c^2_0)(c^0_0)(c^0_0)^2$	0	$(c^2_2)^2(c^2_2)^2$	1	Yes			
$(c^0_0)^2(c^0_0)^2$	$-\frac{1}{3}$		0	No			

the same, i.e., $j_3 = j_4$, or else must form a term in the expansion of $(c_0^0 + c_0^2)^2$. This second case is related to ϕ_0^0 and ϕ_0^2 producing the same spacetime fermion field, ϕ_1^2, when separately twisted by $\phi_{K/4}^0$. (Note however that $\phi_1^2 \times \phi_{K/4}^0 = \phi_0^0$ only.) Following this rule, neither the states corresponding to $(c_0^2)(c_0^0)(c_2^2)(c_2^4)$ and $(c_2^2)(c_2^4)(c_0^2)(c_0^0)$, (which transform between each other under twisting by $\phi_{K/4}^0\phi_{K/4}^0\phi_{K/4}^0\phi_{K/4}^0$ nor those associated with $(c_0^2)(c_0^4)(c_2^2)(c_2^4)$ and $(c_2^2)(c_2^4)(c_0^2)(c_2^4)$, survive the projections as either spacetime bosons or fermions. However, for completeness we include the partition function in the B_4-sector columns of Table 3.1. We define the associated states as either spacetime bosons or fermions simply by the value of $m_3 = m_4$. This is academic, though, because the states do not survive the $\mathbb{Z}_3$ projections.
(In Table 3.2, columns one and seven give the lowest mass^2 of a state with center column $\mathbb{Z}_3$ charge in the appropriate sector. For the D_4-sector states, under $(\phi_{K/4}^0)^{D-2}$ twistings, mass^2 values in column two transform into mass^2 values in column six of the same row and vice-versa.)

In the $K = 4$ case unlike $K = 2$, we find that the $\mathbb{Z}_3$ projection in the Ramond sector wipes out complete spinor fields, not just some of the modes within a given spin field. This type of projection does not occur in the Ramond sector for $K = 2$ since there are no fermionic states with fractional mass^2 values in the $D = 10$ model. Note that our $\mathbb{Z}_3$ GSO projections relate to the $\mathbb{Z}_3$ symmetry pointed out in [31] and briefly commented on after Eqs. (3.2a, 3.2b).

For $K = 8$, a more generalized $\mathbb{Z}_5$ projection holds true for all sectors. For the $K = 16$ theory, there are too few terms and products of string functions to determine if a $\mathbb{Z}_9$ projection is operative. In the $K = 4$ case, the value of our LM (RM) Q_3 charges for states surviving the projection is set by demanding that the massless spin-2 state $\varepsilon^{\mu}_{-1/3}\bar{\varepsilon}^{\bar{\nu}}_{-1/3}|0\rangle$ survives. In the A, B, (and C for $K = 8$, 16) sectors, these projections result in states with squared masses of $0 +$ integer, $\frac{1}{2} +$ integer, and $\frac{3}{4} +$ integer, respectively.

3.2. *The Unique Role of the Twist Field,* $\phi_{K/4}^{K/4}$. As Table 3.1 indicates for the particular case of $K = 4$, the massless A-sector space-time fermion in the fractional superstring theory is created in light-cone gauge by a $(D - 2)$-dimensional tensor product of $(\phi_{K/4}^{K/4})$ fields (with associated string function character $(c_{K/4}^{K/4})^{D-2}$) acting on the vacuum. In this section we examine whether other consistent models are

Table 3.2. Mass Sectors as Function of $\mathbb{Z}_3$ Charge

Lowest M^2	M^2 mod 1	Sector	$\mathbb{Z}_3$ Charge	Sector	M^2 mod 1	Lowest M^2
0	0	A_4	$Q_3 = 1$	B_4	$\frac{6}{12}$	$\frac{6}{12}$
$-\frac{1}{12}$	$\frac{11}{12}$	D_4	$Q_3 = 0$	D_4	$\frac{5}{12}$	$\frac{5}{12}$
$-\frac{2}{12}$	$\frac{10}{12}$	B_4	$Q_3 = -1$	A_4	$\frac{4}{12}$	$\frac{4}{12}$
$-\frac{3}{12}$	$\frac{9}{12}$	D_4	$Q_3 = 1$	D_4	$\frac{3}{12}$	$\frac{3}{12}$
$-\frac{4}{12}$	$\frac{8}{12}$	A_4	$Q_3 = 0$	B_4	$\frac{2}{12}$	$\frac{2}{12}$
$\frac{7}{12}$	$\frac{7}{12}$	D_4	$Q_3 = -1$	D_4	$\frac{1}{12}$	$\frac{1}{12}$

possible if one generalizes the twist field (as $\phi^{K/4}_{K/4}$ is referred to in parafermion models), to another that could fulfill its role of creating massless spacetime fermions or if $\phi^{K/4}_{K/4}$ uniquely qualifies for this task. When it is demanded that $\phi^{K/4}_{K/4}$ and the $\varepsilon \equiv \phi^1_0$ field of reference [16, 7, 10] be used, we can derive the critical dimensions of possible models by observing that $K = 2, 4, 8$, and 16 are the only levels for which

$$k(\phi^1_0)/h(^{K/4}_{K/4}) \in \mathbb{Z} \, . \tag{3.2.1}$$

If we assume (as in [16]) that the operator $(\phi^{K/4}_{K/4})^\mu$ acting on the (tachyonic) vacuum produces a massless spacetime spinor vacuum along the direction μ, and $(\phi^1_0)^\mu$ produces a massless spin-1 state, then for spacetime supersymmetry (specifically $N = 2$ SUSY for fractional type II theories and $N = 1$ SUSY for fractional heterotic) $h(\phi^1_0)/h(\phi^{K/4}_{K/4})$ must equal the number of transverse spin modes, i.e.,

$$\begin{aligned} h(\phi^1_0) &= (D-2)h(\phi^{K/4}_{K/4}) \, , \\ \frac{2}{K+2} &= (D-2)\frac{K/8}{K+2} \, . \end{aligned} \tag{3.2.2}$$

Hence,

$$D = 2 + \frac{16}{K} \in \mathbb{Z} \, . \tag{3.2.3}$$

Thus, from this one assumption, the possible integer spacetime dimensions are determined along with the possible levels K. Perhaps not coincidentally, the allowed dimensions are precisely the ones in which classical supersymmetry is possible. This is clearly a complementary method to the approach for determining D followed in [16, 7, 10].

Demanding Eq. (3.2.1) guarantees spin-1 and spin-1/2 superpartners at

$$\text{mass}^2 = \text{mass}^2(\text{vacuum}) + h(\phi^1_0) = \text{mass}^2(\text{vacuum}) + (D-2) \times h(\phi^{K/4}_{K/4}) \, . \tag{3.2.4}$$

A priori simply demanding the ratio be integer is not sufficient to guarantee spacetime supersymmetry. However, in the previous subsections it proves to be; the masslessness of the (spin-1, spin-1/2) pair occurred automatically.

$\mathrm{m}^2(\text{spin} -1)$	$=$	$m^2(\text{spin} -1/2)$
$h(\phi^1_0) \Uparrow$		$(D-2) \times h(\phi^{K/4}_{K/4}) \Uparrow$
	$m^2(\text{vacuum})$	

Fig. 3.1. Supersymmetry of Lowest Mass States of the Fractional Open String

In fractional superstrings, the primary field $\phi^{K/4}_{K/4} \equiv \phi^{K/4}_{-K/4}$ for $K = 4, 8$, and 16, with related partition function $Z^{K/4}_{K/4} = \eta c^{K/2}_{K/2}$, is viewed as the generalization of $\phi^{1/2}_{1/2}$ at $K = 2$. Are there any other parafermion operators at additional levels-K that could be used to transform the bosonic vacuum into a massless fermionic vacuum and bring about local spacetime supersymmetric models? The answer is that by demanding masslessness of the (spin-1, spin-1/2) pair,[14] there is clearly no other choice for $K < 500$.

The proof is short. We do not assume first that the massless spin-1 fields are a result of the ϕ_0^1 fields. Rather, the necessity of choosing ϕ_0^1 appears to be the result of the uniqueness of $\phi_{K/4}^{K/4}$.

Proof. Assume we have a consistent (modular invariant) closed fractional superstring theory at level-K with supersymmetry in D dimensional spacetime, ($N = 2$ for fractional type II theories and $N = 1$ SUSY for fractional heterotic). Let the massless left (right) spin-1 field be $(\phi_{m_1}^{j_1})^{\mu}|\text{vacuum}\rangle$. This requires that $\phi_{m_1}^{j_1}$ have conformal dimension

$$h(\phi_{m_1}^{j_1}) = c_{\text{eff}}/24 = (D-2)\frac{K}{8(K+2)}\,. \tag{3.2.5}$$

Thus, the twist field $\phi_{m_2}^{j_2}$ that produces the spinor vacuum along one of the $D-2$ transverse dimensions must have conformal dimension

$$h(\phi_{m_2}^{j_2}) = \frac{K}{8(K+2)}\,. \tag{3.2.6}$$

For $K < 500$ the only primary fields with dimension $K/[8(K+2)]$ are the series of $\phi_{K/4}^{K/4}$ for $K \in 2\mathbb{Z}$, and the accidental solutions ϕ_0^2 for $K = 48$, ϕ_0^3 for $K = 96$, and $\phi_{7/2}^{9/2}$ for $K = 98$. With $m = 0$, it is clear that the $K = 48$ and 96 fields could not be used to generate spacetime fermions. The $K = 98$ case could not be used because there is no candidate field at that level whose conformal dimension is a multiple of (3.2.6) (and thus no replacement for $\varepsilon \equiv \phi_0^1$). (A proof of the uniqueness of $\phi_{K/4}^{K/4}$ for all K is being prepared by G.C.)

Confirmation of $\phi_{K/4}^{K/4}$ as the spin-1/2 operator, though, does not immediately lead one to conclude that ϕ_0^1 is the only possible choice for producing massless boson fields. Table 3.3 shows alternative fields at new levels-$K \neq 2, 4, 8$, or 16 whose conformal dimension is one, two, or four times the conformal dimension of $\phi_{K/4}^{K/4}$. (Note that successful alternatives to ϕ_0^1 would lead to a relationship between level and spacetime dimension differing from Eq. (3.2.3).) However, nearly all alternatives are of the form $\phi_0^{j>1}$ and we would expect that modular invariant models using $\phi_0^{j>1}$ to create massless bosons, would necessarily include tachyonic $(\phi_0^0)^{\mu\mu}|\text{vacuum} >$ states. That is, (although we have not proven this yet), we do not believe valid GSO projections exist which can project away these tachyons while simultaneously keeping the massless graviton and gravitino and giving modular invariance. Further, the remaining fields on the list have $m \neq 0 \pmod K$. Each of

[14] Masslessness of at least the left- and right-moving spin-1 spacetime fields (whose tensor product forms the massless spin-2 graviton in a closed string) is of course required for a consistent string theory. Consistent two-dimensional field theories with

$$\text{low mass of left- (right-)moving spacetime spin } -1 \text{ fields } =$$

$$\text{lowest mass of left- (right-)moving spacetime spin } -1/2 \text{ fields} \equiv M_{\min} > 0$$

may exist (as we discuss below) but, the physical interpretation of such models is not clear, (other than to say they would not be theories with gravity.)

Table 3.3. Fields $\phi^{j_1}_{m_1} \neq \phi^1_0$ with Conformal Dimensions in Integer Ratio with $h(\phi^{K/4}_{K/4})$

K	ϕ^j_m	$h(\phi^j_m)/h(\phi^{K/4}_{K/4})$
12	ϕ^2_0	4
24	ϕ^2_0	2
	ϕ^3_0	4
36	ϕ^7_6	4
40	ϕ^4_0	4
48	ϕ^2_0	1
	ϕ^3_0	2
60	ϕ^5_0	4
80	ϕ^4_0	2
84	ϕ^6_0	4
96	ϕ^3_0	1
112	ϕ^7_0	4
120	ϕ^5_0	2
⋮	⋮	⋮

these would not have the correct fusion rules with itself, nor with $\phi^{K/4}_{K/4}$ to be a spacetime boson.

Lastly, we want to consider the possibility that there is meaning to (non-stringy) two-dimensional field theories that contain neither supergravity nor even gravity. Instead let a model of this type contain only a global supersymmetry. The lowest mass spin-1 and spin-1/2 left- or right-moving fields, $(\phi^1_0)^\mu|\text{vacuum}\rangle$ and $(\phi^{j_3}_{m_3})^{D-2}|\text{vacuum}\rangle$, respectively, would be related by

$$\text{mass}^2(\text{vacuum}) + h(\phi^1_0) = \text{mass}^2(\text{vacuum}) + (D-2)\times h(\phi^{j_3}_{m_3})\,. \qquad (3.2.7)$$

In parafermion CFT's there is only a very small number (12) of potential candidates for $\phi^{j_3}_{m_3}$. (Like $\phi^{K/4}_{K/4}$ these twelve are all of the form $\phi^{j_3}_{\pm m_3}$.) We are able to reduce the number of candidates down to this finite number very quickly by proving no possible candidate could have $j_3 > 10$, independent of the level-K. We demonstrate this as follows:

Any potential level-K candidate $\phi^{j_3}_{m_3}$ must satisfy the condition of

$$\frac{K}{K+2}\left[j_3(j_3+1)-2\right] \leqq (m_3)^2 \leqq (j_3)^2 \leqq K^2/4\,. \qquad (3.2.8)$$

By parafermion equivalences (1.1), $|m| \leqq j \leqq K/2$ can be required for any level-K fields. The other half of the inequality, $K/(K+2)\left[j_3(j_3+1)-2\right] \leqq (m_3)^2$ results from the weak requirement that the conformal dimension of the candidate spin-1/2 field $\phi^{j_3}_{m_3}$, creating the fermion ground state along one spacetime direction cannot be greater than the conformal dimension of ϕ^1_0, i.e., $h(\phi^{j_3}_{m_3}) \leqq h(\phi^1_0)$.

From Eq. (3.2.8), we can determine both the minimum and maximum values of K, for a given j, (independent of m). These limits are $K_{\min} = 2j_3$ and $K_{\max} = \mathrm{int}\,[2(j_3)^2/(j_3 - 2)]$. Thus the number of different levels-K that can correspond to the field $\phi^{j_3}_{m_3}$ is $\mathrm{int}\,[(5j_3 - 2)/(j_3 - 2)]$. This number quickly decreases to six as j_3 increases to 10 and equals 5 for $j_3 > 10$. For a given j_3, we will express the levels-K under consideration as $K_i = 2j_3 + i$. Also, we find that from $K_{\min} = 2j_3$, the weak constraint on m_3 implies that we need only consider $\phi^{j_3}_{m_3 = \pm j_3}$ fields.

Thus, our search reduces to finding fields $\phi^{j_3}_{m_3}$ whose conformal dimensions satisfy

$$\frac{h(\phi^1_0)}{h(\phi^{j_3}_{\pm j_3})} = \frac{\dfrac{2}{K_i + 2}}{\dfrac{j_3(j_3+1)}{K_i + 2} - \dfrac{(j_3)^2}{K_i}} \in \mathbb{Z}\,. \tag{3.2.9}$$

It is easily shown that there are no solutions to Eq. (3.2.9) for $i = 0$ to 4 and $j_3 > 10$. As a result, we have reduced our search for possible alternative sources of fermionic ground states to only $\phi^{j_3}_{\pm j_3}$ with $0 < j_3 \leqq 10$. Within this range of j_3, a computer search reveals the following complete set of $\phi^{j_3}_{\pm j_3}$ fields that obey Eq. (3.2.9), as shown in Table 3.4.

The sets of solutions for $j_3 = \frac{1}{2}$, 1, and 2 are related. The existence of a set of solutions, $\{i = 1, 2, \text{and } 4\}$, for any one of these j_3 implies identical sets $\{i\}$ for the

Table 3.4. Potential Alternatives, $\phi^{j_3}_{m_3}$, to $\phi^{K/4}_{K/4}$ for Spin Fields

j_3	$\pm m_3$	K	i	$h(\phi^1_0)$	$h(\phi^{j_3}_{m_3})$	$D = \dfrac{h(\phi^1_0)}{h(\phi^{j_3}_{m_3})} + 2$
1/2	1/2	2	1	1/2	1/16	10**
		3	2	2/5	1/15	8
		5	4	2/7	2/35	7
1	1	3	1	2/5	1/15	8
		4	2	1/3	1/12	6**
		6	4	1/4	1/12	5
3/2	3/2	9	6	2/11	1/11	4
2	2	5	1	2/7	2/35	7
		6	2	1/4	1/12	5
		8	4	1/5	1/10	4**
5/2	5/2	25	20	2/27	2/27	3
3	3	9	3	2/11	1/11	4
		18	12	1/10	1/10	3
4	4	16	8	1/9	1/9	3**
6	6	18	6	1/10	1/10	3
10	10	25	5	2/27	2/27	3

remaining two j_3 as well. The known $\phi^{K/4}_{K/4}$ solutions (marked with a **) correspond to the $i = 1, 2$, and 4 elements in the $j_3 = \frac{1}{2}$, 1, and 2 sets respectively. Whether this pattern suggests anything about the additional related $\phi^{j_3}_{\pm j_3}$ in these sets, other than explaining their appearance in the above table, remains to be seen.

The set of distinct, physically relevant fields can be further reduced. There is a redundancy in the above list. Among this list, for all but the standard $\phi^{K/4}_{K/4}$ solutions, there are two fields at each level, with distinct values of j_3. These pairs are pairs are related by the field equivalences (1.1):

$$\phi^{1/2}_{\pm 1/2} \equiv \phi^{1}_{\mp 1} \quad \text{at } K = 3\,, \tag{3.2.10a}$$

$$\phi^{1/2}_{\pm 1/2} \equiv \phi^{2}_{\mp 2} \quad \text{at } K = 5\,, \tag{3.2.10b}$$

$$\phi^{1}_{\pm 1} \equiv \phi^{2}_{\mp 2} \quad \text{at } K = 6\,, \tag{3.2.10c}$$

$$\phi^{3/2}_{\pm 3/2} \equiv \phi^{3}_{\mp 3} \quad \text{at } K = 9\,, \tag{3.2.10d}$$

$$\phi^{3}_{\pm 3} \equiv \phi^{6}_{\mp 6} \quad \text{at } K = 18\,, \tag{3.2.10e}$$

$$\phi^{5/2}_{\pm 5/2} \equiv \phi^{10}_{\mp 10} \quad \text{at } K = 25\,. \tag{3.2.10f}$$

Because ϕ^{j}_{m} and ϕ^{j}_{-m} have identical partition functions and $\phi^{j}_{-m} \equiv (\phi^{j}_{m})^{\dagger}$ we can reduce the number of possible alternate fields in half, down to six. (Note that we have not been distinguishing between $\pm$ on m anyway.)

If we want models with *minimal* super Yang–Mills Lagrangians we can reduce the number of fields to investigate further. Such theories exist classically only in $D_{\mathrm{SUSY}} = 10, 6, 4, 3$ (and 2) spacetime. Thus we can consider only those $\phi^{j_3}_{\pm j_3}$ in the above list that have integer conformal dimension ratios of $D_{\mathrm{SUSY}} - 2 = h(\phi^{1}_{0})/h(\phi^{j_3}_{\pm j_3}) = 8, 4, 2$, and 1. This would reduce the fields to consider to just the three new possibilities for $D = 4$, and 3 since there are no new additional for $D = 10$ or 6.

Section 4: Conclusions

A viable and consistent generalization of the superstring would be an important development. Our work has shown that the fractional superstring has many intriguing features that merit further study. The partition functions for these theories are found to have simple origins when derived systematically through the factorization approach of Gepner and Qiu. Furthermore, using this affine/theta-function factorization of the parafermion partition functions, we have related the A_K-sector containing the gravitation and gravitino with the massive sectors, B_K and C_K. A bosonic/fermionic interpretation of the B_K-subsectors was given. Apparent "self-cancellation" of the C_K-sector was shown, the meaning of which is under further investigation by G. C. A possible GSO projection was found, adding hope that the partition functions have a natural physical interpretation. Nevertheless, fundamental questions remain concerning the ghost system and current algebra, which prevent a definite conclusion as to whether or not these are consistent theories. However, even if the theories are ultimately shown to be inconsistent, we believe that this program will at least provide interesting identities and new insight into the one case we know is consistent, $K = 2$. In other words, viewed in this more general context, we may better understand what is special about the usual superstring.

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