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Kodaira–Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes

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Abstract. We develop techniques to compute higher loop string amplitudes for twisted $N = 2$ theories with $\hat{c} = 3$ (i.e. the critical case). An important ingredient is the discovery of an anomaly at every genus in decoupling of BRST trivial states, captured to all orders by a master anomaly equation. In a particular realization of the $N = 2$ theories, the resulting string field theory is equivalent to a topological theory in six dimensions, the Kodaira–Spencer theory, which may be viewed as the closed string analog of the Chern–Simons theory. Using the mirror map this leads to computation of the 'number' of holomorphic curves of higher genus curves in Calabi–Yau manifolds. It is shown that topological amplitudes can also be reinterpreted as computing corrections to superpotential terms appearing in the effective 4d theory resulting from compactification of standard 10d superstrings on the corresponding $N = 2$ theory. Relations with $c = 1$ strings are also pointed out.

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1. Introduction

Despite the fact that string theory has been investigated very intensively in particular in the past decade, many of its fundamental principles and symmetries remain as elusive as ever. This lack of understanding of the fundamental principles renders questions of selection of vacua and non–perturbative aspects of string theory out of reach. Actually the problem runs deeper: Not only the problem has to do with understanding the underpinnings of string theory, but also not even many *perturbative* computations are practical, even though in principle many should be computable.

There are some exceptions to the above: First of all, thanks to the matrix models and topological theories, for non-critical strings with dimension $d \leq 2$ one can compute the partition function of the string theory to all order in perturbation theory summarized as solutions to interesting equations belonging to integrable hierarchies. Nevertheless such computations are usually viewed as toy models, not necessarily of relevance to more realistic critical string theories. One of the more useful results coming from these theories was the realization that there is an alternative topological string reformulation of bosonic strings. The word topological signifies the fact that in these theories, before coupling to gravity, the correlation functions are independent of the worldsheet metric. Actually the topological framework is more general than the conventional view of bosonic strings as there are some topological string theories which do not correspond to bosonic strings formulated as matter coupled to gravity. To see this, one has to note that the most interesting subclass of topological theories can be obtained by *twisting* an $N = 2$ superconformal theory, and in such case, the string BRST operator will correspond to the supercharge $Q = G^+$ and the b operator will correspond to the supercharge G^-. That this is more general than the usual bosonic string set up is easy to see from the fact that for the standard formulation of bosonic string the b-cohomology is trivial but for the $N = 2$ theory the G^- cohomology is generally non-trivial. The question naturally arises as to whether the non-triviality of the b–cohomology introduces new phenomena for bosonic strings. We will see in this paper that the non-triviality of the b–cohomology has dramatic consequences in string theory. The b–cohomology elements can be used to form Q–trivial perturbations of the theory, that nevertheless do not decouple. In other words, we shall find an *anomaly in decoupling of BRST trivial states from the physical amplitudes.*

Anomalies of various kinds have played a key role in the development of quantum field theories and string theory. The existence of anomalies means that a computation that on formal grounds would be expected to be zero turns out to be non–zero due to subtleties of the quantum field theory (QFT) in question. For example, the famous $U(1)$ chiral anomaly, explains why the mass of the meson singlet in massless QCD is non-vanishing, and the existence of conformal anomalies in 2d QFT's leads to the fact that the critical dimension of string theory is 26 (or 10) rather than 0. The anomalies are in one way or another related to topological aspects of the theory in question and have been one of the most fruitful areas of interaction between physics and mathematics. All these anomalies can be related to index computations in mathematics which can in turn be effectively understood in the physical set up in terms of 1d supersymmetric sigma–models. The topological strings obtained from twisting the supersymmetric sigma–model may be viewed as a more fancy 2d version of these index theorems which combine the geometry of moduli of Riemann surfaces with the geometry of target space. Viewed in this way, it is perhaps ironic that the very object usually employed to compute anomalies has itself anomalies!

Topological string theories that are obtained from twisting an $N = 2$ superconformal theory have a central charge $\hat{c}$ which can be viewed as the complex dimension of these theories. It turns out that topological string partition functions vanish for all genus (except $g = 1$) unless the critical dimension $\hat{c} = 3$ is achieved. As far as topological theories obtained from twisting *unitary* $N = 2$ theories are concerned there are very few other cases of interest. In particular for unitary twisted theories with $\hat{c} > 3$ (with integral charges) not only the partition function, but all the correlation functions vanish as well. For $\hat{c} < 3$ one must make special choices of operator insertions to have non-vanishing amplitudes. It is thus clear that the most interesting case is the case of $\hat{c} = 3$.

There are two other reasons to be interested in this particular value of $\hat{c}$. One reason is that perhaps the most interesting case of non-critical bosonic string corresponds to strings propagating in 2 dimensions, which is the lower critical dimension of bosonic strings, and this turns out to be related to a topological theory with $\hat{c} = 3$. The other reason to be interested in this particular value of $\hat{c}$ is that in constructing more or less realistic superstring models compactifying from 10 dimensions down to 4, one has to introduce a 6 dimensional internal theory with $\hat{c} = 3$ such as is the case for a Calabi–Yau 3-fold. It is thus exciting that the critical topological theory may be related to more or less realistic string compactifications and indeed we will see that the *topological amplitudes of* $\hat{c} = 3$ *topological string theories effectively compute superpotential terms in the effective low energy field theory of 4–dimensional theories obtained by compactifying the superstring on the corresponding internal theory.* This is an exciting link which thus makes the computation of topological amplitudes more than just an academic exercise.

As if these are not enough reasons to consider critical topological string theories there are many more: In a particular realization of the critical topological strings, the classical limit of the string field theory turns out to describe the classical deformation of the complex structure of Calabi–Yau manifolds (and the related variation of Hodge structure), i.e. the Kodaira–Spencer theory. This relation can be summarized by writing an action whose classical solution correspond to all possible deformation of the complex structure of the Calabi–Yau manifold. This field theory we call the *Kodaira–Spencer (KS) theory of gravity.* It is a gravitational theory in 6 real dimensions with vacua being Calabi–Yau 3–folds and which gauges the complex structure of the manifold. The *Kodaira–Spencer theory can be viewed as the closed string field theory for the critical topological string on a Calabi–Yau.* This is thus a rather simple realization of a closed string field theory which may be helpful for further understanding of closed string field theory in more general cases. One can also consider the quantum Kodaira–Spencer theory, i.e. the higher loops on it which are the same as the partition function at higher genus of topological strings. In particular at one–loop the partition function can be related to an appropriate combination of determinants of various operators which turns out to be related to the Ray–Singer holomorphic torsion. In this context the anomaly in decoupling of BRST–trivial states at one–loop becomes identical to the Quillen anomaly. Thus the higher genus anomaly that we have found in the string set up may be viewed as a generalization of the holomorphic Quillen anomaly for the Kodaira–Spencer theory to higher loops. As far as we know no analog of Quillen type anomaly was previously known for higher loops, and our derivation of the anomaly relies heavily on string theory techniques.

The partition function of the critical topological strings in another realization, which is the mirror transformed version of the KS theory, at the classical level 'counts' the number of holomorphic maps from sphere to the Calabi–Yau manifold. The counting of holomorphic maps from Riemann surfaces of genus g gets 'mirror mapped' to the g–th loop computation in the quantum Kodaira-Spencer theory.

The open string version of the critical topological string theories is also rather interesting. In particular, in one version of these theories (the 'A' version) the string field theory one obtains is the ordinary Chern-Simons theory in 3 real dimensions. By mirror map these should be related (in certain cases) to computation of open strings on 3–complex–dimensional Calabi–Yau manifolds.

In this paper we develop techniques for computations of correlation functions of twisted $N = 2$ theories coupled to gravity with $\hat{c} = 3$. The fact that one can actually compute the integral of certain realistic string amplitudes over the moduli space and write the answer in a closed form is a pleasant surprise. In particular in more or less realistic string theories there are no known computations beyond the tree and one–loop level that can be done in the string theory. For example the bosonic string theories do not make sense beyond tree level (the amplitudes diverge due to tachyons).

Superstring theories have vanishing partition function (due to supersymmetry) to all orders in perturbation theory, and there is a general method to formulate the computations of scattering amplitudes (modulo subtleties with the question of integration over supermoduli space). But no method to explicitly carry them out exists in these theories. In fact there is no reason why they should be simple at all given the presumption that essentially all infinitely many stringy modes should be relevant for such computations, and thus probably it is too much to expect an exact solution.

The idea which leads to the computation of these amplitudes is as follows: Formally the partition function at genus g of these theories would depend holomorphically on parameters t^i which characterize moduli of the theory. However we will find that, even though the conjugate fields, which are in the b-cohomology, are Q trivial, they do not decouple, and so we end up with $\bar{\partial}_{\bar{\imath}} F_g \neq 0$. The fact that the integrand of this dependence is a total derivative on moduli space allows us to go to the boundary to pick up the contributions which will thus involve lower genus computations. In this way we get a recursive equation which we solve for the $\bar{t}^{\bar{\imath}}$ dependence of F_g. The holomorphic dependence cannot be fixed from the anomaly equation alone, however from the fact that they are modular forms of appropriate weight, and making heavy use of the properties of F_g at the boundary of the moduli space (making use of the KS theory) they can be determined up to a few constants. In explicit examples, in particular the quintic 3–fold, for low genus we fix these constants using the mirror map by using the interpretation of leading terms as counting holomorphic maps.

The organization of this paper is as follows: In Sect. 2 we review aspects of $N = 2$ theories and their twisting. This is a basic section for review of old material but some of it from a new light. This includes a review of the geometry of vacua captured by tt^* equations (which in a special case corresponds to special geometry). We also discuss examples of $N = 2$ theories obtained from sigma–models on Calabi–Yau manifolds. These theories admit two different ways to twist, depending on whether the physical fields correspond to deformation of Kähler classes (A–model) or complex structure (B–model). We also discuss coupling of these theories to gravity, i.e. how to get string theories from them, and the notion of the critical dimension. We also point out why the topological partition functions are sections of line bundles and why inserting physical fields corresponds to taking covariant derivatives. Also we show how one can choose *topological coordinates* for moduli (as well as a trivialization for the line bundle) so that as far as topological observables are concerned we can replace covariant derivatives with ordinary ones. This turns out to explain the ansatz used in construction of mirror map.

In Sect. 3 we derive the basic anomaly equation. This includes the anomaly both for partition function as well as correlation functions. It is shown how this equation can be rewritten as a master equation for the full partition function (i.e. summed over all g) of the theory. We will see that the integrability of the master equation is true but non–trivial and is a consequence of tt^* equations.

In Sect. 4 some aspects of open string theory are discussed. This includes discussion of tt^* equation in this context, as well as derivation of anomaly equation at one–loop and aspects of the anomaly equation at higher loops.

In Sect. 5 we discuss what topological strings compute for the A– and B–models. In particular for the B–model we derive the Kodaira–Spencer theory as the string field theory for topological strings. This includes discussion of the symmetries of KS theory as well as the background (in)dependence of it with respect to a choice of base point for complex structure of the manifold. We use the Batalin-Vilkovisky (BV) formalism to quantize the KS theory. We also discuss the one–loop computation of the KS theory and show how the computation is equivalent to computation of a particular combination of the Ray–Singer torsion for the Calabi–Yau manifold. Also discussed there is the behaviour of the partition function of the KS theory near the boundary of moduli space of complex structures — a result which will imply that there are only a finite number of coefficients needed to determine the purely holomorphic dependence of F_g on moduli parameters. In the context of A–model we show that the topological partition function in a particular limit ($\bar{t} \to \infty$) computes the number of holomorphic maps (or more generally the Euler class of a particular bundle on the moduli space of holomorphic maps). A particularly important case discussed there is the contribution of constant maps to the topological string amplitude.

In Sect. 6 we discuss how to solve the recursive anomaly equation by introducing an auxiliary space consisting of the massless modes of the conformal theory. We show that certain Feynman graph rules involving fields corresponding to this auxiliary space can be used to solve the anomaly equation recursively. The vertices of this theory involve n–point functions of lower genus topological theory, and the propagator involves a covariantly defined prepotential and its first two derivatives. These rules can be summarized as a path–integral (which in our case is just a finite dimensional integral) over the auxiliary space. The emergence of this way to solve the anomaly equation is somewhat mysterious, but we try to understand it in the context of the Kodaira–Spencer theory.

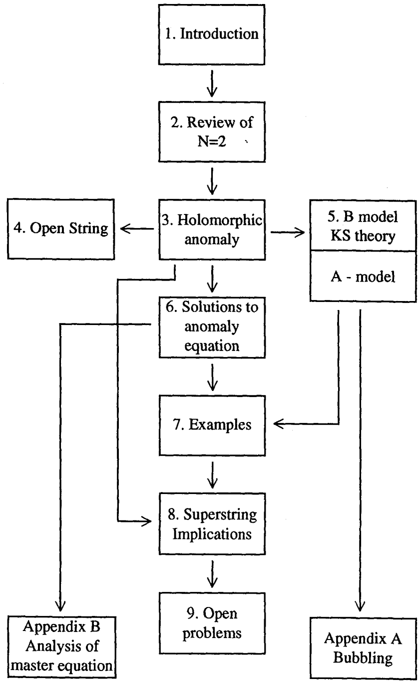
Section 7 is the experimental verification of the paper. In that section we give examples of computation of topological amplitudes, including orbifolds and the quintic 3–fold. In particular for the quintic we compute the genus 2 partition function explicitly and use the mirror map to translate it to the 'counting' of holomorphic maps from genus 2 to the quintic.

Section 8 is where topological strings meet realistic string models. We show how the partition function of topological strings can be reinterpreted as particular computations in conventional type II and open superstrings compactified on the corresponding internal theory. In particular we show that the closed and open string versions of the topological theory compute the dependence of the coefficients of particular terms in the superpotential on the moduli of the internal theory. In the context of open strings in particular this term will have a bearing on the question of gaugino condensates and is worth further investigation for its phenomenological implications for supersymmetry breaking. Also in this section we show that the one–loop computation of threshold corrections for heterotic strings in the context of $(2,2)$ compactifications can be directly related to the one–loop amplitude of topological strings. Using properties established for this amplitude we show that quite independently of which Calabi–Yau manifold one chooses the effective unification scale is rather sensitive to the change of volume of the manifold and the dependence is such as to push the unification scale up as we increase the volume of Calabi–Yau from Planck scale. The sign is fixed by the fact that $c_2 > 0$ for any Calabi–Yau manifold.

Finally in Sect. 9 we discuss open problems and prospects for future work.

In appendix A we discuss computation of contribution of bubblings of sphere to topological amplitudes. In appendix B we present some preliminary analysis on the master anomaly equation.

Perhaps it is useful to summarize the organization of this paper with the following flow chart.



2. Review of Twisted N=2 Theories

In this section we review aspects of twisted $N = 2$ theories. This section will also serve to set some of the notation we will later use, as well as motivate some of the issues which are discussed later in the paper. In Subsection 2.1 we discuss the topological nature of $N = 2$ theories by reviewing the structure of its vacua and chiral rings. We then discuss the geometry of vacuum bundle as a function of moduli of $N = 2$ theories (tt^* equations). We then consider examples of $N = 2$ theories in the context of sigma–models. Next we specialize some of the discussions to the Calabi–Yau 3-folds and review special geometry. In Subsection 2.4 we discuss how to make a string theory out of twisted $N = 2$ theories, which is known as 'coupling to topological gravity.' Also discussed there is why Calabi–Yau 3-folds enjoy a special status among such string models (i.e. why dimension 3 is critical). We will also discuss how the partition function of these theories are not numbers but rather sections of bundles, and how inserting chiral fields is equivalent to taking *covariant* derivatives of the partition function; a fact which will be heavily used in the rest of the paper. Also discussed there is the fundamentally important notion of 'canonical coordinates' which turns out to explain the ansatz used in the construction of mirror maps. We will then give a formal argument for the decoupling of anti-chiral fields from correlation functions, but argue why this formal argument cannot be correct by showing its inconsistency with special geometry, which leads us to the notion of anomalies discussed in Sect. 3.

2.1 Vacuum geometry and twisting of $N = 2$ theories. $N = 2$ supersymmetric theories in 2 dimensions have a very rich structure. We will restrict our attention below to superconformal ones even though most of what we say can be easily generalized to massive $N = 2$ theories (we will use the notations and the results of [1], [2] which the reader may consult for more detail).

Superconformal $N = 2$ theories have four supercharges: Two conjugate left–movers ($G^{\pm}$) and two conjugate right–movers ($\overline{G}^{\pm}$), and two $U(1)$ currents, one left–moving (J) and one right–moving ($\overline{J}$). The $\pm$ sign over G's signifies their $U(1)$ charge with respect to the corresponding current. Among the important commutation relations of the $N = 2$ algebra are the zero-mode commutators, which we denote by the same label as the fields:

$$(G^{\pm})^2 = 0 \, ,$$
$$\{G^{+}, G^{-}\} = 2H_L \, ,$$
$$[G^{\pm}, H_L] = 0 \, ,$$

where H_L denotes the left–moving hamiltonian, and similarly for the right–movers. Also all left–moving operators (anti-)commute with the right–moving ones. From the nilpotency of the G's it follows that we can define the notion of G cohomologies both for the fields and for the states. If we wish to get a finite dimensional space for cohomology group we need to consider suitable addition of left– and right–moving $G's$. There are two inequivalent choices, up to conjugation, and they are given by

$$Q_1 = G^{+} + \overline{G}^{+} \, ,$$

$$Q_2 = G^{+} + \overline{G}^{-} \, .$$

As far as the cohomology states are concerned Q_1 and Q_2 and their conjugates all give rise to the *same* space, spanned by the supersymmetric ground states of the theory ($H_L = H_R = 0$). However as far as the cohomology of the fields are concerned, i.e., fields which satisfy

$$[Q, \phi] = 0 \qquad \phi \sim \phi + [Q, \Lambda]$$

even though they can be set into 1-1 correspondence with ground states of the theory, they are not equivalent with each other as operators. The cohomology operators for Q_1 are called (c, c), i.e., (chiral, chiral) fields and those of Q_2 are called (c, a), i.e., (chiral, anti-chiral) fields (where the two entries correspond to the cohomology condition for left– and right–moving charges respectively). Those of $Q_1^{\dagger}$ and $Q_2^{\dagger}$ are the conjugate fields and are called (a, a) and (a, c) fields respectively. Since the discussion for the two choices of Q's is essentially identical, as they differ only by a convention dependent choice of sign ($\overline{J} \to -\overline{J}$), we will restrict ourselves to Q_1 and its conjugate. We will also drop the subscript from Q_1 and denote it simply by Q. Also, to simplify terminology we will call the (c, c) fields simply chiral fields and the (a, a) fields the anti-chiral fields.

Let us choose a basis for chiral fields representing the Q–cohomology by ϕ_i, and denote the conjugate anti-chiral fields by $\overline{\phi}_i$. The $N = 2$ algebra implies that the (left,right) dimension of ϕ_i, $(h_i, \overline{h}_i)$ is half its charge $(q_i, \overline{q}_i)$

$$(h_i, \overline{h}_i) = \frac{1}{2}(q_i, \overline{q}_i)$$

and that the range for the q_i are bounded by the central charge $\hat{c}$ of the $N = 2$ algebra:

$$0 \leq q_i, \overline{q}_i \leq \hat{c} \, .$$

The dimensions of the anti-chiral fields $\overline{\phi}_i$ are the same as ϕ_i but their $U(1)$ charge is minus that of ϕ_i.

The chiral fields form a ring, the *chiral ring*, defined by

$$\phi_i \phi_j = C_{ij}^k \phi_k + [Q, \cdot\,] \,.$$

Using the $N = 2$ algebra it is easy to see that this definition of the ring is independent of the points of insertion of the fields on the worldsheet. Sometimes we will view C_{ij}^k as a matrix C_i with component $(C_i)_j^k$. The corresponding ring for anti-chiral fields differs only by complex conjugation of $\overline{C_{ij}^k} = (C_{ij}^k)^*$.

As is well known, viewing chiral fields as the first component of a superfield, we can modify the action by perturbing with them:

$$t^i \int d^2z\, d^2\theta^+ \phi_i + \overline{t}^i \int d^2z\, d^2\theta^- \overline{\phi}_i = t^i \int d^2z\, \phi_i^{(2)} + \overline{t}^i \int d^2z\, \overline{\phi}_i^{(2)} \,,$$

where $\phi_i^{(2)} = \{G^-, [\overline{G}^-, \phi_i]\}$ and t^i are complex parameters. If we wish to have a unitary theory we need $\overline{t}^i = (t^i)^*$ (later in the paper we will relax this condition). It is known [3] that the only criterion needed for preserving the conformal invariance is that $(h_i, \overline{h}_i) = (1/2, 1/2)$, i.e. that the charge of ϕ_i be $(1, 1)$.

As mentioned before there is a 1-1 correspondence between the chiral fields and the supersymmetric ground states of the theory, which follows by general considerations of QFT relating operators to states. However it is more useful to do this rather explicitly, which along the way leads to the notion of defining a topological theory. If we consider a hemisphere (see Fig. 1) with the field ϕ_i inserted on it then one is tempted to identify the state obtained at the boundary of this region by the path integral, as the cohomologically non-trivial state representing the supersymmetric ground state corresponding to ϕ_i.

However this is not correct: One reason for this is that the state we get at the boundary is in the wrong Hilbert space, i.e. the NS sector. For the supersymmetric vacuum we need to be in the Ramond sector. Another difficulty is that we need to argue that the state we get is annihilated by Q and to do this we have to make sure that Q is a scalar charge (especially when we have more non-trivial Riemann surfaces). To solve both these problems one *twists* the supersymmetric theory to obtain a *topological* theory, by introducing a background gauge field A which couples to the $U(1)$ current [4]

$$S \to S + \int J\bar{A} + \bar{J}A \,,$$

and one sets $A = \omega/2$, $\bar{A} = \overline{\omega}/2$, where ω is the spin connection. Introducing this gauge field has the effect of shifting the spin of charged fields by half their charge.

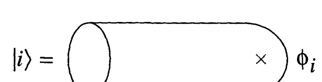


Fig. 1. Inserting the chiral field ϕ_i on the hemisphere and doing the twisted path integral on it will result in a state $|i\rangle$ at the boundary. This state is BRST equivalent to a ground state and can be made an exact ground state by pulling the neck infinitely long.

Thus Q becomes a scalar charge. At the same time $G^-(z)$ and $\overline{G}^-(\overline{z})$ become spin (2,0) and (0,2) currents respectively. Introducing this twisting, on the hemisphere of Fig.1, has the effect of converting the state obtained at the boundary to a state in the Ramond sector, which is annihilated by Q (where we use the fact that ϕ_i commutes with Q). Also the dimensions of all the fields will shift by $h \rightarrow h - \frac{q}{2}$, thus ϕ_i becomes dimension zero and $\overline{\phi}_i$ becomes dimension (1, 1) for marginal directions. We could obtain the exact ground state, not just a state cohomologically equivalent to a ground state, simply by doing the path integral on the hemisphere with the neck pulled infinitely long. We denote the corresponding ground state by $|i\rangle$. There is a canonical vacuum which is obtained by not inserting any field at all. This we will denote by $|0\rangle$. Note that we can write

$$|i\rangle = \phi_i|0\rangle + Q|\cdot\rangle$$

by using the fact that moving ϕ_i to the boundary is a Q–trivial operation. Also note that

$$\phi_i|j\rangle = (C_i)_j^k|k\rangle + Q|\cdot\rangle \ .$$

We can consider also the conjugate twist (the *anti-topological* theory) in which $Q^\dagger$ becomes a scalar. In this case we can parametrize *the same vacua* using the anti-chiral fields, which we denote by $|\overline{i}\rangle$. We must thus have a change of basis transformation relating the two:

$$\langle\overline{i}| = \langle j|M_{\overline{i}}^j$$

which by CPT satisfies $MM^* = 1$. Note that we have thus two natural inner products, the topological one η and the hermitian one g defined respectively by

$$\eta_{ij} = \langle j|i\rangle \ ,$$

$$g_{i\overline{j}} = \langle\overline{j}|i\rangle \ ,$$

which satisfy

$$g_{i\overline{k}} = \eta_{ij}M_{\overline{k}}^j \ .$$

Also note that in the topological theory

$$\langle\phi_i\phi_j\phi_k\rangle = \langle 0|\phi_i\phi_j\phi_k|0\rangle = \langle i|\phi_j|k\rangle = C_{jk}^l\langle i|l\rangle = C_{jk}^l\eta_{il} = C_{jki}$$

which implies that C_{ijk} is totally symmetric in indices (for indices of even fermion number).

We are interested in seeing how the structure of vacua and chiral fields deform as we perturb the theory by marginal chiral fields. As discussed the parameter space is locally given by $(t^i, \overline{t}^i)$. We would like to study the geometry of the *vacuum bundle*, i.e. how the collection of vacuum states $\{|i(t,\overline{t})\rangle\}$ varies as a function of the parameters and in particular find the dependence of C_{ijk}, g and η on $(t^i, \overline{t}^i)$. These are studied in [2] with the following results: First, using the fact that insertion of anti-chiral fields in the action modifies the theory by Q-trivial terms, it follows that

$$\overline{\partial}_l C_{ijk} = 0 \tag{2.1}$$

i.e., C_{ijk} is a symmetric holomorphic function of moduli. One introduces a connection on the vacuum bundle so that $D_i|j\rangle \equiv (\partial/\partial t^i - A_i)|j\rangle$ and $D_{\overline{i}}|j\rangle$ are orthogonal to all the vacua. Then the following equations, the tt^* equations, hold

$$[D_i, D_j] = [\overline{D}_i, \overline{D}_j] = [D_i, \overline{C}_j] = [\overline{D}_i, C_j] = 0 ,$$

$$[D_i, C_j] = [D_j, C_i] \quad [\overline{D}_i, \overline{C}_j] = [\overline{D}_j, \overline{C}_i] ,$$

$$[D_i, \overline{D}_j] = -[C_i, \overline{C}_j] . \tag{2.2}$$

One can also arrange, by a judicious choice of coordinates, for η to be constant [5]. It is also possible to choose the holomorphic (or topological) gauge [1], in which $A_{\bar{i}} = 0$ and in which

$$A_i = -g\partial_i g^{-1} . \tag{2.3}$$

In this gauge, which is the natural gauge chosen by the twisted path–integral, the chiral vacuum states $|i(t)\rangle$ depend holomorphically on the moduli. In this gauge the third line of Eq. (2.2) can be written as

$$\overline{\partial}_j(g\partial_i g^{-1}) = [C_i, g(C_j)^\dagger g^{-1}] . \tag{2.4}$$

We would like to digress slightly to discuss some ambiguities in the twisted path–integral. Ambiguities arise in the normalization of path–integral when there are zero modes to be absorbed. When we twist a topological theory by coupling the $U(1)$ current to a background gauge field, the axial $U(1)$ current $(J + \overline{J})$ becomes anomalous, and so there are fermion zero modes to absorb. In fact to be precise, using the fact that the OPE of J's have a central term

$$J(z)J(0) \sim \frac{\hat{c}}{z^2} ,$$

in genus g the twisting will give rise to a charge violation of

$$\Delta(q, \overline{q}) = \hat{c}(g-1, g-1) . \tag{2.5}$$

This ambiguity in how to absorb the zero mode translates to the ambiguity in defining the normalization of the chiral states $|i\rangle$. Since they are all related by applying the operators ϕ_i on $|0\rangle$ it suffices to discuss ambiguities for $|0\rangle$. We can choose the absorption of fermion zero modes to be consistent with the holomorphic dependence of $|0\rangle$ on moduli. But we cannot completely fix the ambiguity. Consider the line bundle $\mathcal{L}$ over the moduli space, generated by the vacuum state $|0\rangle$. Then what we are saying is that a holomorphic choice of normalization of the twisted path–integral is equivalent to a choice of holomorphic section of $\mathcal{L}$. Two different normalizations of the path-integral give differently normalized vacua as:

$$|0\rangle \to f(t^i)|0\rangle .$$

Note that this freedom in redefining the normalization of $|0\rangle$ holomorphically translates to a change in the connection $A^0_{i0} \to A^0_{i0} + \partial_i f$. Since we have chosen a holomorphic gauge

$$A^0_{i0} = \frac{\partial_i \langle \overline{0}|0\rangle}{\langle \overline{0}|0\rangle} = -\partial_i K,$$

[1] This choice of gauge is implicit in Eq. (2.1).

where

$$\exp(-K) = \langle\overline{0}|0\rangle \ .$$

The fact that $|0\rangle$ is a section of $\mathcal{L}$ translates to the statement that the genus zero partition function Z_0 (with operators inserted to avoid vanishing by charge conservation) will be a section of $\mathcal{L}^2$ (in addition to having properties induced from insertion of operators). Similarly by sewing axioms of topological theory it follows that Z_g is a section of $\mathcal{L}^{2-2g}$. Needless to say, all physically interesting quantities should be independent of how we choose to fix this normalization ambiguity.

2.2 Examples. Even though our discussions in the paper will be for the general case we will occasionally specialize the results to some interesting classes of $N = 2$ SCFT. Those which we will use most in this paper are the supersymmetric sigma–models. It is known that the sigma–model on a Kähler manifold M gives rise to an $N = 2$ QFT [6]. The action is given by

$$S = \int d^2z \left[\omega_{i\bar{j}}\partial X^i\overline{\partial X}^j + \omega^*_{i\bar{j}}\overline{\partial}X^i\partial\overline{X}^j\right] + \text{Fermionic terms},$$

where $\omega_{i\bar{j}}$ denotes a complexified Kähler class. If we denote an integral basis for $H^{(1,1)}(M, Z)$ by ω_i, we have

$$\omega = t^i\omega_i$$

and thus t^i parametrize the moduli of this theory. The fermionic terms in the action are there to make the above supersymmetric. Apart from the kinetic term, the fermionic terms include the four fermion interaction term

$$\int R_{i\bar{j}}{}^{k\bar{l}}\chi^i\overline{\chi}^j\psi_k\overline{\psi}_l$$

which will prove crucial for us later on.

One can twist the fermion number, as discussed in the previous section, to obtain a topological theory [4]. The effect on the action is only to modify the spin of the fermions, making the χ's scalar, the ψ_i is a (1,0) form and $\overline{\psi}_j$ is a (0,1) form. To obtain the observables of this theory it is convenient to go to the large volume limit first $t^i, \overline{t}^i \to \infty$. In this limit the Hilbert space of the theory can be represented by differential forms on M where the (left, right) $U(1)$ charge of the state can be identified with (holomorphic, anti-holomorphic) degree of the form. Moreover on this Hilbert space we get the following dictionary

$$G^+ \leftrightarrow \partial \ ,$$

$$\overline{G}^+ \leftrightarrow \overline{\partial} \ ,$$

and so $Q_1 = G^+ + \overline{G}^+ = d$ and thus the observables ϕ_i in this theory are in 1-1 correspondence with the cohomology elements of M represented by

$$\phi^{(a)} = A^{(a)}_{i_1\dots i_r\bar{j}_1\dots\bar{j}_s}\chi^{i_1}\cdots\chi^{i_r}\overline{\chi^{j_1}}\cdots\overline{\chi^{j_s}} \ .$$

The chiral ring for $t, \overline{t} \to \infty$ is the same as the cohomology ring, but for finite t's it is in general a deformation of the cohomology ring of M, taking into account the holomorphic instantons from sphere to M, i.e., rational curves on M. This deformed ring is called the quantum cohomology ring of M. The precise form of how the instantons contribute to this ring will very much depend on the first Chern class of M. The most interesting case is when $c_1(M) = 0$, i.e., the Calabi–Yau case. In this case, which is also the case needed to obtain a conformal theory out of the sigma–model, the instanton of arbitrary large degree affect the chiral ring. In particular if one is interested in the quantum cohomology ring for a 3-fold Calabi–Yau, if we let i, j, k denote three (1,1) classes, then the ring structure coefficients are given by

$$C_{ijk}(t) = \sum_r r_i r_j r_k d_{r_1 \ldots r_n} \frac{q_1^{r_1} \ldots q_n^{r_n}}{1 - q_1^{r_1} \ldots q_n^{r_n}} ,$$

where $n = \dim H^{(1,1)}(M)$, $q_r = \exp(-t^r)$, and $d_{r_1 \ldots r_n}$ are the number of primitive holomorphic instantons of degree $(r_1, \ldots, r_n)$. The denominator is the contribution of multi-coverings of primitive instantons [7, 8]. Note that the structure constants of this ring depend on the choice of the Kähler class, but are independent of the complex structure of M.

Now, as discussed in the previous section, superconformal theories have two natural rings [1], the (c, c) and (a, c) and thus also there exists two ways to twist the theory. In particular in the Calabi–Yau case for which both the fermion number and the axial fermion number are conserved, we can twist in two different ways, depending on which of these rings we wish to be the physical ring. The choice discussed above corresponds to twisting the fermion number current and gives rise to (say) the (c, c) ring as being the topological one. The other choice of twisting, corresponding to axial fermion number twisting has also been studied [9, 10]. Again it turns out to be easier to study the model in the large volume limit. In this limit again the Hilbert space can be identified with anti-holomorphic forms wedged with holomorphic vectors, i.e.,

$$\mathcal{H} = \bigoplus_{p,q} \wedge^p \overline{T}_M^* \otimes \wedge^q T_M , \tag{2.6}$$

where in here and in the following $T_M, \overline{T}_M$ denote the holomorphic and anti-holomorphic tangent bundles respectively and $T_M^*, \overline{T}_M^*$ denote the holomorphic and anti-holomorphic cotangent bundles. We can obtain from this Hilbert space the Hilbert space of forms simply by contracting the vector indices with holomorphic n-form which always exists for a Calabi–Yau n-fold. This converts the (q, p) sector above to a differential form of degree $(n - q, p)$. On this Hilbert space $\mathcal{H}'$ the dictionary for the supercharges turn out to be

$$G^+ = \frac{1}{2}(\partial^\dagger + \bar{\partial}) ,$$

$$\bar{G}^- = \frac{1}{2}(\bar{\partial} - \partial^\dagger) . \tag{2.7}$$

Note that $Q_2 = G^+ + \overline{G}^- \leftrightarrow \bar{\partial}$ and so again the observables can be identified with the cohomology elements of M. For later use in the paper we need also expression for left– and right–moving fermion numbers. Unlike the previous case, the fact that (2.7) mixes the holomorphic and anti–holomorphic degrees in this non–trivial way, implies that the fermion numbers in this case are not simply identified as the left and right degrees of form, as that would lead to a wrong commutation relation with G^+. To fix this, we should recall [11] that for Kähler manifolds there is an $sl(2)$ action on the forms, generated by wedging with the Kähler class k, contracting with k which we represent by $k^\dagger$ and the shifted total degree of the form $(p+q-n)/2$. Also we have

$$[k,\partial^\dagger]=i\bar\partial \qquad [k^\dagger,\bar\partial]=-i\partial^\dagger\ ,$$
$$[k^\dagger,\partial]=i\bar\partial^\dagger \qquad [k,\bar\partial^\dagger]=-i\partial\ .$$

Using this we can write $F_{L,R}$ as

$$F_{L,R}=\frac{1}{2}(ik-ik^\dagger\pm(p-q))\ . \tag{2.8}$$

The reader can check that the right–hand side (r.h.s.) is CPT odd and has the correct commutation properties with the G's.

Even though the chiral fields are again in 1-1 correspondence with the cohomology elements, the chiral ring for this twisting is very different from the previous one. Let Ω denote the holomorphic n–form. For example if the Calabi–Yau is a 3–fold, then the structure constants for the marginal directions, which are parametrized by elements of $H^{(2,1)}(M)$ can be written as

$$C_{ijk}=-\int_M \Omega\wedge\partial_i\partial_j\partial_k\Omega\ .$$

Note that these structure constants are independent of the Kähler class so they continue to hold for finite volume as well. But they do depend on the complex structure of M, which is parametrized by elements of $H^{(2,1)}(M)$ as will be discussed in great detail later in this paper.

So to summarize, we see that in the first case of twisting, i.e. the (c,c) twisting, the topological correlation functions are sensitive to the Kähler class of the manifold and compute the rational curves in the Calabi–Yau. This twisting is called the *A–twisting* or the *Kähler twisting*. On the other hand, in the case of (a,c) twisting we see that the topological correlation functions are only sensitive to the complex structure of the manifold which is encoded in how the holomorphic three form varies (or the variation of Hodge structure). This twisting is called the *B–twisting* or the *complex twisting*.

NOTE: For convenience of keeping the same notation in the rest of the paper when we deal with the B– or A–model we denote the supercharge always by $Q=G^++\overline{G}^+$ by a trivial change of conventions on the right–moving $U(1)$ charge if necessary. This will not cause confusion as we rarely talk about both models at the same time.

2.3 Special geometry and Calabi–Yau 3*–folds.* In a unitary superconformal theory there is only one chiral primary with $q=0$, namely the identity operator 1. We consider the normalized metric[2]

$$G_{i\bar j}=\left.\frac{g_{i\bar j}}{\langle\bar 0|0\rangle}\right|_{\beta=1}\ , \tag{2.9}$$

(here and in the following the indices i, $\bar j$ are restricted to the marginal directions in coupling constant space whereas indices a, $\bar b$ go through all chiral primaries). It is easy to see that $G_{i\bar j}$ is equal to the usual Zamolodchikov metric [12]. Indeed the definition of the tt^* metric can be written as

$$\langle\bar b|a\rangle_\beta=\langle R^\dagger_{\bar b}(\beta)R_a(0)\rangle_{\text{sphere}},$$

[2] Here β is the perimeter of the circle used to define the states $|i\rangle$ (cf. Fig. 1).

where R_a is the operator creating the Ramond vacuum $|a\rangle$ out of the $SL(2,\mathbf{C})$–invariant vacuum. Now, the topological map $|a\rangle \longrightarrow \phi_a$ differs only for the overall normalization from the *unitary* spectral flow operator U which maps the R sector into the NS one. Since the unitary operator U preserves inner products, we have

$$\langle \bar{\phi}_{\bar{b}}(\beta)\phi_a(0)\rangle_{\text{sphere}} = \frac{\langle \bar{b}|a\rangle_\beta}{\langle \bar{0}|0\rangle_\beta}\ , \tag{2.10}$$

where we used that the correct (unitary) normalization of U is just the one for which $\langle 1\rangle_{\text{sphere}} = 1$. Equation (2.10) is true for any chiral primary fields. In the particular case in which ϕ_i has charge 1 (and dimensions $(1/2, 1/2)$) from (2.9) we get

$$\langle \bar{\phi}_{\bar{j}}(z)\phi_i(0)\rangle_{\text{sphere}} = \frac{G_{i\bar{j}}}{z\bar{z}}\ . \tag{2.11}$$

Let $\Phi_i(z,\theta)$ be the $N = 2$ chiral superfield whose first component is ϕ_i. From (2.11) and $N = 2$ supersymmetry we get[3]

$$\langle \Phi_i(z_1,\theta_1)\ \bar{\Phi}_{\bar{j}}(z_2,\theta_2)\rangle_{\text{sphere}} = \frac{G_{i\bar{j}}}{\tilde{z}_{12}\tilde{z}^*_{12}}\left[1 + \frac{\theta^-_{12}\theta^+_{12}}{\tilde{z}_{12}}\right]\left[1 + \frac{\bar{\theta}^-_{12}\bar{\theta}^+_{12}}{\tilde{z}^*_{12}}\right]\ .$$

Then, if $\phi^{(2)}_i \equiv \int d^2\theta\, \Phi_i$ is the marginal operator multiplying the coupling t^i in the action, one has

$$G_{i\bar{j}} = \langle \phi^{(2)}_i(1)\,\bar{\phi}^{(2)}_{\bar{j}}(0)\rangle_{\text{sphere}}\ ,$$

which is the original definition of the Zamolodchikov metric [12].

The Zamolodchikov metric $G_{i\bar{j}}$ has remarkable geometric properties. The most interesting situation is when $\hat{c} = 3$. In this case the metric $G_{i\bar{j}}$ satisfies a set of constraints which define the so–called *special (Kähler) geometry*. A hermitian metric $G_{i\bar{j}}$ is said to be *special Kähler* if:

i) It is a restricted Kähler metric, i.e. a Kähler metric such that the corresponding Kähler form is 2π times the Chern class of a line bundle $\mathcal{L}$. Locally this means

$$\begin{aligned} &G_{i\bar{j}} = \bar{\partial}_{\bar{j}}\partial_i K, \\ &\text{with} \quad \|1\|^2_{\mathcal{L}} = e^{-K} \end{aligned}\ , \tag{2.12}$$

where 1 is a local holomorphic section trivializing $\mathcal{L}$.

ii) There is a holomorphic symmetric tensor C_{ijk} with coefficients in $\mathcal{L}^2$ satisfying[4]

$$\bar{\partial}_{\bar{\imath}} C_{jkl} = 0 \qquad D_i C_{jkl} = D_j C_{ikl}\ , \tag{2.13}$$

such that the Riemann curvature of $G_{i\bar{j}}$ reads

$$R_{i\bar{j}k}{}^{l} \equiv -\bar{\partial}_{\bar{j}}\Gamma^l_{ki} = G_{k\bar{j}}\delta_i{}^l + G_{i\bar{j}}\delta_k{}^l - e^{2K}C_{ikn}G^{n\bar{n}}C^*_{\bar{j}\bar{m}\bar{n}}G^{\bar{m}l}\ . \tag{2.14}$$

It follows from the tt^* equations that the condition *i)* is satisfied by the Zamolodchikov metric $G_{i\bar{j}}$ of any critical $N = 2$ theory , with the bundle $\mathcal{L}$ identified with

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[3] We use the shorthand notation $\theta^\pm_{12} = \theta^\pm_1 - \theta^\pm_2$ and $\tilde{z}_{12} = z_1 - z_2 - \theta^+_1\theta^-_2 - \theta^-_1\theta^+_2$.

[4] Here D_i is covariant both with respect the Christoffel connection of $G_{i\bar{j}}$ and the canonical connection on the bundle $\mathcal{L}$, i.e. $A_i = -\partial_i K$.

the vacuum line bundle defined by the identity operator. Moreover, if $\hat{c} = 3$ the condition *ii)* holds as well with

$$C_{ijk} = \langle 0|\phi_i(\infty)\,\phi_j(1)\,\phi_k(0)|0\rangle_{\text{top.}}\ . \tag{2.15}$$

$\hat{c} = 3$ is crucial here because only in this case C_{ijk} is non-vanishing for marginal fields (1+1+1=3).

In view of Eq. (2.12), the first assertion is equivalent to saying that $G_{i\bar{j}}$ is Kähler with potential

$$K = -\log\langle\bar{0}|0\rangle\ . \tag{2.16}$$

Let us show this. The index 0 will denote the identity operator, while $i, j, \ldots = 1, \ldots, m$ denote the marginal directions (i.e. chiral primary fields with charge 1). Then $U(1)$ charge conservation gives

$$g_{0\bar{k}} = g_{k\bar{0}} = 0 \qquad \left(gC_i^\dagger g^{-1}\right)_k^{\ 0} = 0\ .$$

Let us project the tt^* equation in the identity sector

$$-\bar{\partial}_{\bar{j}}\partial_i \log\langle\bar{1}|1\rangle = \left[\bar{\partial}_{\bar{j}}(g\partial_i g^{-1})\right]_0^{\ 0} = (C_i)_0^{\ k} g_{k\bar{l}} C^{*\ \bar{l}}_{\bar{j}\bar{0}} g^{\bar{0}0} = g_{i\bar{j}}/g_{0\bar{0}} \equiv G_{i\bar{j}}\ ,$$

where we used the definition of the identity operator i.e.

$$C_{i0}^{\ \ k} = C_{0i}^{\ \ k} = \delta_i^{\ k}\ . \tag{2.17}$$

This shows *i)*. To show *ii)* let us notice that, if $\hat{c} = 3$, the tensor defined by Eq. (2.15) has all the required properties: From (2.15) we see that it is a section[5] of $\mathrm{Sym}^3 T^* \otimes \mathcal{L}^2$: Indeed ϕ_i is an operator valued section of T while $|0\rangle$ and $\langle 0|$ are Hilbert space–valued sections of $\mathcal{L}$. C_{ijk} is holomorphic because, as we saw in Sect. 2.1, the topological 3–point function has this property. Finally, it satisfies the first condition in (2.13) as a consequence of Eq. (2.1). It also satisfies the second condition in (2.13) because it differs from the corresponding equation in (2.2) only by the fact that in $D_i C_j$ the derivative should also act on the j index by the Christoffel connection but that is clearly symmetric in its indices. Here it is crucial that the tt^* connection is equal to the Zamolodchikov connection plus the canonical connection for the bundle $\mathcal{L}$ as it follows from the equation (compare eqs. (2.9), (2.16))

$$g_{i\bar{j}} = e^{-K} G_{i\bar{j}}\ , \tag{2.18}$$

the definition of the tt^* connection (2.3), and the definitions of the Zamolodchikov and line bundle connections which are given by $\Gamma^l_{ki} \equiv G_{k\bar{m}}\partial_i G^{\bar{m}l}$ and $-\partial_i K$, respectively.

Now we are ready to show the main identity, Eq. (2.14). Using the well–known formula for the Riemann curvature in Kähler geometry, we have

$$\begin{aligned}
-R_{i\bar{j}k}^{\ \ \ \ l} &= \bar{\partial}_{\bar{j}}\left(G_{k\bar{m}}\partial_i G^{\bar{m}l}\right) = \\
&= \bar{\partial}_{\bar{j}}\left(g_{k\bar{m}}\partial_i g^{\bar{m}l}\right) - \left(\bar{\partial}_{\bar{j}}\partial_i K\right)\delta_k^{\ l} = \\
&= [C_i, \bar{C}_{\bar{j}}]_k^{\ l} - G_{i\bar{j}}\,\delta_k^{\ l} = \\
&= e^{2K} C_{ikn} G^{n\bar{n}} C^*_{\bar{j}\bar{m}\bar{n}} G^{\bar{m}l} - G_{k\bar{j}}\delta_i^{\ l} - G_{i\bar{j}}\delta_k^{\ l},
\end{aligned} \tag{2.19}$$

[5] Here and below T denotes the $(1,0)$ tangent bundle of the coupling constant (moduli) space.

where we used (2.18), (2.17) and the tt^* equations together with the CPT constraint $\eta^{-1}g = (g^{-1})^t\eta^*$.

Special geometry originally was discovered in two seemingly unrelated contexts: The geometry of periods on a Calabi–Yau 3–fold [13] and $N = 2$ supergravity in four dimensions [14]. The ground state geometry of $\hat{c} = 3$ superconformal theories combines these two topics together in a natural way. In the present paper we shall use quite heavily the relationship of special geometry with the complex geometry of Calabi–Yau 3–folds. In order to be self–contained, we recall the basic facts about this connection.

Before doing this, it is convenient to formulate the above special geometry in a slightly more abstract way. Since special geometry is equivalent to the tt^* geometry for a family of $\hat{c} = 3$ superconformal theories, we shall use the terminology arising in this last context.[6]

Consider the 'improved' connection

$$\nabla_i = D_i - C_i, \qquad \bar{\nabla}_{\bar{j}} = \bar{D}_{\bar{j}} - \bar{C}_{\bar{j}} , \tag{2.20}$$

acting on the vector bundle $\mathcal{V}$ of ground states of equal left–right charge. The tt^* equations are equivalent to the statement that the 'improved' connection is flat. Hence we can identify all fibers of $\mathcal{V}$ with the one at a given base point by parallel transport with respect to this improved connection. In this way (apart for aspects related to global monodromies) we can see $\mathcal{V}$ as a product bundle with fiber the fixed ground state vector space V. In this gauge, the 'improved' derivatives ∇_i and $\bar{\nabla}_{\bar{j}}$ reduce to the ordinary ones ∂_i and $\bar{\partial}_{\bar{j}}$. To the fibers of $\mathcal{V}$ we can give a real structure by declaring real the ground states which are mapped into themselves by CPT. Since

$$(C_i)_{\bar{k}}^{\bar{l}} M_{\bar{l}}{}^{m} = M_{\bar{k}}{}^{k}(C_i)_k^{\ l} ,$$

the real structure is invariant under parallel transport, i.e. the fixed vector space V has a natural real structure. We fix once and for all a basis of V whose elements $|\alpha\rangle$ ($\alpha = 1, \ldots, 2m{+}2$) are real vectors. In this basis CPT acts by the usual complex conjugation. The tt^* metric is not invariant under parallel transport by the ∇–connection; however, if q is the $U(1)$ charge operator, the following real skew–symmetric (symplectic) metric

$$Q_{\alpha\beta} = \langle\alpha|(-1)^{q+3/2}|\beta\rangle$$

is invariant because the matrix $C_i Q$ is skew–symmetric. In the gauge in which the 'improved' connection vanishes, this symplectic form is just a constant matrix; we can choose our real basis $\{|\alpha\rangle\}$ so that it is the standard symplectic unit E.

At a given point in coupling space, the ground state space V admits a decomposition into subspaces corresponding to states having definite $U(1)$ charges. However, as we change the couplings t^i, this charge decomposition changes, since parallel transport by the ∇–connection does not preserve charge. This is obvious from (2.20) since the matrix C_i, representing multiplication by the field ϕ_i, increases the charge by 1. Special geometry just describes how the states of given charge rotate in the fixed space V as we vary the couplings.

[6] Without losing any real generality, we can also assume that all the chiral primary fields have integral $U(1)$ charges. This assumption will be implicit throughout the paper. For a discussion of 'special geometry' in presence of fractional charges, see Ref. [15]

At a given point in coupling constant space V decomposes into a one–dimensional subspace corresponding to $|0\rangle$ having degree[7] 0, an m–dimensional subspace spanned by the vectors $\phi_i|0\rangle$ having degree 1, and their dual subspaces (with respect to the symplectic form Q) having degrees 3 and 2, respectively. As we vary t, the states of degree 0 form a line subbundle of the trivial vector bundle with fiber the fixed space V. This line subbundle is just our vacuum bundle $\mathcal{L}$. In the same way, the states of degree 1, $\{\phi_i|0\rangle\}$, span the fibers of the vector bundle $(T\otimes\mathcal{L})$, those of degree 2 the dual space $(T\otimes\mathcal{L})^*$, and finally those of degree 3 the dual line bundle $\mathcal{L}^*$. Thus we have the charge decomposition

$$V = \mathcal{L} \,\oplus\, (T\otimes\mathcal{L}) \,\oplus\, (T\otimes\mathcal{L})^* \,\oplus\, \mathcal{L}^* \,. \tag{2.21}$$

This decomposition satisfies four main properties. Let ξ and ζ be two sections of V with definite degrees; then

1. $\xi^t E\zeta = 0$ unless $l(\xi) + l(\zeta) = 3$.
2. $(-1)^{l(\xi)}\xi^\dagger E\xi > 0$. Indeed, comparing with the definition of g, we see that this is just the squared norm of the vacuum state corresponding to ξ.
3. $\partial_i\xi$ is a sum of two pieces, one with $l = l(\xi)$ and one with $l = l(\xi) + 1$. This property is evident from (2.20) which also gives

$$\partial_i\xi\Big|_{l(\xi)+1} = -C_i\xi \,. \tag{2.22}$$

4. $\mathcal{L}$ is a *holomorphic* line subbundle of V. Indeed, $\bar\nabla_{\bar j}$ acting on a degree 0 state produces a pure degree 0 state; hence the flat connection $\bar\nabla_{\bar j}$ induces a holomorphic structure on $\mathcal{L}$. Since in the present gauge the ∇–connection is trivial, this is just the canonical holomorphic structure for a subbundle of V.

Working in the symplectic basis, the only non–trivial datum is how the original ground states $|\phi_a\rangle$ are written in terms of the symplectic ones $|\alpha\rangle$, i.e. we must know the coefficients of the expansion

$$|\phi_a\rangle = V^\alpha_a|\alpha\rangle \,. \tag{2.23}$$

From these coefficients we can easily recover the tt^* metric

$$g_{a\bar b} = (-1)^{l_a} V_b^\dagger E V_a \,, \tag{2.24}$$

while the matrices C_i can be extracted from (2.22) which can be rewritten as

$$\partial_i V^\alpha_a = -(C_i)_a{}^b V^\alpha_b + \text{terms with lower charge} \,.$$

Giving V^α_a is equivalent to giving the decomposition in Eq. (2.21). The matrix V^α_a is restricted by the above four conditions. Conversely, given any decomposition (2.21) satisfying these conditions we can construct a metric g satisfying the tt^* equations with respect to the C_i defined by Eq. (2.22) and having all the properties discussed above. Indeed the tt^* equations are equivalent to the flatness of the connection ∇, which is automatic in such a construction.

[7] For later convenience, we define the 'degree' l of a ground state to be the $U(1)$ charge of the corresponding NS state, i.e. $l = q + 3/2$.

In fact, it is enough to know $V^\alpha \equiv V_0^\alpha$, i.e. how the vacuum line bundle $\mathcal{L}$ sits in V. Indeed from (2.17) one has

$$\partial_i V^\alpha = -C_{i0}^k V_k^\alpha + \ldots = -V_i^\alpha \bmod . V^\alpha ,$$

so we can read V_i^α (i.e. the degree 1 subbundle) from the derivatives of V^α. The degree 2 and 3 subbundles then can be recovered by duality. Acting $\bar{\nabla}$ on both sides of Eq. (2.23), we see that V^α should be a holomorphic function of the t's; this is property 4. above. Then the above four conditions are automatically satisfied if and only if V^α is holomorphic and satisfies[8]

$$V^t E V = V^t E\, \partial_i V = 0, \qquad V^t E V > 0 . \tag{2.25}$$

So given any holomorphic function $V^\alpha(t)$ satisfying (2.25) we construct a special geometry. In particular, from (2.24) we see that the Kähler potential is

$$e^{-K} = g_{0\bar{0}} = V^t E V . \tag{2.26}$$

Consider $\partial_i \partial_j \partial_k V^\alpha$. The component of top degree is given[9] by $-(C_i C_j C_k)_0^\rho V_\rho^\alpha$. Hence

$$V^t\, E \partial_i \partial_j \partial_k V = -(C_i C_j C_k)_{00} \equiv -C_{ijk} , \tag{2.27}$$

which is the most convenient way to define C_{ijk}.

The above discussion applies to any $N = 2$ conformal model with $\hat{c} = 3$. Now we specialize to the B–model based on a Calabi–Yau 3–fold M (which we assume to be simply–connected). In this case the chiral primary fields of $U(1)$ charge q are in one–to–one correspondence with the elements of $H^{3-q,q}(M)$. This follows from Eq. (2.8) and the fact that for a simply connected Calabi–Yau 3–fold the relevant vacua correspond to primitive cohomology classes in degree 3, which are annihilated both by k and $k^\dagger$. All these spaces are subspaces of the ordinary de Rham group $H^3(M, \mathbf{C})$; since the de Rham cohomology depends only on the topology of M, this group is independent of the couplings t^i (which control the complex structure of M). $H^3(M, \mathbf{C})$ can be seen as a fixed space while the definite charge subspaces $H^{3-q,q}(M)$ do move as we move the t^i's. Then the constant space $H^3(M, \mathbf{C})$ is easily identified with the space V of the abstract $N = 2$ theory. Notice that this space has a natural real structure, namely a class in $H^3(M, \mathbf{C})$ is *real* iff it belongs to $H^3(M, \mathbf{R})$; this structure coincides with the one defined by CPT in the B–model. Then the charge decomposition is identified with the Hodge decomposition $H^3(M) = \oplus_q H^{3-q,q}(M)$. On $H^3(M)$ there is a natural symplectic form given by

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta ,$$

which is also independent of t^i and real since it is topologically defined. With respect to this pairing and real structure, the Hodge decomposition satisfies 1. and 2. (the Riemann bilinear relations). That the Hodge decomposition also satisfies conditions 3. and 4. is a consequence of the Kodaira–Spencer theory of complex deformations.

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8 In particular, any special manifold is a Legendre submanifold of a complex contact manifold.

9 The index ρ labels the unique chiral primary field of charge 3, normalized so that $\langle \rho \rangle = 1$.

Let μ_i be the element[10] of $H^1(M, T_M)$ associated with an infinitesimal variation δt^i of the complex structure, and let ω be any harmonic $(3-q, q)$ form. Then

$$\partial_i \omega = \mu_i \wedge \omega + \beta_i \ , \tag{2.28}$$

where β_i is a closed $(3-q, q)$ form, and μ_i acts on forms as contraction for the vector index and exterior multiplication for the form index. Thus $\mu_i \wedge \omega$ is a $(2-q, q+1)$ form. Equation (2.28) is nothing else than condition 3. The same argument applied to $\bar{\partial}_{\bar{j}} \omega$ implies condition 4.

According to our previous discussion, we can recover the ground state geometry for the B–model provided we know how the space $H^{3,0}(M)$ (which we identified with the line subbundle $\mathcal{L}$) sits in $H^3(M, \mathbf{C})$, i.e. if we know for each point in moduli space which de Rham class corresponds to the $(3,0)$ form Ω. Choosing Ω to depend holomorphically on t^i, we can rewrite Eqs. (2.26) and (2.27) in the form

$$\begin{aligned} e^{-K} &= \int_M \bar{\Omega} \wedge \Omega \ , \\ C_{ijk} &= - \int_M \Omega \wedge \partial_i \partial_j \partial_k \Omega \ . \end{aligned} \tag{2.29}$$

As a symplectic basis of vacua we can take the states associated to the (real) 3–forms which are dual to a canonical set of 3–cycles, i.e. a set of cycles γ_α ($\alpha = 1, \ldots, 2m+2$) such that their intersection pairing has the canonical form

$$\#(\gamma_\alpha, \gamma_\beta) = \delta_{\beta, \alpha+m+1} - \delta_{\beta+m+1, \alpha} \ .$$

Then our basic vector V^α becomes

$$V^\alpha = \int_{\gamma_\alpha} \Omega \ , \tag{2.30}$$

so that V^α is just given by the periods of the holomorphic $(3,0)$ form. Equations (2.26) and (2.27) allow to write all the relevant geometric quantities in terms of these periods. One can also show that the metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is equal to the Weil–Petersson metric on the Calai–Yau moduli space. To see this, consider Eq. (2.28) with ω replaced by Ω. Since $H^{3,0}(M)$ is one–dimensional, β_i is cohomologous to $f_i \Omega$, where f_i is some holomorphic function of the moduli t^i. Using the first of eqs. (2.29), we have

$$\begin{aligned} G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K &= -\partial_i \bar{\partial}_{\bar{j}} \log \int_M \bar{\Omega} \wedge \Omega \\ &= - \frac{\int \bar{\partial}_{\bar{j}} \bar{\Omega} \wedge \partial_i \Omega}{\int \bar{\Omega} \wedge \Omega} + \frac{\int \bar{\partial}_{\bar{j}} \bar{\Omega} \wedge \Omega \ \int \bar{\Omega} \wedge \partial_i \Omega}{\left(\int \bar{\Omega} \wedge \Omega\right)^2} \ . \end{aligned} \tag{2.31}$$

Now, Eq. (2.28) together with type considerations give

$$\begin{aligned} &\int_M \bar{\Omega} \wedge \partial_i \Omega = f_i \int_M \bar{\Omega} \wedge \Omega, \\ &\int_M \bar{\partial}_{\bar{j}} \bar{\Omega} \wedge \partial_i \Omega = \int_M \left(\bar{\mu}_{\bar{j}} \wedge \bar{\Omega}\right) \wedge \left(\mu_i \wedge \Omega\right) + f_i \bar{f}_{\bar{j}} \int_M \bar{\Omega} \wedge \Omega \ . \end{aligned}$$

[10] T_M denotes the holomorphic tangent bundle (sheaf) of the Calabi–Yau manifold M.

Inserting these back into (2.31) we get

$$G_{i\bar{j}} = -\frac{1}{\int_M \bar{\Omega} \wedge \Omega} \int_M (\bar{\mu}_{\bar{j}} \wedge \bar{\Omega}) \wedge (\mu_i \wedge \Omega) \equiv (\bar{\mu}_{\bar{j}}\,,\, \mu_i)\,, \tag{2.32}$$

where $(\cdot,\cdot)$ is the inner product on the bundle $\Omega^{0,1} \otimes T_M$ induced by (any) Calabi–Yau metric on M. (The last equality in (2.32) is most easily seen by writing down the explicit index structure of the various tensors involved.) By definition, the r.h.s. of (2.32) is the Weil–Petersson metric on the Calabi–Yau moduli space. Comparing this result with Eq. (2.9), we also see that

$$g_{i\bar{j}} = -\int_M (\bar{\mu}_{\bar{j}} \wedge \bar{\Omega}) \wedge (\mu_i \wedge \Omega)\,.$$

We stress that Ω is well defined by the above conditions only up to multiplication by a holomorphic function of the t^i's. Clearly, two Ω's differing for such a factor define the same Hodge decomposition; therefore this ambiguity is immaterial in all the above discussion and has no physical consequence (in particular the Zamolodchikov metric is independent of these choices). Instead the tensor C_{ijk} gets multiplied by $f(t)^2$ when $\Omega \to f(t)\Omega$. This behaviour reflects the basic fact that C_{ijk} is a tensor with coefficients in the line bundle $\mathcal{L}^2$.

For future reference, we give an alternative expression for the B–model 3–point functions C_{ijk}. As before, we start from the basic identity

$$\partial_i \Omega = \mu_i \wedge \Omega + \dots\,, \tag{2.33}$$

where the dots stand for a closed form of type $(3,0)$. Taking a second derivative

$$\partial_i \partial_j \Omega = \mu_i \wedge \mu_j \wedge \Omega + \dots\,, \tag{2.34}$$

where now the dots denote closed forms of type $(2,1)$ and $(3,0)$. From (2.34) and considerations of type we see that

$$\int_M \Omega \wedge \partial_i \partial_j \Omega = 0\,.$$

Taking the derivative of this identity, we get

$$\int_M \partial_i \Omega \wedge \partial_j \partial_k \Omega = -\int_M \Omega \wedge \partial_i \partial_j \partial_k \Omega = C_{ijk}\,. \tag{2.35}$$

Note that replacing the derivatives in the l.h.s. of (2.35) by their explicit expressions (2.33) and (2.34) the omitted terms do not contribute because of type considerations. So for our present purposes we can ignore them.

We introduce the following notation: A prime $'$ means contraction of the vector indices with the holomorphic 3–form Ω, while a superscript $^\vee$ means contraction of the form indices with the unique holomorphic 3–vector dual to Ω. Obviously these two operations are each others inverse $(A')^\vee = A$, and $(B^\vee)' = B$. In this notation (2.33) and (2.34) read

$$(\partial_i \Omega)^\vee = \mu_i + \dots\,,$$

$$\partial_i \partial_j \Omega = (\mu_i \wedge \mu_j)' + \dots = \left[(\partial_i \Omega)^\vee \wedge (\partial_j \Omega)^\vee\right]' + \dots\,.$$

Replacing the second equation in (2.35) we get the formula we are looking for[11]

$$C_{ijk} = \int_M \partial_i \Omega \wedge \left[(\partial_j \Omega)^\vee \wedge (\partial_k \Omega)^\vee\right]' = \int_M \partial_i \Omega \wedge (\partial_j \Omega)^\vee \wedge \partial_k \Omega \,. \tag{2.36}$$

The case of a Calabi–Yau n–fold ($n > 3$) is rather similar. As we saw, the Zamolodchikov metric $G_{i\bar{j}}$ is Kähler also in this case, again with potential $-\log\langle\bar{0}|0\rangle$. On the other hand the basic identity for its curvature, Eq. (2.14) is going to change. To get the corresponding identity for an n–fold, we have to go through the same steps as in (2.19). All the steps are unchanged, except for the last one. Then the following formula is valid for all n (we use capital latin letters to denote charge 2 chiral primaries)

$$-R_{i\bar{j}k}{}^{l} = e^K C^I_{ik}\, g_{I\bar{J}}\, C^{*\,\bar{J}}_{\bar{j}\bar{m}} G^{\bar{m}l} - G_{k\bar{j}}\delta^l_i - G_{i\bar{j}}\delta^l_k \,.$$

2.4 Coupling twisted $N = 2$ theory to gravity. Bosonic string theory is in many ways like a twisted $N = 2$ theory [16, 17, 18]. It has a *scalar* supercharge $Q_{BRST} = Q + \overline{Q}$, which is the usual BRST operator. It has anti-ghosts, $b, \bar{b}$ of spin (2,0) and (0,2), with the property

$$Q^2 = b_0^2 = 0 \,,$$
$$\{Q, b_0\} = H_L \,,$$

and it has two $U(1)$'s, $G, \bar{G}$ corresponding to the left and right ghost numbers. Identifying

$$2 j_{BRST} \leftrightarrow G^+ \,,$$
$$b \leftrightarrow G^- \,,$$
$$bc \leftrightarrow J \,,$$

and similarly for right-movers. Thus the notion of a physical state in the bosonic string becomes exactly the same as that of a chiral state in the twisted theory. Thus we can define coupling of twisted $N = 2$ theory to gravity by integrating correlation functions of chiral fields over moduli space of Riemann surface, with the insertion of G^-'s folded with $3g - 3$ Beltrami differentials. In particular the partition function of the twisted $N = 2$ theory coupled to gravity at genus $g > 1$, F_g, can be defined by[12]

$$F_g = \int_{\mathcal{M}_g} \left\langle \prod_{k=1}^{3g-3} \left(\int G^- \mu_k\right)\left(\int \bar{G}^- \bar{\mu}_k\right)\right\rangle ,$$

where μ_i denote the Beltrami differentials, and $\mathcal{M}_g$ denotes the moduli space of genus g Riemann surfaces. For F_1 the answer can be written using the corresponding analysis of the bosonic string case [19]. To do this note that in bosonic string one inserts $bc\bar{b}\bar{c}$ to absorb the ghost zero modes. This is translated in the twisted theory to the insertion of left and right fermion number currents. Also, to fix the normalization it

[11] Recall that $\wedge$ means exterior product with respect to the form indices and *contraction with respect to the vector indices.*

[12] For the case of $g = 2$ one has to put a factor of 1/2 in front because all the $g = 2$ curves have a $\mathbf{Z}_2$ symmetry.

is best to write the answer in the operator formulation which is particularly convenient for the torus:

$$F_1 = \frac{1}{2}\int \frac{d^2\tau}{\tau_2}\mathrm{Tr}\left[(-1)^F F_L F_R q^{H_L}\bar{q}^{H_R}\right] , \tag{2.37}$$

where the factor of 1/2 in front takes care of the fact that there is a $\mathbf{Z}_2$ reflection symmetry for all tori (this normalization is different from the one used in [20]). For genus 0, the 0,1, and 2 point functions are zero, as is the case in bosonic string, and the three point functions can be written as

$$\langle \phi_i \phi_j \phi_k \rangle = C_{ijk} \ .$$

In other words the chiral fields ϕ_i which after twisting have dimension zero play the same role as $c\bar{c}V_i$ in the usual bosonic strings, and $\phi_i^{(2)}$ plays the same role as V_i.

It is a rather nice property of twisted *unitary* $N = 2$ theories that F_g thus defined is finite and thus well defined. The only potential divergence would have come from the regions near the boundary of moduli space of Riemann surfaces. But in such cases, the fact that the propagator on a long tube is given by $G_0^- \frac{1}{L_0+\bar{L}_0}\bar{G}_0^-$ and that it annihilates the massless modes, implies that only the massive modes propagate and thus the integrand in F_g is exponentially small in these regions (the coefficient of exponent being fixed by the first non-vanishing eigenvalue of $L_0 = \bar{L}_0$).

Despite an almost complete parallel between bosonic string and twisted $N = 2$ theories coupled to gravity, there are two notable differences. The first one is that the ghost number violation in bosonic string at genus g is universal and is given by $3g - 3$, whereas for twisted $N = 2$ theories it is given by (2.5) as $\hat{c}(g - 1)$. In particular we see that $\hat{c} = 3$ is a critical case in that it gives the same degree of charge violation as bosonic string. So in particular this suggests that Calabi–Yau 3-folds are a specially interesting class to consider [21]. Note that only for $\hat{c} = 3$ the F_g has a chance to be non-zero for $g > 1$, by $U(1)$ charge conservation. For all the other values of $\hat{c}$, the only way to get a non-zero result is by introducing other correlators. The correlators involving chiral fields may be used to prevent vanishing of correlation functions only for $1 < \hat{c}$; For $\hat{c} \leq 1$ the charges of all $\phi_i^{(2)}$ are negative (the maximum being $\hat{c} - 1$) and so cannot be used to balance charges. In these cases, which happen to be intensively studied in connection with matrix models, one needs to include the full topological gravity multiplet and construct gravitational descendants which give rise to non-vanishing correlation functions [21, 22, 23]. Also for $\hat{c} > 3$ one needs fractional chiral fields ϕ_i with charges between $0 < q < 1$ in order to have a chance of balancing the charges (gravitational descendants do not help in this case as they contribute $+N$ to the charge violation condition). In particular for Calabi-Yau manifolds, which have no fractional chiral states, with dimension bigger than 3 all the correlations vanish, and thus the theory is not very interesting (except possibly for three point functions on the sphere with $\sum q_i = n$ and F_1 which is computable and non-zero in general). For a Calabi-Yau 2-fold, which is either $K3$ or T^4, the situation is hardly more interesting: all the correlation functions vanish in the case of T^4 due to too many fermion zero modes and on $K3$ because of charge conservation[13]. Finally for the case of one–dimensional Calabi–Yau manifolds only F_1 is non–zero

[13] In the case of $K3$ in principle there was a chance that the genus g correlation function for $g > 1$ with $g - 1$ insertions of the highest charge chiral field which balances the charge lead to non-vanishing of the amplitude. But in fact using the techniques in this paper, i.e. the holomorphic anomaly equation in this case, one can show the amplitude still vanishes even with this insertion.

(as studied in [20]), due to charge conservation unless one introduces gravitational descendants.

So clearly as far as the twisted sigma–models coupled to gravity are concerned, the most interesting case is the 3–fold Calabi–Yau case which will be the focus of our examples. Actually we will consider the more general possibility of a unitary SCFT with $\hat{c} = 3$ with integral $U(1)$ charges for chiral fields. Many of the results we will discuss can be easily generalized to other similar cases and would be interesting to study.

The other major difference between the bosonic string and critical topological strings, even if we choose $\hat{c} = 3$, is that in the case of topological strings obtained by twisting unitary $N = 2$ theories, the G^- cohomology is generically non–trivial whereas absolute b–cohomology is always trivial in the bosonic strings. This, as we will see in the rest of the paper is a crucial difference, and it leads to the anomaly discussed in detail in the next section.

The case of Calabi–Yau 3–fold as a string theory has already been studied in [24] for both open and closed strings. In particular it was discovered there that in the case of the open string theory, the target space physics is equivalent to three dimensional Chern–Simons theory. In the case of closed strings there were some puzzles raised which we resolve in connection with our discussion on the Kodaira–Spencer theory in Sect. 5.

2.5 Properties of n–point functions and the holomorphicity paradox. Consider the n–point functions of the $N = 2$ twisted theory coupled to gravity at genus g

$$C^g_{i_1 i_2 \ldots i_n} = \\ = \int_{\mathcal{M}_g} \Big\langle \int \phi^{(2)}_{i_1} \cdots \int \phi^{(2)}_{i_n} \prod_{k=1}^{3g-3} (\int G^- \mu_k)(\int \bar{G}^- \bar{\mu}_k) \Big\rangle . \tag{2.38}$$

We would like to relate this to F_g. At first sight one may think that $C^g_{i_1 i_2 \ldots i_n}$ can be written simply as

$$C^g_{i_1 i_2 \ldots i_n} = \partial^n_{i_1 i_2 \ldots i_n} F_g \, .$$

However this formula cannot possibly be true since, as it is evident from its definition (2.38) (and discussed in Sect. 2.1), F_g is a section of $\mathcal{L}^{2-2g}$ and $C^g_{i_1 i_2 \ldots i_n}$ is a section of the *non–trivial* vector bundle $\mathrm{Sym}^n T^* \otimes \mathcal{L}^{2-2g}$ sitting over coupling space. Therefore acting with ∂_{i_k}'s on F_g makes no sense at all. Geometrically it is clear that the correct relation should have the form

$$C^g_{i_1 i_2 \ldots i_n} = \mathcal{D}_{i_n} \ldots \mathcal{D}_{i_1} F_g \, , \tag{2.39}$$

where $\mathcal{D}_i$ is some suitable connection compatible with the transition functions for the appropriate bundles. On $\mathrm{Sym}^n T^* \otimes \mathcal{L}^{2-2g}$ there is a natural connection D_i, i.e. the one induced by the Zamolodckikov connection on T plus the canonical connection on $\mathcal{L}$, see Section 2.3. It is natural to guess that (2.39) holds with $\mathcal{D}_i$ replaced by this natural connection D_i. This is what we will presently argue. This will imply that the following recursion relation (for $2g + n - 3 > 0$) holds

$$e^{2(1-g)K} G^{\bar{j}_1 i_1} G^{\bar{j}_2 i_2} \ldots G^{\bar{j}_{n-1} i_{n-1}} C^g_{i_1 i_2 \ldots i_{n-1} i_n} = \\ = \partial_{i_n} \left(e^{2(1-g)K} G^{\bar{j}_1 i_1} G^{\bar{j}_2 i_2} \ldots G^{\bar{j}_{n-1} i_{n-1}} C^g_{i_1 i_2 \ldots i_{n-1}} \right) . \tag{2.40}$$

To show this, there are two things that we will have to argue: One is covariantization with respect to the Zamolodchikov metric and the other is covariantization with respect to the natural connection on $\mathcal{L}$. Both will arise from contact terms. First let us discuss how the covariantization with respect to the Zamolodchikov metric arises. This actually has been done in full generality for marginal operators of conformal theory in [25] leading to the following contact term in our case:

$$\phi_i^{(2)}(z)\,\phi_j^{(2)}(0) \sim \delta^2(z)\Gamma_{ij}^k\phi_k^{(2)}(0)\,,$$

where Γ_{ij}^k is defined in Sect. 2.3 and is the connection for the Zamolodchikov metric. This amounts to the first thing we wished to show. However we would also like to rederive this result using the tt^* machinery: In defining the amplitude in (2.38) we have to be careful to regularize the computation by making sure that two operators do not get closer to each other than distance ϵ. However, when we take the derivative of F_g with respect to t^i this region is not excluded. Therefore the difference between the explicit meaning of the correlation and the derivative with respect to t^i will include the regularization of the integration of $\phi_i^{(2)}$ in a small neighborhood of $\phi_j^{(2)}$ for all j's. Since this is a local computation we may as well do it on a hemisphere, where we can apply tt^* equations. In such a case the integral of $\phi_i^{(2)}$ over the hemisphere including the field $\phi_j^{(2)}$ in it minus the one with $\phi_j^{(2)}$ outside of the hemisphere, is equal to, as far as the topological states are concerned [2]

$$\begin{aligned} G_{-1}^-\overline{G}_{-1}^-\partial_i|j\rangle - \phi_j^{(2)}\partial_i|0\rangle &\to \left[(A_i)_j^k\phi_k^{(2)} - (A_i)_0^0\cdot\phi_j^{(2)}\right]|0\rangle \\ &= [(g^{-1})^{k\bar{j}}\partial_i g_{j\bar{j}} + \partial_i K\delta_j^k]\phi_k^{(2)}|0\rangle = \Gamma_{ij}^k\phi_k^{(2)}|0\rangle\,. \end{aligned}$$

Therefore we see that the insertion of $\phi_i^{(2)}$ in the correlation is equivalent to

$$\partial_i - \Gamma_i$$

as was to be shown.

Now we turn to the second covariantization, i.e., with respect to the line bundle connection on $\mathcal{L}$. This arises from the hidden contact term between the term we added to the action to twist the supersymmetric theory

$$\frac{1}{2}\int J\bar{\omega} + c.c. = \frac{1}{2}\int R\varphi + \text{total} \quad \text{derivatives}$$

and the operator $\phi_i^{(2)}$, as J has a contact term with it. Here ω is the spin connection and φ is the scalar which bosonizes the $U(1)$ current. To study this term, again we use the fact that it is local, and that we can thus first study the case of the hemisphere to which we can apply tt^* considerations. The same argument as above leads to the contact term

$$(A_i)_0^0\cdot 1 = -\partial_i K\cdot 1\,.$$

More generally, since the contribution due to this term is proportional to $\int R$, and the above computation was done on hemisphere with net $\int R = 2\pi$, we can write the above term more generally as

$$-\partial_i K\frac{R}{2\pi}\,.$$

On genus g surface this leads to an integrated contact term

$$-\partial_i K(2-2g)\ ,$$

and so insertion of $\int \phi_i^{(2)}$ in the correlation amount to covariantization also with respect to the line bundle $\mathcal{L}^{2-2g}$ which leads us to our final answer for the operator insertion,

$$\int \phi_i^{(2)} \to \partial_i - \Gamma_i - (2-2g)\partial_i K\ . \tag{2.41}$$

Note that this is consistent with the fact that the insertion of $\int \phi_i^{(2)}$ should not lead to any further ambiguities in fixing the normalization of the path integral. In particular $\exp[(2g-2)K]\cdot F_g$ is independent of such ambiguity as far as holomorphic derivatives are concerned as the K varies precisely to compensate the ambiguity of F_g as discussed in Sect. 2.1. Note in particular that the symmetry in exchange of $\phi_i^{(2)}$ is consistent with the above relation with covariantization since

$$D_{i_1}D_{i_2}\dots D_{i_{n-3}}C^{(g)}_{i_{n-2}i_{n-1}i_n}$$

is symmetric in all its n indices. This is a consequence of $[D_i, D_j]=0$ (i.e. the curvature of the natural connection has type $(1,1)$) (for the case of genus 0 we also need the fact that $D_i C_{jkl}$ is totally symmetric in its four indices, see Eq. (2.13)).

Now that we have understood how covariantizations arise we are going to present a formal argument for decoupling of anti-chiral operators $\bar\phi_{\bar i}^{(2)}$ from the correlation functions. Let us consider the correlations on the sphere:

$$C_{i_1 i_2\dots i_n} = \left\langle \phi_{i_1}(0)\,\phi_{i_2}(1)\,\phi_{i_3}(\infty)\int\phi_{i_4}^{(2)}\cdots\int\phi_{i_n}^{(2)}\right\rangle . \tag{2.42}$$

If the BRST–trivial states do decouple from the physical amplitudes, then the n–point functions $C_{i_1\dots i_n}$ should depend holomorphically on the couplings t^i. Indeed,one has

$$\begin{aligned}\bar\partial_{\bar j} C_{i_1 i_2\dots i_n} &=\\ &= \int\sqrt{g}\,d^2z\left\langle \phi_{i_1}(0)\,\phi_{i_2}(1)\,\phi_{i_3}(\infty)\int\phi_{i_4}^{(2)}\cdots\int\phi_{i_n}^{(2)}\oint_{C'_z}G^+\oint_{C_z}\bar G^+\bar\phi_{\bar j}(z)\right\rangle ,\end{aligned} \tag{2.43}$$

where C_z and C'_z are small contours enclosing the point z. Now we can deform the C'_z countour around the other operator insertions. Since

$$\oint_{C_w} G^+\phi_i(w) = 0,$$

$$\oint_{C_w} G^+\phi_i^{(2)} = \partial\phi_i^{(0,1)} = d\phi_i^{(0,1)}\ ,$$

we get

$$\begin{aligned}\bar\partial_{\bar j} C_{i_1 i_2\dots i_n} &= -\sum_{k=4}^{n}\int\sqrt{g}d^2z\left\langle\phi_{i_1}(0)\phi_{i_2}(1)\phi_{i_3}(\infty)\int\phi_{i_4}^{(2)}\cdots\times\right.\\ &\qquad\left.\times\int d\phi_{i_k}^{(0,1)}\cdots\int\phi_{i_n}^{(2)}\oint_{C_z}\bar G^+\bar\phi_{\bar j}(z)\right\rangle\\ &\overset{\text{formally}}{=} 0\ ,\end{aligned} \tag{2.44}$$

since $\int d\phi_i^{(0,1)} = 0$. This formal manipulation can be extended to the n–point functions at genus g (2.38). Since the BRST variation of $(\int G^- \mu_k)$ produces the energy momentum tensor folded in the Beltrami differential, the additional terms arising from the deformation of the contour have the form of the derivative of some correlation function with respect to the moduli of the complex surface just as it happens in the bosonic string theory; hence they are also expected to vanish upon integration over the (compactified) moduli space of genus g Riemann surfaces $\mathcal{M}_g$.

However, this consequence of the decoupling of BRST–trivial states is in contradiction with what we know from tt^* geometry as we will now see. This contradiction leads to a paradox that will be resolved by the discovery of a new 'holomorphic' anomaly, which will be discussed in the next section. The point is that holomorphicity of the n–point functions (for $n > 3$) is not consistent with the recursion relation (2.39). Indeed, $\bar{\partial}_{\bar{j}}$ does not commute with D_i: Rather $[\bar{\partial}_{\bar{j}}, D_i]$ is the non–vanishing curvature of the natural connection. For instance, consider the 4–point function $C_{ijkl} = D_l C_{ijk}$. Since $\bar{\partial} C_{ijk} = 0$, we have

$$\begin{aligned}\bar{\partial}_{\bar{m}} C_{ijkl} = \bar{\partial}_{\bar{m}} D_l C_{ijk} = [\bar{\partial}_{\bar{m}}, D_l] C_{ijk} = \\ = 2G_{l\bar{m}} C_{ijk} - (R_{l\bar{m}i}{}^n C_{njk} + 2 \text{ permutations}) ,\end{aligned}$$

where $R_{l\bar{m}i}{}^n$ is the curvature of the Zamolodchikov metric. To see that the r.h.s. is indeed not zero, replace this curvature by its explicit expression given by special geometry.

As we shall see in the next section, in higher genus the situation is even worse, since there the partition function F_g is also not holomorphic.

What is the way out of this paradox, i.e. where is the loop–hole in the naive argument around Eq. (2.44)? The point is that although it is true that we remain with a sum of terms each with an operator $\int d\phi_i^{(0,1)}$ inserted (cf. (2.44)), these terms do not vanish upon integration over the Riemann surface, because the corresponding integral gets a non–trivial boundary term when the field $\phi_i^{(0,1)}$ approaches a point where some other operator is inserted. Indeed the n–point function on the sphere should be written more invariantly as an integral over the moduli space $\mathcal{M}_{0,n}$ of a sphere with n punctures. The configurations where two points get close together make the boundary of this space. Then, taking $\bar{\partial}_{\bar{j}}$ of the n–point function and deforming contours as in Eq. (2.44) we get the integral over $\mathcal{M}_{0,n}$ of an *exact* top form. But, since the boundary of $\mathcal{M}_{0,n}$ is not empty (for $n > 3$), this does not mean that the integral itself vanishes. However, $\bar{\partial}_{\bar{j}} C_{i_1 \ldots i_n}$ may get a contribution only from the boundary of $\mathcal{M}_{0,n}$. Since the boundary corresponds to two operators colliding, we see that the n–point function may fail to be holomorphic only because of contact terms. This is precisely what we found by the explicit computation above.

In the next section we shall see how this holomorphic anomaly appears in higher genus. There again we shall find that $\bar{\partial}_{\bar{j}} C^g_{i_1 \ldots i_n}$ gets contribution only from the boundary of the moduli space of genus g surfaces with n punctures $\mathcal{M}_{g,n}$. However, since in this case the boundary has more components, new interesting phenomena will appear.

2.6 Canonical coordinates and special coordinates. Before turning to the next section we would like to make one comment about covariantization which will be both useful for us later as well as clarifying the relation to some work already in the literature: In topological theories it is well known that the insertion of chiral fields can be represented by ordinary derivatives [26]. This is also implicitly used in the discussion of counting of holomorphic curves using the topological sigma–models [4]. What we have said so far seems to be at odds with these works. In fact not only is there no contradiction but actually clarifying the relation of our work to these will explain some of the ansatz made in the study of the mirror map [7]. The point is that if we start from a unitary $N = 2$ theory which we denote by S_0, twist it and then perturb *only* by topological fields

$$S = S_0(t_0, \bar{t}_0) + \int d^2z d^2\theta\, \delta t^i \phi_i \ ,$$

there are no infinities and one could be naive in taking field insertions on the world sheet. In other words we do not have to prevent points approaching each other by cutting out small discs around each field insertion. In fact as we mentioned it was rather important in the derivation of various properties of the topological correlations to take ordinary derivatives of the partition function to obtain topological correlations. It turns out that this is possible as long as we are only interested in topological correlations *and for a fixed* $\bar{t}_0^i$. This is possible because of (2.2), in other words, the fact that

$$[D_i, D_j] = 0$$

means that for a fixed base point $(t_0, \bar{t}_0)$, shifting t alone can be accomplished by ordinary derivative, i.e., we can choose a gauge in which $D_i \to \partial_i$. This being true means that we are taking a choice of coordinates on the moduli space as well as a gauge for the line bundle on the moduli space so that

$$\partial_{i_1} \ldots \partial_{i_n} \Gamma^k_{ij} \Big|_{(t_0, \bar{t}_0)} = 0 = \partial_{j_1} \ldots \partial_{j_r} K \Big|_{(t_0, \bar{t}_0)} \ . \tag{2.45}$$

It turns out that these conditions *fix* the choice of coordinates and line bundle section $|0\rangle$ up to linear transformation, and can be done for *arbitrary* Kähler manifolds with arbitrary line bundles on it having real analytic metrics.

Let us first talk about the line bundle. Consider a local section near t_0 and let the norm of this section be e^{-K}. In arbitrary coordinate system K has the following expansion $K(z, \bar{z}) = K_0(z) + \bar{z}^m F_{\bar{m}}(z) + o(\bar{z}^2)$, where we take the t_0 to correspond to $z = 0$. By redefining the choice of the local holomorphic section we can get rid of K_0 which is purely holomorphic. With this choice of local section all holomorphic derivatives of K are equal to zero at the origin. So such a section exists. Moreover it is unique up to multiplication by a constant because any z dependence will give rise to a non-constant K_0 which will thus violate the condition that holomorphic derivatives of $\partial_i K$ vanish at the origin.

Now we show that the same can be done for the Christoffel connection. On a Kähler manifold there is locally a Kähler potential, which we again denote by K, and which can be expanded as above, and with a choice of gauge K_0 can be chosen to be zero. The expansion for the metric follows from the expansion for Kähler potential $G_{k\bar{m}} = \partial_k F_{\bar{m}}(z) + o(\bar{z})$. Making the *holomorphic* change of variables $z^i \to t^i$ such that

$$\frac{\partial z^k}{\partial t^i} \partial_k F_{\bar{m}}(z) = C_{i\bar{m}} = const \ , \tag{2.46}$$

or, explicitly, $t^i = C^{i\bar{m}} F_{\bar{m}}(z)$, one reduces metric to the form $G_{k\bar{m}} = C_{i\bar{m}} + o(\bar{z})$. The ambiguity in t-coordinates is in the choice of constant matrix $C_{i\bar{m}}$ and it is parametrized by $GL(n)$. In t-coordinates holomorphic Christoffel symbols vanish at the origin ($\bar{z} = 0$) together with all holomorphic derivatives as was to be shown. We shall call the coordinates t^i and the choice of local trivialization of the line bundle in which (2.45) holds *canonical coordinates* with respect to the base–point t_0.

To relate to some comments made in the literature, we would like to draw attention to a natural base point in the case of A–model, and that is infinite volume $\bar{t}_0 = \infty$. In this case the gauge choice (2.45) implies that we must have $\partial_i K\Big|_{\bar{t}\to\infty}$ vanish which explains the gauge choice made in finding the mirror map in [7] (in particular the normalization of the holomorphic three form one needs is in the gauge where $\partial_i K = -\partial_i \log\langle\bar{0}|0\rangle\Big|_{\bar{t}\to\infty} = 0$). It is in this gauge that the path integral for the A–model is given by a sum over holomorphic maps and thus this is the right gauge in order to count these maps. We will now discuss this in more detail.

The crucial property of the canonical coordinates with base point at infinity[14] is that, for an appropriate choice of the matrix $C_{i\bar{m}}$, they coincide with the special coordinates (in the sense of special geometry). Since Eq. (2.45) completely characterizes the canonical coordinates, it is enough to show that the special coordinates satisfy this equation with $\bar{t}_0 = \infty$. For convenience, we show this in the context of the B–model, using the periods of the holomorphic 3–form. The argument can be easily extended to the general case using the more abstract methods of Sect. 2.3. The key formula is (2.26)

$$e^{-K} = \varpi^\dagger E \varpi \ , \tag{2.47}$$

where ϖ^α are the periods in a symplectic basis. ϖ depends holomorphically on the moduli. Let s^i $(i = 1, 2, \ldots, m)$ be the 'special coordinates' and put $X_i = X_0 s_i$, where X_0 is a holomorphic coordinate along the fiber of $\mathcal{L}$. Then the periods take the form

$$\varpi^t = (X_0, X_i, \partial_{X_0}\mathcal{F}, \partial_{X_i}\mathcal{F}) \ ,$$

where $\mathcal{F}$ is a holomorphic function of X_0, X_i homogeneous of degree 2. We have introduced the homogeneous coordinates X_I $(I = 0, 1, \ldots, m)$ because ϖ takes value in the line bundle $\mathcal{L}$; then X_0 corresponds to the freedom in the choice of trivialization of $\mathcal{L}$. The condition that ϖ is a section of $\mathcal{L}$ also explains the homogeneity condition on $\mathcal{F}$. We want to take the limit $\bar{s}^j \to \infty$ in Eq. (2.47) while keeping s^i generic. We need the behaviour of the periods $\varpi^*(\bar{s})$ as $\bar{s} \to \infty$. This behaviour is described by the Schmid orbit theorems [27]: As $s \to \infty$ one has

$$\mathcal{F}(X_0, X_i) = \frac{d_{ijk} X_i X_j X_k}{X_0} + c X_0^2 + O(e^{2\pi i s}) \ ,$$

for some non–degenerate numbers d_{ijk}. Given the 'factorized' form of Eq. (2.47), taking the limit $\bar{s}^j \to \infty$ while keeping s^i fixed is a well defined procedure. More precisely, we make $\bar{s}^i \to \bar{\lambda}\bar{s}^i$ and send $\bar{\lambda}$ to infinity. In this limit one has (up to exponentially small terms)

$$\varpi^\dagger = \bar{X}_0(1, \bar{\lambda}\bar{s}^i, -\bar{\lambda}^3 \bar{d}_{ijk}\bar{s}^i\bar{s}^j\bar{s}^k + 2c^*, 3\bar{d}_{ijk}\bar{\lambda}^2\bar{d}_{ijk}\bar{s}^j\bar{s}^k) \ .$$

Therefore

$$e^{-K} = \sum_{r=0}^{3} \bar{\lambda}^r A_r \ , \tag{2.48}$$

---

[14] By 'point at infinity' we mean the following. For the A–model a point where the volume of the Calabi–Yau manifold is infinite, i.e. the weak coupling limit. For the B–model we mean a degeneration point in the complex moduli space around which the nilpotent part of the monodromy is maximal. As it is well known, this is 'infinite volume' from the mirror viewpoint.

with

$$A_3 = |X_0|^2 \bar{d}_{ijk} \bar{s}^i \bar{s}^j \bar{s}^k \ ,$$
$$A_2 = -3|X_0|^2 \bar{d}_{ijk} s^i \bar{s}^j \bar{s}^k \ ,$$
$$A_1 = \bar{X}_0 \bar{s}^i \partial_i \mathcal{F} \ ,$$
$$A_0 = \bar{X}_0 \mathcal{F} - 2c^* |X_0|^2 \ .$$

Notice that in the special coordinates A_3 and A_2 take a universal form (that is, they are independent of $\mathcal{F}$). From (2.47), (2.48) one has

$$K = -\log X_0 - \log \bar{X}_0 - \log[\bar{\lambda}^3 d_{\bar{i}\bar{j}\bar{k}} \bar{s}^i \bar{s}^j \bar{s}^k - 3\bar{\lambda}^2 d_{\bar{i}\bar{j}\bar{k}} s^i \bar{s}^j \bar{s}^k] + O(\bar{\lambda}^{-1}) \ ,$$

and then the Zamolodchikov metric reads

$$G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K = \frac{1}{\bar{\lambda}^2} L_{i\bar{j}}(\bar{s}) + O(\bar{\lambda}^{-3}) \ , \tag{2.49}$$

where $L_{i\bar{j}}(\bar{s})$ is a non–degenerate anti–holomorphic matrix. On the other hand,

$$K = -\log X_0 + (\text{antiholomorphic}) + O(\bar{\lambda}^{-1}) \ . \tag{2.50}$$

Thus, taking the base point $\bar{t}_0$ to be at infinity, the canonical gauge for $\mathcal{L}$ defined by the second of (2.45) is just $X_0 = 1$, which is the standard gauge in special geometry. Moreover as $\bar{\lambda} \to \infty$ the $(1,0)$ part of the Christoffel connection D_i becomes the trivial one. Indeed using (2.49),

$$D_i = G \partial_i G^{-1} = \partial_i + O(\bar{\lambda}^{-1}) \ , \tag{2.51}$$

which shows that the special coordinates s^i satisfy (2.45) at infinity and hence can be identified with the canonical coordinates t^i with respect to this base point.

3. Holomorphic Anomaly

In the topological theory, the BRST invariance would imply that partition functions and correlation functions are holomorphic on the moduli space of the theory since variation with respect to the anti-holomorphic moduli $\bar{t}^i$ inserts the BRST trivial operator $\bar{\phi}^{(2)}_{\bar{i}} = \{G^+, [\overline{G}^+, \bar{\phi}_{\bar{i}}]\}$. This indeed is the case for the Yukawa coupling. However we saw in the previous section that the holomorphicity is, in general, not consistent with the covariance on the moduli space. This means that there is something wrong with the assumption on the BRST invariance. What we saw there is reminiscent of the chiral anomaly in the Yang-Mills theory where one finds that it is not possible to preserve both the vector and the chiral gauge invariances of the theory. Thus we call this phenomenon the holomorphic anomaly. In this section, we will uncover a subtle breakdown of the BRST invariance in the twisted $N = 2$ model coupled to the gravity, and rederive the non-holomorphicity we found in the previous section as a special case.

3.1 Homolorphic anomalies of partition functions. Let us first examine the partition function F_g for $g \geq 2$. The naive BRST invariance would imply $\bar{\partial}_{\bar{i}} F_g = 0$. We are going to show that this is not the case. The derivative with respect to $\overline{t}^i$ is generated by an insertion of the anti-chiral field $\overline{\phi}_{\bar{i}}$ as

$$
\begin{aligned}
\frac{\partial}{\partial \overline{t}^i} F_g &= \int_{\mathcal{M}_g} [dm] \int d^2 z \langle \oint_{C_z} G^+ \oint_{C'_z} \overline{G}^+ \overline{\phi}_{\bar{i}}(z) \prod_{a=1}^{3g-3} \int \mu_a G^- \int \overline{\mu}_a \overline{G}^- \rangle_{\Sigma_g} \\
&= \int_{\mathcal{M}_g} [dm] \sum_{b,\bar{b}=1}^{3g-3} \langle \int \overline{\phi}_{\bar{i}} \int 2\mu_b T \int 2\overline{\mu}_{\bar{b}} \overline{T} \prod_{a \neq b} \int \mu_a G^- \prod_{\bar{a} \neq \bar{b}} \int \overline{\mu}_{\bar{a}} \overline{G}^- \rangle_{\Sigma_g} \\
&= \int_{\mathcal{M}_g} [dm] \sum_{b,\bar{b}=1}^{3g-3} 4 \frac{\partial^2}{\partial m_b \partial \overline{m}_{\bar{b}}} \langle \int \overline{\phi}_{\bar{i}} \prod_{a \neq b} \int \mu_a G^- \prod_{\bar{a} \neq \bar{b}} \int \overline{\mu}_{\bar{a}} \overline{G}^- \rangle_{\Sigma_g} \, .
\end{aligned}
\tag{(3.1)}
$$

In the first line of this equation, the contours C_z and C'_z are around the point z where the anti-chiral $\overline{\phi}_{\bar{i}}$ is inserted. We then moved these contours around the Riemann surface Σ_g, and picked up the commutators, $\oint_{C_w} G^+ \cdot G^-(w) = 2T(w)$ and $\oint_{C_w} \overline{G}^+ \cdot \overline{G}^-(\bar{w}) = 2\overline{T}(\bar{w})$. The insertions of T and $\overline{T}$ are then converted into the derivatives with respect to the moduli $m, \bar{m}$ of Σ_g. Using the Cauchy theorem, we can reduce the r.h.s. to an integral on the boundary of the moduli space $\mathcal{M}_g$.

The boundary of $\mathcal{M}_g$ consists of $([\frac{1}{2}g] + 1)$ irreducible components $\mathcal{D}_g^r$ ($r = 0, 1, ..., [\frac{1}{2}g]$) each of which consists of surfaces with nodes. Surfaces belonging to $\mathcal{D}_g^0$ are such that they become connected surfaces of genus $(g-1)$ with two punctures upon removal of the nodes. On the other hand, $\mathcal{D}_g^r$ ($r \geq 1$) consists of surfaces which become, upon removal of the nodes, two disconnected surfaces, one of genus r and one of genus $(g-r)$, each with one puncture.

A surface which sits in the neighbourhood of $\mathcal{D}_g^0$ has a long tube which becomes a node as the surface approaches $\mathcal{D}_g^0$. Thus we can choose coordinates near $\mathcal{D}_g^0$ as 4–tuple (τ, m', z, w), where τ is the length and the twist of the tube and it serves as a transverse coordinate to $\mathcal{D}_g^0$ (the surface approaches $\mathcal{D}_g^0$ as $\tau \to \infty$), while (m', z, w) are moduli of a genus–$(g-1)$ surface with two punctures (where z and w denote the moduli corresponding to the two punctures) which is obtained by removing the node from the surface.

The contribution of the boundary component $\mathcal{D}_g^0$ to $\overline{\partial}_{\bar{i}} F_g$ is given as follows. Because of the second–order derivative in r.h.s. with respect to m_b and $\overline{m}_{\bar{b}}$, at the boundary we will be left with a derivative in the direction normal to $\mathcal{D}_g^0$. In the coordinates (τ, m', z, w), the normal derivative is expressed as $\frac{\partial}{\partial \mathrm{Im}\, \tau}$. In the limit $\tau \to \infty$, the Beltrami-differentials $\mu^{(z)}$ and $\mu^{(w)}$ associated to the moduli z and w become localized near the punctures, i.e.

$$
\int \mu^{(z)} G^- \to \oint_{C_z} G^- \, ,
$$

while those associated to the moduli m' reduces to the Beltrami-differentials μ' on the genus–$(g-1)$ surface Σ_{g-1}. Thus the contribution of $\mathcal{D}_g^0$ is given by

$$
\begin{aligned}
&\int_{\mathcal{D}_g^0} [dm', dz, dw] \frac{\partial}{\partial \mathrm{Im}\, \tau} \langle \int_{\Sigma_g} \overline{\phi}_{\bar{i}} \oint_{C_z} G^- \oint_{C'_z} \overline{G}^- \oint_{C_w} G^- \oint_{C'_w} \overline{G}^- \times \\
&\qquad\qquad \times \prod_{a=1}^{3g-6} \int_{\Sigma_{g-1}} \mu'_a G^- \int_{\Sigma_{g-1}} \overline{\mu}'_a \overline{G}^- \rangle_{\Sigma_g} \, .
\end{aligned}
\tag{3.2}
$$

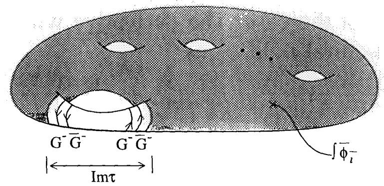

Fig. 2. Contributions from the boundary of moduli space where $\overline{\phi}_{\bar{i}}$ is outside the long tube vanishes.

Let us examine the integrand of (3.2). Since the operator $\overline{\phi}_{\bar{i}}$ is integrated over the entire surface Σ_g it either sits on the tube which will be stretched out in the limit $\tau \to \infty$ or lies outside of the tube which becomes the genus–$(g-1)$ surface Σ_{g-1} in this limit. When $\overline{\phi}_{\bar{i}}$ sits outside of the tube (see Fig. 2), states which propagate on the tube are projected onto the ground states in the limit $\tau \to \infty$. Since the ground states are generated by the chiral fields, the effect of a node on the degenerate surface can be represented by insertions of $\phi_j(z)$ and $\phi_k(w)$ on the points z and w where the node is attached, and the node itself is replaced by the ground state metric η^{jk}. In the coordinates (τ, m, z, w), the integrand becomes

$$\frac{\partial}{\partial \mathrm{Im}\,\tau}\eta^{jk}\Big\langle \oint_{C_z} G^- \oint_{C'_z} \overline{G}^- \phi_j(z) \oint_{C_w} G^- \oint_{C'_w} \overline{G}^- \phi_k(w) \times \\ \times \int_{\Sigma_{g-1}} \overline{\phi}_{\bar{i}} \prod_{a=1}^{3g-6} \int_{\Sigma_{g-1}} \mu'_a G^- \int_{\Sigma_{g-1}} \mu'_a \overline{G}^- \Big\rangle_{\Sigma_{g-1}} .$$

This turns out to be zero since the correlation function in the above is defined on Σ_{g-1} and does not depend on the coordinate τ. Thus, when $\overline{\phi}_{\bar{i}}$ lies outside of the tube, there is no contribution from the component $\mathcal{D}_g^0$ to $\overline{\partial}_{\bar{i}} F_g$.

Let us turn to the case when $\overline{\phi}_{\bar{i}}$ sits on the tube (see Fig. 3). Suppose $\overline{\phi}_{\bar{i}}$ is away from both ends of the tube. In this case, states on both sides of $\overline{\phi}_{\bar{i}}$ on the tube are projected onto the ground states. Thus the effect of the node is represented by an insertion of

$$\phi_j(z)\eta^{jj'}\langle j'| \int \overline{\phi}_{\bar{i}} |k'\rangle \eta^{k'k}\phi_k(w)$$

on Σ_{g-1}. Here the integral $\int \overline{\phi}_{\bar{i}}$ is over the tube away from both ends. Since

$$\langle j|\overline{\phi}_{\bar{i}}|k\rangle = \langle \bar{j}|\bar{\phi}_{\bar{i}}|\bar{k}\rangle M_j^{\bar{j}} M_k^{\bar{k}} = \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j'} G^{\bar{k}k'} \eta_{j'j}\eta_{k'k}$$

is independent of the position of $\overline{\phi}_{\bar{i}}$, we can replace the integral by the multiplication of the volume of the domain of the integral which can be approximated by the volume $\mathrm{Im}\,\tau$ of the tube when $\tau \to \infty$. When $\overline{\phi}_{\bar{i}}$ is close to one of the ends of the tube, the amplitude does not scale like the volume $\mathrm{Im}\,\tau$, and such a configuration can be neglected in this approximation. The integrand of (3.2) then becomes

$$\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \Big\langle \oint_{C_z} G^- \oint_{C'_z} \overline{G}^- \phi_j(z) \oint_{C_w} G^- \oint_{C'_w} \overline{G}^- \phi_k(w) \times \\ \times \prod_{a=1}^{3g-6} \int \mu'_a G^- \int \overline{\mu}'_a \overline{G}^- \Big\rangle_{\Sigma_{g-1}} \tag{3.3}$$

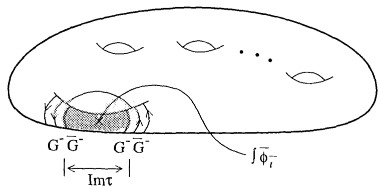

Fig. 3. The contribution from the boundary of moduli space comes from the configuration where $\overline{\phi}_{\bar{\imath}}$ is on the long tube of length $\mathrm{Im}\tau$ as $\tau \to \infty$.

where the volume factor $\mathrm{Im}\,\tau$ is cancelled by the normal derivative $\frac{\partial}{\partial \mathrm{Im}\,\tau}$. This remains finite in the limit $\tau \to \infty$.

We need to integrate (3.3) over the boundary component $\mathcal{D}_g^0$ which is parametrized by $m' \in \mathcal{M}_{g-1}$ and $z, w \in \Sigma_{g-1}$. Since the interchange of the two points z and w does not change the complex structure of the punctured surface, we should include a factor of $(1/2)$ if we are to integrate z and w over the entire surface Σ_{g-1} without a constraint. The contribution of the boundary component $\mathcal{D}_g^0$ to $\overline{\partial}_{\bar{\imath}} F_g$ is then expressed as

$$\frac{1}{2}\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}} e^{2K} G^{j\bar{\jmath}} G^{k\bar{k}} \int_{\mathcal{M}_{g-1}} [dm'] \Big\langle \int \phi_j^{(2)} \int \phi_k^{(2)} \prod_{a=1}^{3g-6} \int \mu'_a G^- \int \overline{\mu}'_a \overline{G}^- \Big\rangle_{\Sigma_{g-1}} . \tag{3.4}$$

The expression (3.4) can be further simplified by the condition $\hat{c} = 3$. In general, when $\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}} \neq 0$, the left and the right $U(1)$ charges of the three chiral fields ϕ_i, ϕ_j and ϕ_k should sum up to be $\hat{c}$,

$$q_j + q_k + q_i = \overline{q_j} + \overline{q_k} + \overline{q_i} = \hat{c} .$$

In the present situation, $\hat{c} = 3$ and $q_i = \overline{q_i} = 1$. Therefore we must have $q_j + q_k = \overline{q}_j + \overline{q}_k = 2$. Furthermore, if $q_j = 0$ or $\overline{q}_j = 0$, $\phi_j^{(2)} = \{G^-, [\overline{G}^-, \phi_j]\} = 0$ since a chiral state with $q = 0$ is annihilated by both G^+ and G^-. Therefore we can restrict j and k in (3.3) to those with $(q_j, \overline{q}_j) = (q_k, \overline{q}_k) = (1,1)$. These are the ones which correspond to the marginal deformations of the twisted $N = 2$ model, and we can replace the insertions of $\int \phi^{(2)}$ in (3.4) by covariant derivatives D. The contribution (3.4) of the boundary component $\mathcal{D}_g^0$ to $\overline{\partial}_{\bar{\imath}} F_g$ is then expressed as

$$\frac{1}{2}\overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}} e^{2K} G^{j\bar{\jmath}} G^{k\bar{k}} D_j D_k F_{g-1} . \tag{3.5}$$

Let us turn to the other boundary components $\mathcal{D}_g^r$ ($r = 1, \ldots, [\frac{1}{2}g]$). A surface in the neighborhood of $\mathcal{D}_g^r$, has a long tube which connects two disconnected surfaces Σ_r and Σ_{g-r} of genus r and genus $(g-r)$. Thus we can choose coordinates near $\mathcal{D}_g^r$ as 5–tuple (τ, m', z, m'', w) where τ characterizes the tube connecting the two surfaces, and $(m', z) \in \mathcal{M}_{r,1}$ and $(m'', w) \in \mathcal{M}_{g-r,1}$. As in the case of $\mathcal{D}_g^0$ discussed in the above, a non-vanishing contribution to $\overline{\partial}_{\bar{\imath}} F_g$ comes from the region where the amplitude scales like $\mathrm{Im}\,\tau$. This is the case when the operator $\overline{\phi}_{\bar{\imath}}$ is on the tube (see Fig. 4). The factor $\mathrm{Im}\,\tau$ is cancelled by the derivative operator $\frac{\partial}{\partial \mathrm{Im}\,\tau}$ and the effect

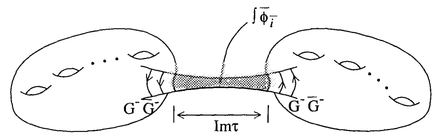

Fig. 4. Another component of the boundary of moduli space where the Riemann surface splits to two Riemann surfaces connected by a long tube; to get a nonvanishing contribution $\overline{\phi}_{\bar{i}}$ is inserted on the tube.

of the tube is represented by the operator

$$\phi_j(z)\eta^{jj'}\langle j'|\overline{\phi}_{\bar{i}}|k'\rangle\eta^{k'k}\phi_k(w) = \overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\phi_j(z)\phi_k(w) ,$$

where $\phi_j(z)$ is inserted on Σ_r and $\phi_k(z)$ is on Σ_{g-r}. The contribution of $\mathcal{D}_g^r$ to $\overline{\partial}_{\bar{i}}F_g$ is then given by

$$\begin{aligned}
&\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\int_{\mathcal{M}_r}[dm']\langle\int\phi_j^{(2)}\prod_{a=1}^{3r-3}\int\mu'_aG^-\int\overline{\mu}'_a\overline{G}^-\rangle_{\Sigma_r}\times\\
&\quad\times\int_{\mathcal{M}_{g-r}}[dm'']\langle\int\phi_k^{(2)}\prod_{a=1}^{3(g-r)-3}\int\mu''_aG^-\int\overline{\mu}''_a\overline{G}^-\rangle_{\Sigma_{g-r}}=\\
&=\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}D_jF_rD_kF_{g-r}\ .
\end{aligned}$$

Extra care is required when g is even and $r = \frac{1}{2}g$. In this case there is a $\mathbf{Z}_2$ symmetry between the two surfaces Σ_r and Σ_{g-r}, and we must include a factor $(1/2)$ to take into account this symmetry. The contributions of the boundary components $\mathcal{D}_{g,k}$ ($k = 1, \ldots, [\frac{1}{2}g]$) are then

$$\sum_{r=1}^{[\frac{1}{2}g]}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}D_jF_rD_kF_{g-r}$$

if g is odd and

$$\sum_{r=1}^{\frac{1}{2}g-1}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}D_jF_rD_kF_{g-r}+\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}D_jF_{\frac{1}{2}g}D_kF_{\frac{1}{2}g}$$

if g is even. They can be summarized in a single equation as

$$\frac{1}{2}\sum_{r=1}^{g-1}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}D_jF_rD_kF_{g-r}\ .$$

By combining this with (3.5), we obtain the holomorphic anomaly of the genus-g partition function as

$$\overline{\partial}_{\bar{i}}F_g=\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\left(D_jD_kF_{g-1}+\sum_{r=1}^{g-1}D_jF_rD_kF_{g-r}\right)\ . \qquad (3.6)$$

This gives a recursion relation for F_g with respect to the genus g. In fact, it is possible to solve this equation iteratively, and we will present a systematic method to do so in Sect. 6.

The holomorphic anomaly equations of F_g's for all $g \geq 2$ can be combined into a single equation by introducing a formal sum of F_g's as

$$\mathcal{F}(\lambda; t, \bar{t}) = \sum_{g=1}^{\infty} \lambda^{2g-2} F_g \ . \tag{3.7}$$

Since each F_g is a section of a line-bundle $\mathcal{L}^{2-2g}$ over the moduli space of the topological theory, $\mathcal{F}(\lambda; t, \bar{t})$ should be regarded as a function on the total space of $\mathcal{L}$, with λ being a coordinate on the fiber of $\mathcal{L}$. We then consider the following equation:

$$\left(\overline{\partial}_{\bar{i}} - \overline{\partial}_{\bar{i}} F_1\right) \exp(\mathcal{F}) = \frac{\lambda^2}{2} \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \hat{D}_j \hat{D}_k \exp(\mathcal{F}) \ , \tag{3.8}$$

where

$$\begin{aligned} \hat{D}_j \mathcal{F}(\lambda; t, \bar{t}) &\equiv \sum_g \lambda^{2g-2} D_j F_g \\ &= \sum_g \lambda^{2g-2} (\partial_j - (2g-2) \partial_j K) F_g \\ &= (\partial_j - \partial_j K \lambda \partial_\lambda) \mathcal{F}(\lambda; t, \bar{t}) \ . \end{aligned}$$

By expanding both-hand sides of (3.8) in power series of λ and by comparing each term in the expansion, we recover the holomorphic anomaly equation (3.6). We call this the *master anomaly equation* of the topological string theory. It is satisfying to see that the holomorphic anomalies for all $g \geq 2$ are summarized in a single equation. Later we will further improve this equation to incorporate the genus–1 anomaly equation.

As we will solve the holomorphic anomaly equation (3.6) later, it is instructive to check the integrability of the equation here. Since the holomorphic anomaly is summarized in the master equation (3.8), it is sufficient to prove

$$\begin{aligned} &[d_{\bar{i}}, d_{\bar{j}}] = 0 \ , \\ &d_{\bar{i}} = \overline{\partial}_{\bar{i}} - \overline{\partial}_{\bar{i}} F_1 - \frac{\lambda^2}{2} \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \hat{D}_j \hat{D}_k \ . \end{aligned}$$

By using the special geometry relation

$$[\overline{\partial}_{\bar{i}}, D_j]^l_k = -G_{\bar{i}j} \delta^l_k - G_{\bar{i}k} \delta^l_j + C_{jkm} \overline{C}_{\bar{i}\bar{l}\bar{m}} e^{2K} G^{m\bar{m}} G^{\bar{l}l} \tag{3.9}$$

and the properties of the Yukawa coupling

$$\begin{aligned} \overline{C}_{\bar{i}\bar{j}\bar{k}} = \overline{C}_{\bar{j}\bar{i}\bar{k}}, \quad D_{\bar{i}} \overline{C}_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}} \overline{C}_{\bar{i}\bar{k}\bar{l}} \ , \\ \partial_i \overline{C}_{\bar{j}\bar{k}\bar{l}} = 0, \end{aligned}$$

we find the commutator to be

$$\begin{aligned} [d_{\bar{i}}, d_{\bar{j}}] = \lambda^2 \overline{C}_{\bar{i}\bar{k}\bar{l}} e^{2K} G^{k\bar{k}} G^{l\bar{l}} (\partial_k \overline{\partial}_{\bar{j}} F_1 - \frac{1}{2} \mathrm{Tr} C_k \overline{C}_{\bar{j}}) \partial_l - (\bar{i} \leftrightarrow \bar{j}) + \\ + \frac{\lambda^2}{2} \overline{C}_{\bar{i}\bar{k}\bar{l}} e^{2K} G^{k\bar{k}} G^{l\bar{l}} D_k \partial_l \overline{\partial}_{\bar{j}} F_1 - (\bar{i} \leftrightarrow \bar{j}) \ . \end{aligned}$$

That the r.h.s. of this equation is zero is the consequence of the holomorphic anomaly equation[15] of F_1

$$\partial_i \overline{\partial}_{\bar{j}} F_1 = \frac{1}{2} \mathrm{Tr}\, C_i \overline{C}_{\bar{j}} - \frac{\chi}{24} G_{i\bar{j}} \,, \tag{3.10}$$

where $\chi = \mathrm{Tr}(-1)^F$. Substituting this into the above, we obtain

$$[d_{\bar{i}}, d_{\bar{j}}] = \frac{\lambda^2}{4} \left[\overline{C}_{\bar{i}\bar{k}\bar{l}} e^{2K} G^{k\bar{k}} G^{l\bar{l}} \mathrm{Tr}(D_k C_l) \overline{C}_{\bar{j}} - (\bar{i} \leftrightarrow \bar{j}) \right] = 0 \,.$$

Here we also used $D_i C_{jkl} = D_j C_{ikl}$. It is curious to see that both the special geometry relation and the genus–1 anomaly equation play important roles in proving the consistency of the holomorphic anomaly at $g \geq 2$. This in fact is not without a reason. We will see later that the special geometry relation can be regarded as a holomorphic anomaly equation at genus–0, and the anomalies at all genera including $g = 0$ and 1 can be described in a single framework.

Now that we found the BRST–trivial operator $\{G^+, [\overline{G}^+, \overline{\phi}_{\bar{i}}]\}$ does not decouple from F_g, one might wonder whether F_g is sensitive to still other types of BRST-trivial deformations of the topological theory. The (c, c) field ϕ_i which generate the truly marginal deformation of the topological theory satisfies

$$[G^+, \phi_i] = [\overline{G}^+, \phi_i] = 0,$$

and the (a, a) field $\overline{\phi}_{\bar{i}}$ which is complex conjugate to ϕ_i obeys

$$[G^-, \overline{\phi}_{\bar{i}}] = [\overline{G}^-, \overline{\phi}_{\bar{i}}] = 0 \,.$$

However the topological theory realized by the twisted $N = 2$ model may also contain a (a, c) field $\widetilde{\phi}$ subject to

$$[G^-, \widetilde{\phi}] = [\overline{G}^+, \widetilde{\phi}] = 0$$

and its conjugate (c, a) field. Thus we would like to know if F_g is sensitive to a deformation generated by these operators. We show here that, in fact, the operator $\{G^+, [\overline{G}^-, \widetilde{\phi}]\}$ and its conjugate decouple from F_g.

If we insert such an operator on Σ_g, we can deform the contour of G^+ surrounding the operator $\widetilde{\phi}$ and pick up the commutator of G^+ with $\int \mu_a G^-$ $(a = 1, ..., 3g - 3)$ inserted on Σ_g. The commutator produces the energy-momentum tensor T which is then converted into a derivative with respect to the moduli m,

$$\int_{\mathcal{M}_g} [dm] \int d^2 z \langle \oint_{C_z} G^+ \oint_{C'_z} \overline{G}^- \widetilde{\phi}(z) \prod_{a=1}^{3g-3} \int \mu_a G^- \int \overline{\mu}_{\bar{a}} \overline{G}^- \rangle =$$

$$= \int_{\mathcal{M}_g} [dm] \sum_{b=1}^{3g-3} 2 \frac{\partial}{\partial m_b} \int d^2 z \langle \int_{C'_z} \overline{G}^- \widetilde{\phi}(z) \prod_{a \neq b} \int \mu_a G^- \prod_{\bar{a}=1}^{3g-3} \int \overline{\mu}_{\bar{a}} \overline{G}^- \rangle \,.$$

Due to the derivative with respect to m_b, this becomes an integral on the boundary of $\mathcal{M}_g$. It turns out that the boundary term vanishes for the following reason. So far

[15] The holomorphic anomaly equation for F_1 here differs by a factor $(1/2)$ to the one presented in [20] due to the different normalization of F_1.

we have not touched $\int \overline{\mu}_a \overline{G}^-$ in the right-moving sector, and there are still $(3g-3)$ of them on Σ_g. In the neighbourhood of $\mathcal{D}_g^r$ ($r = 0, 1, ..., [\frac{1}{2}g]$), one of them becomes a contour integral of $\overline{G}^-$ around the tube which becomes a node on $\mathcal{D}_g^r$. Namely we have the $\overline{G}^-$ charge inserted on the tube. As the surface approaches the boundary, states which propagate on the tube are projected onto the ground states, all of which are annihilated by $\overline{G}^-$. It does not matter whether the operator $[\overline{G}^-, \widetilde{\phi}]$ is on or off the tube since it anti-commutes with $\overline{G}^-$. In this way, the boundary term vanishes due to the $\overline{G}^-$ charge which comes from one of $\int \overline{\mu}_a \overline{G}^-$. Since there is no boundary term, the operator $\{G^+, [\overline{G}^-, \widetilde{\phi}]\}$ decouples from F_g. Similarly F_g is invariant under the deformation generated by (c, a) fields.

In the case of the topological sigma–model of A–type described in Sect. 2, the (c, c) and (a, a) fields generate deformations of the Kähler class on the Calabi-Yau manifold M while the (a, c) and (c, a) fields correspond to deformations of the complex structure. The result here suggests that F_g in this case is independent of the complex structure of M, but depends on the Kähler class on M. The anti-holomorphic dependence of F_g on the Kähler moduli is determined by the holomorphic anomaly equation (3.6). The situation is opposite in the case of the B–model. In this case, F_g does not depend on the Kähler moduli of M. Especially F_g is independent of the volume of M. This fact becomes important in Sect. 5.

3.2 Holomophic anomalies of correlation functions. So far, we have studied the holomorphic anomaly of partition functions. Let us now turn to correlation functions $C^{(g)}_{i_1 \cdots i_n}$ of the chiral fields given by

$$\begin{aligned} C^{(g)}_{i_1 \cdots i_n} &= \int_{\mathcal{M}_g} \langle \prod_{r=1}^{n} \int \phi^{(2)}_{i_r} \prod_{a=1}^{3g-3} \int \mu_a G^- \int \overline{\mu}_a \overline{G}^- \rangle \\ &= D_{i_1} \cdots D_{i_n} F_g \,. \end{aligned}$$

As in the case of the partition function F_g, the derivative $\overline{\partial}_{\overline{i}}$ brings down the BRST trivial operator $\{G^+, [\overline{G}^+, \overline{\phi}_{\overline{i}}]\}$, and the commutators of G^+ and $\overline{G}^+$ with G^- and $\overline{G}^-$ in $C^{(g)}_{i_1, \ldots, i_n}$ generate second–order derivatives with respect to $(m, \overline{m}) \in \mathcal{M}_g$ and $(z_r, \overline{z}_r) \in \Sigma_g$, where z_r ($r = 1, \ldots, n$) are the positions of the chiral fields ϕ_{i_r}. We can then apply the Cauchy theorem to reduce the computation to a boundary integral. The boundary in this case consists of two types; one is the boundary of the moduli space $\mathcal{M}_{g,n}$ of a genus–g surface with n–punctures. Another contribution arises in a limit when one of the chiral fields ϕ_{i_r} approaches $\overline{\phi}_{\overline{i}}$.

The computation on the boundary of the first–type is a straightforward generalization of the one for $\overline{\partial}_{\overline{i}} F_g$ we did in the above. The boundary of the moduli space $\mathcal{M}_{g,n}$ consists of irreducible components $\mathcal{D}^{(0)}_{(g,n)}$ and $\mathcal{D}^{(r,s)}_{(g,n)}$ each of which consists of surfaces with punctures and nodes. Here, for $\mathcal{D}^{(r,s)}_{(g,n)}$, r and s run from 0 to g and from 0 to n respectively, $\mathcal{D}^{(0,0)}_{(g,n)}$ and $\mathcal{D}^{(0,1)}_{(g,n)}$ are empty, and $\mathcal{D}^{(r,s)}_{(g,n)}$ is identified with $\mathcal{D}^{(g-r,n-s)}_{(g,n)}$. Surfaces belonging to $\mathcal{D}^{(0)}_{(g,n)}$ become connected surfaces of genus $(g-1)$ with $(n+2)$ punctures upon removal of the nodes (see Fig. 5). On the other hand, $\mathcal{D}^{(r,s)}_{(g,n)}$ consists of surfaces which become, upon removal of the nodes, two disconnected surfaces, one of genus r with $(s+1)$ punctures and another of genus $(g-r)$

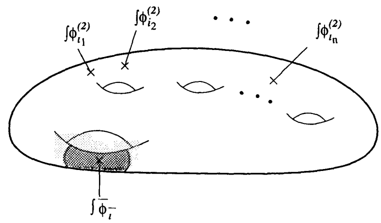

Fig. 5. One of the boundary components of moduli space with fields inserted. The contribution again comes from the insertion of the $\overline{\phi}_{\bar{i}}$ on the tube.

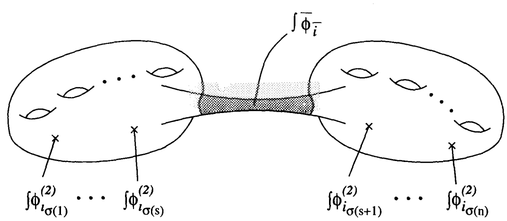

Fig. 6. The contribution from another component of moduli space where again the operator $\overline{\phi}_{\bar{i}}$ is inserted on the tube.

with $(n - s + 1)$ punctures (see Fig. 6). As in the case of the partition function F_g, contributions of these boundary components to $\overline{\partial}_{\bar{i}} C^{(g)}_{i_1 \cdots i_n}$ come from the region where the operator $\overline{\phi}_{\bar{i}}$ sits on the tube which becomes the node at the boundary of $\mathcal{M}_{g,n}$, and are expressed as

$$\begin{aligned} &\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} C^{(g-1)}_{jki_1\cdots i_n} + \\ &+\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \sum_{r=0}^{g}\sum_{s=0}^{n} \frac{1}{s!(n-s)!} \sum_{\sigma\in S_n} C^{(r)}_{ji_{\sigma(1)}\cdots i_{\sigma(s)}} C^{(g-r)}_{ki_{\sigma(s+1)}\cdots i_{\sigma(n)}} , \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &C^{(0)}_{i_1\cdots i_n} = D_{i_1}\cdots D_{i_{n-3}} C_{i_{n-2} i_{n-1} i_n} \quad (n \geq 3) , \\ &C^{(0)} = 0, \quad C^{(0)}_i = 0, \quad C^{(0)}_{ij} = 0 . \end{aligned}$$

The boundary of the second type arises since there is a singularity in the operator product of ϕ_j $(j = i_1, ..., i_n)$ and $\overline{\phi}_{\bar{i}}$,

$$\phi_j(z)\overline{\phi}_{\bar{i}}(w) \sim \frac{G_{j\bar{i}}}{|z-w|^2} \quad (z \to w) . \tag{3.12}$$

How we regularize this divergence is a part of the definition of the theory. In the perturbed $N = 2$ theory given by the action $S = S_0(t_0, \bar{t}_0) + \delta t^i \int \phi^{(2)}_i + \delta t^{\bar{i}} \int \overline{\phi}^{(2)}_{\bar{i}}$, we assume that the original theory with the action S_0 has the $N = 2$ superconformal invariance, which in particular means that the theory is finite. In order to perturb the theory while maintaining the superconformal symmetry, we must specify how to deal with the short distance singularity between $\phi_j^{(2)}$ and $\overline{\phi}_{\bar{i}}^{(2)}$,

$$\phi_j^{(2)}(z)\overline{\phi}_{\bar{i}}^{(2)}(w) \simeq 4\partial_z\overline{\partial}_{\bar{z}}\phi_j(z)\overline{\phi}_{\bar{i}}(w) \simeq \frac{4G_{j\bar{i}}}{|z-w|^4} \quad (z \to w)\,. \tag{3.13}$$

This divergence which arises from this short distance singularity is power in $|z-w|$ and is not universal. Thus we can simply subtract it away (one can renormalize the divergence into the cosmological constant if one wishes). Once we subtract the singularity in the operator product between ϕ_j and $\overline{\phi}_{\bar{i}}$, the boundary of the second type does not contribute to $\overline{\partial}_{\bar{i}}C^{(g)}_{i_1\cdots i_n}$.

This is the case when the world-sheet is a flat infinite plane. When the world-sheet is compact, there are subleading divergences in (3.12) and (3.13) which generate non-vanishing contributions for the boundary of the second type. The subleading divergences depend linearly on the curvature of Σ, and they can be derived from the short distance expansion of the Green's function on Σ. We can also understand this effect from the topological field theoretical point of view as follows. Let us choose a metric on Σ as $|\nu(z)|^4$, where ν is a meromorphic $\frac{1}{2}$–differential on Σ with a pole and g–zeros at the Riemann divisor. Since the theory is conformally invariant, we are free to use any metric we like. In this metric, the curvature has delta-function like singularities each of which carries $\int R = \pm 4\pi$ ($+4\pi$ at the pole of ν and -4π at the zeros of $\nu(z)$). When the operators $\phi_j(z)$ and $\overline{\phi}_{\bar{i}}(w)$ are away from the support of the curvature, there is no contribution from the boundary of the second type since the computation is the same as in the case of the flat infinite plane. On the other hand, near the curvature singularity, we must take into account the fact that, due to the twisting, there is an operator $e^{\pm\varphi}$ inserted there where φ is the bosonized $U(1)$ current. The operator e^{φ} is the chiral field of the maximum charge $(3,3)$ (corresponding to the holomorphic 3–form on the Calabi-Yau 3–fold), and $e^{-\varphi}$ is its conjugate anti-chiral field. Thus we can evaluate the boundary term as in the case of the boundary of the first type discussed in the above. We then obtain $\pm 2\sum_{s=1}^{n} G_{\bar{i}i_s}C^{(g)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n}$ from each of the curvature singularities. We should also take into account the effect of the punctures on the surface. This can be done most easily by noting that the final result should be linear in the integral of the curvature $\int R = -2\pi(2g-2+n-1)$ on the genus–g surface with $(n-1)$–punctures. The contribution from the boundary of the second type is then

$$-(2g-2+n-1)\sum_{s=1}^{n} G_{\bar{i}i_s}C^{(g)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n}\,. \tag{3.14}$$

By combining (3.11) and (3.14), we obtain

$$\begin{aligned}\overline{\partial}_{\bar{i}}C^{(g)}_{i_1\cdots i_n} &= \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}C^{(g-1)}_{jki_1\cdots i_n} + \\ &+ \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\sum_{r=0}^{g}\sum_{s=0}^{n}\frac{1}{s!(n-s)!}\sum_{\sigma\in S_n} C^{(r)}_{ji_{\sigma(1)}\cdots i_{\sigma(s)}}C^{(g-r)}_{ki_{\sigma(s+1)}\cdots i_{\sigma(n)}} - \\ &- (2g-2+n-1)\sum_{s=1}^{n} G_{\bar{i}i_s}C^{(g)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n}\,.\end{aligned} \tag{3.15}$$

Especially when $n = 0$, this equation reduces to the anomaly equation (3.6) of F_g. The derivation of this equation is valid also for $g = 0$ ($n \geq 4$) and $g = 1$ ($n \geq 2$). The anomaly equation in the case of $g = 1, n = 1$ is given by (3.10) and is slightly different from the above[16].

To understand the structure of this equation better, let us take a look at the simplest case of $g = 0, n = 4$. In this case, the equation becomes

$$\begin{aligned}\overline{\partial}_{\overline{i}} C_{i_1 i_2 i_3 i_4} =& \overline{C}_{\overline{j}\overline{i}\overline{k}} e^{2K} G^{j\overline{j}} G^{k\overline{k}} \left(C_{j i_1 i_2} C_{k i_3 i_4} + C_{j i_1 i_3} C_{k i_4 i_2} + C_{j i_1 i_4} C_{k i_2 i_3} \right) - \\ & - G_{\overline{i} i_1} C_{i_2 i_3 i_4} - G_{\overline{i} i_2} C_{i_3 i_4 i_1} - G_{\overline{i} i_3} C_{i_4 i_1 i_2} - G_{\overline{i} i_4} C_{i_1 i_2 i_3} \, .\end{aligned}$$

We can rederive this equation by computing the $\overline{t}^{\overline{i}}$-derivative of $C_{i_1 i_2 i_3 i_4} = D_{i_1} C_{i_2 i_3 i_4}$ directly by using the holomorphicity of the Yukawa coupling $\overline{\partial}_{\overline{i}} C_{ijk} = 0$ and the special geometry relation (3.9) for the commutator $[\overline{\partial}_{\overline{i}}, D_j]$. In general, at $g = 0$, one can deduce the anomaly equation (3.15) $n \geq 4$ from the special geometry relation and the holomorphicity of C_{ijk} by mathematical induction in n. Similarly the anomaly equation (3.15) for $g \geq 1$ is a consequence of the special geometry and the holomorphic anomaly (3.6) of F_g. Thus we come to view that *the special geometry is also one of the aspects of the holomorphic anomaly in the topological string theory.*

Previously we found that the holomorphic anomalies of the partition functions F_g ($g \geq 2$) can be summarized in the form of the master anomaly equation (3.8). It is also possible to combine them with the anomalies of the correlation functions (3.15) into a single set of equations. It turns out that the equations also contain the genus-1 anomaly equation (3.10). For this purpose, we introduce the following object:

$$W(\lambda, x; t, \overline{t}) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} + \left(\frac{\chi}{24} - 1 \right) \log \lambda \, , \tag{3.16}$$

where $C^{(g)}_{i_1 \cdots i_n} = 0$ for $(2g - 2 + n) \leq 0$. This may be regarded as a generating function for the correlation functions. Because of the $\log \lambda$ term in r.h.s., $\exp(W)$ transforms like a section of $\mathcal{L}^{(\frac{\chi}{24} - 1)}$. Let us consider the following equation:

$$\begin{aligned}&\frac{\partial}{\partial \overline{t}^{\overline{i}}} \exp(W) = \\ &= \left[\frac{\lambda^2}{2} \overline{C}_{\overline{i}\overline{j}\overline{k}} e^{2K} G^{j\overline{j}} G^{k\overline{k}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\overline{i}j} x^j \left(\lambda \frac{\partial}{\partial \lambda} + x^k \frac{\partial}{\partial x^k} \right) \right] \exp(W) \, .\end{aligned} \tag{3.17}$$

Substituting (3.16) into the above and expanding it in powers of λ and x^i's, one recovers the anomaly equation (3.15) for the correlation functions. One also finds that the genus-1 equation (3.10), which in the case of $\hat{c} = 3$ can be written as

$$\partial_i \overline{\partial}_{\overline{j}} F_1 = \frac{1}{2} C_{ikl} \overline{C}_{\overline{j}\overline{k}\overline{l}} e^{2K} G^{k\overline{k}} G^{l\overline{l}} - \left(\frac{\chi}{24} - 1 \right) G_{i\overline{j}}$$

is also contained in this equation. Here the sums over k and l are over those with $(q_k, \overline{q}_k) = (q_l, \overline{q}_l) = (1, 1)$.

[16] The genus-1 one-point function may be included in the above equation if we allow the substitution $(2g - 2) C^{(g)} \to (\chi/24 - 1)$ for $g \to 1$.

Equation (3.17) will prove to be crucial in Sect. 6 when we solve the anomaly equation (3.6) and derive explicit expressions for F_g.

Since the anomaly equation is summarized in (3.17), one may try to solve it directly. However we must also remember that W has the structure of (3.16) with

$$C^{(g)}_{i_1\cdots i_n} = \begin{cases} D_{i_1}\cdots D_{i_n}F_g & \text{for } g\geq 1 \\ D_{i_1}\cdots D_{i_{n-3}}C_{i_{n-2}i_{n-1}i_n} & \text{for } g=0 \end{cases}$$

and

$$C^{(g)}_{i_1\cdots i_n}=0 \quad \text{for } \ 2g-2+n\leq 0\,.$$

This property of W can also be summarized in a single equation as

$$\begin{aligned}&\left[\frac{\partial}{\partial t^i}+\Gamma^k_{ij}x^j\frac{\partial}{\partial x^k}+\partial_i K(\frac{\chi}{24}-1-\lambda\frac{\partial}{\partial\lambda})\right]\exp(W)=\\ &=\left[\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{2g-2}D_iC^{(g)}_{i_1\cdots i_n}x^{i_1}\cdots x^{i_n}\right]\exp(W)\\ &=\left[\frac{\partial}{\partial x^i}-\partial_iF_1-\frac{1}{2\lambda^2}C_{ijk}x^jx^k\right]\exp(W)\,.\end{aligned} \tag{3.18}$$

The two Eq.s (3.17) and (3.18) combined, are equivalent to all the holomorphic anomaly equations. In Appendix B, we analyse the two equations directly to all order in g. The order-by-order solution of the anomaly equation is presented in Sect. 6.

Recently Witten [28] discussed the implication of the holomorphic anomaly, which we had previously announced in [20], to the background (in)dependence of the string theory. There he also derived two equations, one involving $\partial_{\bar{t}^{\bar{i}}}$ and the other involving ∂_{t^i}, for some finite dimensional quantum system associated to the Calabi-Yau manifold which resemble the two equations (3.17) and (3.18), derived in the above. It would be interesting to understand the precise connection between them.

4. Comments on the Open String Version

The topological field theories obtained by twisting $N=2$ supersymmetry can also be defined on Riemann surfaces Σ having boundaries. In order to preserve topological invariance, one has to impose appropriate Q–invariant boundary conditions. Generally speaking, in the open string case the methods of Sects. 2 and 3 are much less powerful than in the closed one: The reason being that all our arguments rest on manipulations involving the *two* scalar supercharges of the twisted theory. In the open case the boundary condition is chosen so that the combination $Q=G^++\overline{G}^+$ is preserved; however the other combination $G^+-\overline{G}^+$ does not leave the boundary condition invariant and hence is not conserved any longer. Then some of the maneuvers do not extend to the open case. In particular in the open case the curvature of the Zamolodchikov metric does not satisfy the special geometry relation. This is related to the fact, that while closed strings compactified on a Calabi–Yau 3–fold lead to $N=2$ space–time supergravity, in the open case they lead to $N=1$. In the first case special geometry is implied by space–time supersymmetry [14], while the only requirement from $N = 1$ supersymmetry is that the Zamolodchikov metric should be Kähler.

A particular realization of open strings satisfying the appropriate boundary conditions can be described in the context of the sigma–models. The corresponding boundary conditions, with either A– or B–twisting, were described in ref. [24]. In the A–model one picks a Lagrangian submanifold $L_i \subset M$ for each component C_i of $\partial\Sigma$. Let TL_i and NL_i be the tangent and normal bundles of L_i in M. Then one requires that the boundary C_i is mapped into the submanifold L_i; at the boundary the normal derivative of the bosonic field X takes values in $X^*(NL_i)$; instead χ and the pullback of ψ to L_i take value in $X^*(TL_i)$. For the B–model one requires that the normal derivative of X to vanish on $\partial\Sigma$, and that θ vanishes on the boundary as well as the pullback of $\star\rho$ to $\partial\Sigma$.

Just as the topological field theory on surfaces without boundaries defines a closed string theory, if we allow boundaries the topological model will define an open string theory. We can also couple to this string theory target–space gauge fields by introducing Chan–Paton factors as usual, i.e. coupling the gauge fields to charges which propagate along the boundary. In the B–model this results in a coupling of the open string to a rank N holomorphic bundle E over the Calabi–Yau 3–fold M having structure group $U(N)$.

Given the deep analogy between the open and closed cases, it may be appropriate to pause a while to discuss how the results of Sections 2 and 3 get modified in the open case.

4.1 tt^* in the open string case. Let us start with the open analog of the tt^* equation. We consider the following geometry (see Fig. 7): a flat strip of width π and length L with a half–disk attached at one end. On the boundary, except for the segment l at the opposite end, we impose the appropriate open string boundary condition, as discussed above. On the circle arc we insert the open string topological observable $\mathcal{O}_\alpha$.

The topological path integral in this geometry — viewed as a functional of the boundary values of the fields on the segment l — defines a state in the open string Hilbert space which we call $|\alpha\rangle$. Notice that this state is automatically in the Ramond sector. To see this, observe that the twisting introduces an extra holonomy factor for the fermions equal to

$$\exp\left[\pm\frac{i}{2}\oint\omega\right], \tag{4.1}$$

where ω is the spin–connection. Given that the boundary in Fig. 7 has a geodesic deviation of π, (4.1) gives an additional factor (-1) which transforms the NS sector into the R one. Just as in the closed case, for each topological state $|\alpha\rangle$ we can find a representative which is an actual vacuum for the untwisted theory defined in the

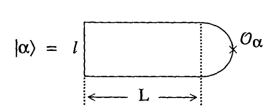

Fig. 7. By inserting the topological observable $\mathcal{O}_\alpha$ on the circle arc of the open string world sheet and doing the twisted path integral on the half–disk we get a state $|\alpha\rangle$ at the boundary. If we take $L \to \infty$ the state thus obtained is a ground state in the open string Ramond sector.

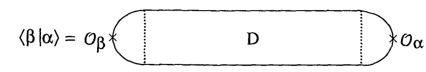

Fig. 8. Open string topological metric $\eta_{\alpha\beta}$ can be defined by gluing the two topological path-integrals on the two half–discs with the chiral operators inserted at the two end boundaries.

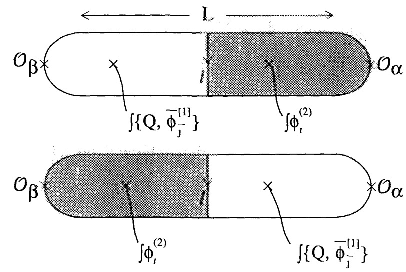

Fig. 9. The computation of the curvature of the tt^* metric in the path integral formulation involves the difference of the two path–integrals shown here.

strip. This vacuum is obtained simply by letting $L \to \infty$ in the definition of $|\alpha\rangle$. If θ is the CPT operation for the untwisted theory, $|\bar\alpha\rangle \equiv \theta|\alpha\rangle$ is also a vacuum. This allows us to introduce in the open case a real structure matrix $M_{\bar\beta}{}^{\alpha}$ analogous to that for the closed case, and then a hermitian tt^* metric

$$g_{\alpha\bar\beta} = \eta_{\alpha\gamma}\, M_{\bar\beta}{}^{\gamma},$$

where $\eta_{\alpha\gamma}$ is the open case topological metric, defined by the topological path integral performed in the geometry of Fig. 8.

Going through the same argument used in the closed case, we introduce the natural *metric* connection

$$A_{i\ \alpha\beta} = \langle\beta|\partial_i|\alpha\rangle, \qquad \overline{A}_{\bar j\ \alpha\beta} = \langle\beta|\bar\partial_{\bar j}|\alpha\rangle\ .$$

Again topological invariance implies that $\overline{A}_{\bar j\alpha\beta} = 0$ and hence[17]

$$(A_i)_\alpha^{\ \beta} = -g_{\alpha\bar\gamma}\partial_i g^{\bar\gamma\beta}\ .$$

We wish to compute the curvature of this connection. Repeating word–for–word the closed case analysis [2], we see that the curvature can be represented by the $L \to \infty$ limit of the difference of the two contributions represented in Fig. 9.

In both cases we perform the path–integral with the twisted action on a long strip with half disks attached to the two ends on whose boundaries we insert the topological observables $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, respectively. In the first term the integral of $\phi_i^{(2)}$ over the right half of the 'rounded strip' D is also inserted while the insertion[18] of $\{Q, \overline{\phi}_{\bar j}^{[1]}\}$ is integrated over the left half. In the second term the two halves interchange their role. Let us consider the first contribution. By topological invariance, we can deform the

[17] Notice that this equation is consistent with the fact that the combined Zamolodchikov metric for open *and* closed string operators should be Kähler. As mentioned in the text, this is the only condition on the Zamolodchikov metric which is expected for the open case.

contours such that Q acts on $\phi_i^{(2)}$, giving $d\phi_i^{(1)}$. Then the integral in the right half of D gives just the line integral $\int_l \phi_i^{(1)}$ where l is the segment separating the two halves of Fig. 9[19]. The second term can be handled in the same way. But this time we get $-\int_l \phi_i^{(1)}$ because the orientation is the opposite one. Then the difference is just

$$-R_{i\bar{j}\alpha\beta} = \left\langle \mathcal{O}_\beta(+\infty) \int\limits_l \phi_i^{(1)} \int\limits_D d^2z\, \overline{\phi}_{\bar{j}}^{[1]}(z)\, \mathcal{O}_\alpha(-\infty) \right\rangle_{\text{strip}},$$

where $\overline{\phi}_{\bar{j}}^{[1]}$ is integrated over the full 'rounded strip' D, and the limit $L \to \infty$ is implied. Then the open version of the tt^* equations read

$$\overline{\partial}_{\bar{j}}\left(g_{\alpha\bar{\gamma}}\partial_i g^{\bar{\gamma}\beta}\right) = \left\langle \mathcal{O}^\beta(+\infty) \int\limits_l \phi_i^{(1)} \int\limits_D d^2z\, \overline{\phi}_{\bar{j}}^{[1]}(z)\, \mathcal{O}_\alpha(-\infty) \right\rangle_{\text{strip}} \Bigg|_{L\to\infty}. \tag{4.2}$$

Equation (4.2) is much less useful than its closed counterpart (2.4) because it is not in the form of a closed differential equation for the metric $g_{\alpha\bar{\beta}}$. However, Eq. (4.2) can be used to relate the holomorphic anomaly for the open case to the tt^* metric $g_{\alpha\bar{\beta}}$ much in the same spirit as we did in Sect. 3 for the closed case.

4.2 Holomorphic anomaly at one–loop. In the open case the one–loop partition function for the topological theory coupled to gravity is given by the following quantity:

$$F_1 = \int_0^\infty \frac{dL}{L} \mathrm{Tr}\left[(-1)^F F e^{-LH}\right], \tag{4.3}$$

which is represented by a path integral (see Fig. 10) over a flat cylinder of length π and perimeter L with the Fermi current integrated along a generator l.

Taking the derivative of F_1 with respect to the complex modulus t^i and going through the standard manipulations, we get

$$\partial_i F_1 = \int_0^\infty dL \left\langle \int\limits_l (G^- + \overline{G}^-) \int\limits_{l'} \phi_i^{(1)} \right\rangle_{\text{cylinder}}^L. \tag{4.4}$$

By definition the r.h.s. of (4.4) is the one 'point' function

$$\left\langle \int_l \phi_i^{(1)} \right\rangle_{1-\text{loop}}$$

for the (open) topological theory coupled to gravity. This quantity, being topological, is a holomorphic function of the t^i's except possibly for anomalies associated to failure in the decoupling of Q–exact states. Thus $\overline{\partial}_{\bar{j}}\partial_i F_1$ measures the holomorphic anomaly at one–loop for the open case.

---

[18] Here and below $\overline{\phi}_{\bar{j}}^{[1]}$ is defined to be equal to $\frac{1}{2}[(G^+ - \overline{G}^+), \overline{\phi}_{\bar{j}}]$.

[19] The integral $\int \phi_i^{(1)}$ along the other components of the boundary vanishes because of the boundary conditions.

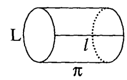

Fig. 10. Open one-loop partition function, is represented by a cylinder with perimeter L. The integration over moduli involves integration over L with the Fermion number current inserted on the line l

Let us compute $\overline{\partial}_{\bar{\jmath}}\partial_i F_1$. We have

$$
\begin{aligned}
&\overline{\partial}_{\bar{\jmath}}\partial_i F_1 = \\
&= \int_0^\infty dL \left\langle \int_l (G^- + \overline{G}^-) \int_{l'} \phi_i^{(1)} \int d^2z \, \{Q, \overline{\phi}_{\bar{\jmath}}^{[1]}(z)\} \right\rangle_{\text{cylinder}}^L + \dots \\
&= -\int_0^\infty dL \left\langle \int_l (T + \bar{T}) \int_{l'} \phi_i^{(1)} \int d^2z \, \overline{\phi}_{\bar{\jmath}}^{[1]}(z) \right\rangle_{\text{cylinder}}^L + \dots ,
\end{aligned}
$$

where $\dots$ stands for the contribution from the contact term between $\phi_i^{(1)}$ and $\overline{\phi}_{\bar{\jmath}}^{[1]}$. In the next section we shall introduce much more powerful techniques to deal with such contact terms in one–loop stringy computations. For this reason we defer the discussion of such terms until we have developed the right tools.

The insertion of the operator $\int_l (T + \bar{T}) \equiv H$ is equivalent to taking the derivative with respect L. Then we have

$$
\begin{aligned}
&\overline{\partial}_{\bar{\jmath}}\partial_i F_1 = \\
&= -\int_0^\infty dL \frac{d}{dL} \left\langle \int_{l'} \phi_i^{(1)} \int d^2z \, \overline{\phi}_{\bar{\jmath}}^{[1]}(z) \right\rangle_{\text{cylinder}}^L + \dots \\
&= -\left\langle \int_{l'} \phi_i^{(1)} \int d^2z \, \overline{\phi}_{\bar{\jmath}}^{[1]}(z) \right\rangle_{\text{cylinder}} \Bigg|_{L\to\infty} + \dots \\
&= -\int d^2z \lim_{L\to\infty} \mathrm{Tr}\left[(-1)^F \int_{l'} \phi_i^{(1)} \overline{\phi}_{\bar{\jmath}}^{[1]}(z) e^{-LH} \right] + \dots ,
\end{aligned}
\tag{4.5}
$$

where the contact–like contribution from the boundary at $L = 0$ is absorbed in the dots to be discussed in the next section.

As $L \to \infty$ only the vacuum contributions survive in (4.5), and the trace in the Hilbert space can be replaced by a trace over the open string vacua. Then

$$
\begin{aligned}
&\overline{\partial}_{\bar{\jmath}}\partial_i F_1 = \\
&= -\sum_{\alpha\beta} (-1)^{F_\alpha} \eta^{\alpha\beta} \left\langle \mathcal{O}_\alpha(+\infty) \int_l \phi_i^{(1)} \int d^2z \, \overline{\phi}_{\bar{\jmath}}^{[1]}(z) \, \mathcal{O}_\beta(-\infty) \right\rangle_{\text{strip}}^L \Bigg|_{L\to\infty} + \dots \\
&= \mathrm{tr}\left[(-1)^F R_{i\bar{\jmath}} \right] + \dots ,
\end{aligned}
\tag{4.6}
$$

where in the last step we used the open tt^*, Eq. (4.2). This is the anomaly equation we are looking for. It can be rewritten as

$$\partial_i \overline{\partial}_{\bar{j}} \big(F_1 - \mathrm{tr}[(-1)^F \log g]\big) = \dots \ , \tag{4.7}$$

where ... again denotes the short distance contributions that will be described from a more geometrical perspective in the next section.

4.3 The holomorphic anomaly at higher loops. At higher loops the situation with anomaly is similar to the one–loop case, although more complicated. Consider, for instance, the case of a surface Σ with $h+1$ boundaries and genus 0 (see Fig. 11).

In this case we have $3h-6$ real moduli, and F_h^0 is given by

$$F_h^0 = \int_{\mathcal{M}_h^0} \Big\langle \prod_{k=1}^{3h-6} \int \mu_k (G^- + \overline{G}^-) \Big\rangle_h^0 \ , \tag{4.8}$$

where $\mathcal{M}_h^0$ is the moduli space of genus zero surfaces with $h+1$ boundaries and μ_k are the corresponding Beltrami differentials. Taking the derivative $\overline{\partial}_{\bar{j}} F_h^0$ inserts in the r.h.s. of Eq. (4.8) the operator $\int d^2z \, \{Q, \overline{\phi}_{\bar{j}}^{[1]}\}$. Integrating Q by parts we get a sum of terms in which Q acts on $\int \mu_k (G^- + \overline{G}^-)$ resulting in an insertion of $\int \mu_k (T + \bar{T})$, which is then replaced by a derivative with respect to the corresponding modulus m^k.

Then the r.h.s. of Eq. (4.8) is reduced to a sum of contributions from the boundary of the moduli space $\mathcal{M}_h^0$. This boundary has many components. There are components like those in Fig. 12 which corresponds to open surfaces with a smaller number of boundaries and involving a sum over intermediate open string vacua $|\alpha\rangle$, but also components as the one in Fig. 13 in which the degeneration of the surface involves a sum over the closed string vacua $|i\rangle$. Collecting all contributions we get the anomaly formula for the open case which will involve the derivative of the lower h, g partition function with respect to open or closed string couplings with the operator $\int \overline{\phi}_{\bar{j}}^{[1]}$

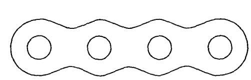

Fig. 11. An open string diagram with no handles $g = 0$ and $h = 5$ boundaries.

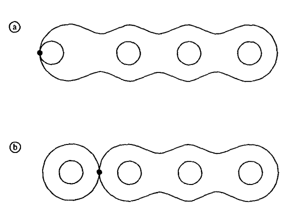

Fig. 12. The boundary of open string worldsheet may involve degeneration of the surface connected by long strips, represented here conformally by a black dot. The intermediate state on the long strip is an open string state. These contributions lead to insertion of open string vertices on the lower h Riemann surfaces.

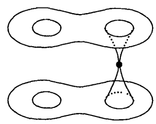

Fig. 13. The degeneration may also include contributions where the intermediate state is a closed string state.

inserted in the lower genus amplitude (cf. Eq. (4.6)). Because of this operator insertion, in the open string case the holomorphic anomaly is a much less powerful tool than in the closed string case, and the anomaly equation has not the form of a recursion relation for the F_h^g's.

This additional operator insertion in the anomaly equation is problematic in the sense that it is not topological, and hence it seems that the result of its insertion for a general model cannot be computed by TFT methods. The tt^* methods are somehow more powerful: e.g. on the strip they allow to compute such a correlation in terms of the derivatives of the metric $g_{\alpha\bar{\beta}}$, see Eq. (4.2). It is plausible that all the lower h and g correlations arising in the computation of $\overline{\partial}_{\bar{j}} F_h^g$ can be computed in a similar fashion by an extension of the tt^* idea to geometries other than the strip.

5. What are the Topological Amplitudes Computing?

In Sect. 2 we discussed two classes of examples of twisted $N = 2$ theories coupled to gravity, the A– and the B–model and discussed what they compute at the tree level. In this section we give an interpretation of what the topological amplitudes are computing in these two cases after coupling to 2d gravity, i.e. the higher genus interpretation of the amplitudes for these theories. We will first discuss the case of the B–model, where we will see that the target space field theory interpretation of the model is related to a theory of gravity on Calabi–Yau 3–folds which for reasons to be explained we will call the Kodaira–Spencer theory of gravity. The tree level amplitudes in this case are related to the classical theory of variation of Hodge structures, i.e. the special geometry that we discussed in Section 2. The one–loop amplitude of this theory is related to the holomorphic Ray–Singer torsion. The geometric meaning of higher–loop amplitudes is less clear, though can be formally defined and may be viewed as quantum corrections to special geometry. In the case of the A–model, the target space field theory interpretation is far more difficult. It should be again a theory of gravity on Calabi–Yau manifolds, but a very non-standard one, which requires the loop space of Calabi–Yau even for the formulation of the theory. However the interpretation of what the A–model is computing is rather simple for any genus g. In fact the A–model, in the limit $\bar{t} \to \infty$, computes the number (or the appropriate Euler character) of holomorphic maps from a genus g surface to the Calabi–Yau. In this sense the A– and B– models have complementary virtues. The meaning of the computations are more clear in the A–model but the formulation of the target space theory is very clear for the B–model. We will use both models, in conjunction with mirror symmetry, later in the paper to solve explicit examples at higher loops.

In Subsections 5.1-5.9 we discuss the case of B–model and KS gravity, and in Subsections 5.10-5.13 we discuss the case of the A–model.

5.1 Kodaira–Spencer Theory as a String Field Theory of the B–Model. The computations of the topological B–model before coupling to gravity, can be related to classical questions in variation of Hodge structure, i.e. the complex structure of Calabi-Yau and how it varies. In the language of sigma models this is related to the fact that the B–model topological theory is independent of the volume of the manifold. Rescaling the volume to infinity implies that in the topological B–theory, not coupled to gravity, the path–integral configurations are dominated by constant maps, thus leading to classical geometry questions, and in particular the questions of variation of Hodge structure of Calabi–Yau 3-folds. As we will discuss later in the section this is essentially true (modulo a crucial subtlety) even after we couple to $2d$-gravity, where we discuss the closed string field theory of the B–model. Before doing this we wish to discuss some mathematical aspects of the Kodaira–Spencer theory of deformations of complex structure which turn out to correspond to target space physics of the B–model. In other words we will argue why the g–th loop correction for the Kodaira–Spencer theory is the same as $F_g(t^i, \bar{t}^i)$ defined in Sect. 2. We will explicitly check this correspondence at genus zero and one, and also show that in the case of genus one our anomaly coincides with the Quillen anomaly for the Ray–Singer torsion.

5.2 Deformations of complex structure. As it was discussed in Sect. (2.1) the observables in the B–model are in one to one correspondence with cohomology elements $H^p(\wedge^q T_M)$, where T_M is the holomorphic tangent bundle. The two forms $\phi_A^{(2)}$ are possible perturbations of the Lagrangian. In case $p = 1$, $q = 1$ operators $\phi_A^{(2)}$ for $A \in H^{(0,1)}(T_M)$ correspond to marginal deformations of the B–model and are in one to one correspondence with deformations of complex structure of Calabi–Yau 3–fold M. In the spirit of string theory one expects that $A \in \Omega^{(0,1)}(T_M)$ should be the basic field in the field theory in question. This field theory is closely related to the mathematical theory of deformations of complex structures. Before proceeding further we first review some elements of this theory.

The complex structure on manifold M is determined by the $\bar{\partial}$ operator. To the first order the change of complex structure is described by deformation of $\bar{\partial}$ operator $\bar{\partial} \to \bar{\partial} + A^i \partial_i$ [29]. This is a deformation of $\bar{\partial}$ operator acting on functions. One can describe not only the infinitesimal deformations of complex structure but a finite one. The new complex structure is described by requiring that functions satisfying the condition

$$(\bar{\partial} + A^i \partial_i) f = 0 \ , \tag{5.1}$$

are holomorphic in the new complex structure. In other words the kernel of the deformed $\bar{\partial}$ coincides with kernel of (5.1). The integrability condition

$$\bar{\partial}(\bar{\partial} f + A^i \partial_i f) = (\bar{\partial} A^j + A^i \partial_i A^j) \partial_j f = 0$$

is equivalent to the Kodaira–Spencer (KS) equation [29]

$$\bar{\partial} A + \frac{1}{2}[A, A] = 0 \ . \tag{5.2}$$

Once again A is $(0,1)$ form with coefficients in $(1,0)$ vector fields and the brackets $[\,,\,]$ mean the commutator of two vector fields and wedging. Two solutions of (5.2) correspond to the same complex structure if they differ by a diffeomorphism. In the linear approximation the Kodaira Spencer equation reduces to $\bar{\partial} A = 0$. The solution is defined modulo diffeomorphisms generated by vector fields $A \to A + \bar{\partial}\epsilon$, and thus A has to be a cohomology element. The ambiguity in the choice of cohomology representative is promoted to the ambiguity in the solution of Kodaira Spencer equation.

Before fixing the ambiguity in question let us mention that for any Calabi–Yau manifold there is an isomorphism

$$' : \quad \Omega^{(0,p)}(\wedge^q T_M) \to \Omega^{(3-q,p)}(M) \tag{5.3}$$

given by the product with the holomorphic $(3,0)$ form. Sometimes we use the notation $(A \cdot \Omega) = A'$. Without lack of generality we impose the constraint

$$\partial A' = 0 \ . \tag{5.4}$$

To fix the ambiguity, $A \to A + \bar{\partial}\epsilon$, we impose the gauge condition

$$\bar{\partial}^\dagger A' = 0 \ . \tag{5.5}$$

This gauge condition requires the choice of metric on the Calabi–Yau manifold. It will be clear later that these conditions fix the solution uniquely.

Let A, B be $(0,1)$ forms with the coefficients in vector fields which satisfy the gauge condition $\partial A' = \partial B' = 0$. It was proven by Tian [30] (see also [31]) that

$$[A, B]' = \partial(A \wedge B)' \ . \tag{5.6}$$

Later we will need the generalization of this lemma where A, B belong to $\Omega^p(\wedge^q T_M)$ [32]. Using this lemma we can rewrite the KS equation in Tian form

$$\bar{\partial}A' + \frac{1}{2}\partial(A \wedge A)' = 0.$$

The tangent space to the moduli space of complex structures is given by $H^{(0,1)}(T_M)$. Let A_1 be an infinitesimal deformation of complex structure satisfying conditions (5.5), (5.4). Then for any A_1 one can "exponentiate" the deformation of complex structure by constructing the solution to the KS equation

$$A = \sum_{n=1}^{\infty} \epsilon^n A_n,$$

where ϵ is a formal expansion parameter (we put $\epsilon = 1$ later). We will show that it is possible to get a unique solution of the KS equation satisfying the gauge condition $\bar{\partial}^\dagger A' = 0$ such that A'_n is ∂–exact for $n > 1$. Note that this latter condition automatically implies that we can use Tian's form of the KS equation. This choice means that A'_1 is a harmonic form, which we will call *massless*, and A'_n for $n > 1$ can be written as a linear combination of eigenstates of Laplacian with positive eigenvalue. We will call these states the *massive* states.

Let us see how we can construct the solution recursively (following the work of [30, 33]) making sure that at each stage $\partial A'_n = 0$ and that A'_n is ∂–exact for $n > 1$. Let A_1 satisfy the gauge condition (5.5) together with constraint (5.4). Thanks to Tian's Lemma the equation for A'_2 becomes

$$\bar{\partial}A'_2 + \frac{1}{2}\partial(A_1 \wedge A_1)' = 0.$$

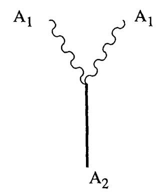

Fig. 14. The first order perturbation computation for solving the KS equation in Tian's gauge. Two massless modes represented by wavy lines join to give a massive mode whose propagator is represented by a solid line.

Note that the solution to this equation for A_2' is unique up to addition of $\bar{\partial}\nu$. In order to get rid of this ambiguity we will consider the gauge condition $\bar{\partial}^\dagger A_2' = 0$. Then the solution can be written as

$$A_2' = -\bar{\partial}^\dagger \frac{1}{\Delta} \partial (A_1 \wedge A_1)' , \tag{5.7}$$

where

$$\Delta = 2[\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial}]$$

is the Laplacian. To see that the above is a solution, first note that it is well defined, because ∂ annihilates the kernel of Δ. Then acting by $\bar{\partial}$ and using the fact that $\bar{\partial}(A_1 \wedge A_1)' = 0$ (because A_1 is $\bar{\partial}$ closed and $\bar{\partial}$ commutes with the operation $'$ since Ω is holomorphic) one checks that it is a solution to the equation. It also satisfies the conditions of being ∂–exact (because ∂ and $\bar{\partial}^\dagger$ anticommute for a Kähler manifold) and $\bar{\partial}^\dagger$ closed. The fact that there is always a solution to the above equation is also known as $\partial\bar{\partial}$-Lemma [11][20]. This in particular means that with the gauge condition we have chosen

$$\frac{-1}{2\bar{\partial}} \partial \equiv -\bar{\partial}^\dagger \frac{1}{\Delta} \partial \tag{5.8}$$

and it can be viewed as a propagator for massive modes. Equation (5.7) describes interaction between two massless modes and a massive one and then further propagation of the massive state. It can be represented as the diagram of Fig. 14.

The equation for the next iteration becomes $\bar{\partial} A_3' + \partial (A_2 \wedge A_1)' = 0$. The second term in this equation is $\bar{\partial}$ closed $\bar{\partial}\partial(A_2 \wedge A_1)' \sim \partial([A_1, A_1] \wedge A_1)' \sim [[A_1, A_1], A_1] = 0$, and therefore one may use the above propagator again,

$$A_3' = 2\bar{\partial}^\dagger \frac{1}{\Delta} \partial (A_1 \wedge (\bar{\partial}^\dagger \frac{1}{\Delta} \partial (A_1 \wedge A_1)')^\vee)' ,$$

where $(A')^\vee = A$. Note that this solution satisfies the required conditions. Again this contribution has a clear interpretation. Two massless states go to a massive one (as before), but now the propagator receives corrections due to the coupling with the massless state in the background. This contribution corresponds to the diagram of Fig. 15.

[20] The $\partial\bar{\partial}$–Lemma reads: if ω is any $\bar{\partial}$ closed form and ω is also ∂ exact, then $\omega = \partial\bar{\partial}\phi$.

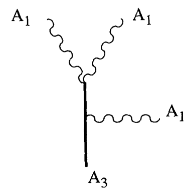

Fig. 15. Second order perturbation computation for solving the KS equation. Here two massless modes join to give a massive mode, emit a massless mode which finally gives rise to the massive mode A_3. Note that the only propagators involve massive modes, and the massless modes are like the background fields.

It is already clear that $\partial\bar{\partial}^\dagger/\Delta$ is a propagator for the massive states for the field theory in question. The massless modes play the role of the background. It is quite remarkable that the KS equation reproduces the perturbation series of a ϕ^3 theory.

At the n−th iteration step all $A_1, \ldots A_{n-1}$ satisfy the conditions $\partial A_1' = \cdots = \partial A_{n-1}' = 0$ and the KS equation becomes

$$\bar{\partial} A_n' + \frac{1}{2}\sum_{i=1}^{n} \partial(A_{n-i} \wedge A_i)' = 0 \,. \tag{5.9}$$

The second term of this equation is $\bar{\partial}$ closed. This follows from the equations satisfied for $\bar{\partial} A_i'$ dictated by induction and the Jacobi identity for the $(0,1)$ forms with coefficients in vector fields and Tian's lemma

$$\partial([A,B] \wedge C)' + \partial([C,A] \wedge B)' + \partial([B,C] \wedge A)' = 0 \ . \tag{5.10}$$

It follows from the above arguments therefore that Eq. (5.9) has a solution and it is ∂–exact. The perturbation theory described above is convergent in some open neighborhood of the origin [30].

We just proved that for any $x \in H^{(0,1)}(T_M)$ there is a map $x \longrightarrow A[x]$ given by the solution of the KS equation, with $A_1 = x$. This choice of terminology is consistent with the definition of x^i given in Sect. 3, and can basically be viewed as shifting the complex structure of the Calabi-Yau labeled by $(t, \bar{t}) \longrightarrow (t{+}x, \bar{t})$. For later convenience we will write $A[x] = x + A(x)$. Decomposition into x and $A(x)$ is quite natural. A cohomology element x represents a *massless* mode while $A(x) = \sum_{n=2}^{\infty} A_n$ contains the *massive* modes of the field.

Under the deformation of complex structure the holomorphic $(3,0)$ form gets changed. For an infinitesimal deformation the deformed holomorphic form is equal to $\Omega_0 + x'$. For the finite deformations the holomorphic $(3,0)$ form mixes with $(2,1)$, $(1,2)$ and $(0,3)$ and it satisfies the equation

$$\bar{\partial}\Omega + \frac{1}{2}\partial(\Omega^\vee \wedge A)' = 0 \ ,$$

where prime and check are defined with respect to the fixed holomorphic three form Ω_0. It follows from Tian's lemma that the deformed holomorphic $(3,0)$ form is given as follows [33]

$$\Omega = \Omega_0 + A' + (A \wedge A)' + (A \wedge A \wedge A)' \ . \tag{5.11}$$

Coordinates in $H^{(0,1)}(T_M)$, denoted by x, may serve as affine coordinates on some open neighborhood of the moduli space of complex structures (see also [34]) thanks to Tian's mapping. These coordinates are in fact very special (not to be confused with special coordinates except for the particular case of base point at infinity, as discussed in Sect. 2.6) and corresponds to the canonical coordinates discussed in full generality in Sect. 2.6. In this coordinate the Kähler potential is given as follows:

$$e^{-K(x,\bar{x})} = \int_M \Omega_0 \wedge \bar{\Omega}_0 + \int_M A' \wedge \bar{A}' + \int_M (A \wedge A)' \wedge (\bar{A} \wedge \bar{A})' + \int_M (A \wedge A \wedge A)' \wedge (\bar{A} \wedge \bar{A} \wedge \bar{A})' \,, \tag{5.12}$$

where $A[x] = x + A(x)$. Taking into account that $A(x) = O(x^2)$ and x and $A(x)$ are orthogonal to each other we get the expansion

$$e^{-(K(x,\bar{x})-K_0)} = 1 + x^i \bar{x}^j G^{(0)}_{i\bar{j}} + O(x^2 \bar{x}^2) \,.$$

It immediately follows from this expansion that in these coordinates $\partial_i K = 0 = \Gamma^k_{ij}$ vanishes at the origin together with all holomorphic derivatives. Therefore in these coordinates the covariant holomorphic derivatives at the origin coincide with the ordinary derivatives

$$D_i D_j \ldots D_k F = \partial_i \partial_j \ldots \partial_k F \ .$$

This property is very important and was the defining property of canonical coordinates discussed in full generality in Sect. 2.6. Let us clearly state that the canonical coordinates are uniquely determined by the point in the moduli space (the origin of the coordinate system) and the choice of the basis in $H^{(0,1)}(T_M)$.

It is instructive to consider the example of canonical coordinates in the case $T^2 \times T^2 \times T^2$, where T^2 is a two dimensional torus. The complex structure of each torus is described by one complex parameter τ_i. One can carry out the construction of canonical coordinates for each torus separately. Let us parametrize each torus using coordinates (σ_1, σ_2), where (σ_1, σ_2) runs over unit square. In this parametrization ∂, $\bar{\partial}$ are given as follows:

$$\partial = \frac{1}{(\bar{\tau} - \tau)}(\bar{\tau}\partial_1 - \partial_2) \ , \quad \bar{\partial} = \frac{1}{(\bar{\tau} - \tau)}(-\tau\partial_1 + \partial_2) \,.$$

Now, let us choose the base point $(a, \bar{a})$. The holomorphic flat coordinate x around $(a, \bar{a})$ is defined as follows $\bar{\partial}(\tau) = \bar{\partial}(a) + x\partial(a)$. It implies the relation $x(\tau)$

$$x = \frac{\tau - a}{\tau - \bar{a}} \,,$$

i.e. the upper–half plane gets mapped into the open unit disk. The Kähler potential in this coordinate is equal to

$$\begin{aligned} e^{-K(x,\bar{x})} &= \int \prod_i (dz_i - x_i d\bar{z}_i) \wedge (d\bar{z}_i - \bar{x}_i dz_i) = \\ &= \prod_i (1 - x_i \bar{x}_i)(a_i - \bar{a}_i) = \prod_i (\tau_i - \bar{\tau}_i) \left| i \frac{a_i - \bar{a}_i}{\tau_i - \bar{a}_i} \right|^2 \,. \end{aligned} \tag{5.13}$$

The x dependence is quite remarkable. It is clear that all derivatives with respect to x are proportional to $\bar{x}$ and therefore identically equal to zero at the origin. The factor inside the absolute value is the gauge factor $f(a_i) = -i\prod_i(a_i - \bar{a}_i)/(\tau_i - \bar{a}_i)$.

5.3 Kodaira–Spencer theory as the string field theory. So far we have discussed what seems to be a perturbative field theory which describes the perturbation of complex structure of Calabi–Yau manifolds starting from a base–point. Since the B–model describes the deformation of the complex structure, the effective string field theory of the B–models must be this underlying field theory, which we shall call the Kodaira–Spencer theory of gravity. We have two options in writing this field theory: We can either use the Kodaira-Spencer equation in the Tian gauge to write the action giving rise to these equations, or directly use the rules for constructing closed string field theory along general lines discussed in the literature (see [35] for a thorough review of the literature). We will follow the first line and see why it is the same as the second.

To write an action[21] we first need to fix some data: the point P (which we sometimes denote also by $(t_0^i, \bar{t}_0^i)$) in the moduli space of complex structures (background) and a cohomology element $x \in H^{(0,1)}(T_M)$. The physical field A in the KS theory is a $(0,1)$ form with coefficients in vector fields which is also constrained to satisfy condition $\partial A' = 0$. For reasons that will be clear in a moment we assume that A includes only massive modes. This means that A lies in the subspace $\mathcal{H} \subset \Omega^{(0,1)}(T_M)$ orthogonal to $H_{\bar{\partial}}^{(0,1)}(T_M)$, or in other words

$$\int_M A' \wedge \bar{z}' = 0$$

for any $\bar{z} \in H_{\bar{\partial}}^{(1,0)}(T^*)$. Thanks to constraint (5.4), this definition is independent of the choice of representative in cohomologies.

The Kodaira–Spencer action is given as follows:

$$\lambda^2 S(A, x|P) = \frac{1}{2}\int_M A' \frac{1}{\partial}\bar{\partial}A' + \frac{1}{6}\int_M ((x + A) \wedge (x + A))'(x + A)' , \tag{5.14}$$

where λ^2 is the coupling constant. In spite of the non-local kinetic term this action is well defined. Indeed, it follows from the $\partial\bar{\partial}$-Lemma that $\bar{\partial}A' = \partial\bar{\partial}v$ and therefore $\partial^{-1}\bar{\partial}A' = \bar{\partial}v + \partial\rho + z$, where ρ and z summarize the ambiguities and $z \in H_{\bar{\partial}}^{(1,0)}(T^*)$. The condition that A' is massive together with the constraint it satisfies implies that ρ and z do not contribute to the action which therefore is well defined. Note that to define the action we did not use the metric on Calabi–Yau manifold. We just used its complex structure[22]. This is just like the Chern–Simons theory. Thus the KS theory is a topological theory (or more properly it could be called a holomorphic topological theory in the sense that it does depend on the complex structure of the Calabi–Yau). Varying the KS action with respect to A we recover the Kodaira–Spencer equation in Tian's form

$$\bar{\partial}A' + \frac{1}{2}\partial((x + A) \wedge (x + A))' = 0 . \tag{5.15}$$

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[21] For an example of string field theory for topological theories see [36].

[22] To see that the action is well defined and independent of the choice of metric on M, we can also use the ∂ constraint to write $A' = \partial\phi$ and substitute it in the action to get a local action for ϕ.

The existence of this action explains the fact that in the perturbation expansion for $A(x)$ discussed before one naturally gets Feynman rules of some field theory. In fact they are nothing but the tree level diagrams of KS theory. Note that the propagator for KS action $\bar{\partial}^{-1}\partial$ is given by (5.8) in the appropriate gauge.

We now wish to see why the action (5.14) is the same as what we would have gotten from the target space theory of the B–model. For this, we employ the arguments of Witten [24]. He used the fact that volume perturbation for the Calabi–Yau is BRST trivial in the B–model set up, to take the infinite volume limit. In this case the worldsheet configurations *for a fixed worldsheet modulus* is dominated by constant maps. But as noted in [24] this is not the full story. The reason is that we are discussing a theory of 2d gravity which means we are integrating over the moduli of Riemann surfaces. No matter how large a volume of Calabi–Yau we choose if we go close enough to the boundary of the moduli space we can get finite action. In other words the worldsheets which will have finite action are the ones concentrated in long thin tubes, which means that we are going to end up with an ordinary field theory as an *exact* field theory of string model (i.e. all the stringy massive modes are irrelevant because of topological triviality of these modes). Indeed this argument applies even taking into account potential anomalies, because as discussed in Sect. 3 there is no anomaly for the decoupling of the Kähler-moduli in the B–model.

So to fix the string field we have to recall that the field in question should have charge $(1,1)$ which in our case translates to the fact that A should belong to $\bar{T}^*_M \wedge T_M$. Let us also recall the dictionary developed in Sect. 2: In the large volume limit operator $\bar{\partial}$ is identified with BRST operator $\bar{\partial} = Q = G_0^+ + \bar{G}_0^+$, while $\partial = G_0^- - \bar{G}_0^- = b_0^-$. The string field A' should satisfy two constraints

$$\partial A' = b_0^- A' = 0 \quad \text{and} \quad (L_0 - \bar{L}_0)A' = (\Delta - \bar{\Delta})A' = 0 \ . \tag{5.16}$$

In case of the KS theory the second constraint is a trivial consequence of Kählerian geometry and amazingly the first condition is precisely Tian's condition which led to the simplification and proof of integrability of the KS equation in the case of Calabi–Yau 3-fold. In order to borrow the machinery of closed string field theory we need to find an expression for $c_0^- = c_0 - \bar{c}_0$. However there is no such object just because the b-cohomology is not trivial. What is true instead is that on the *massive* states of the theory, we can in fact define a

$$c_0^- = \frac{1}{\partial} = \frac{\partial^\dagger}{\Delta}$$

which satisfies

$$\{c_0^-, b_0^-\} = 1 \ ,$$

and we are thus forced to write down the action *only for the massive modes*. Therefore, the kinetic piece of the KS action coincides with the free part of the standard string field theory action

$$\frac{1}{2}\int A' \frac{1}{\partial}\bar{\partial}A' = \frac{1}{2}(A', c_0^- Q A') \ .$$

The gauge $\bar{\partial}^\dagger A' = 0$ is nothing else but the Siegel gauge in which both $b_0^- = \partial$ and $b_0^+ = \bar{\partial}^\dagger$ annihilates the physical fields. In this gauge the propagator takes the familiar form

$$\frac{b_0^+ b_0^-}{(L_0 + \bar{L}_0)} = \frac{\bar{\partial}^\dagger \partial}{\Delta} \ .$$

Magically enough this is identical with the Kodaira–Spencer kinetic term and the propagator. The cubic interaction term is quite standard and gives rise to the interaction term of the Kodaira–Spencer action.

Thus the KS action is nothing else but the closed string field theory action at least up to cubic order. One of the main difficulties of the closed string theory is the absence of a decomposition of the moduli space of Riemann surfaces compatible with Feynman rules. To avoid this problem one should introduce higher string vertices and as a result the closed string field theory becomes non–polynomial (see [35] and references there). The contribution to these higher string vertices comes entirely from the internal domains of the moduli space of Riemann surfaces. Quantized KS theory is defined as the large volume limit of topological sigma–model and as a topological theory it gets contribution entirely from the boundary of moduli space of Riemann surfaces. Therefore, the higher vertices should be absent in quantized KS theory. It is quite satisfactory that we thus end up with precisely the KS theory as the string field theory of the B–model[23]. This is further confirmed in Sect. 5.4 where we will find that the KS theory, with the ghost fields added, already satisfies the BV master equation and needs no further corrections.

Let us now discuss the gauge symmetries of Kodaira–Spencer theory. As a string field theory we certainly expect it to have such symmetries. Being a theory of gravity the Kodaira–Spencer theory should be invariant under diffeomorphisms (we will make this statement precise in a moment). Put differently, the fact that the variation of $\bar\partial$ can also be affected by diffeomorphisms, and we do not wish to take this as a physical variation, we need to consider the theory as a gauge theory with respect to the diffeomorphism group. The kinetic part of the action is clearly invariant under the shift of A by $\bar\partial$-exact term which means $\delta A = \bar\partial\epsilon = Q\epsilon$. This linearized gauge transformation can be extended to a full non-linear gauge transformation which turns out to be nothing else but an *Ω-preserving* diffeomorphism

$$z^i \longrightarrow z^i + \epsilon^i(z,\bar z)\,.$$

The condition that ϵ is a Ω preserving diffeomorphism means that it satisfies the constraint $\partial\epsilon' = 0$. The full gauge transformation of the Kodaira-Spencer field A, which can be deduced from the variation of $\bar\partial$ under the diffeomorphism, is given as follows

$$\delta A = \bar\partial\epsilon - [\epsilon,(x+A)]\,,$$

and using Tian's lemma it can be rewritten in a more familiar form $\delta A' = \bar\partial\epsilon' - \partial(\epsilon\wedge(x+A))'$. One can verify that this transformation is a symmetry of the action. Indeed the variation of the action is equal to

$$\begin{aligned}\lambda^2\delta S = &-\int_M A'\bar\partial((x+A)\wedge\epsilon)' + \frac{1}{2}\int_M ((x+A)\wedge(x+A))'\bar\partial\epsilon' \\ &-\frac{1}{2}\int_M ((x+A)\wedge(x+A))'\partial((x+A)\wedge\epsilon)'\,.\end{aligned} \tag{5.17}$$

[23] It is amusing to note that the closed string field theory of $N = 2$ strings [37] also has a cubic action, the 4-real-dimensional action for the Plebanski equation describing Ricci–flat Kähler metric in 2 complex dimensions, which is very similar to the KS action given above.

The first two terms cancel each other as can be seen by integrating by parts. The vanishing of the third term follows from the Jacobi identity. Indeed, the last term can be rewritten as follows:

$$\int_M ([(x+A),\epsilon]\wedge(x+A))'(x+A)' = \frac{1}{2}\int_M ([(x+A),(x+A)]\wedge\epsilon)'(x+A)' = \frac{1}{2}\int_M ([(x+A),(x+A)]\wedge(x+A))'\epsilon' = 0\;. \tag{5.18}$$

To formulate the KS theory we fixed some data: a point in the moduli space P and the cohomology element x. Note that the fact that x cannot be written as part of the kinetic term is because of the ∂^{-1} in the kinetic term, which renders the appearance of x meaningless. So the KS theory *does not have the degree of freedom to shift the complex structure* as a dynamical field in the theory. Instead the existence of the coupling with x *as a background field* in the interaction term is there to take care of this. One may ask how the theory changes if we choose a different base point P. We parametrize the position of the base point P in canonical coordinates $P = P(t,\bar{t})$. Ignoring the holomorphic anomaly the KS action depends only on t and is independent of $\bar{t}$. The shift in t coordinate can be achieved by shifting the field A by the solution of the KS equation (let $A_0(x)$ be the solution of KS equation). Then, consider the following identity:

$$\begin{aligned}\lambda^2 S(A+A_0(x),x|t,\bar{t}) = \int_M A'\frac{1}{\partial}\left(\bar{\partial}A_0' + \frac{1}{2}\partial(x+A_0)\wedge(x+A_0))'\right) + \\ \frac{1}{2}\int_M A_0'\frac{1}{\partial}\bar{\partial}A_0' + \frac{1}{6}\int_M ((x+A_0)\wedge(x+A_0))'(x+A_0)' + \\ \frac{1}{2}\int_M A'\frac{1}{\partial}\left[\bar{\partial}A' + \partial((x+A_0)\wedge A)'\right] + \frac{1}{6}\int_M (A\wedge A)'A'\;.\end{aligned} \tag{5.19}$$

The first term vanishes due to the equation of motion. The second and the third terms are naturally combined into the classical KS action evaluated on the solution of KS equation. The two remaining terms have an interpretation as the KS action around the *new background*. Indeed the combination in the square brackets coincides with the deformed $\bar{\partial}$ operator around the new background. There is still one subtlety, the *prime* operation is defined with respect to the old background. In the new background the prime operation should be defined by contraction with the deformed holomorphic 3–form given by (5.11). Noticing, that only projection on $(3,0)$–forms contributes to the action one can replace the prime operation around the old background by the prime operation around the new background. As a result of this formal manipulation we obtain the relation

$$S(A+A_0(x),x|t,\bar{t}) = S(A_0(x),x|t,\bar{t}) + S(A,0|t+x,\bar{t})\;. \tag{5.20}$$

In the original definition of the KS theory t and $\bar{t}$ are complex conjugate to each other. Without the holomorphic anomaly, the KS action is independent of $\bar{t}$ and one can replace $S(\;|t+x,\bar{t})$ by $S(\;|t+x,\bar{t}+\bar{x})$. If such arguments were true they imply the background independence of the KS theory or background independence of the corresponding closed string field theory. The dependence of the KS action on $\bar{t}$ destroys background independence. In other words the holomorphic anomaly governs the background dependence of the KS action (see also discussion in [28]). In the presence of the holomorphic anomaly relation (5.19) may serve as the definition of the KS action where the condition $t = \bar{t}^*$ is relaxed.

We now come to a puzzle raised by Witten in his study of this theory [24]. It was pointed out in [24] that the fact that the three point function C_{ijk} is not zero seems to be at odds with the fact that there is no obstruction to deforming by the marginal operators. The resolution of this puzzle in the context of the KS theory is simply that the massless fields, i.e. the string modes, *are not dynamical fields* and so there is no reason that the classical value of action is independent of their expectation value (as we will discuss in more detail below). Thus the fact that the kinetic term cannot be defined unless we delete the massless modes means in particular that C_{ijk} may be non-zero even if the massless modes can be given arbitrary expectation value.

Since KS theory is a field theory in six dimensions, it is not easy to explicitly compute higher loop amplitudes. In particular this 6-dimensional field theory looks highly non-renormalizable from the simple power counting argument. It is quite remarkable that topological string theory of the B–model provides a prescription to quantize the Kodaira-Spencer theory. The properly regularized Kodaira-Spencer theory should satisfy

$$e^{W(\lambda,x|t,\bar{t})} = \int DAe^{S(x,A|t,\bar{t})} \ ,$$

where the effective action $W(\lambda, x|t,\bar{t})$ was defined in Sect. 3. We also introduce the notation $x = x^i\mu_i$, where μ_i is some basis in $H^{(0,1)}(T)$. Even though the r.h.s. of this equation is to be properly defined at higher loops it is well defined as it stands for the tree level. Let us prove this relation at least at the tree level. Later in this section we will see that it also continues to hold at one–loop.

At the tree level, the contribution of the path-integral simply gives rise to the classical action evaluated for solutions to the field equations. Let us denote this action by $S_0(x, A_0|t,\bar{t})$, where $A_0(x)$ is such that $A_0(x)+x$ satisfies the KS equation (expanded about the base point $(t,\bar{t})$). Thus we need to show

$$W_0(x|t,\bar{t}) = \lambda^2 S_0(x, A_0(x)|t,\bar{t}) \ , \tag{5.21}$$

where W_0 is the tree level contribution to W (i.e. the coefficient of λ^{-2}). Note that in the x-coordinate which is a canonical one, W_0 of Sect. 3 is defined by the condition

$$\partial_i\partial_j\partial_k W_0 = C_{ijk}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} C^0_{ijks_1\ldots s_n} x^{s_1}\ldots x^{s_n}$$

and also W_0 has no linear or quadratic dependence on x. We see simply from the definition of S_0 that up to $O(x^3)$ they are thus equal. We need to show that it holds to all orders. Let us compare the third derivatives of both sides of (5.21). The third derivative of the classical action is given as follows:

$$\begin{aligned}\frac{d^3S_0}{dx^i dx^j dx^k} = {} & \left[(\delta_A S)\partial_i\partial_j\partial_k A\right] + \left[(\delta_A^2 S)\partial_i A + (\delta_A\partial_i S)\right]\partial_j\partial_k A + \\ & \left[(\delta_A^3 S)\partial_i A\partial_j A\partial_k A + 3(\delta_A^2\partial_i S)\partial_j \right.\\ & \left. A\partial_k A + 3(\delta_A\partial_i\partial_k S)\partial_j A + \partial_i\partial_j\partial_k S\right] \ ,\end{aligned} \tag{5.22}$$

where δ_A is the variational derivative with respect to A and $\partial_i = \partial/\partial x^i$ and symmetrization with respect to ijk is implicit. The first two terms vanish: the first one

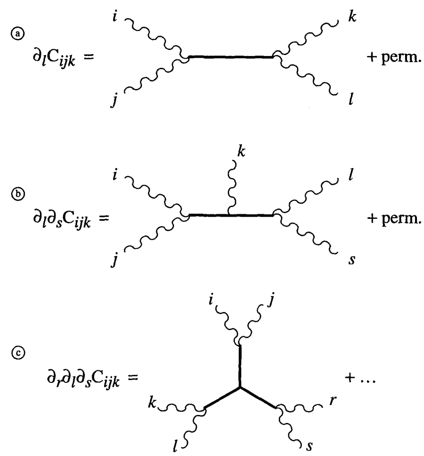

Fig. 16. Tree level computations in the KS theory as a function of the background fields (the wavy lines which represent the massless modes). As argued in the text the n–point functions at the tree level can also be computed by taking appropriate number of derivatives of the Yukawa coupling. Here the four point function (a), five point function (b), and six point function (c), are represented and can be most easily computed by taking the first, second and third derivative of Yukawa couplings respectively.

vanishes because $\delta_A S = 0$ by the equations of motion which is the definition of $A_0(x)$. The second term vanishes by taking the derivative of $\delta_A S$, along the classical solution, with respect to x_i and expanding to the third order term. Finally the last term can be rewritten as

$$\frac{d^3 S_0}{dx^i dx^j dx^k} = \int \left((\mu_i + \partial_i A_0) \wedge (\mu_j + \partial_j A_0)\right)' \wedge (\mu_k + \partial_k A_0)' = C_{ijk}(x) ,$$

where the last equality follows from the alternative definition of Yukawa coupling discussed in Sect. 2 (see (2.36)). This proves Eq. (5.21).

$W_0(x|t, \bar{t})$ may be viewed as the effective action for the massless modes x having integrated out the massive modes. It is quite amazing that integrating the massive modes has only the effect of taking derivatives of the Yukawa coupling. For example (see Fig. 16) the four point function gives rise to $\partial_l C_{ijk}$, the five point function to $\partial_s \partial_l C_{ijk}$ and the six point function to $\partial_r \partial_s \partial_l C_{ijk}$.

In fact we will now use this fact to estimate the behavior of the partition function at genus g of the KS theory to all loops, as we approach the boundary of moduli space. This will be needed in conjunction with the anomaly equation to constrain the global properties of the partition function of the topological string theory and will be heavily used in the context of solving explicit examples.

This is done by estimating the leading divergence of each diagram as we approach the boundary of moduli space. To do this we need the estimate of the propagator and the three point interaction of the massive modes (the massless modes do not propagate in loops). Let us denote the leading divergence of the propagator by P, of the massive vertex by V_{MMM}, and of the vertex with two massless and one massive mode as V_{ttM}. Using the topology of ϕ^3 graphs at g loops we estimate the Kodaira–Spencer partition function F_g to behave as

$$F_g \sim P^{3g-3} (V_{MMM})^{2g-2} . \tag{5.23}$$

We want to express this in terms of C_{ttt}, the leading divergence in the Yukawa coupling for the massless modes *written in the canonical coordinate* t. The n–point functions at tree level, are given by $\partial_t^n C_{ttt}$. Using the tree–level KS perturbation theory, we learn that the four point function of the massless modes behaves as

$$\partial_t C_{ttt} \sim P\,(V_{ttM})^2,$$

while the six point function goes like

$$\partial_t^3 C_{ttt} \sim P^3\,(V_{ttM})^3\,V_{MMM}\ .$$

Eliminating V_{ttM} from these two equations and using (5.23) we learn that

$$F_g \sim \frac{[\partial_t^3 C_{ttt}]^{2g-2}}{[\partial_t C_{ttt}]^{3g-3}}\ . \tag{5.24}$$

Note that the estimate (5.24) is independent of the definition of the canonical coordinate t or the gauge for the line bundle $\mathcal{L}$ as it should be.

5.4 BV formalism and closed string field theory. In this section we quantize the KS action using the BV formalism which is particularly well suited to string theory. The interpretation of the KS theory as string field theory turns out to be very useful. In this interpretation the KS field A is identified with the string field. But in string theory there are 'ghost' states, which mean that we are not restricted to ghost number $(1,1)$. Translated to the geometry of Calabi-Yau, this means that we should broaden the range of A so that $A \in \Omega^{(0,p)}(\wedge^q T_M)$; the ghost counting coincides with the fermion counting and is equal to $F_L + F_R = (p + q - 3)$. The original KS field A' has ghost number 2.

The consistent scheme for quantization string field theory is given by the Batalin–Vilkovisky (BV) formalism [38], [39]. In the Batalin-Vilkovisky formalism one has to relax the condition for the ghost numbers of string field and include all possible fields with arbitrary ghost numbers. The fields A with ghost numbers $q(A) \leq 2$ are called fields, while the fields A^* with ghost numbers $q(A) > 2$ are called antifields. The space of functionals of fields-antifields is equipped with an odd antibracket $\{\ ,\ \}$. The BRST symmetry is a canonical transformation in the antibracket. The BRST variations of the fields are given as follows:

$$\delta_{BRST}\mathcal{A} = \{\mathcal{A}, S\}\ .$$

The original action is replaced by full action which depends on both fields and antifields. The full action satisfies two conditions. When all antifields are set to zero the full action reduces to the original one. The full action also satisfies the Batalin-Vilkovisky master equation

$$\frac{1}{2}\{S,S\} = \hbar \Delta S\ , \tag{5.25}$$

where Δ is the natural Laplacian on the space of fields-antifields. The r.h.s. of (5.25) is a contribution coming from the path integral measure. At the classical level ($\hbar = 0$) the Batalin-Vilkovisky equation is nothing else but the condition that full action is gauge invariant. The gauged fixed action is determined by an odd functional $\Psi(A)$ and is given by $S_\Psi(A) = S(A, A^* = \delta\Psi/\delta A)$.

In the case of the KS theory the full space of fields is a subspace $\mathcal{H}$ of $\oplus_{p,q}\Omega^{(0,p)}(\wedge^q T_M)$ satisfying the constraints (5.16). The space

$$\oplus_{p+q\leq 2}\Omega^{(0,p)}(\wedge^q T_M)$$

is the space of fields, while

$$\oplus_{p+q>2}\Omega^{(0,p)}(\wedge^q T_M)$$

is the space of antifields. Note that not all (p,q) are allowed, and the projection of $\mathcal{H}$ on $\oplus\Omega^{(0,p)}(\wedge^3 T_M)$ is empty. Taking into account that both fields and antifields satisfy constraints (5.16) we get exactly the same number of fields and antifields. Fields and antifields are paired with each other

$$A \in \Omega^{(0,p)}(\wedge^q T_M) \longleftrightarrow A^* \in \Omega^{(0,3-p)}(\wedge^{(2-q)} T_M) \ ,$$

and obey opposite statistics. The odd bracket structure on the space of field-antifields is given by

$$\{A_p^q(z), A_{\tilde{p}}^{\tilde{q}*}(w)\} = \delta_{p+\tilde{p},3}\delta_{q+\tilde{q},2}\Omega^{-1}\partial\delta(z,w)\bar{\Omega} \ ,$$

where $\delta(z,w)$ is the delta function on the Calabi–Yau manifold, defined as follows:

$$\int_M \delta(x,y)\Omega(x)\wedge\bar{\Omega}(x) = 1 \ .$$

This structure is promoted to a canonical antibracket on the space of functionals and formally may be written as follows:

$$\{F,L\} = \sum\int\left(\partial\left(\frac{\delta F}{\delta A}\right)^{\vee}\frac{\delta L}{\delta A^*} - \frac{\delta F}{\delta A^*}\partial\left(\frac{\delta L}{\delta A}\right)^{\vee}\right)^{\vee} \ .$$

It is quite remarkable that the full KS action is given by the same expression as the original KS action, but without any restrictions on the ghost numbers. Indeed, the ghost number conservation requires that either all fields in the action are elements of $\Omega^{(0,1)}(T_M)$, or at least one field has ghost number greater than 2 and therefore this field is an antifield. When all antifields are put to zero the only contribution to the action comes from the original field $A\in\Omega^{(0,1)}(T_M)$. It is a tedious but straightforward check that the full action is invariant under the nonlinear gauge transformation. The proof is based on a generalized Tian Lemma (5.6) for arbitrary (p,q) forms and the generalized Jacobi identity (5.10).

The naive definition of the Laplacian turns out to be the correct one:

$$\Delta = \int\left(\frac{\delta}{\delta A^*}\partial\left(\frac{\delta}{\delta A}\right)^{\vee}\right)^{\vee} .$$

To verify that this definition is indeed covariant one has to take into account that $\delta A_p^q(x)/\delta A_r^s(y) = \delta_{p,r}\delta_{q,s}\delta(x,y)\Omega\wedge\bar{\Omega}$. Now we can check whether the full Kodaira Spencer action $S(A,A^*)$ satisfies the master equation (5.25). The gauge invariance of the full action implies that l.h.s of (5.25) is equal to zero. The r.h.s. can be easily computed and it is equal to

$$\Delta S \sim \int \partial(\Omega A_0^1)\wedge\bar{\Omega} = 0 \ .$$

Indeed, $\partial(\Omega A_0^1) = \partial(A_0^1)' = 0$ due to constraint (5.16). The above discussion implies that quantum corrections are not needed for maintaining the gauge invariance of the KS theory.

5.5 Open string field theory. In the case of the open string, the resulting string field theories were studied in detail by [24]. There it is shown that the space–time physics of the A–model, defined on the non–compact Calabi–Yau 3–fold T^*L (where L is any *real* 3–fold), is equivalent to the usual Chern–Simons field theory on the real three–manifold L. Instead the B–model is classically equivalent to the following field theory on the original Calabi–Yau manifold M:

$$S = \frac{1}{2}\int_M \Omega \wedge \mathrm{Tr}\left(B \wedge \overline{\partial} B + \frac{2}{3} B \wedge B \wedge B\right) , \tag{5.26}$$

where the field B is a one–form on M of type $(0,1)$ taking values in End(E) and Ω is the holomorphic $(3,0)$ form. The classical solutions of (5.26) are the possible inequivalent holomorphic structures we can put on the bundle E. We thus see the space–time interpretation of the *closed* B–model string, i.e. the Kodaira–Spencer theory is very reminiscent of (5.26); in particular, the classical solutions will correspond to the possible inequivalent holomorphic structures we can put on the manifold M itself. To make this analogy even more striking it turns out that the KS action itself may be viewed as a CS action where the gauge group of the open string is replaced by an infinite dimensional group of Ω–preserving diffeomorphisms of the 3–fold. This point we will now explain.

Let us consider a 6–real–dimensional symplectic manifold M which consists of a 3-dimensional base space X and a 3-dimensional internal space Y. This symplectic manifold may be regarded as an "analytic continuation" of a Calabi-Yau manifold, where we relate the complex coordinates $(z, \overline{z})$ of the Calabi-Yau to a pair of real coordinates (x, y) of M $(x \in X, y \in Y)$. The Kähler structure on the Calabi-Yau is inherited on M as the symplectic structure, the holomorphic and the anti-holomorphic 3-forms on the Calabi-Yau become the volume forms on the base X and on the fiber Y. There is also analog of the $'$-operation on M which is realized by a multiplication by the volume form on the fiber. Consider the Lie algebra $\mathcal{L}$ of the volume preserving vector fields (satisfying condition $d_y A' = 0$) along the fiber with coefficients in 1-forms on the base. We also assume that the space of $\mathcal{L}'$ is orthogonal to H_1 on M. An invariant Killing form for this Lie algebra is given as follows:

$$TrAB = \int_Y d^3y A' \frac{1}{d_y} B' \, .$$

In this notation it is easy to see that KS action coincides with CS action for $\mathcal{L}$,

$$\lambda^2 S(A,0) = \frac{1}{2}\int_X d^3x TrA \wedge d_x A + \frac{1}{3}\int_X d^3x TrA \wedge A \wedge A \, .$$

5.6 Kodaira-Spencer theory at one–loop. In this section we will discuss the computation of Kodaira-Spencer theory partition function at one–loop. In order to do this, and in view of more general applications, we will first discuss the *holomorphic Ray-Singer Torsion.*

5.7 Holomorphic Ray-Singer torsion. Consider a Kähler manifold M with a holomorphic vector bundle V on it equipped with a norm and a connection compatible with it. Let $\overline{\partial}_V$ denote the del-bar operator coupled with the vector bundle acting [24]

$$\overline{\partial}_V : \qquad \wedge^p\overline{T}^* \otimes V \to \wedge^{p+1}\overline{T}^* \otimes V ,$$

where p runs from 0 to $\dim(M) - 1$. Let $\Delta_V = \Delta_1 + \Delta_2$ denote the corresponding Laplacian where $\Delta_1 = \overline{\partial}_V\overline{\partial}_V^\dagger$ and $\Delta_2 = \overline{\partial}_V^\dagger\overline{\partial}_V$. Let us consider the spectrum of $\Delta_V^{(p)}$ acting on $\wedge^p\bar{T}^* \otimes V$. By Hodge decomposition we can find the non-zero spectrum of the Laplacian by finding the spectra of $\Delta_1^{(p)}$ and $\Delta_2^{(p)}$. Note that the spectra of $\Delta_1^{(p)}$ and $\Delta_2^{(p-1)}$ are the same, as are the spectra of $\Delta_2^{(p)}$ and $\Delta_1^{(p+1)}$. Let us denote the spectrum of $\Delta_2^{(p)}$ by $\{\lambda_{p,p+1}\}$. In constructing a determinant of the Laplacian acting on forms of all degree it is natural to consider an alternating product of spectra raised to the power of ± 1 depending on the parity of the form, deleting the zero modes. However this will just give the net answer 1, because the spectra of Laplacian coming from $\Delta_1^{(p)}$ will cancel with those of $\Delta_2^{(p-1)}$ and those from $\Delta_2^{(p)}$ will cancel with those of $\Delta_1^{(p+1)}$. To avoid this trivial cancellation we can consider instead

$$\prod_{p=0}^{n-1}\{\lambda_{p,p+1}\}^{-(-1)^p} .$$

This can also be written, taking into account the Hodge decomposition, as

$$I(V) = \prod(\det{}'\Delta_V^{(p)})^{(-1)^p p} , \tag{5.27}$$

where $'$ denotes deleting the zero modes. The appropriately regularized $I(V)$ in the case that V is a flat bundle is known as the holomorphic Ray-Singer torsion for this vector bundle [40]. We will use the same terminology even if V is not flat. The main theorem in [40] asserts that for flat bundles $I(V_1)/I(V_2)$ is independent of the choice of Kähler metric on M though it does depend on the choice of complex structure on M (the case considered in [40] is when there are no zero modes)[25]. Morally speaking we should think of $I(V)$ as the $\prod(\det{}'\overline{\partial}_V^{(p)})^{(-1)^p}$. Note that formally one may write

$$\log I(V) = \int_\epsilon^\infty \frac{ds}{s}\mathrm{Tr}'(-1)^p p \, \exp(-sH) , \tag{5.28}$$

where the Tr' is over all degree forms in the positive eigenspace of H where $H = \Delta_V$. This integral is regularized by taking s to run from $\epsilon > 0$ to ∞.

The main technique to compute the Ray-Singer holomorphic torsion has been recently developed in connection with Quillen's holomorphic anomaly [41]. Consider a family of complex structures on M parametrized by a complex parameter t. Let us assume that there are no jumps in the zero modes of $\overline{\partial}_V$. Choose a holomorphic basis for the zero modes of $\overline{\partial}_V$ and let $d_p = \log(\det g^{(p)})$ denote the determinant of

[24] We are abusing the notation of denoting the section of the bundle and the bundle both by V.

[25] We expect that a generalization of Ray-Singer torsion exists where one relaxes the condition for the flatness of the bundles and allows arbitrary holomorphic bundles, in which case one should expect that for a fixed Kähler structure the precise choice of the Kähler metric would be immaterial for the holomorphic torsion.

the inner product in the subspace of $\wedge^p T^* \otimes V$ of the kernel of $\overline{\partial}_V$. Then it turns out that using the Quillen anomaly in this context one can show [41]

$$\partial\overline{\partial}[\log I(V)] = \partial\overline{\partial}\sum_p(-1)^p\, d_p + 2\pi i \int_M \mathrm{Td}(T)\mathrm{Ch}(V)\Big|_{(1,1)} , \tag{5.29}$$

where T is the tangent bundle of M viewed as a bundle over M times the complex moduli space, Td denotes the Todd class

$$\mathrm{Td}[T] = \det\left[\frac{R/2\pi i}{1-\exp(-R/2\pi i)}\right]$$

where R is the curvature form for the tangent bundle, and $\mathrm{Ch}(V) = \mathrm{tr}\exp(F/2\pi i)$ denotes the Chern class of the vector bundle V, viewed as a bundle over M times the complex moduli space where F is the curvature of V. The symbol $|_{(1,1)}$ in the above formula means that we take the $(n+1, n+1)$ form of the integrand and integrate over M to be left with a $(1,1)$ form on the complex moduli. The basic idea behind (5.29) is that, if we ignore the zero modes that are present, if we integrate both sides over 1 dimensional complex moduli, the left–hand side (l.h.s) gives $2\pi i$ times the total number of zero modes of $\overline{\partial}$ (weighted with $\pm$ sign) and r.h.s. is the family's index for the $\overline{\partial}_V$ operator, and thus counts precisely the same as the l.h.s. The main non-trivial content of (5.29) is that it is true even *before* integration over moduli space (this can also be argued using the integrated version by taking various interesting limits). The terms corresponding to the determinant of the norm of the zero modes in (5.29) is also familiar from the Quillen anomaly and come about because we are dealing with the determinant of Laplacian with the zero modes deleted (see e.g. [42]).

5.8 KS theory at one–loop and RS torsion. Having developed the notion of the RS torsion, we are now ready to compute the partition function of the KS theory at one–loop. In fact we will be more general as the computation can be carried out in the B–model version of any Calabi-Yau n–fold and not just the 3–fold. From the formula for F_1 given by (2.37) it is possible to extract the large volume behaviour, in which case F_L and F_R as noted in Sect. 2 are given by

$$F_{L,R} = \frac{1}{2}(i(k-k^\dagger) \pm (p-q)) ,$$

inserting $F_L F_R$ in the trace, and using the $sl(2)$ invariance of Kähler manifolds we can replace $-(k-k^\dagger)^2$ in the trace with $(p+q-n)^2$, and noting that insertions of p^2 or q^2 alone in the trace are independent of the moduli (as they would be index computations) leads us to the statement that the insertion of $F_L F_R$ is equivalent (as far as the moduli dependence is concerned) with insertion of $p \cdot q$. Now using the form of F_1 and comparison with (5.28) leads us to

$$F_1 = \frac{1}{2}\sum_q (-1)^q q \,\log I(\wedge^q T^*) .$$

Now, according to [20] we have a formula for the $\partial\overline{\partial}$ anomaly of F_1. On the other hand, using the Quillen anomaly discussed above for $I(V)$, we can compute the anomaly in another way. The fact that the two are the same is a very interesting check on these ideas, and in particular is the 'mirror' version of the conformal theory statement of the anomaly. There were two terms in the anomaly discussed in [20], as there are two terms for the anomaly (5.29). The first term in each of these two is the same, and simply is the contribution of the volume of zero modes to the anomaly. The more subtle term is the second one which comes from the contact terms both in string theory and in the computation of Quillen anomaly. As shown in [20] the second term there is $\chi(M)\cdot G/24$, where G is the Kähler form for the Zamolodchikov metric on moduli space. Therefore we wish to prove the following equation:

$$\boxed{2\pi i \int_M \mathrm{Td}(T) \sum_{p=0}^{n} (-1)^p p\, \mathrm{Ch}(\wedge^p T^*)\Big|_{(1,1)-\text{part}} = \frac{1}{12}\chi(M)\, G} \, . \tag{5.30}$$

We start by recalling a few facts [43]. First of all,

$$\mathrm{Td}(T) \sum_{p=0}^{n} (-1)^p \mathrm{Ch}(\wedge^p T^*) = c_n(T)$$

(T^* is the cotangent bundle). This is (a special case of) Theorem 10.1.1 in [43]. Now we apply the Hirzebruch argument to our case. Let γ_i be the eigenvalues of the curvature form. Consider the identity (notation as in the proof of Th. 10.1.1)

$$\sum_{p=0}^{n} (-1)^p x^p\, \mathrm{Ch}(\wedge^p \xi) = \prod_{i=1}^{n} (1 - x e^{-\gamma_i}) \, . \tag{5.31}$$

One has

$$\sum_{p=0}^{n} (-1)^p p\, \mathrm{Ch}(\wedge^p \xi) = \frac{\partial}{\partial x} \sum_{p=0}^{n} (-1)^p x^p \mathrm{Ch}(\wedge^p \xi)\Big|_{x=1} .$$

Using the identity (5.31), the r.h.s. becomes

$$n \prod_j (1 - e^{-\gamma_j}) - \sum_j \prod_{i \neq j} (1 - e^{-\gamma_i}) \, .$$

Imitating the proof of the quoted theorem, we consider ($\xi \mapsto T^*$)

$$\mathrm{Td}(T) \sum_{p=0}^{n} (-1)^p p\, \mathrm{Ch}(\wedge^p T^*) = n\, c_n(T) - \sum_j \frac{\gamma_j}{1 - e^{-\gamma_j}} \prod_{k \neq j} \gamma_k \, . \tag{5.32}$$

Now,
$$\frac{\gamma_j}{1 - e^{-\gamma_j}} = 1 + \frac{1}{2}\gamma_j + \frac{1}{12}\gamma_j^2 + \dots ,$$

where ... means higher degree. Inserting this expansion in (5.32) we get

$$\mathrm{Td}(T) \sum_{p=0}^{n} (-1)^p p\, \mathrm{Ch}(\wedge^p T^*) = \frac{n}{2} c_n(T) - c_{n-1}(T) - \frac{1}{12} c_n(T) c_1(T) + \dots \, .$$

We have to take the $(n+1, n+1)$ component of the r.h.s. which is

$$-\frac{1}{12} c_n(T)\, c_1(T) \, .$$

Now using the fact that by the discussion in Sect. 2, $c_1(T) = -c_1(T^*) = -G/2\pi i$, and the fact that $\int c_n(T) = \chi(M)$, we get (5.30) which is what we wished to show.

5.9 One–loop topological open string amplitudes. If we consider the open string version of the $N = 2$ twisted model coupled to gravity, as mentioned before, it turns out that the space of vacua is related to a choice of a holomorphic vector bundle V over M [24]. In such a case taking the large volume limit in the B-version of the model would lead us to (5.28). Thus the one-loop partition function of the open string is exactly the holomorphic Ray–Singer torsion, $F_1 = I(V)$, and is thus computable again using the Quillen anomaly (5.29).

Note in particular the computation in Sect. 4 of the one–loop open string amplitude gives the same answer as the first term in the Quillen anomaly for Ray–Singer torsion. The contact terms were not considered in Sect. 4, but since they can be computed in this field theory setup, they must be the same as the ones leading to the index integral.

5.10 The geometrical information encoded in F_g for the A–model. In this subsection we describe the geometrical information encoded in F_g for the A–model defined on a Calabi–Yau 3–fold M. As discussed in Sect. 2.2. the A–model action reads

$$S = \sum_i t^i \int (\omega_i)_{\alpha\bar\beta}\, \partial X^\alpha \bar\partial \bar X^{\bar\beta} + \sum_i \bar t^i \int (\omega_i)_{\alpha\bar\beta}\, \partial \bar X^{\bar\beta} \bar\partial X^\alpha + \text{fermions},$$

where the integral forms ω_i span $H^{1,1}(M)$. As we know from the discussion in Section 3, F_g is not a holomorphic section of $\mathcal{L}^{2-2g}$, and therefore F_g depends on a choice of a base point $\bar t^i$ in the moduli space $H^{1,1}(M, \mathbf{C})$. The meaning of F_g is particularly transparent if we choose the base point to be at infinity, i.e. to correspond to positive infinite volume. Then we set $\bar t^i = \bar t m^i$, where $\sum_i m^i \omega_i$ is a positive Kähler form ω, and then send $\bar t \to +\infty$. Of course, in this process the t^i's are still kept arbitrary. Since in the weak coupling (= infinite volume) limit the A–model correlations reduce to classical geometric objects, this is the choice of base point for which the geometric nature of F_g is more evident.

Indeed, as $\bar t \to +\infty$ the action becomes

$$S = \bar t \int \omega_{\alpha\bar\beta}\, \partial \bar X^{\bar\beta} \bar\partial X^\alpha + \dots\,. \tag{5.33}$$

Then all finite action configurations satisfy $\bar\partial X^\alpha = 0$, i.e. correspond to holomorphic maps from the Riemann surface Σ_g to the Calabi–Yau space M. Thus the g–loop amplitudes for the A–model with base point at infinity are exactly given by sums over holomorphic maps X from genus g surfaces to the Calabi–Yau space M of the form

$$F_g = \sum_n^a N^g_{n_1 n_2 \dots n_h} q_1^{n_1} q_2^{n_2} \cdots q_h^{n_h},$$

where, as in Sect. 2, $q_k = \exp[-t^k]$ and $n_k = \int X^* \omega_k$. The coefficients $N^g_{n_1 n_2 \dots n_h}$ are related to the 'number' of maps in the given topological class as we will discuss below. This means that we can use the A–model partition functions to 'count' the number of such maps, or equivalently the number of genus g holomorphic curves lying on M. This counting was done for the special cases $g = 0$ and $g = 1$ in Ref. [7] and [20], respectively.

In general, given a Riemann surface Σ_g the existence of a holomorphic map of a given degree into M depends on the complex structure of M. Because of the absence of mixed anomalies (Section 3), F_g is independent of the complex structure of M. Then in order to get the number of curves from the A–model it is crucial that we integrate over the complex moduli of Σ_g, i.e. that the A–model is coupled to topological gravity. Then the 'number' of holomorphic maps $\Sigma_g \to M$ summed over the moduli space of $\mathcal{M}_g$ is independent of the complex structure of M.

In order to extract from F_g the number of maps of a given type, we need to know for each kind of map (including multi–covers and singular ones) how the coefficient $N^g_{n_1 n_2 \ldots n_h}$ in the q–expansion of F_g is related to the actual number of holomorphic curves. This requires doing an explicit path–integral around an instanton of the given type. The rest of this section and Appendix A are dedicated to such path–integral computations. In fact, this section is rather technical. We will limit ourselves to a smooth manifold M and not deal with spaces such as orbifolds, though many of the techniques we discuss can be easily adapted to such cases. The limit $\bar{t} \to \infty$ is implicit throughout.

We recall that F_g is given by $F_g = \int_{\mathcal{M}_g} \mathcal{Z}_g$, where $\mathcal{Z}_g$ is the following top form over $\mathcal{M}_g$ (for $g \geq 2$):

$$(3g-3)!\,(2\pi i)^{3g-3}\,\mathcal{Z}_g = \\ = \left\langle \left[dm^a \Big(\int \mu_a \psi_\alpha \partial X^\alpha \Big) \bigwedge d\bar{m}^{\bar{b}} \Big(\int \bar{\mu}_{\bar{b}} \bar{\psi}_{\bar{\beta}} \bar{\partial} \bar{X}^{\bar{\beta}} \Big) \right]^{3g-3} \right\rangle_g , \tag{5.34}$$

and m^a are coordinates on $\mathcal{M}_g$ associated to the Beltrami differentials μ_a.

5.11 Contribution to F_g from an isolated genus g curve. If F_g has to 'count' the number of genus g curves lying on the Calabi–Yau manifold M, in particular it should be true that the contribution to F_g from an *isolated* such curve $\mathcal{C}_g$ is given by

$$\exp\left[-\sum_i t^i \int_{\mathcal{C}_g} \omega_i \right] , \tag{5.35}$$

with coefficient 1. Here we check explicitly this property of F_g. The assumption of $\mathcal{C}_g$ being isolated is rather unrealistic; for $g > 1$ the holomorphic curves typically belong to multi–parameter families. Below we shall drop this assumption.

Let $\mathcal{T}_g$ be the Teichmüller space of genus g curves. Clearly counting holomorphic maps $\Sigma_g(m) \to \mathcal{C}_g$ for $m \in \mathcal{M}_g$ is equivalent to counting holomorphic maps *homotopic to the identity* but with $m \in \mathcal{T}_g$. We shall take this second viewpoint[26]. We take as base point in $\mathcal{T}_g$ the point corresponding to the complex structure of $\mathcal{C}_g$ (for some choice of marking); hence for $m^a = 0$ we have a holomorphic map $\Sigma_g(0) \to \mathcal{C}_g \subset M$ homotopic to the identity. By the general argument around (5.33) the contribution from $\mathcal{C}_g$ to F_g has support at $m^a = 0$, so in the following we take m^a to be very small.

[26] It is also convenient (although not necessary) to modify the metric $\omega_{\alpha\bar{\beta}}$ in a tubular neighborhood of $\mathcal{C}_g$ such that the induced metric on $\mathcal{C}_g$ has constant curvature. We are free to do this, since a deformation of the metric preserving the Kähler class is a D–term perturbation which does not affect any topological quantities.

Our action can be rewritten as

$$S = \frac{1}{2}(t+\bar{t})\int d^2z\,\omega_{\alpha\bar{\beta}}\,\partial_\mu X^\alpha \partial^\mu \bar{X}^{\bar{\beta}} + \frac{1}{2}(t-\bar{t})\int X^*\omega + \text{fermions}]\,, \tag{5.36}$$

where ω is the Kähler form of M. We are interested in the limit $\bar{t} \to +\infty$ at t fixed. The second term in the r.h.s. of (5.36) is independent of the smooth map X, as long as its image is in the homology class of $\mathcal{C}_g$. Hence the minimum of the action in this topological class is obtained by minimizing the first term, i.e. by the corresponding harmonic map. Here we are interested only in m^a small. In this case the harmonic map has the form $X(z)^\alpha + \delta X^\alpha$, where $X(z)^\alpha$ is the map $\Sigma_g(0) \to \mathcal{C}_g$. We can decompose the variation δX^α into a component perpendicular to $\mathcal{C}_g$ and one along $\mathcal{C}_g$. The component perpendicular is an element of $H^0(\Sigma_g(0), T_M)$ and hence vanishes by the rigidity assumption. Then, to the first order, our harmonic map can still be seen as map $X\colon \Sigma_g(m) \to \mathcal{C}_g$. It is a theorem by Schoen and Yau [44] that there exists a unique harmonic map $\Sigma_g(m) \to \mathcal{C}_g$ (homotopic to the identity). Neglecting higher orders in m^a, the value of the action at the extrema is then

$$S_{\min} = \frac{1}{2}(t+\bar{t})E(m^a,\bar{m}^b) + \frac{1}{2}(t-\bar{t})d,$$

where $E(m^a,\bar{m}^b) = \int d^2z\, g_{i\bar{j}}\,\partial_\mu X^i \partial^\mu \bar{X}^{\bar{j}}$ is the Schoen–Yau "energy" as a function of the moduli and

$$d = \int X^*\omega = \frac{1}{t}\sum_i t^i \int_{\mathcal{C}_g} \omega_i,$$

is the 'degree' of $\mathcal{C}_g$.

Of course, one has

$$E(m^a,\bar{m}^b) \geq d,$$

with equality if and only if the corresponding harmonic map is holomorphic, which happens only for $m^a = 0$. The function $E(m^a,\bar{m}^b)$ is the Kähler potential for the 'Weil–Petersson' (WP) metric $W_{a\bar{b}}$ at the base point, i.e. [45]

$$\partial_a \bar{\partial}_b E\Big|_{m=0} = W_{a\bar{b}}\ .$$

More precisely, $W_{a\bar{b}}$ is the usual Weil–Petersson metric if we have chosen the metric on M so that the induced metric on $\mathcal{C}_g$ has constant curvature (see previous footnote). Otherwise, $W_{a\bar{b}}$ is some metric on $\mathcal{T}_g$; our computations below are valid for any choice of the metric.

Consider the Schoen–Yau solution $X(z,\bar{z})$. By definition, this smooth function is holomorphic with respect the complex structure defined by $\mathcal{C}_g$. Then, applying the Kodaira–Spencer machinery to the variation of the complex structure of the Riemann surface, we see that

$$\bar{\partial}X - \Phi\,\partial X = 0\,, \tag{5.37}$$

where Φ is the KS vector defining the complex structure of $\mathcal{C}_g$ in terms of that of $\Sigma_g(m)$.

Now, let us consider the derivative $\partial_{m^a}X$ of X at $m=0$. If μ_a is the Beltrami differential corresponding to an infinitesimal variation of the moduli δm^a, one has

$\partial_{m^a}\Phi = \mu_a$. Then, taking the derivative of (5.37), we get

$$\partial_{m^a}\partial_{\bar z}X\Big|_{m=0} = \mu_a\partial_z X\Big|_{m=0} + \Phi\Big|_{m=0}\partial_{m^a}\partial X = \mu_a,$$

where we used the fact that at $m = 0$ the Schoen–Yau map is the identity, i.e. $X(z)|_{m=0} = z$, and $\Phi|_{m=0} = 0$. The same argument give $\bar\partial_{\bar m^a}\bar\partial X = 0$.

Then[27]

$$\partial_a\bar\partial_{\bar b}E\Big|_{m=0} = \int_\Sigma \omega_{\alpha\bar\beta}\,\partial_{m^a}\bar\partial X^\alpha \wedge \bar\partial_{\bar m^b}\partial\bar X^{\bar\beta}\Big|_{m=0} = (\mu_b, \mu_a),$$

where $(\cdot,\cdot)$ is the Hodge inner product on $\bar K \otimes K^{-1}$ with respect to the metric $\gamma_{z\bar z}$ on Σ_g induced by the imbedding in M, i.e. $\gamma_{z\bar z} = \omega_{\alpha\bar\beta}\partial_z X^\alpha \partial_{\bar z}\bar X^{\partial\beta}$. If we choose this metric to be constant curvature, this inner product is (by definition) the Weil–Petersson metric on $\mathcal{M}_g$.

As $\bar t \to \infty$, the contribution of F_g from the curve $\mathcal{C}_g$ is concentrated at the point $m = 0$ in moduli space. Hence we can assume m to be small. In this case, one has

$$\begin{aligned} e^{-S_{\rm bos}}\Big|_{\bar t\to\infty} &\approx e^{-t\,d}\exp\Big[-\frac{1}{2}\bar t\, W_{a\bar b}\, m^a \bar m^b\Big]\Big|_{\bar t\to\infty} \\ &= \left(\frac{2\pi}{\bar t}\right)^{3g-3}\big(\det W_{a\bar b}\big)^{-1} e^{-d\,t}\,\delta(m_a)\delta(\bar m_b) \\ &= \left(\frac{2\pi}{\bar t}\right)^{3g-3} e^{-d\,t}\delta_W(m)\ , \end{aligned} \tag{5.38}$$

where $\delta_W(m)$ is covariant δ–function for the WP metric, i.e. such that

$$\int d\mu_{WP} f(m)\delta_W(m-a) = f(a)\ ,$$

where $d\mu_{WP}$ is the WP volume form.

From (5.38) we see that only the identity map contributes to the integral. In the pre–exponential factor in (5.34) we can replace $X^\alpha(z)$ by this identity map.

Let $i\colon \mathcal{C}_g \to M$ be the imbedding map and $\eta_{A\,\alpha}$ ($A = 1,\dots,3(g-1)$) be a basis[28] of $H^0(\mathcal{C}_g, K\otimes i^*T^*)$ orthonormal in the sense that

$$\int_\Sigma \omega^{\alpha\bar\beta}\,\bar\eta_{\bar B\bar\beta}\wedge\eta_{A\,\alpha} = \delta_{A\bar B}\ .$$

Let μ_a be the Beltrami's corresponding to the moduli m^a, chosen to be harmonic with respect to the metric $\gamma_{z\bar z}$. Then consider the quantity

$$\omega_{\alpha\bar\beta}\bar\mu_{\bar a}\bar\partial\bar X^{\bar\beta}dz$$

---

[27] There are two terms in the definition of $E(m,\bar m)$. However, since their difference is just the topological invariant d, for computing the variation of $E(m,\bar m)$ we can replace their sum by twice the first term.

[28] In fact, since the map $X\colon \Sigma_g(0) \to \mathcal{C}_g$ is the identity, η_A are just the ordinary quadratic differentials on Σ_g.

it belongs to $H^0(\mathcal{C}_g, K \otimes i^*T^*)$ and hence has an expansion in terms of the η_A basis of the form $\bar{B}_{\bar{a}}{}^A \eta_A$ for some coefficients $B_{\bar{a}}{}^A$.

Let us expand $\psi(z)$ as[29]

$$\psi(z) = \sqrt{t}\, \psi^A \eta_A(z)\ .$$

Then

$$dm^a\Big(\int \mu_a \psi_\alpha\, \partial X^\alpha\Big) = \sqrt{t}\, \psi^B dm^a B_a^{\bar{A}} \Big(\int \omega^{\bar{\beta}\alpha}\, \bar{\eta}_{\bar{A}\,\bar{\beta}} \wedge \eta_{B\,\alpha}\Big)$$
$$= \sqrt{t}\, dm^a B_a^{\bar{A}}\, \psi^B\, \delta_{\bar{A}B}\ .$$

Then the expression

$$\left[dm^a\Big(\int \mu_a \psi_\alpha \partial X^\alpha\Big) \bigwedge d\bar{m}^{\bar{b}}\Big(\int \bar{\mu}_{\bar{b}} \bar{\psi}_{\bar{\beta}} \bar{\partial} \bar{X}^{\bar{\beta}}\Big)\right]^{3g-3} \Bigg|_{\text{zero-modes}}$$

after the integration over the ψ zero–modes becomes

$$(3g-3)!\, |\bar{t}|^{3g-3} |\det[B]|^2 \prod dm_a d\bar{m}_{\bar{b}} = (3g-3)! |\bar{t}|^{3g-3} d\mu_{WP}\ . \tag{5.39}$$

Here the last equality follows since $|\det[B]|^2$ is nothing else than $\det[W]$, where W is the WP metric. Indeed,

$$(\mu_a, \mu_b) = \int \omega^{\alpha\bar{\gamma}} (\omega_{\alpha\bar{\beta}} \bar{\mu}_{\bar{a}} \bar{\partial} \bar{X}^{\bar{\beta}})(g_{\bar{\gamma}\delta} \mu_b \partial X^\delta)$$
$$= \bar{B}_{\bar{a}}^A B_b^{\bar{B}} \int \omega^{\bar{\gamma}\alpha} \eta_{A\ \alpha} \wedge \bar{\eta}_{\bar{B}\ \bar{\gamma}} = (\bar{B}B^t)_{\bar{a}b}\ .$$

Finally from (5.34), (5.38), and (5.39), we get

$$\mathcal{Z}_g \Big|_{\text{isolated curve}} = e^{-d\ t} \delta_W(m) d\mu_{WP}\ . \tag{5.40}$$

By definition of $\delta_W(m)$, the integral of the r.h.s. in any domain of $\mathcal{M}_g$ containing our base point $m = 0$ is just $\exp[-d\,t]$, that is the contribution of an isolated genus g curve to $\mathcal{F}_g$ is given by Eq. (5.35).

5.12 Contribution to F_g from a continuous family of curves. Typically the holomorphic maps are not isolated but belong to a family. We have to say how we 'count' instantons in this case. In general a direct path–integral computation is quite hard. However, general principles [46] lead to an abstract formula for $N^g_{n_1,\dots,n_h}$ which is valid in full generality. In the A–model on a Calabi–Yau 3–fold this formula is as follows. Assume we have a family of holomorphic maps from genus g surfaces to M,

$$f_s\colon \Sigma_g(s) \to M\ , \qquad\qquad s \in \mathcal{S}\ ,$$

where $\mathcal{S}$ is the space of parameters for the family. Over $\mathcal{S}$ we define the bundle $\mathcal{V}$ whose fiber at s is the vector space

$$\mathcal{V}\Big|_s = H^0(\Sigma_g(s), K \otimes f_s^* T_M^*)\ ,$$

[29] The factor $\sqrt{t}$ arises because physically we have to normalize the Fermi zero–modes with respect to the true metric $(t + \bar{t})g_{i\bar{j}}$, rather than with respect to the reference one $g_{i\bar{j}}$.

and let $r = \mathrm{rank}(\mathcal{V})$. Then the contribution of this family to F_g reads

$$\exp\left[-t^a \int_\Sigma f_s^* \omega_a\right] \int_{\mathcal{S}} c_r(\mathcal{V}) \,, \tag{5.41}$$

that is the coefficient is just the integral over the moduli space $\mathcal{S}$ of the Euler class of the bundle $\mathcal{V}$. It is using this abstract formulation that Aspinwall and Morrison [8] were able to prove the formula for contribution from multi–covers in genus zero that we mentioned in Sect. 2.

5.13 Contribution to F_g from constant maps. We wish to compute the limit of F_g when $\bar{t}^{\bar{\jmath}}$ *and* t^i go to infinity. The result of this computation will be needed below to fix part of the ambiguities arising in the solution of the anomaly equation. In this limit only the constant maps contribute to F_g.

The moduli space of constant maps from a genus g surface to M is given by $X = \mathcal{M}_g \otimes M$. There are three χ zero–modes spanning the fiber of the vector bundle[30] $\pi_2^* T_M$ over X, while the $3g$ ψ zero–modes span the fiber of the vector bundle

$$\mathcal{V} = \pi_1^* \mathcal{H} \otimes \pi_2^* T_M^* \,,$$

where $\mathcal{H}$ is the Hodge bundle over $\mathcal{M}_g$ (i.e. the bundle whose fiber at m is $H^0(\Sigma_g(m), K)$).

Then the general formula [46] gives

$$F_g\Big|_{t,\bar{t}\to\infty} = \int_X c_{3g}(\mathcal{V}) \,. \tag{5.42}$$

It is easy to recover (5.42) by a direct path integral computation. In order to do this, we introduce some notation. Let ω_A $(A = 1, \ldots, g)$ be a basis of holomorphic one–forms on $\Sigma_g(m)$, and the $N^{\bar{A}B}$ be the inverse matrix of $N_{A\bar{B}} = \mathrm{Im}\,\Omega_{AB} = \int \omega_A \wedge \bar{\omega}_B$. Then we put

$$A_{a\,AB} = \int (\mu_a \omega_A) \wedge \omega_B \,, \tag{5.43}$$

and $\mathcal{A}_{AB} = dm^a A_{a\ AB}$. From the theory of variations of Hodge structure (which is essentially the same thing as the tt^* equations) we know that the curvature of the Hodge bundle is $P_{A\bar{B}a\bar{b}} dm^a \wedge d\bar{m}^{\bar{b}}$, where[31]

$$P_{A\bar{B}a\bar{b}} = -A_{aAC} N^{C\bar{D}} A^*_{\bar{b}\bar{D}\bar{B}} \,.$$

Then the curvature of $\mathcal{V}$ is given by

$$\mathcal{R}_{(A,\alpha)}^{(B,\beta)} = \left[\delta_\alpha^{\ \beta} P_{A\bar{C}\,a\bar{b}} N^{\bar{C}B} dm^a \wedge d\bar{m}^{\bar{b}} + \delta_A^{\ B} R_{\alpha\bar{\gamma}\sigma\bar{\rho}}\, G^{\bar{\gamma}\beta} dx^\sigma \wedge d\bar{x}^{\bar{\rho}}\right] . \tag{5.44}$$

[30] π_i is the projection into the i–th factor space of $X = \mathcal{M}_g \otimes M$.

[31] Notice that if you interpret A_{aAB} as the 3–point function on the sphere, this is just the tt^* equation for the curvature.

As $t, \bar{t} \to \infty$ the theory gets coupled in a weaker and weaker way, and we can use perturbation theory (that is free fields). Then the derivatives of the scalars can be eliminated using the free contraction [47]

$$\langle \partial X^\alpha(z) \bar{\partial} \bar{X}^{\bar{\beta}}(\bar{w}) \rangle_g = G^{\bar{\beta}\alpha} \omega_A(z) N^{A\bar{B}} \bar{\omega}_{\bar{B}}(\bar{w}) \,. \tag{5.45}$$

We denote the $3g$ ψ zero modes, by ψ^A_α ($\alpha = 1,2,3,\ A = 1,\ldots,g$) with

$$\psi_\alpha(z) = \psi^A_\alpha \, \omega_a(z) \,. \tag{5.46}$$

In addition we have three *constant* zero–modes χ^α. The extra Fermi zero–modes are absorbed by 3 factors of ($\int R_{\alpha\bar{\beta}}{}^{\gamma\bar{\delta}} \chi^\alpha \bar{\chi}^{\bar{\beta}} \, \psi_\gamma \wedge \bar{\psi}_{\bar{\delta}}$) extracted from the exponential of the action. Using (5.46) and the definition of $N_{A\bar{B}}$ we get

$$\int R_{\alpha\bar{\beta}}{}^{\gamma\bar{\delta}} \chi^\alpha \bar{\chi}^{\bar{\beta}} \, \psi_\gamma \wedge \bar{\psi}_{\bar{\delta}} = R_{\alpha\bar{\beta}}{}^{\gamma\bar{\delta}} \chi^\alpha \bar{\chi}^{\bar{\beta}} \, \psi^A_\gamma \bar{\psi}^{\bar{B}}_{\bar{\delta}} \, N_{A\bar{B}} \,.$$

Then using (5.43) and (5.46), the contribution to (5.34) from the constant maps is reduced[32] to an integral over the zero–modes ψ^A_α, χ^α and x^α of the following quantity

$$\frac{(-i)^3}{3!(3g-3)!\,(2\pi i)^{3g}} \left[\psi^A_\alpha A_{AB} \bar{\psi}^{\bar{C}}_{\bar{\beta}} A^*_{\bar{C}\bar{D}} G^{\bar{\beta}\alpha} N^{B\bar{D}} \right]^{3g-3} \left(R_{\alpha\bar{\beta}}{}^{\gamma\bar{\delta}} \chi^\alpha \bar{\chi}^{\bar{\beta}} \psi^A_\gamma \bar{\psi}^{\bar{B}}_{\bar{\delta}} N_{A\bar{B}} \right)^3 .$$

After integrating away the χ's, we remain with the integral over the ψ's and the bosons of

$$\frac{(-1)^g}{(3g)!\,(2\pi i)^{3g}} \left(\left[G^{\alpha\bar{\beta}} P_{A\bar{B}\,a\bar{b}} dm^a \wedge d\bar{m}^b + N_{A\bar{B}} R^{\alpha\bar{\beta}}{}_{\gamma\bar{\delta}} dx^\gamma \wedge d\bar{x}^{\bar{\delta}} \right] \psi^A_\alpha \bar{\psi}^{\bar{B}}_{\bar{\beta}} \right)^{3g} . \tag{5.47}$$

Comparing with (5.34) we see that

$$\begin{aligned} \lim_{t,\bar{t}\to\infty} F_g \equiv \lim_{t,\bar{t}\to\infty} \int_{\mathcal{M}_g} \mathcal{Z}_g &= \\ = (-1)^g \int_{\mathcal{M}_g \otimes M} \det\left[\frac{\mathcal{R}}{2\pi i}\right] &= \int_{\mathcal{M}_g \otimes M} c_{3g}(\mathcal{V}) \,, \end{aligned} \tag{5.48}$$

which is Eq. (5.42).

The class $c_{3g}(\mathcal{H} \otimes T^*_M)$ can be related to the Chern classes of T_M and $\mathcal{H}$ using the 'splitting principle.' We use x_i (resp. y_a) to denote the 'eigenvalues' of the curvature of the bundle $\mathcal{H}$ (resp. T^*_M). We start from the identity

$$\begin{aligned} \prod_{i=1}^g \prod_{a=1}^3 (x_i + y_a) &= \prod_{a=1}^3 \sum_{r=0}^g y_a^{g-r} \sigma_r(x_i) \\ &= \sum_{r_1,r_2,r_3=0}^g y_1^{g-r_1} y_2^{g-r_2} y_3^{g-r_3} \sigma_{r_1}(x_i) \sigma_{r_2}(x_i) \sigma_{r_2}(x_i) \,, \end{aligned} \tag{5.49}$$

[32] As always, non–zero modes cancel by topological invariance.

where σ_r are the elementary symmetric functions. For $c_{3g}(\mathcal{V})$ we are interested in the terms in (5.49) homogeneous of degree 3 in the y's. For $g > 2$ they are given by

$$\begin{aligned}\sum_{r_1+r_2+r_3=3} & y_1^{r_1} y_2^{r_2} y_3^{r_3} \sigma_{g-r_1}(x_i)\sigma_{g-r_2}(x_i)\sigma_{g-r_3}(x_i) \equiv \\ & \equiv [\sigma_1(y)]^3 \sigma_{g-3}(x)[\sigma_g(x)]^2 + \\ & + \sigma_1(y)\sigma_2(y)[\sigma_{g-2}(x)\sigma_{g-1}(x)\sigma_g(x) - 3\sigma_{g-3}(x)\sigma_g(x)^2] + \\ & + \sigma_3(y)[\sigma_{g-1}(x)^3 - 3\sigma_{g-2}(x)\sigma_{g-1}(x)\sigma_g(x) + 3\sigma_{g-3}(x)\sigma_g(x)^2].\end{aligned}$$

In particular, this identity says that the component of $c_{3g}(\mathcal{H}\otimes T_M^*)$ which is a $(3,3)$–form on M reads

$$c_3(T_M^*)\left[c_{g-1}^3 - 3c_{g-2}\, c_{g-1}\, c_g + 3c_{g-3}\, c_g^2\right] + \text{ terms proportional to } c_1(T_M^*),$$

where $c_k \equiv c_k(\mathcal{H})$. Since for a Calabi–Yau manifold $c_1(T_M) = 0$, we have

$$\lim_{t,\bar{t}\to\infty} F_g = -\chi(M) \int_{\mathcal{M}_g} \left\{c_{g-1}^3 - 3c_{g-2}\wedge c_{g-1}\wedge c_g + 3c_{g-3}\wedge c_g^2\right\} . \tag{5.50}$$

The integral in the r.h.s. can be simplified. Indeed, as it is well known, the c_k's are not all independent. Let $\mathbf{H}$ be the de Rham bundle (i.e. the bundle over $\mathcal{M}_g$ with fiber $H^1(\Sigma_g, \mathbf{C})$). Obviously, $\mathcal{H}$ is a holomorphic subbundle[33] of $\mathbf{H}$ and we have the exact sequence

$$0 \to \mathcal{H} \to \mathbf{H} \to \mathcal{H}^* \to 0 .$$

Now, from the tt^* geometry discussed in Sect. 2, we know that $\mathbf{H}$ comes with a natural flat connection — the ∇–connection — and hence

$$1 = c(\mathbf{H}) = c(\mathcal{H})\, c(\mathcal{H}^*) .$$

Then we have the following relation between the c_k's:

$$\left(1 + \sum_{k=1}^{g} c_k\right)\left(1 + \sum_{h=1}^{g}(-1)^h c_h\right) = 1 . \tag{5.51}$$

In particular, Eq. (5.51) gives the c_{2k}'s as polynomials in the c_{2m+1}. The first few relations are

$$c_2 = \frac{1}{2}(c_1)^2, \qquad c_4 = c_1 c_3 - \frac{1}{8}(c_1)^4 . \tag{5.52}$$

Equating the two sides of Eq. (5.51) in degree $4g$ we get

$$c_g^2 = 0 , \tag{5.53}$$

while in degree $4(g-1)$ we get

$$2c_g c_{g-2} - c_{g-1}^2 = 0 .$$

[33] This is the same situation we encountered in Sect. 2.1 in the context of special geometry in Section 2.1. The trivial bundle $\mathbf{H}$ plays here the same role as $H^3(M, \mathbf{C})$ in Sect. 2.

Using these two relations Eq. (5.50) reduces to

$$\boxed{\lim_{t,\bar{t}\to\infty} F_g = \frac{1}{2}\,\chi(M) \int\limits_{\mathcal{M}_g} c_{g-1}^3 \,.} \tag{5.54}$$

The integrals $\int_{\mathcal{M}_g} c_{g-1}^3$, can be easily computed if we know the Chow ring of $\mathcal{M}_g$. In fact, by definition our c_k are represented in the Chow ring by the tautological classes λ_k (notations as in ref. [48]).

The Chow ring of $\mathcal{M}_g$ is explicitly known for $g = 2$ [48] and for $g = 3$ [49]. In particular, for $g = 2$ Th. 10.1 of ref. [48] gives

$$\int_{\mathcal{M}_2} (c_1)^3 = \frac{1}{2880}\,,$$

while for $g = 3$ using Eq. (5.52) and Table 10 of ref. [49] we have

$$\int_{\mathcal{M}_3} (c_2)^3 = \frac{1}{8}\int_{\mathcal{M}_3} (c_1)^6 = \frac{1}{8\cdot 90720} = \frac{1}{725760}.$$

Then

$$\begin{aligned} F_g\Big|_{t,\bar{t}\to\infty} &= \frac{\chi(M)}{5760} && \text{for } g = 2 \\ &= \frac{\chi(M)}{1451520} && \text{for } g = 3\,. \end{aligned} \tag{5.55}$$

6. Solution to the Anomaly Equation and Feynman Rules for F_g

In this section, we develop a systematic method to solve the holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r D_k F_{g-r} \right) . \tag{6.1}$$

6.1 Feyman rules at $g = 2, 3$. As a warm–up, let us start with the genus–2 case,

$$\bar{\partial}_{\bar{i}} F_2 = \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1 \right) . \tag{6.2}$$

Interestingly enough, a key to solving this equation lies in a genus-0 object. Because the Yukawa–coupling $\overline{C}_{\bar{i}\bar{j}\bar{k}}$ is totally symmetric in its indices and satisfies

$$D_{\bar{i}} \overline{C}_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}} \overline{C}_{\bar{i}\bar{k}\bar{l}}\,,$$

we can always *integrate* the Yukawa coupling locally as

$$\overline{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} \bar{\partial}_{\bar{k}} S\,, \tag{6.3}$$

where S is a local section of $\mathcal{L}^{-2}$. In fact, in all the examples we will discuss later, it is possible to construct S globally on the moduli space of the topological theories.

We will present such constructions later in this section[34]. To simplify the expressions below, we use the following notation:

$$S_{\bar{i}} \equiv \bar{\partial}_{\bar{i}} S \, , \qquad S_{\bar{i}}^{j} \equiv \bar{\partial}_{\bar{i}} S^{j} , \quad \text{where} \quad S^{j} \equiv G^{j\bar{j}} S_{\bar{j}} \, . \tag{6.4}$$

In this notation,

$$\overline{C}_{\bar{i}}^{jk} = \bar{\partial}_{\bar{i}} S^{jk} \, , \tag{6.5}$$

where

$$\overline{C}_{\bar{i}}^{jk} \equiv \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} , \qquad S^{jk} \equiv G^{j\bar{j}} S_{\bar{j}}^{k} \, .$$

We now solve the genus-2 equation (6.2) by "integration-by-parts." We first rewrite (6.2) using (6.5) as

$$\bar{\partial}_{\bar{i}} \left[F_2 - \frac{1}{2} S^{jk} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) \right] = -\frac{1}{2} S^{jk} \bar{\partial}_{\bar{i}} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) \, .$$

The r.h.s. can be evaluated using the holomorphic anomaly of F_1 and the special geometry relation for $[\bar{\partial}_{\bar{i}}, D_j]$ as

$$\begin{aligned} &-\frac{1}{2} S^{jk} \bar{\partial}_{\bar{i}} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) = \\ &= -\frac{1}{2} \overline{C}_{\bar{i}}^{mn} S^{jk} \left(\frac{1}{2} C_{nmjk} + C_{mnj} \partial_k F_1 + C_{jkm} \partial_n F_1 \right) + \frac{\chi}{24} S_{\bar{i}}^{j} \partial_j F_1 \, . \end{aligned}$$

Now we repeat the integration-by-parts.

$$\begin{aligned} &\bar{\partial}_{\bar{i}} \Bigg[F_2 - \frac{1}{2} S^{jk} \left(D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1 \right) + \\ &\qquad + \frac{1}{4} S^{mn} S^{jk} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^{j} \partial_j F_1 \Bigg] = \\ &= \frac{1}{4} S^{mn} S^{jk} \bar{\partial}_{\bar{i}} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^{j} \bar{\partial}_{\bar{i}} \partial_j F_1 \, . \end{aligned}$$

It turns out that the r.h.s. of this equation can also be written as a total derivative with respect to $\bar{t}^{i}$. By using the genus–1 anomaly and the special geometry, we find

$$\begin{aligned} &\frac{1}{4} S^{mn} S^{jk} \bar{\partial}_{\bar{i}} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^{j} \bar{\partial}_{\bar{i}} \partial_j F_1 = \\ &= \bar{\partial}_{\bar{i}} \Bigg[S^{jk} S^{pq} S^{mn} \left(\frac{1}{8} C_{jkp} C_{mnq} + \frac{1}{12} C_{jpm} C_{kqn} \right) - \\ &\qquad - \frac{\chi}{48} S^{j} C_{jkl} S^{kl} + \frac{\chi}{24} \left(\frac{\chi}{24} - 1 \right) S \Bigg] \, . \end{aligned}$$

[34] In refs. [50], a solution to (6.3) is constructed using particular coordinates on the moduli space. However S constructed there does not behave nicely under the modular transformation, and thus is not globally defined on the moduli space. The explicit expressions of S we obtain in various examples later differ from those obtained in these references.

Thus the iteration stops here. We have converted the genus-two anomaly equation (6.2) into the following form:

$$
\begin{aligned}
&\bar\partial_{\bar i} F_2 = \\
&= \bar\partial_{\bar i}\Bigg[\frac{1}{2} S^{jk} D_j \partial_k F_1 + \frac{1}{2} S^{jk} \partial_j F_1 \partial_k F_1 - \frac{1}{8} S^{jk} S^{mn} C_{jkmn} - \\
&\quad - \frac{1}{2} S^{jk} C_{jkm} S^{mn} \partial_n F_1 + \frac{\chi}{24} S^j \partial_j F_1 + \\
&\quad + \frac{1}{8} S^{jk} C_{jkp} S^{pq} C_{qmn} S^{mn} + \frac{1}{12} S^{jk} S^{pq} S^{mn} C_{jpm} C_{kqn} - \\
&\quad - \frac{\chi}{48} S^j C_{jkl} S^{kl} + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right) S\Bigg] .
\end{aligned}
\tag{6.6}
$$

Now one can easily integrate this equation as

$$
\begin{aligned}
F_2 = {} & \frac{1}{2} S^{ij} C^{(1)}_{ij} + \frac{1}{2} C^{(1)}_i S^{ij} C^{(1)}_j - \frac{1}{8} S^{jk} S^{mn} C_{jkmn} - \\
& - \frac{1}{2} S^{ij} C_{ijm} S^{mn} C^{(1)}_n + \frac{\chi}{24} S^i C^{(1)}_i + \\
& + \frac{1}{8} S^{ij} C_{ijp} S^{pq} C_{qmn} S^{mn} + \frac{1}{12} S^{ij} S^{pq} S^{mn} C_{ipm} C_{jqn} - \\
& - \frac{\chi}{48} S^i C_{ijk} S^{jk} + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right) S + f_2(t) ,
\end{aligned}
\tag{6.7}
$$

where we used the notation $C^{(g)}_{i_1 \cdots i_n} = D_{i_1} \cdots D_{i_n} F^{(g)}$. This equation can be expressed graphically as in Fig. 17

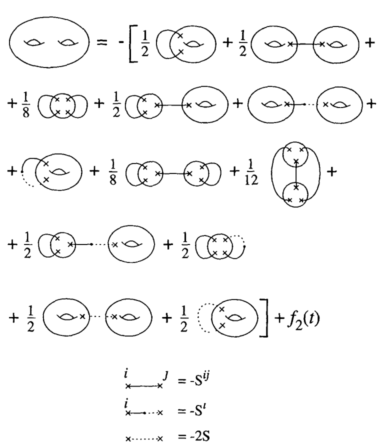

Fig. 17. The terms obtained by solving the anomaly equation for genus 2 have a strong resemblance to Feynman graphs with correct symmetry factors and with an appropriate definition of vertices. This correspondence can be made precise as discussed in the text. We identify the solid lines with the massless moduli modes and the dotted lines with the dilaton field. These graphs fix F_2 up to a holomorphic function of moduli represented by $f_2(t)$.

Here $f_2(t)$ is some meromorphic object which is not fixed at this stage. Since both F_2 and S are sections of $\mathcal{L}^{-2}$ and C_{ijk} is a section of $\mathcal{L}^2 \times \mathrm{Sym}^3 T^*$ on the moduli space, f_2 must be a meromorphic section of $\mathcal{L}^{-2}$. Although we cannot determine f_2 from the holomorphic anomaly alone, the holomorphicity gives rather stringent constraints on f_2 and, in many cases, almost uniquely determines it. In the case of the topological sigma–model, we can exploit the geometric meaning of F_2 studied in Sect. 4 to fix f_2. In the next section, we will demonstrate this procedure in various examples.

This method also works in the case of $g = 3$. After six iterations of integration-by-parts, we obtain

$$\begin{aligned} F_3 =& \frac{1}{2} S^{ij} C^{(2)}_{ij} + C^{(1)}_i S^{ij} C^{(2)}_j + (\frac{\chi}{24} + 2) S^i C^{(2)}_i + \\ &+ 2F_2 S^i C^{(1)}_i - \frac{1}{2} S^{ij} C_{ijk} S^{kl} C^{(2)}_l - \frac{1}{4} S^{ij} S^{kl} C^{(1)}_{ijkl} - \\ &- \frac{1}{2} S^{ij} C^{(1)}_{ijk} S^{kl} C^{(1)}_l - \frac{1}{4} S^{ij} S^{kl} C^{(1)}_{ik} C^{(1)}_{jl} + \\ &+ \cdots \text{(it would take five more pages to write them all)} \cdots + f_3(t) \, . \end{aligned} \tag{6.8}$$

Here $f_3(t)$ is a meromorphic section of $\mathcal{L}^{-4}$. Genus 3 contribution is presented in Fig. 18.

One may observe that Eqs. (6.7) and (6.8) have a strong resemblance to the Feynman rule (see Fig. 17). Consider a finite dimensional quantum system with $(-S^{ij})$ as a propagator connecting the indices i and j, C_{ijk}, C_{ijkl},...as classical vertices, $C^{(1)}_i$, $C^{(1)}_{ij}$,... as one-loop corrected vertices etc, and compute two- and three-loop partition functions according to the Feynman rule. If we multiply an overall factor of (-1) after the computation, we reproduce all the terms in (6.7) and (6.8) including all the symmetry factors, except for those containing S^i, S and the holomorphic sections f_2 and f_3.

The terms involving S^i and S can also be recovered if we introduce one more degree of freedom φ and extend the Feynman rule as follows. The propagators are given as

$$K^{ij} = -S^{ij}, \quad K^{i\varphi} = -S^i, \quad K^{\varphi,\varphi} = -2S \tag{6.9}$$

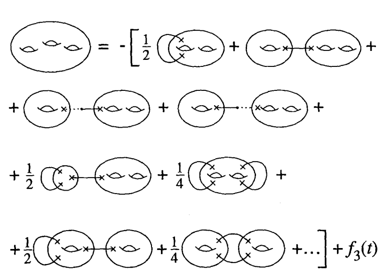

Fig. 18. Some of the Feynman graphs which emerge in solving the genus 3 anomaly equation.

and the vertices are given by

$$\widetilde{C}^{(g)}_{i_1\cdots i_n,\varphi^{m+1}} = (2g-2+n+m)\widetilde{C}^{(g)}_{i_1\cdots i_n,\varphi^m} \,,$$
$$\widetilde{C}^{(g)}_{i_1\cdots i_n} = C^{(g)}_{i_1\cdots i_n} \,, \qquad \widetilde{C}^{(1)}_{\varphi} = \frac{\chi}{24} - 1 \,, \tag{6.10}$$
$$\widetilde{C}^{(0)}_{\varphi^m} = 0, \quad \widetilde{C}^{(0)}_{i,\varphi^m} = 0, \quad \widetilde{C}^{(0)}_{ij,\varphi^m} = 0, \quad \widetilde{C}^{(1)} = 0 \,.$$

Compute two- and three-loop partition functions using this Feynman rule and multiply the overall factor of (-1) after the computation. By adding the meromorphic sections f_2 and f_3, we recover the expressions (6.7) and (6.8). The definition (6.10) of the vertices reminds us of the puncture equation in topological gravity. In fact, we will now identify the variable φ with the dilaton which is the first topological descendant of the puncture operator [4] $\sigma_1(P)$. All the other topological descendants decouple from the correlation functions simply by the $U(1)$ charge conservation and thus *the only non-vanishing correlation functions involve those of marginal fields and the dilaton field.* So far we have only discussed the marginal fields. To properly discuss the dilaton field coupling we need to enlarge the field space from that of pure topological theory. However luckily the correlation for the dilaton field can quite generally be eliminated from correlation functions by the recursion relations. In fact the first equation in (6.10) is precisely the general recursion relation of [4] and so φ is indeed the dilaton field.

6.2 Feynman rules for arbitrary g. The emergence of the Feynman rule is rather mysterious from the way we discovered it at $g = 2$ and 3. It would be extremely difficult to prove the Feynman rule for $g \geq 4$ by using the method in the above since the number of iterations would grow exponentially. Thus we will develop another technique which enables us to prove the Feynman rule directly for all g. We will do so by reducing the Feynman rule for F_g to the Schwinger–Dyson equation of the finite dimensional system. In Sect. 3, we introduced the generating function $W(\lambda, x; t, \bar{t})$ for $C^{(g)}_{i_1\cdots i_n}$ which satisfies

$$\frac{\partial}{\partial \bar{t}^{i}} \exp(W) =$$
$$= \left[\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\left(\lambda\frac{\partial}{\partial\lambda} + x^k\frac{\partial}{\partial x^k}\right)\right]\exp(W) \,.$$

To prove the Feynman rule, it is more useful to consider a generating function $\widetilde{W}(\lambda, x, \varphi, t, \bar{t})$ for the vertices $\widetilde{C}^{(g)}_{i_1\cdots i_n,\varphi^m}$ of the Feynman rule,

$$\widetilde{W}(\lambda, x, \varphi; t, \bar{t}) = \sum_{g=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\lambda^{g-1}\widetilde{C}^{(g)}_{i_1\cdots i_n;\varphi^m}x^{i_1}\cdots x^{i_n}\varphi^m \,.$$

By the definition of the vertices (6.10), $\widetilde{W}$ is related to W as

$$\begin{aligned}
&\widetilde{W}(\lambda, x, \varphi; t, \bar{t}) = \\
&= \sum_{g=0}^{\infty} \sum_{n}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} \left(\frac{1}{1-\varphi}\right)^{2g-2+n} + (\frac{\chi}{24} - 1) \log\left(\frac{1}{1-\varphi}\right) \\
&= W\left(\frac{\lambda}{1-\varphi}, \frac{x}{1-\varphi}; t, \bar{t}\right) - \left(\frac{\chi}{24} - 1\right) \log \lambda .
\end{aligned}$$

Thus $\widetilde{W}$ satisfies the equation

$$\frac{\partial}{\partial \bar{t}^{i}} \exp(\widetilde{W}) = \left[\frac{\lambda^2}{2} \overline{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi}\right] \exp(\widetilde{W}) . \tag{6.11}$$

It turns out that there is another function of x^i and φ which satisfies almost the same equation as (6.11). It is given as follows:

$$Y(\lambda, x, \varphi; t, \bar{t}) = -\frac{1}{2\lambda^2} \left(\Delta_{ij} x^i x^j + 2\Delta_{i\varphi} x^i \varphi + \Delta_{\varphi\varphi} \varphi^2\right) + \frac{1}{2} \log(\frac{\det \Delta}{\lambda^2}). \tag{6.12}$$

Here Δ is an inverse of the propagator K defined by (6.9), i.e.

$$\begin{aligned}
S^{ij} \Delta_{jk} + S^{i} \Delta_{k\varphi} &= -\delta^{i}_{k} , \\
S^{ij} \Delta_{j\varphi} + S^{i} \Delta_{\varphi\varphi} &= 0 , \\
S^{i} \Delta_{ij} + 2S \Delta_{j\varphi} &= 0 , \\
S^{i} \Delta_{i\varphi} + 2S \Delta_{\varphi\varphi} &= -1 .
\end{aligned} \tag{6.13}$$

Thus Y may be regarded as a kinetic term for the finite dimensional system of x^i and φ. The most important properties of these inverse propagators are

$$\begin{aligned}
\bar{\partial}_{\bar{i}} \Delta_{jk} &= \overline{C}_{\bar{i}}^{mn} \Delta_{mj} \Delta_{nk} + G_{\bar{i}j} \Delta_{k\varphi} + G_{\bar{i}j} \Delta_{j\varphi} , \\
\bar{\partial}_{\bar{i}} \Delta_{j\varphi} &= \overline{C}_{\bar{i}}^{mn} \Delta_{mj} \Delta_{n} + G_{\bar{i}j} \Delta_{\varphi\varphi} , \\
\bar{\partial}_{\bar{i}} \Delta_{\varphi\varphi} &= \overline{C}_{\bar{i}}^{mn} \Delta_{m\varphi} \Delta_{n\varphi} ,
\end{aligned}$$

which we can derive from (6.4) and (6.5). Just as the anomaly equations for $C^{(g)}_{i_1 \cdots i_n}$ are encoded in (6.11), the above equations for Δ's can be written as a differential equation for Y.

$$\frac{\partial}{\partial \bar{t}^{i}} \exp(Y) = \left[-\frac{\lambda^2}{2} \overline{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi}\right] \exp(Y) . \tag{6.14}$$

Now we consider the following integral.

$$Z = \int dx d\varphi \exp(Y + \widetilde{W}) . \tag{6.15}$$

Although this integral itself may be divergent, we can compute its perturbative expansion with respect to λ. The integral Z may be regarded as a partition function of a finite dimensional quantum system with dynamical variables x^i and φ, and the perturbative expansion of Z can be evaluated using the standard technique of the Feynman rule as

$$\begin{aligned}\log Z = &\lambda^2\left[F_2 - \frac{1}{2}S^{ij}C^{(1)}_{ij} - \frac{1}{2}C^{(1)}_i S^{ij}C^{(1)}_j + \cdots\right] + \\ &+\lambda^4\left[F_3 - \frac{1}{2}S^{ij}C^{(2)}_{ij} - C^{(1)}_i S^{ij}C^{(2)}_j + \cdots\right] + \\ &+\lambda^6\left[F_4 - \frac{1}{2}S^{ij}C^{(3)}_{ij} - C^{(1)}_i S^{ij}C^{(3)}_j - \frac{1}{2}C^{(2)}_i S^{ij}C^{(2)}_j + \cdots\right] + \\ &+\cdots+\lambda^{2g-2}\left[F_g - \frac{1}{2}S^{ij}C^{(g-1)}_{ij} - \frac{1}{2}\sum_{r=1}^{g-1}C^{(r)}_i S^{ij}C^{(g-r)}_j + \cdots\right] + \cdots ,\end{aligned} \tag{6.16}$$

where $(\cdots)$ in the coefficient of λ^2 represents the terms in the r.h.s. of (6.7), $(\cdots)$ in the coefficient of λ^4 represents those in (6.8), and so on.

Previously we found, by the iterative method, that the coefficients of λ^2 and λ^4 in the perturbative expansion of Z are holomorphic in t. We can now prove the holomorphicity of Z to all order in the perturbation as the Schwinger–Dyson equation of the finite dimensional system. By using (6.11) and (6.14), we obtain

$$\begin{aligned}\frac{\partial}{\partial\bar{t}^i}Z = &\int dx d\varphi\, e^Y\left[\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j\partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right]e^{\widetilde{W}} + \\ &+\int dx d\varphi\, e^{\widetilde{W}}\left[-\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j\partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right]e^Y \\ =&\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\int dx d\varphi\left[\frac{\partial}{\partial x^j}\left(e^Y\frac{\partial}{\partial x^k}(e^{\widetilde{W}})\right) - \frac{\partial}{\partial x^j}\left(e^{\widetilde{W}}\frac{\partial}{\partial x^k}(e^Y)\right)\right] - \\ &- G_{\bar{i}j}\int dx d\varphi\,\frac{\partial}{\partial\varphi}\left[x^j e^{Y+\widetilde{W}}\right] .\end{aligned}$$

The point is that the integrand in the r.h.s. of this equation is the total derivative with respect to x^i and φ. In the perturbative expansion, we are free to perform the integration-by-part and drop boundary terms since integrals involved in the perturbation are all Gaussian. Thus we have derived

$$\frac{\partial}{\partial\bar{t}^i}Z = 0 .$$

As is evident from the expansion (6.16), the holomorphicity of Z means that we can express F_g as a meromorphic section f_g of $\mathcal{L}^{2-2g}$ minus a sum over the Feynman graphs constructed from the propagators (6.9) and the vertices (6.10).

6.3 Construction of propagators. So far we have assumed that there is a global section S of $\mathcal{L}^{-2}$ which satisfies (6.3). Now we are going to construct such an object. The important ingredient is again the special geometry relation

$$R_{i\bar{j}l}{}^{k} = -\overline{\partial}_{\bar{j}}\Gamma^k_{il} = G_{i\bar{j}}\delta^k_l + G_{k\bar{j}}\delta^k_i - C_{ilm}\overline{C}_{\bar{j}}^{km} . \tag{6.17}$$

Since $G_{i\bar{j}} = \partial_i \overline{\partial}_{\bar{j}} K$, this can be rewritten as

$$\overline{\partial}_{\bar{i}}[S^{jk} C_{klm}] = \overline{\partial}_{\bar{i}} \left[\partial_l K \delta^j_m + \partial_m K \delta^j_l + \Gamma^j_{lm} \right] .$$

This can be easily integrated as

$$S^{ij} C_{jkl} = \delta^i_l \partial_k K + \delta^i_k \partial_l K + \Gamma^i_{kl} + f^i_{kl} , \tag{6.18}$$

where f^i_{kl} is some meromorphic object which should compensate for the non-covariance of $\partial_k K$ and Γ^i_{kl} in the r.h.s. We can express f^i_{kl} as

$$f^i_{kl} = \delta^i_l \partial_k \log f + \delta^i_k \partial_l \log f - \sum_{a=1}^{n} v_{l,a} \partial_k v^{i,a} + \widetilde{f}^i_{kl} ,$$

where f is a meromorphic section of $\mathcal{L}$, $\{v^{i,a}\}_{a=1,\ldots,n}$ (n is the dimensions of the moduli space) are meromorphic tangent vectors which are linearly independent almost everywhere on the moduli space, $v_{i,a}$ are inverse of $v^{i,a}$ ($\sum_a v_{i,a} v^{j,a} = \delta^i_j$) and $\widetilde{f}^i_{kl}$ is a meromorphic section of $T \times \mathrm{Sym}^2 T^*$. In general, (6.18) has $\frac{1}{2} n^2(n+1)$ equations for $\frac{1}{2} n(n+1)$ variables S^{ij} and it is over-determined when $n > 1$. Thus we should make an appropriate choice of $\widetilde{f}^i_{kl}$ to ensure that (6.18) is solvable with respect to S^{ij}.

The situation is much simpler in the one-modulus case since there is only one equation in (6.18) and there is no constraint on $\widetilde{f}^1_{11}$. In order to construct F_g by using the Feynman rule, (6.7) and (6.8) for example, we do not need the most general solution to $\overline{\partial} S^{11} = \overline{C}^{11}_{\bar{1}}$ since any holomorphic ambiguity in S^{11} is absorbed into the holomorphic section f_g which we add to F_g at the end of the computation. Thus we can, for example, choose $\widetilde{f}^1_{11} = 0$. With this choice, S^{11} becomes

$$S^{11} = \frac{1}{C_{111}} \left[2\partial \log(e^K |f|^2) - (G_{1\bar{1}} v)^{-1} \partial (v G_{1\bar{1}}) \right] . \tag{6.19}$$

To find S^i, we need to integrate

$$\overline{\partial}_{\bar{i}} S^j = G_{\bar{i}k} S^{jk} .$$

Substituting (6.19) into this, we obtain

$$\begin{aligned} \overline{\partial} S^1 &= \frac{1}{C_{111}} \left[2\partial \log(e^K |f|^2) G_{1\bar{1}} - v^{-1} \partial (v G_{1\bar{1}}) \right] \\ &= \frac{1}{C_{111}} \overline{\partial} \left[(\partial \log(e^K |f|^2))^2 - v^{-1} \partial (v \partial K) \right] . \end{aligned}$$

A special solution to this equation can be easily found as

$$S^1 = \frac{1}{C_{111}} \left[\left(\partial \log(e^K |f|^2) \right)^2 - v^{-1} \partial \left(v \partial \log(e^K |f|^2) \right) \right] . \tag{6.20}$$

Finally we need to find S which satisfies

$$\bar{\partial} S = G_{1\bar{1}} S^{\bar{1}} \ . \tag{6.21}$$

A special solution to this equation is given by

$$\begin{aligned} S = &\left[S^1 - \frac{1}{2} D_1 S^{11} - \frac{1}{2} (S^{11})^2 C_{111} \right] \partial \log(e^K |f|^2) + \\ &+ \frac{1}{2} D_1 S^1 + \frac{1}{2} S^{11} S^1 C_{111} \ . \end{aligned} \tag{6.22}$$

Let us check that this indeed satisfies (6.21). We first note that the following special combination of S^1 and S^{11} is holomorphic

$$\begin{aligned} &\bar{\partial} \left[S^1 - \frac{1}{2} D_1 S^{11} - \frac{1}{2} (S^{11})^2 C_{111} \right] = \\ &= G_{1\bar{1}} S^{11} - \frac{1}{2} [\bar{\partial}, D_1] S^{11} - \frac{1}{2} (G^{1\bar{1}})^2 \partial \overline{C}_{\bar{1}\bar{1}\bar{1}} - \overline{C}_{\bar{1}}^{11} S^{11} C_{111} \\ &= 0 \ , \end{aligned}$$

where we used the special geometry relation[35] (6.17), the definitions of S^1 and S^{11} and $\partial \overline{C}_{\bar{1}\bar{1}\bar{1}} = 0$. Now it is straightforward to check Eq. (6.21) as

$$\begin{aligned} \bar{\partial} S = &G_{1\bar{1}} S^1 - \frac{1}{2} G_{1\bar{1}} D_1 S^{11} - \frac{1}{2} G_{1\bar{1}} (S^{11})^2 C_{111} + \\ &+ \frac{1}{2} [\bar{\partial}, D_1] S^1 + \frac{1}{2} D_1 S^1_{\bar{1}} + \frac{1}{2} \overline{C}_{\bar{1}}^{11} S^1 C_{111} + \frac{1}{2} S^{11} S^1_{\bar{1}} C_{111} \\ = &G_{1\bar{1}} S^1 + \frac{1}{2} [\bar{\partial}, D_1] S^1 + \frac{1}{2} \overline{C}_{\bar{1}}^{11} S^1 C_{111} \\ = &G_{1\bar{1}} S^1 \ . \end{aligned}$$

Here we once again used the special geometry relation[36].

To summarize, in the one-modulus case, the propagators S^{11} , S^1, and S are given as

$$\begin{aligned} S^{11} &= \frac{1}{C_{111}} \partial \log \left[2 \partial \log(e^K |f|^2) - (G_{1\bar{1}} v)^{-1} \partial (v G_{1\bar{1}}) \right] \ , \\ S^1 &= \frac{1}{C_{111}} \left[\left(\partial \log(e^K |f|^2) \right)^2 - v^{-1} \partial \left(v \partial \log(e^K |f|^2) \right) \right] \ , \\ S &= \left[S^1 - \frac{1}{2} D_1 S^{11} - \frac{1}{2} (S^{11})^2 C_{111} \right] \partial \log(e^K |f|^2) + \\ &\quad + \frac{1}{2} D_1 S^1 + \frac{1}{2} S^{11} S^1 C_{111} \ . \end{aligned}$$

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[35] $D_1 S^{11} = (\partial - 2\Gamma_1^{11} - 2\partial K) S^{11}$. Therefore $[\bar{\partial}, D_1] S^{11} = 2 G_{1\bar{1}} S^{11} - 2 C_{111} \overline{C}_{\bar{1}}^{11} S^{11}$.

[36] $D_1 S^1 = (\partial - \Gamma_1^{11} - 2\partial K) S^1$. Therefore $[\bar{\partial}, D_1] S^1 = -C_{111} \overline{C}_{\bar{1}}^{11} S^1$.

In the multi-moduli case, (6.18) gives

$$S^{ij}C_{jkl} = (\delta^i_l\partial_k + \delta^i_k\partial_l)\log(e^K|f|^2) - \sum_{a=1}^{n} v_{l,a}G^{i\bar{i}}\partial_k(v^{m,a}G_{m\bar{i}}) + \widetilde{f}^i_{kl} \,. \tag{6.23}$$

In order to obtain an expression for S^{ij} from this equation, we need to "invert" the Yukawa coupling. Although we do not know if it is possible to do so in general, it is certainly possible for the A-model discussed in Sect. 4. In this model, each chiral field corresponds to a Kähler form in the target space and, in the large volume limit, the Yukawa coupling C_{ijk} is given as an intersection of the three Kähler forms. There is a distinguished Kähler modulus t^1 in this model corresponding to an overall scaling of the target space metric. In the large volume limit $t^1 \to \infty$, the Yukawa coupling C_{ij1} then gives the inner product of the two Kähler forms k_i and k_j, and it is non-degenerate as an $n \times n$ matrix, $\det(C_{ij1})_{i,j=1,\ldots,n} \neq 0$. Since $\det(C_{ij1})$ is holomorphic in t, this means that $\det(C_{ij1})$ should be non-zero almost everywhere on the moduli space. Therefore we can invert C_{ij1} in (6.23) to find an expression for S^{ij}, provided we made an appropriate choice of $\widetilde{f}^i_{kl}$.

As in the one-modulus case, we substitute (6.23) into $\overline{\partial}_{\bar{i}}S^j = G_{\bar{i}i}S^{ij}$ to obtain

$$\begin{aligned}\overline{\partial}_{\bar{i}}[S^j]C_{jkl} =& G_{\bar{i}l}\partial_k\log(e^K|f|^2) + G_{\bar{i}k}\partial_l\log(e^K|f|^2)+\\ &- \sum_{a=1}^{n} v_{l,a}\partial_k(v^{m,a}G_{\bar{i}m}) + \widetilde{f}^i_{kl}G_{\bar{i}i} \,.\end{aligned}$$

This can be easily integrated as

$$\begin{aligned}S^iC_{ijk} =& \partial_j\log(e^K|f|^2)\partial_k\log(e^K|f|^2) - \sum_{a=1}^{n} v_{k,a}\partial_j\left[v^{l,a}\partial_l\log(e^K|f|^2)\right] +\\ &+ \widetilde{f}^l_{jk}\partial_l\log(e^K|f|^2) + \widetilde{f}_{jk} \,.\end{aligned} \tag{6.24}$$

Here $\widetilde{f}_{jk}$ is a meromorphic section of Sym^2T^*. As in the case of S^{ij} in (6.23), with an appropriate choice of $\widetilde{f}_{jk}$, we can invert the Yukawa coupling in the above and obtain an expression for S^i.

To complete the Feynman rule, we need S which satisfies

$$\overline{\partial}_{\bar{i}}S = G_{\bar{i}i}S^i \,. \tag{6.25}$$

A special solution to this equation is given by

$$\begin{aligned}S =& \frac{1}{2n}\left[(n+1)S^i - D_jS^{ij} - S^{ij}S^{kl}C_{jkl}\right]\partial_i\log(e^K|f|^2)+\\ &+ \frac{1}{2n}\left(D_iS^i + S^iS^{jk}C_{ijk}\right) \,.\end{aligned} \tag{6.26}$$

Let us check that this satisfies (6.25). As in the case of one-modulus, the following combination of S^i and S^{ij} is holomorphic due to the special geometry relation and

$\overline{\partial}_{\bar{i}} C_{jkl} = 0,$

$$\begin{aligned}
&\overline{\partial}_{\bar{i}} \left[(n+1)S^j - D_k S^{jk} - S^{jk}S^{mn}C_{jmn}\right] = \\
&= (n+1)G_{\bar{i}k}S^{jk} - [\overline{\partial}_{\bar{i}}, D_k]S^{jk} - G^{j\bar{j}}G^{k\bar{k}}\partial_k \overline{C}_{\bar{i}\bar{j}\bar{k}} - \\
&\qquad - \overline{C}_{\bar{i}}^{jk} S^{mn} C_{kmn} - \overline{C}_{\bar{i}}^{mn} S^{jk} C_{kmn} \\
&= 0\,.
\end{aligned}$$

We can then compute $\overline{\partial}_{\bar{i}} S$ as

$$\begin{aligned}
\overline{\partial}_{\bar{i}} S =& \frac{n+1}{2n} S_{\bar{i}} + \\
&+ \frac{1}{2n}\left([\overline{\partial}_{\bar{i}}, D_j]S^j + D_j S_{\bar{i}}^j + S_{\bar{i}}^j S^{kl} C_{jkl} + S^j \overline{C}_{\bar{i}}^{kl} C_{jkl} - S_{\bar{i}}^j S^{kl} C_{jkl}\right) \\
=& G_{\bar{i}j} S^j\,.
\end{aligned}$$

Here again, we used the special geometry relation for $[\overline{\partial}_{\bar{i}}, D_j]$.

Thus we have prepared all the ingredients we need for the Feynman rule of F_g. In the next section, we will construct F_g explicitly in several examples.

7. Examples — The Experimental Evidence

In this section we show how to compute a higher loop partition function F_g (for small g) for some examples. We will elaborate in details how the perturbation theory developed in previous section works. The simplest and most trivial example would be a three dimensional complex torus. In this case there is nothing to compute. All loop partition functions are identically equal to zero due to fermion zero modes. The simplest way to get a non-zero answer is to orbifoldize the model. Below we will consider two examples — the $\mathbf{Z}_3 \otimes \mathbf{Z}_3$ orbifold model and the quintic — in detail, and comment on some other models also at the end.

7.1. Orbifold. Let us start with some definitions. The $\mathbf{Z}_3 \otimes \mathbf{Z}_3$ orbifold is obtained by dividing $T^2 \times T^2 \times T^2$, with each torus having a $\mathbf{Z}_3$ symmetry, by the discrete group generated by $g = \mathrm{diag}\,(1, \omega, \omega^2)$ and $h = \mathrm{diag}\,(\omega, \omega^2, 1)$. This model has 3 untwisted Kähler moduli corresponding to the moduli of each of the tori and 81 corresponding to the blow up modes. This orbifold is rigid and has no complex moduli. The Euler characteristic $\chi = 168$. We will denote the Kähler moduli of each of the three tori by τ_a $(a = 1, 2, 3)$. The Kähler potential is given as follows:

$$e^{-K(\tau_i, \bar{\tau}_i)} = i \prod_{i=1}^{3} (\tau_i - \bar{\tau}_i)\,.$$

The only non zero component of Yukawa coupling is $C_{123} = 1$. Zamolodchikov's metric is diagonal and is equal to

$$G_{a\bar{b}} = -\frac{\delta_{ab}}{(\tau_a - \bar{\tau}_a)^2}\,.$$

We also need the genus one partition function, which is equal to

$$F_1 = -\kappa \sum_a log(\tau_a - \bar{\tau}_a)|\eta^2(\tau_a)|^2 \quad ,$$

where $\kappa = 4$ for this orbifold. In spite of the fact that it is easy to solve the equations for F_2 and F_3 directly, we first review the ingredients of perturbation technique. In case of orbifold the equations for different components of propagator S^{ab}, S^a and S are very simple,

$$\bar{\partial}_c S^{ab} = -\frac{1}{(\tau_c - \bar{\tau}_c)^2} \quad , \quad \bar{\partial}_b S^a = -\frac{S^{ab}}{(\tau_b - \bar{\tau}_b)^2} \quad , \quad \bar{\partial}_a S = -\frac{S^a}{(\tau_a - \bar{\tau}_a)^2} \quad , \tag{7.1}$$

where (abc) is a permutation of (123) and $S^{ab} = 0$ for $a = b$. Integrating these equations we obtain

$$\begin{aligned} S^{ab} &= -\left(\frac{1}{(\tau_c - \bar{\tau}_c)} + 2\frac{\eta'(\tau_c)}{\eta(\tau_c)}\right) , \\ S^a &= \left(\frac{1}{(\tau_b - \bar{\tau}_b)} + 2\frac{\eta'(\tau_b)}{\eta(\tau_b)}\right)\left(\frac{1}{(\tau_c - \bar{\tau}_c)} + 2\frac{\eta'(\tau_c)}{\eta(\tau_c)}\right) , \\ S &= -\prod_a \left(\frac{1}{(\tau_a - \bar{\tau}_a)} + 2\frac{\eta'(\tau_a)}{\eta(\tau_a)}\right) , \end{aligned} \tag{7.2}$$

where (abc) is a permutation of (123). At every integration step the holomorphic piece was fixed by modular invariance. For example, integrating the equation for S^{ab} we obtain $S^{ab} = -1/(\tau_c - \bar{\tau}_c)^2 + f(\tau_c)$. The untwisted moduli space of Kähler structures for this orbifold is the product of three copies of fundamental domain in the upper half plane modulo a symmetry group. The condition of modular invariance fixes $f(\tau) = 2\eta'(\tau)/\eta(\tau)$. Similar arguments lead to the answers (7.2). It is easy to verify that all diagrams give rise to the same type of contribution and therefore F_2 is proportional to S (this is in fact a peculiarity of the orbifold example). One can also solve the equation for genus two directly. In this case the anomaly equation reads

$$\bar{\partial}_a F_2 = -\frac{1}{2}\frac{1}{(\tau_a - \bar{\tau}_a)^2}\partial_b F_1 \partial_c F_1 \; .$$

Taking into account the explicit form of F_1, one can easily integrate the above equation[37]

$$F_2 = \frac{1}{2\kappa}\prod_a \partial_a F_1 = \frac{\kappa^2}{2}\prod_a \left(\frac{1}{(\tau_a - \bar{\tau}_a)} + 2\frac{\eta'(\tau_a)}{\eta(\tau_a)}\right) \; .$$

[37] If we considered the non-abelian orbifold obtained by the above one modded out by a further symmetry which permutes the three tori, we would have ended up with one untwisted modulus and then F_g would be a modular function of weight $6g - 6$ with respect to this modulus. In this case even at genus 2 we would have to fix the coefficient of holomorphic contribution to F_2 as there is a modular form of weight 6.

The equation for genus 3 is given as follows:

$$\begin{aligned}\bar\partial_a F_3 =& \frac{1}{2}\frac{1}{(\tau_a-\bar\tau_a)^2}[(\partial_b+\frac{2}{\tau_b-\bar\tau_b})(\partial_c+\frac{2}{\tau_c-\bar\tau_c})F_2+ \\ &\partial_b F_1(\partial_c+\frac{2}{\tau_c-\bar\tau_c})F_2+\partial_c F_1(\partial_b+\frac{2}{\tau_b-\bar\tau_b})F_2]\ .\end{aligned} \tag{7.3}$$

As usual (abc) is a permutation of (123). After substitution the genus-two solution obtained above, Eq. (7.3) becomes

$$\begin{aligned}\bar\partial_a F_3 =& \frac{1}{4\kappa}\frac{1}{(\tau_a-\bar\tau_a)^2}\partial_a F_1[(\partial_b+\frac{2}{\tau_b-\bar\tau_b})\partial_b F_1(\partial_c+\frac{2}{\tau_c-\bar\tau_c})\partial_c F_1+ \\ &(\partial_b F_1)^2(\partial_c+\frac{2}{\tau_c-\bar\tau_c})\partial_c F_1+(\partial_c F_1)^2(\partial_b+\frac{2}{\tau_b-\bar\tau_b})\partial_b F_1]\ .\end{aligned}$$

The expression in brackets does not depend on τ_a and therefore we only need to solve the equation

$$\bar\partial_a f = \frac{1}{4\kappa}\frac{1}{(\tau_a-\bar\tau_a)^2}\partial_a F_1\ . \tag{7.4}$$

We must be careful at this point since the solution is not unique. The reason for this is the existence of modular form of weight four $\eta''(\tau)/\eta(\tau)-3(\eta'(\tau)/\eta(\tau))^2$. The general solution for (7.4) is given as follows:

$$f=\sum_{a=1}^{3} x(\partial_a F_1)^2+(\frac{1}{8\kappa}-\kappa x)(\partial_a+\frac{2}{\tau_a-\bar\tau_a})\partial_a F_1\ ,$$

where x is an arbitrary parameter. The condition of permutation symmetry forces the coefficients in front of the two terms to be equal to each other. As a result we obtain a fully symmetric solution for genus-three partition function

$$\begin{aligned}F_3 =& \frac{1}{8\kappa(\kappa+1)}\sum_{a=1}^{3}(\partial_a+\frac{2}{\tau_a-\bar\tau_a})\partial_a F_1(\partial_{a+1}+\frac{2}{\tau_{a+1}-\bar\tau_{a+1}})\partial_{a+1}F_1(\partial_{a+2}F_1)^2+ \\ &+C_0\prod_{a=1}^{3}\left(\frac{\eta''(\tau_a)}{\eta(\tau)}-3\left(\frac{\eta'(\tau_a)}{\eta(\tau)}\right)^2\right)\ ,\end{aligned}$$

where C_0 is an arbitrary constant and it can not be determined from the anomaly equation.

This phenomenon persists at every genus whenever there is a modular form of appropriate weight. Unfortunately we do not know the asymptotic behavior of F_g for the orbifold to fix the ambiguity. An analysis along the line of Sect. 5 done for the orbifolds would be needed to fix this ambiguity.

7.2 Quintic. Quintic hypersurface can be described as the vanishing locus of a homogeneous polynomial of degree 5 of five variables $W(x_\imath)=0$ which determines the embedding of complex 3–fold in $\mathbf{P}^4$. This Calabi Yau 3–fold has 101 complex moduli, all these moduli can be thought as the coefficients of the polynomial, and 1 Kähler moduli, which can be thought as the Kähler class of $\mathbf{P}^4$.

To construct the mirror manifold one starts with a 1-parameter subfamily of quintic hypersurfaces given by

$$W(x_i) \equiv \sum x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \ .$$

All these M_ψ hypersurfaces are invariant under the discrete group $\mathbf{Z}_5^3$. Following the construction of [51] one may obtain the mirror family W_ψ by dividing out the discrete symmetries.

The mirror manifold W has only 1 complex modulus and 101 Kahler moduli. One can describe the mirror family W_ψ using the same complex parameter ψ as for M_ψ (for construction of mirror map see [7]). The variations of ψ can be identified with deformations of complex structure of the mirror W. The multiplication of ψ by a fifth root of unity $\alpha = e^{2i\pi/5}$ can be always undone by appropriate change of variables x_i and therefore $\psi \to \alpha\psi$ is a modular transformation. All physical observables are invariant under $\psi \to \alpha\psi$. The modular parameter ψ describes a degenerate Calabi–Yau 3–fold only for $\psi = 1$ and $\psi = \infty$. For $\psi = 1$ the corresponding Calabi–Yau is conifold, while for $\psi = \infty$ the corresponding Calabi–Yau manifold is a singular quintic. In spite of the fact that ψ and $\alpha\psi$ correspond to the same complex structure the point $\psi = 0$ is a regular point corresponding to one of Gepner's model.

The purpose of this section is to present the computations of numbers of holomorphic curves of low genus in the quintic hypersurface. To be more precise there are no holomorphic isolated curves for genus bigger than one. The numbers we compute are in fact the Euler characteristics of the corresponding families as discussed in Sect. 5.10. We will follow the following logic in this section. We first compute the elements of the diagram technique for fixed ψ but in the limit $\bar{\psi} \to \infty$. We discuss the holomorphic ambiguity by requiring regularity $F_g(\psi)$ everywhere except $\psi = 1$ and $\psi = \infty$. Then by making a mirror transform and expanding in instantons we extract the numbers in question.

The holomorphic three form Ω is taken in the gauge

$$\Omega = 5\psi \frac{x_4 dx_0 dx_1 dx_2}{\partial W/\partial x_3} \ .$$

In the same gauge the Yukawa coupling is equal to

$$C_{\psi\psi\psi} = -\int \Omega \wedge \frac{\partial^3 \Omega}{\partial \psi^3} = \left(\frac{2\pi i}{5}\right)^3 \frac{5\psi^2}{1-\psi^5} \ .$$

Different components of the propagator are expressed in terms of the Kähler potential, Zamolodchikov's metric and two sections $f \in \mathcal{L}$ and $v \in T^*$ (see formulas (6.19), (6.20) and (6.22)). The Kähler potential always enters into invariant combination $e^K|f|^2$, while the metric enters in the invariant combination $G_{\psi\bar{\psi}}|v|^2$. As ψ goes to 0, e^K diverges as $|\psi|^{-2}$, while the metric remains finite. The condition of regularity at the origin implies that f should necessarily have a zero at $\psi = 0$, while v remains finite. The regularity condition at the origin and the absence of any additional singularities except possibly at $\psi = \infty$ and $\psi = 1$ implies the following ansatz for f and v

$$f(\psi) = \psi(1-\psi^5)^a \quad , \quad v(\psi) = (1-\psi^5)^b \ ,$$

where a and b are some constants. The precise choice of these sections is irrelevant since any holomorphic ambiguity can be reabsorbed into the section f_2 which we add to the final answer F_2. From the general formulas (6.19), (6.20) and (6.22) one can immediately deduce that for small ψ

$$S^{\psi\psi}\left(\frac{\partial}{\partial\psi}\right)^2 \sim \psi^2\left(\frac{\partial}{\partial\psi}\right)^2 \,, \quad S^{\psi}\frac{\partial}{\partial\psi} \sim \psi\frac{\partial}{\partial\psi} \quad \text{and} \quad S \sim const\,.$$

The behavior of the perturbation series near singularity $\psi = 1$ follows from the general arguments presented in Sect. 5 (see Eq. (5.24)). In order to apply the formula (5.24) we need to find the canonical coordinate near $\psi = 1$ which is an interesting example of how different a canonical coordinate can be from the special coordinate. In fact the canonical coordinate at this point is just

$$t \sim -\log(1-\psi^5)\,,$$

as can be seen from the fact that in this coordinate Γ^t_{tt} and all its holomorphic derivatives go to zero as $\psi \to 1$. Taking into account the explicit form of the Yukawa coupling which in the ψ coordinate behaves as

$$C_{\psi\psi\psi} \sim \frac{1}{(1-\psi^5)}$$

and that

$$C_{ttt} = \left[\frac{\partial\psi}{\partial t}\right]^3 C_{\psi\psi\psi} \sim (1-\psi^5)^2\,,$$

and using formula (5.24) we find that

$$F_g \sim \frac{\left[\partial_t^3 C_{ttt}\right]^{2g-2}}{\left[\partial_t C_{ttt}\right]^{3g-3}} \sim \frac{a_g}{(1-\psi^5)^{2g-2}}$$

as $\psi \to 1$.

To discuss the large $\bar\psi$ limit let us recall that special coordinates of special geometry are nothing else but canonical coordinates around infinity. One may regard the mirror map $\psi \to t$ as a transformation to canonical coordinates. Using the general properties of canonical coordinates we conclude that Zamolodchikov's metric $G_{\psi\bar\psi}$ and Kähler potential $K(\psi,\bar\psi)$ have the following expansion:

$$\begin{aligned} G_{\psi\bar\psi}d\bar\psi =& C\frac{dt}{d\psi}\frac{d\bar\psi}{\bar\psi^2} + o(\bar\psi^{-3})\,,\\ K(\psi,\bar\psi) =& -\log\varpi_0(\psi) + o(\bar\psi^{-1})\,, \end{aligned}$$

where C is some constant and $\varpi_0(\psi)$ is the solution of Picard-Fuchs equation (we are following the notations of [7]). The passage to canonical coordinate implies the change of gauge in such a way that all holomorphic derivatives of K vanish. Namely

$$K \longrightarrow K + \log\varpi_0 + const$$

(see the discussion at the end of Sect. 2). The choice of $const$ is equivalent to the choice of string coupling constant, and we will choose it in such a way that Yukawa coupling has an integral expansion ($const = 3\log(2\pi i/5)$).

In computing the higher genus amplitudes of this example it is convenient first to take the limit $\bar{t} \to \infty$ while fixing t. This is useful because in this limit as discussed in Sect. 5.10, there is some information about the behaviour of F_g (as counting of holomorphic maps of genus g in Calabi–Yau). We use this correspondence to fix the holomorphic ambiguity in integrating the anomaly equation.

To consider the $\bar{t} \to \infty$ we use the results discussed at the end of Sect. 2 to simplify the formulas for different components of the propagator. Indeed, plugging these expansions into (6.19) and (6.20) we obtain the following result:

$$S^{\psi\psi} = \left(\frac{5}{2\pi i}\right)^3 \frac{1-\psi^5}{5\psi^2} \partial_\psi \log\left(\frac{dt}{d\psi} v \left(\frac{f}{\varpi_0}\right)^2\right) ,$$

$$S^{\psi} = \left(\frac{5}{2\pi i}\right)^3 \frac{1-\psi^5}{5\psi^2} \left[\left(\partial_\psi \log(f/\varpi_0)\right)^2 + v^{-1}\partial_\psi v \partial_\psi \log(f/\varpi_0)\right] .$$

There is not much simplification in the expression for S. Namely,

$$S = \left[S^\psi - \frac{1}{2}D_\psi S^{\psi\psi} - \frac{1}{2}(S^{\psi\psi})^2 C_{\psi\psi\psi}\right] \partial_\psi \log(f/\varpi_0) + \frac{1}{2}D_\psi S^\psi + \frac{1}{2}S^{\psi\psi}S^\psi C_{\psi\psi\psi} .$$

In the large volume limit ($\psi \to \infty$) the propagators $S^{\psi\psi} \sim \psi^2$, $S^\psi \sim \psi$ and $S \sim const$ and therefore all F_g go to $const$.

The genus zero and one have already been discussed in [7] [20] respectively. So we consider the genus 2 for which the techniques developed in this paper are crucial. The genus two partition function is given by Eq. (6.7)

$$F_2 = \left(\frac{1}{2}S^{\psi\psi}C^1_{\psi\psi} + \frac{1}{2}C^1_\psi S^{\psi\psi}C^1_\psi - \frac{1}{8}S^{\psi\psi}S^{\psi\psi}C_{\psi\psi\psi\psi} + \ldots\right) + f(\psi) ,$$

where $f(\psi)$ is holomorphic ambiguity. The most general form of the holomorphic ambiguity consistent with the asymptotic behavior of F_g is given as follows:

$$f_2(\psi) = A + \frac{B}{(1-\psi^5)} + \frac{C}{(1-\psi^5)^2} . \tag{7.5}$$

Now we are almost done. We just need to transform F_2 to canonical coordinate t and canonical section for the bundle. Note that F_2 is a section of a line bundle $\mathcal{L}^{-2}$. Taking into account the change in the gauge in going to canonical coordinates we obtain F_2

$$F_2(\psi) \longrightarrow \left(\left(\frac{2\pi i}{5}\right)^3 \varpi_0(\psi(t))\right)^2 F_2(\psi(t)) . \tag{7.6}$$

Note that the ambiguities in the choice of the sections f and v given by two coefficients a and b should simply shift F_2 by a holomorphic function and thus should be possible to absorb in A, B and C. That this should be possible leads to a strong check both for the Feynman graph techniques discussed in Sect. 6 in solving the F_g, as well as for the computer code we wrote. So we set $a = b = 0$ and we are thus left to fix the three unknown coefficients A, B and C.

To do this we need to know the structure of instanton expansion. First of all there are no genuine genus two curves of degree 1, 2 and 3. The contribution of degree 1, 2 and 3 comes entirely from the bubbling of the sphere or a torus (in case of degree 3). We take into account these bubblings and demand that the denominators of coefficients of such terms can at most be $1/5760$ consistent with some characteristic class computation on moduli space of genus 2. Now it is natural to expect that after subtraction of these contributions the rest of the expansion be with integral coefficients as would follow from 'counting' holomorphic curves. Of course there is no guarantee that this is correct to impose, because indeed there are continuous families of holomorphic curves and we are computing the appropriate Euler characters as discussed in Sect. 5, and these could be fractional if the corresponding moduli space has orbifold points. Anyhow to proceed we assume that at least in the case of the quintic these coefficients are integral and we end up uniquely fixing all the coefficients. We obtain $A = -71375/288$, $B = -10375/288$, $C = 625/48$, and get

$$F_2(q) = -\frac{5}{144} + \frac{1}{240}\sum_n^{\infty} \frac{d_n q^n}{(1-q^n)^2} + \sum_r D_r q^r \ , \tag{7.7}$$

where d_n counts the number of holomorphic rational curves of degree n, D_n counts the number of holomorphic curves of genus 2. We found that there is no toroidal bubbling, which can also be argued on physical grounds[38]. In the above search we *did not* impose by hand the large volume behavior $t, \bar{t} \to \infty$ computed in Sect. 5. Indeed it was shown there that the leading term should be $\chi(M)/5760$ which in our case is $-5/144$ in agreement with what we found, thus lending further support to the assumptions we made in fixing the coefficients of bubbling. Moreover the number $1/240$ is also very natural as it is minus the Euler character of moduli space of genus 2 curves. It would be very interesting to understand this. Also the structure of the multi-bubbling is very simple, though different from what has been encountered in genus 0 [7] and 1 [20]. It would also be important to derive this structure. At any rate the results of this computation for D_n are summarized in Table 1.

As we have just seen the knowledge of the instanton expansion allows us to fix holomorphic ambiguity. Holomorphic ambiguity at genus g can be written as follows:

$$f_g(\psi) = \sum_{g=0}^{2g-2} \frac{A_g}{(1-\psi^5)^{2g-2}} \ .$$

In general there are $2g-1$ unknown parameters. To fix this ambiguity uniquely one needs to know the precise structure of the instanton expansion. What is lacking in particular is how the lower genera contribute to genus g (bubbling). Even if this is fixed, to completely fix the A_g we need to know the first few coefficients for the number of holomorphic curves of genus g to fix all the rest.

The asymptotic behavior of $D_n(g)$ (i.e. the coefficient of asymptotic expansion for large n and fixed g) is determined by the structure of singularity around $\psi = 1$. As was argued the asymptotic behavior of F_g as $\psi \to 1$ is given by

$$F_g(t) \to \frac{A_{2g-2}}{(1-\psi^5)^{2g-2}} \left(\left(\frac{2\pi i}{5}\right)^3 \varpi_0(\psi) \right)^{2g-2} \ .$$

[38] If there were toroidal bubbling then we would end up with a moduli space which has as a factor a torus. However, since we have to bring down factors of curvature from the action to absorb the fermion zero modes, and since the curvature vanishes for the torus, we just get zero.

Table 1. # curves of genus g on quintic hypersurface

Degree	$g=0$	$g=1$
n=0	5	50/12
n=1	2875	0
n=2	609250	0
n=3	317206375	609250
n=4	242467530000	3721431625
n=5	229305888887625	12129909700200
n=6	248249742118022000	31147299732677250
n=7	295091050570845659250	71578406022880761750
n=8	375632160937476603550000	154990541752957846986500
n=9	503840510416985243645106250	324064464310279585656399500
...	...	...
large n	$a_0 n^{-3}(\log n)^{-2} e^{2\pi n\alpha}$	$a_1 n^{-1} e^{2\pi n\alpha}$
Degree	$g=2$	g
n=0	-5/144	$-100 \cdot [c^3_{g-1}]$
n=1	0	
n=2	0	
n=3	0	
n=4	534750	
n=5	75478987900	
n=6	871708139638250	
n=7	518546255661726925	
n=8	900673642524236753450000	
n=9	325859687147358266010240500	
...	...	...
large n	$a_2 n(\log n)^2 e^{2\pi n\alpha}$	$a_g n^{2g-3}(\log n)^{2g-2} e^{2\pi n\alpha}$

The last factor $((2\pi i/5)^3 \varpi_0)^{2g-2}$ is nothing else but the gauge transformation. In the limit $\psi \to 1$ this factor tends to a constant and therefore it does not affect the asymptotic behavior. On the other hand the structure of singularity around $\psi = 1$ is dictated by asymptotic behavior of $D_n(g)$ coefficients. Assuming the reasonable ansatz $D_n(g) \sim n^\rho (\log n)^\sigma e^{2\pi n t(1)}$ we immediately get[39]

$$F_g(\psi) \sim \int dn\, n^\rho (\log n)^\sigma e^{-2\pi n(t(\psi)-t(1))} \sim$$
$$\sim \left(\frac{1}{\psi-1}\right)^{\rho+1} \left[\log(\psi-1)\right]^{\sigma-\rho-1} .$$

Comparing the last two formulas we obtain $\rho = 2g-3$ and $\sigma = 2g-2$. Thus the asymptotic behavior of $D_n(g)$ is given as follows:

$$D_n(g) = a_g n^{2g-3} (\log n)^{2g-2} e^{2\pi n\alpha} , \tag{7.8}$$

where a_g and $\alpha = t(1)$ are constants which are not universal in the sense that they depend on the manifold under consideration. Morally speaking the degree of the map n coincides with the notion of the area of the embedding measured in some

[39] The contribution from lower genera is subleading for $\psi \to 1$.

units. In this interpretation D_n is nothing else but a fixed area partition function. Asymptotic dependence (7.8) of $D_n(g)$ on the degree of the map is the same as the area dependence for the $c = 1$ model coupled to gravity. It even reproduces correctly the logarithmic scaling violation [52], which is specific for the $c = 1$ model. This fact is not very surprising; $\hat{c} = 3$ $N = 2$ topological models are closely related to the $c = 1$ model coupled to gravity. In fact it has been shown that a particular $\hat{c} = 3$ twisted $N = 2$ theory is equivalent to the $c = 1$ model coupled to gravity [53]. To see the logarithmic scaling violation, consider F_g as a function of cosmological constant Δ which can be identified with $2\pi(t - \alpha)$. For large areas (n) one can replace the summation by the integral

$$F_g(\Delta) \sim \int dn\, n^{2g-3}(\log n)^{2g-2} e^{-n\Delta} \sim \left(\frac{\Delta}{\log \Delta}\right)^{2-2g} ,$$

The real scaling behavior is determined not by Δ but by $\mu = \Delta/\log\Delta$ exactly like in the $c = 1$ model. The Δ dependence of F_g coincides with t-dependence (up to irrelevant shift and rescaling). Then the logarithmic scaling violation is entirely due to the structure of the canonical map around $\psi = 1$. Indeed, $(\psi - 1) \sim (t-\alpha)/\log(\psi - 1) \sim (t - \alpha)/\log(t - \alpha) \sim \mu$ around $\psi = 1$ (where t here is the canonical coordinate defined for $\bar{t} \to \infty$).

In cases where there are more than one Kähler moduli, fix a direction in the Kähler cone of $H^{1,1}(M, Z)$. Let us denote this direction by $(n_1, ..., n_r)$, where n_i are integers. For large n the asymptotic behavior for $D_{n\cdot(n_1,\dots,n_r)}(g)$ for fixed $(n_1, ..., n_r)$ is thus expected to be given by the expression (7.8). The exact values a_g and α clearly depend on M and the direction chosen in $H^{1,1}$, while the powers $2g - 3$ and $2g - 2$ are expected to be universal. It would be important to check this conjecture in full generality.

7.3 Other examples of Calabi-Yau models. Here we briefly describe the results of genus two calculations for some other Calabi-Yau models. Let us first consider some hypersurfaces in projective spaces. These Calabi–Yau spaces are described as the vanishing loci of quasihomogeneous polynomials which describe (up to deformation) the embedding of Calabi–Yau 3–folds in a weighted projective space

$$\begin{aligned}
k = 5 \ &: \qquad W_0 = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \ , \\
k = 6 \ &: \qquad W_0 = 2z_0^3 + z_1^6 + z_2^6 + z_3^6 + z_4^6 = 0 \ , \\
k = 8 \ &: \qquad W_0 = 4z_0^2 + z_1^8 + z_2^8 + z_3^8 + z_4^8 = 0 \ , \\
k = 10 &: \qquad W_0 = 5z_0^2 + 2z_1^5 + z_2^{10} + z_3^{10} + z_4^{10} = 0 \ .
\end{aligned} \tag{7.9}$$

These models were earlier investigated in connection with $g = 0$ holomorphic maps in [54, 55]. The higher genus computations for these models are parallel to the quintic case. The most general holomorphic ambiguity consistent with asymptotic behavior is given by (7.5) (with 5 replaced by k). Again to fix the ambiguity we must know some additional data (the large volume behavior of genus two partition function fixes only A). In case of the quintic we knew that there are no genuine genus two curves of degree 1, 2 and 3. Now there is no such information available. It is known that there are families of genus two curves of degree 1 and 2 for cases $k = 6$ and $k = 8$. There are no reasons to believe that their contribution to genus two partition function is zero. We denote their contributions by N and M respectively.

As in the quintic case the genus two partition function has the structure

$$F_2(q) = \frac{\chi(M)}{5760} + \frac{1}{240}\sum_n^{\infty} \frac{d_n q^n}{(1-q^n)^2} + \sum_r D_r q^r \ .$$

The coefficient $1/240$ in front of the spherical bubbling is universal and independent of the model. We found that after subtraction the genus zero contribution (bubbling) $\tilde{F}_2(q) = \chi(M)/5760 + \sum_r D_r q^r$ has almost the integral expansion, except for the $k = 6$ model. The results of calculations are summarized in the following q-expansions:

$$\begin{aligned}\tilde{F}_2^{k=6}(q) = &-\frac{17}{480} + N\,q + M\,q^2 + \\ &+ (14735432142 + 18504\,M - 97465842\,N)\,q^3 + \\ &+ \left(\frac{512439449683401}{2} + 239228316\,M - 1652255019168\,N\right)\,q^4 + \\ &+ (3199366969602589296 + 2654549098512\,M - \\ &\qquad - 20399446637531235\,N)q^5 + \\ &+ (34720817411136316872780 + 27042685856051310\,M - \\ &\qquad - 219919127006205233856\,N)q^6 + \cdots\end{aligned} \tag{7.10}$$

for the $k = 6$ model,

$$\begin{aligned}\tilde{F}_2^{k=8}(q) = &-\frac{37}{720} + N\,q + Mq^2 + \\ &+ (2297430758208 + 102816\,M - 2982239872\,N)\,q^3 + \\ &+ (222468094578584808 + 7410413536\,M - 282015713196032\,N)\,q^4 + \\ &+ (15516453237414083197120 + 459069253511168\,M - \\ &\qquad - 19447231842568395440\,N)\,q^5 + \\ &+ (941762378252908894389530784 - 26129248919673002880\,M - \\ &\qquad - 1171714563944600408125440\,N)\,q^6 + \cdots\end{aligned} \tag{7.11}$$

for $k = 8$ and

$$\begin{aligned}\tilde{F}_2^{k=10}(q) = &-\frac{1}{20} + N\,q + M\,q^2 + \\ &+ (2869664890712800 + 1271200\,M - 447052624000\,N)\,q^3 + \\ &+ (3508008133715103890200 + 1143497004000\,M - \\ &\qquad - 529021878501120000\,N)\,q^4 + \\ &+ (3098620653232515436678572256 + 887703919048960000\,M - \\ &\qquad - 457872639654043275150000\,N)\,q^5 + \\ &+ (2385179845759540102344438070862400 + 634572439637621668400000\,M - \\ &\qquad - 346846888907287393959739633664\,N)\,q^6 + \cdots\end{aligned} \tag{7.12}$$

for $k = 10$. In fact we checked that all coefficients are integer up to q^{10}, except for the coefficient q^4 in the $k = 6$ model (provided that N and M are integers). This in particular suggests that there must be continuous families of holomorphic maps in this case where they have at least $\mathbf{Z}_2$ orbifold points, and they contribute a $1/2$ to the coefficient of q^4. It would be interesting to verify this.

Another example which is amusing is the $\mathbf{Z}_3$ orbifold which is obtained by modding out $T^2 \times T^2 \times T^2$ by a diagonal (ω, ω, ω). In this case explicit computation of F_1 shows that it is zero. Now the anomaly formula for F_2 implies that F_2 is purely holomorphic, and indeed this is exactly zero as can be seen by a direct computation of the orbifold model at all g. This in particular means that even though there was room for F_g to be non-zero consistent with the anomaly equation, as there are appropriate holomorphic functions, nevertheless it vanishes.

There are other interesting models that one may wish to consider. A particularly interesting class is where there are *no* marginal operators in the twisted theory. This can happen, for example, in the context of B–models for Calabi–Yau which are rigid. In such cases the F_g is simply a number (up to multiplication by the string coupling constant λ^{2g-2}), and summing over all g will lead to a function $F(\lambda)$. This may be an easier case to study. In particular since there are no marginal directions, there are no anomalies either. A simple realization of this type of model is again given by the $\mathbf{Z}_3 \times \mathbf{Z}_3$ orbifold model discussed in this section, but with the B–twist instead of the A–twist. The $\mathbf{Z}_3$ orbifold in the B–twist is also rigid but in this case one can show again by explicit computation that $F_g = 0$.

Note that in the A–model twisting and for smooth manifolds, we can compute F_g for all g up to exponentially small corrections, in the limit of large volume, as $F_g \to \frac{1}{2}\chi(M)[c_{g-1}^3]$, in terms of some cohomology computation on the moduli of Riemann surfaces. In particular if $\chi(M) \neq 0$ (and barring an accidental zero of $[c_{g-1}^3]$) we see that $F_g \neq 0$. It would be interesting in this connection to study Calabi–Yau manifolds with $\chi = 0$, as this argument also shows that $F_g = 0$ up to exponentially small terms in the large volume limit.

8. Physical Implications of Topological Amplitudes

One of the main motivations to study $N = 2$ SCFT's comes from the fact that they serve as building blocks for string vacua. In this connection particular objects which have natural interpretations for the $N = 2$ SCFT's turn out to also have some interesting phenomenological implications in string models. One such object is the Yukawa coupling. If one considers heterotic strings compactified on a Calabi–Yau 3–fold, with gauge connection identified with the spin connection of Calabi–Yau, then the chiral primary fields of charge 1 give rise to massless generations and the chiral ring coefficients C_{ijk} give the Yukawa couplings between the different generations. Given the fact that Yukawa couplings are simply the three point function of topological gravity, it is natural to expect that all the other computations of twisted $N = 2$ theories coupled to gravity also have similar physical significance for an appropriate string theory. In particular we would like to discuss the significance of F_g in connection with standard string theories. Before we discuss this let us note where we could look for such contributions in the effective field theories arising from string theory.

Let us note that the massless fields $t^i(x^\mu)$ in general end up as the lowest component of chiral superfields from the spacetime point of view. In general in supersymmetric theories we can have F-terms, i.e., superpotential terms, which involve only chiral superfields, i.e. are holomorphic functions in t^i. Now morally we expect F_g to be a holomorphic function of t^i (ignoring the holomorphic anomaly) and so we expect that F_g is a contribution to a superpotential. This observation, together with the fact that F_g is a section of a particular bundle essentially fixes what term we get in the effective Lagrangian. However instead of guessing we will show this more directly below. So before we proceed further, the dictionary we expect is

$$\text{Topological Computations} \leftrightarrow F-\text{terms in field theories.}$$

We will discuss both the case of closed and open strings. Afterwards we consider the computation of threshold corrections for heterotic strings at one-loop and its relation to F_1 and Ray–Singer torsion. The case of the closed string has also been recently discussed in detail by [56].

8.1 Type II string Interpretation. We start by asking which string theory F_g should be related with? Given the fact that it is left–right symmetric, and it is related to the twisting of a supersymmetric sigma–model for closed string theory, one is naturally led to consider type II strings compactified from 10 to 4 on a $\hat{c} = 3$ internal theory. We thus are searching for low energy effective field theory terms that F_g is computing. Compactifying type II on $\hat{c} = 3$ theory gives rise to a low energy field theory in four dimensions with $N = 2$ supergravity. The chiral fields t^i are scalar fields for this supergravity (for aspects of $N = 2$ supergravities that one obtains by compactifying on Calabi-Yau manifolds see [57]). The $N = 2$ supergravity multiplet in particular contains a Maxwell field which is called gravi–photon. We will denote the field strength for this field by T. This field arises from the Ramond–Ramond sector of type II string and the vertex operator for this field, in the limit of vanishing momentum $k \longrightarrow 0$ is proportional to

$$V_T^{\pm\pm} = k_{\pm\pm} S^{\pm} \overline{S}^{\pm} \sigma \overline{\sigma} e^{(-\phi/2)} ,$$

where ϕ is part of the bosonized β, γ field [58] and where S $(\overline{S})$ denote the left–moving (right-moving) 4d spinor vertex operators and σ $(\overline{\sigma})$ denotes the unique vertex operator for the left–moving (right–moving) charge $3/2$ (3/2) Ramond vacuum state for the internal $N = 2$ theory (with $\hat{c} = 3$). Indeed this vertex operator is the same as the FMS [58] spin operator (taking into account the fact that the internal theory is a general $\hat{c} = 3$ rather than flat space). Note that σ, S and $\exp(-\phi/2)$ (together with their right–moving counterparts) generate the spectral flow from the NS sector to the R sector. We will use this vertex operator to go between the twisted theory and the untwisted theory.

There are a number of differences between the twisted theory and the ordinary type II strings. First of all there are more fields in the ordinary theory. In addition to an *untwisted* $N = 2$ SCFT with $\hat{c} = 3$, in type II strings we have the fermionic diffeomorphism ghosts (b, c) of spin $(2, -1)$, the bosonic super–diffeomorphism ghosts (β, γ) of spin $(3/2, -1/2)$, and the space–time fields, which we take to be two complex bosons X^i of spin 0 and two complex fermions ψ^i and their conjugates $\chi_{\bar{i}}$ of spin $1/2$ with $i = 1, 2$. Of course the same content of fields is needed for the right-moving part which we denote by barred fields. If we could *twist* the 1/2 integral spin fields by half a unit, then their spins would be the same as the integral spin fields but with opposite statistics, so they would tend to cancel out of the partition function. In addition we would need to twist the internal $N = 2$ theory which is also the same as shifting the 1/2-integral fermion spins of the internal theory. Both of these can be accomplished by insertion of $(2g-2)$ vertex operators for gravi-photon V_T^{++} (modulo some subtleties mentioned below). The way to see this is that the spin content of fields can be changed by addition to the action of

$$\frac{1}{2}\int R\varphi\,, \tag{8.1}$$

where φ denotes the bosonized version of the fields. We can choose the curvature R to have delta–function like support at $2g-2$ points. But each such point is equivalent to the insertion of V_T^{++} as mentioned before. However, to write it in a conformally meaningful way, given that V_T^{++} is dimension $(1,1)$ we have to integrate it over the surface (which is equivalent to choosing the delta-function support for R by averaging over all points); we thus have found the dictionary that

$$\Big\langle\Big[\int V_T^{++}\Big]^{2g-2}\cdots\Big\rangle_{\text{untwisted}} = \langle\cdots\rangle_{\text{twisted}}\,. \tag{8.2}$$

This means that the determinant of non-zero modes of the extra fields which were not in the original twisted internal $N=2$ theory cancel out, leaving us with the twisted internal theory. However, we have to pay particular attention to the zero modes of the extra fields we have introduced: There are zero modes for b,β and the ψ,χ system that have to be absorbed in order for the partition function not to vanish. Let us first deal with the ghost zero modes.

The b zero modes give rise to the measure over moduli space. In fact if μ_i denote the basis for Beltrami-differentials, we have to insert in the superstring measure a factor of

$$\Big|b(\mu_1)\dots b(\mu_{3g-3})\Big|^2$$

to absorb the b zero modes. For the β zero modes we usually have to insert $2g-2$ factors of $\delta(\beta)\cdot G$, where G is the $N=2$ supersymmetry current for the *full* theory. But that is true for the partition function with no operators inserted. In our case inserting $2g-2$ vertex operators V_T^{++} which are in the $-1/2$ picture means that we need to insert $3g-3$ factors of $\delta(\beta)$. Moreover the fact that β,γ is effectively twisted means that we can choose the same basis for the Beltrami differentials to fold with them. Moreover by charge conservation for the internal twisted theory only the G^- component of the internal theory gives non-vanishing amplitude, so we end up with

$$\Big|\delta\big(\beta(\mu_1)\big)\cdots\delta\big(\beta(\mu_{3g-3})\big)\Big|^2|G^-(\mu_1)\cdots G^-(\mu_{3g-3})|^2\,.$$

With this choice of Beltrami-differentials the zero modes of b and $\delta(\beta)$ give opposite contribution and thus b,c and β,γ completely drop out of the picture, having left us with the twisted $N=2$ theory with $3g-3$ insertions of G^- which is precisely the prescription we had for computing F_g of the twisted string coupled to gravity. However we still have to get rid of the space–time fermion zero modes. There are g of ψ^i (which has spin 1) and one of $\chi_{\bar i}$ (which has spin zero) zero mode for each i (and similarly for the right movers). To absorb the χ zero mode and one ψ zero mode we can insert the operator

$$\epsilon_{ij}\epsilon_{i'j'}\epsilon_{\bar i\bar j}\epsilon_{\bar i'\bar j'}\int\psi^i\chi_{\bar i}\overline{\psi}^{i'}\overline{\chi}_{\bar i'}\int\psi^j\chi_{\bar j}\overline{\psi}^{j'}\overline{\chi}_{\bar j'}\,. \tag{8.3}$$

Note that up to factors of momentum, this operator is precisely the insertion of two graviton vertex operators. We are left to absorb $g-1$ extra zero modes of ψ^i.

Taking into account that after twisting ψ^i has spin 1, one is tempted to introduce $g-1$ operators of the form $\int \psi^i \overline{\psi}^j$, but unfortunately this does not have a well-defined meaning as a vertex operator for the untwisted theory. Instead, motivated by a suggestion from the authors of [56], we can make the insertion of $g-1$ of $\psi^1\psi^2\overline{\psi}^1\overline{\psi}^2$ operators at $g-1$ of the points where we have taken the delta–function curvature singularity. This choice will have the property of absorbing the unwanted ψ zero modes, without getting an operator which does not make sense in the untwisted theory. This is because choosing this position for the $g-1$ curvature singularities will convert $g-1$ of V_T^{++} to V_T^{--} which is the vertex for gravi–photon field with opposite self–duality property. In this way we can absorb all the zero modes and end up with F_g. We thus see, putting all this together, that

$$F_g = \Big\langle \Big[\int V_T^{++}\Big]^{g-1}\Big[\int V_T^{--}\Big]^{g-1} \epsilon_{ij}\epsilon_{i'j'}\epsilon_{\overline{i}\overline{j}}\epsilon_{\overline{i}'\overline{j}'} \int \psi^i\chi_{\overline{i}}\overline{\psi}^{i'}\overline{\chi}_{\overline{i}'} \int \psi^j\chi_{\overline{j}}\overline{\psi}^{j'}\overline{\chi}_{\overline{j}'}\Big\rangle \,. \tag{8.4}$$

Putting the momentum factors this means that F_g is the coefficient in the low energy effective action for a term of the form $R^2(T^2)^{g-1}$. This completes the derivation of the relation between topological partition function and field theory. However we should note that in the above derivation we were somewhat careless in some points: We assumed that we can twist fields simply by adding $\frac{1}{2}\int R\varphi$ term to the action, but as is well known this is true up to boundary terms. The boundary terms are in fact responsible for picking which point on the Jacobian of the twisted field we end up with (i.e. the choice of the flat bundle) — we have to make sure that we end up with the trivial flat bundle tensored with the appropriate power of the canonical bundle. Secondly, a point which is related to this, is the fact that we have to sum over spin structures in the untwisted theory. Somehow this is already taken into account in the twisting, because viewing the twisting as choosing a background gauge field set equal to half the gauge connection is ambiguous up to a choice of a $\mathbf{Z}_2$ bundle, which is just the choice of spin structure. This ambiguity should translate to a sum over spin structure to get a correspondence between the twisted and untwisted theory. To make sure that these points do not affect our argument one will have to go to more detail and check the explicit factors arising in the twisting. Fortunately this has been considered very carefully in [56] using bosonization techniques which confirms the above heuristic arguments.

As argued at the beginning of this section we should expect a term in the superpotential which gives rise to the effective action of the form $R^2(T^2)^{g-1}$. In fact one can find an F-term which gives rise to such a term:

$$[F_g(\mathcal{W}^2)^g]_F \tag{8.5}$$

where $\mathcal{W}^2$ is the square of the Weyl superfield ($\mathcal{W}^2$ is a composite chiral superfield of weight 2), and $[\cdots]_F$ is the F–density for conformal $N=2$ supergravity. Notice that this coupling makes sense since, F_g is a section[40] of $\mathcal{L}^{2-2g}$, which — in the language of conformal $N=2$ tensor calculus [14] — means that it is a chiral field of weight $2-2g$, so that the combination $F_g(\mathcal{W}^2)^g$ has weight 2 and hence defines an invariant F–term [14].

[40] In more traditional terms, F_g is represented in superspace by a homogeneous function of the vector fields X_I of weight $2-2g$. The F_g we use throughout the paper is obtained from this homogeneous function by choosing a gauge for the line bundle $\mathcal{L}$.

However, Eq. (8.5) makes sense only if F_g is a chiral superfield, which happens only if F_g is a holomorphic function of the chiral fields t^i. But as discussed in Sect. 3, F_g is not holomorphic because of anomalies. Then (8.5) cannot be the correct form of the supergravity coupling corresponding to the amplitudes we discussed above. However we have to recall how one deals with a field theory which has flat directions, as is the case here. In such cases there are *inequivalent* vacua determined by what the expectation value of the massless fields are. Suppose we have chosen such an expectation value, which we denote by $(t_0, \bar{t}_0)$. Then we can expand F_g *holomorphically* about this base point. What this means is that we consider (in canonical coordinates)

$$F_g(x + t_0, \bar{t}_0) = \sum_i \frac{1}{n!} x^{i_1} ... x^{i_n} D_{i_1} ... D_{i_n} F_g(t_0, \bar{t}_0) =$$

$$= \sum_i \frac{1}{n!} x^{i_1} ... x^{i_n} \partial_{i_1} ... \partial_{i_n} F_g(t_0, \bar{t}_0) \ .$$

Thus F_g is now a *holomorphic* function of superfields x^i, and we are thinking of $(t_0, \bar{t}_0)$ as a base point for expansion of F_g and not as a superfield. This view of the effective Lagrangian we are presenting is motivated from the fact that in the construction of solutions to the anomaly equation, discussed in Sect. 5, a function W was introduced which was a *holomorphic* function of x^i. So in particular we end up with the superpotential for the $N = 2$ supergravity, including all loop contributions:

$$\sum_g [F_g(x + t_0, \bar{t}_0) \mathcal{W}^2 (\lambda \mathcal{W}^2)^{g-1}]_F = [\mathcal{W}^2 \, W(\lambda \mathcal{W}, x; t_0, \bar{t}_0)]_F \ , \tag{8.6}$$

where here λ^{-1} is a section of $\mathcal{L}^{-1}$ and plays the role of compensating field in the supergravity theory [14] (one–loop contribution can also be included here by addition of a term proportional to $\log \lambda \mathcal{W}$).

8.2 Open superstring interpretation. As discussed in the previous section, we can also consider the twisted $N = 2$ theory for the open strings. It is also natural to ask what is the interpretation of the F_h^g in the low energy effective theory of some superstring theory. The natural superstring theory to look for in this context is the 10-dimensional open superstrings compactified on an internal $N = 2$ SCFT with $\hat{c} = 3$. This theory gives rise to a 4-dimensional low energy theory of $N = 1$ supersymmetric Yang-Mills coupled to supergravity. Actually as is well known to get a consistent theory we need to consider unoriented strings. Also if we wish to avoid anomalies we need to take the gauge group $O(32)$ which brings us to one of the most interesting superstring theories. Our considerations in the following will also apply to the more general gauge group of $O(N)$.

Unoriented strings will have worldsheets which include both orientable and non-orientable surfaces. Let us concentrate on the contribution from orientable surfaces which we have discussed for the twisted $N = 2$ theories. To simplify further let us first consider the case with no handles $g = 0$ with h boundaries. We will use the same idea as in the closed string case, in other words add the extra fields which are present in the superstring compared to the $N = 2$ twisted topological model, and then put appropriate insertions to twist the $\frac{1}{2}$-integral spin fields to obtain integral fields which cancel among each other except for zero modes, which have to be checked separately. The field analogous to the graviphoton in the open string case is the gaugino field,

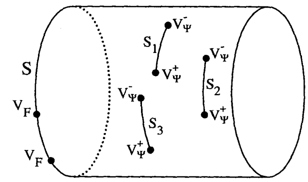

Fig. 19. The worldsheet for open strings with $g = 0$ and $h = 5$ boundaries, two of which are the two boundaries of the cylinder and three of them (S_1, S_2, S_3) are slits on the cylinder. The twisted theory corresponds to putting gaugino vertex operators V_Ψ on the end points of the slits and gauge field vertex operators V_F on the boundary of the cylinder.

which we denote by the vertex operator $V_\Psi^\pm$, at zero momentum. This operator is the spectral flow operator in the internal $N = 2$ SCFT, combined with the operator which twists the spins of β, γ ghosts and space-time fermionic fields ψ, χ. In particular this operator is inserted where we choose curvature singularities of appropriate strength.

Let us consider the open string worldsheet shown in Fig. 19.

This is a cylinder with two boundaries and with $h-2$ slits cut on it. We also mean this geometrically, i.e., that the metric be the flat metric on the cylinder. However, note that this introduces curvature singularities at the two end points of each of the $h-2$ slits. The reason for this is that the zero curvature on the boundary corresponds to π radians, but here at the two end points we get 2π radians of worldsheet. So we insert V_Ψ^+ operators at each of the two end points of the $(h-2)$ slits. This takes care of the twisting of the internal theory; the ghost zero modes also cancel leaving us with the measure for the twisted $N = 2$ theory coupled to gravity. So we only need to consider the space-time fermion zero modes. There are $h-1$ zero modes for each of the two ψ^i and 1 zero mode for each of the $\chi_{\bar{i}}$. The $\chi_{\bar{i}}$ zero modes can be absorbed by adding the operator

$$\epsilon^{\bar{i}\bar{l}}\epsilon_{jk} \oint \chi_{\bar{i}}\psi^j \oint \chi_{\bar{l}}\psi^k .$$

Each of these is the vertex operator of a gauge field V_F (up to momentum factors) at zero momentum. Again as in the closed string case we need to absorb the remaining $h-2$ zero modes for each of the ψ^i. Again, this can be done in a conformally meaningful way only by including them at one of the two end points of each of the $h-2$ slits converting $h-2$ of the V_Ψ^+ operators to V_Ψ^- operators. This will thus conclude absorbing zero modes, and so we end up with

$$F_h^0 = \langle [\oint_{S_i} V_\Psi^+ \oint_{S_i} V_\Psi^-]^{h-2} \oint_S V_F \oint_S V_F \rangle_{untwisted} ,$$

where S_i denote the interior slits and the S denotes one of the two boundaries of the cylinder. Thus we see, taking into account the structure of the insertions at the boundaries in taking the trace, that this gives rise to a term in the effective lagrangian of the form

$$F_h^0 \cdot \mathrm{Tr}\, F^2 [\mathrm{Tr}\, \Psi^2]^{h-2}.$$

The non-orientable worldsheets do not contribute to this amplitude because the absorption of fermion zero modes does not have the right structure.

As discussed in the introduction we expect that the topological theory is computing the coefficient of a superpotential term. Indeed there is a superpotential term which gives rise to the above interaction and that is given by

$$\int d^2\theta F_h^0 (W_\alpha W^\alpha)^{h-1} . \tag{8.7}$$

As discussed in the previous section, the partition function F_h^0 is now going to be a section of $\mathcal{L}^{2-h}$. This is consistent with the fact that W^2 is a section of $\mathcal{L}$ (which is also related to the fact that closed string coupling is the square of the open string coupling constant). The discussion we had regarding the non–holomorphicity of F_g in the closed string case applies word for word in the present situation and we will thus not repeat it.

The appearance of (8.7) as a topological amplitude, which is in principle exactly computable possibly using anomaly techniques discussed for open strings, is very interesting. This is because such an interaction has a strong bearing on the question of gaugino condensates, which has been proposed [59] as a mechanism to break supersymmetry in the context of superstrings! This would be very interesting to pursue in detail. Also the heterotic version of this would have to be investigated [60].

8.3 Threshold corrections for heterotic strings. In the context of heterotic strings the one–loop contribution to threshold correction for gauge coupling is related to the topological amplitude we have been discussing. In fact it has been shown in [61] that the one–loop corrected gauge coupling constant which depends on the moduli of the internal theory can be written as

$$\frac{16\pi^2}{g_a^2(\mu)} = k_a \frac{16\pi^2}{g_{GUT}^2} + b_a \cdot \log \frac{M_{GUT}^2}{\mu^2} + \Delta_a \ , \tag{8.8}$$

where a denotes the gauge group in question, k_a is the level of the group, b_a denotes the contribution of massless modes to the threshold, and Δ_a which includes contribution of internal stringy states is given by

$$\Delta_a = \int \frac{d^2\tau}{\tau_2} \mathrm{Tr}'(-1)^{F_L} F_L Q_a^2 q^{H_L} \bar{q}^{H_R} \ , \tag{8.9}$$

where the trace is in the R-R sector and is over the massive modes of the internal theory, including the right–moving gauge group contribution and the four dimensional modes, Q_a denotes a gauge group generator for the group a, and the integral is over the fundamental domain of moduli of tori. The b_a in the above formula reflects the fact that the zero modes lead to a divergence in the above formula which can be removed by defining a running scale μ, and so

$$b_a = Tr(-1)^{F_L} F_L Q_a^2 \Big|_{massless\ modes}$$

It was shown [61] that Δ_a satisfies an anomaly equation in terms of its dependence on moduli of the internal theory. Moreover it was shown that in the case of identifying gauge connection with the spin connection of the Calabi-Yau, which breaks $E_8 \times E_8$ heterotic string to $E_6 \times E_8$, $\Delta(E_6) - \Delta(E_8)$ satisfies the same anomaly equation as $12 \cdot F_1$, where F_1 is the genus one topological partition function defined in Sect. 2. It would be interesting to show this fact directly and moreover show that they also have the same holomorphic piece, i.e. not only $\partial\bar{\partial}[\Delta(E_6) - \Delta(E_8)] = 12\partial\bar{\partial}F_1$, but that $\Delta(E_6) - \Delta(E_8) = 12F_1$.

In order to argue this, it is worthwhile deriving the more general formula for the behavior of Δ_a even if the internal theory is not $(2,2)$, i.e. when the gauge connection is not identified with the spin connection of the Calabi–Yau but belongs to some bundle V. The bundle V needs to be stable and $\frac{1}{2}c_2(V) = \frac{1}{2}c_2(M)$ for a consistent heterotic string vacuum [62]. To be able to relate Δ_a to what we have computed and in particular to the Ray–Singer torsion, in this generality, we need to take a particular limit, namely the limit of large volume of the Calabi–Yau. We will compute the dependence of Δ_a on the complex moduli of Calabi–Yau in this limit. Actually taking the large volume limit in the case $V = T(M)$ is not a restriction as it is well known that the complex structure dependence and Kähler structure dependence of Δ_a decouple in this case. So for this case our remarks are quite general. We suspect our answer is also independent of this limit in the more general case but we do not have a rigorous argument.

Let us consider the internal theory to be a Calabi–Yau manifold with a vector bundle V on it. Let H denote the holonomy of this bundle. This means that the first E_8 gets broken down to

$$E_8 \to G \times H ,$$

where G is the maximal remaining group for which $G \times H$ can be imbedded in E_8. We will for simplicity of notation take G to be a simple Lie group, otherwise we can do what we are about to do for each simple factor of G. Thus the unbroken gauge group in 4 dimensions is $G \times E_8$. Now the adjoint representation of E_8 breaks under this decomposition to

$$(248) \to \sum_\alpha (R_\alpha, r_\alpha) ,$$

where R_α (r_α) denotes the G (H) representation.

Now consider the limit of infinite volume on the Calabi–Yau with arbitrary complex structure. In this limit the computation of Δ_a is easy to do, because by adapting the argument used in the derivation of the Kodaira–Spencer theory to the present case, the interior part of the moduli space make no contribution to the answer, and only degenerate Riemann surfaces contribute. In the case of moduli of torus, this means that only $\tau_2 \to \infty$ contributes, in which case (and after integrating over τ_1) only the massless modes of the right–moving sector contributes and the internal theory simply becomes the same computation as the Ray–Singer torsion discussed in Sect. 5.7. We thus see that

$$\Delta(G) = \sum_\alpha \mathcal{T}(R_\alpha) I(V_{r_\alpha}) , \tag{8.10}$$

where $I(V_{r_\alpha})$ denotes the Ray-Singer torsion for the vector bundle V with representation r_α and $\mathcal{T}(R_\alpha)$ denotes the index of representation R_α (coming from Q_a^2). Also note that similarly for the unbroken E_8 we have

$$\Delta(E_8) = \mathcal{T}(E_8) I_0 , \tag{8.11}$$

where I_0 denotes the Ray-Singer torsion with the trivial bundle. As discussed in [61] only the difference between the threshold corrections is meaningful, and so physically we should only consider $\Delta(G) - \Delta(E_8)$. This is our general result. Now we specialize to the case where $V = T(M)$, in which case $G = E_6$. We have the decomposition

$$248 \to (78,1) \oplus (27,3) \oplus (\overline{27},\overline{3}) \oplus (1,8) . \tag{8.12}$$

Also we note that since the spin connection is identified with the gauge connection we have

$$I(V_0) = I_0 \qquad I(V_3) = I(T^*) \qquad I(V_{\bar{3}}) = I(T^* \wedge T^*) \,. \tag{8.13}$$

Using the values $\mathcal{T}(E_8) = 30, \mathcal{T}(E_6) = 12, \mathcal{T}(27) = 3$, and making use of (8.13), (8.12), (8.10) and (8.11) we find

$$\begin{aligned}\Delta(E_6) - \Delta(E_8) &= (12-30)I_0 + 3\big(I(T^*) + I(T^* \wedge T^*)\big) = \\ &= 6\big(-3I_0 + \frac{1}{2}I(T^*) + \frac{1}{2}I(T^* \wedge T^*)\big) = \\ &= 6\big(\frac{-3}{2}I_0 + \frac{1}{2}I(T^*) + \frac{1}{2}I(T^* \wedge T^*) - \frac{3}{2}I(T^* \wedge T^* \wedge T^*)\big) = \\ &= \frac{12}{2}\sum_p (-1)^p \left(p - \frac{3}{2}\right) I(\wedge^p T^*) = 12F_1 \,,\end{aligned} \tag{8.14}$$

where we used the fact that $I(T^* \wedge T^* \wedge T^*) = I_0$. This is what we wished to show. Even though we derived this in the context of complex structure dependence of the threshold corrections, by mirror transform, it may also be viewed as the Kähler structure dependence. If we view it in this way we can then use the result of [20] to estimate the dependence of F_1 for large volume of Calabi-Yau. It was shown there that (taking into account the factor of 2 difference in the definition of F_1)

$$F_1 \xrightarrow{k \gg 1} \frac{1}{24}\int_M k \wedge c_2 \,,$$

where k denotes the Kähler class of the Calabi-Yau manifold. Note that (as discussed in [20]) $\int_M k \wedge c_2 > 0$. So we have

$$\Delta(E_6) - \Delta(E_8) = \frac{1}{2}\int_M k \wedge c_2 > 0 \,.$$

Now we can use (8.8) to see that the effect of changing k beyond the Planck scale is the same as getting an $M_{GUT}^{effective}$ according to

$$M_{GUT}^{effective} = M_{GUT} \cdot \exp(\frac{\Delta}{2b}) = M_{GUT} \cdot \exp\left[\frac{\int_M k \wedge c_2}{2b}\right] \,,$$

where in this case $b = 54+3(h_{1,1}+h_{1,2})$. We thus see that the effective grand unification scale for this relatively general class of string compactifications is extremely sensitive to the size of the internal manifold and moreover when we increase the size of the internal manifold above Planck scale it tends to increase exponentially fast!

9. Open Problems

In this section we discuss open problems and directions for future research. Let us first summarize some of the main results of this paper. We have considered $N = 2$ twisted topological strings. The partition function of these theories at a given genus g is formally a ›lomorphic modular function of weight $2g - 2$ on moduli space of the conformal theory. However we find that there is an anomaly and that the partition function is not necessarily holomorphic. What goes wrong with the formal argument of holomorphicity is the assumption that total derivative terms vanish upon integration over the moduli space of Riemann surfaces. Using the geometry of moduli space of Riemann surfaces and the structure of the $N = 2$ twisted theories one can compute the boundary contributions and they turn out to be expressible as (products of) lower genus correlation functions. This can be summarized as a second order linear differential equation, the *master anomaly equation*, for the full partition function of the theory summed over all genera. This recursion relation for the antiholomorphic dependence of the partition function can be solved by introducing Feynman rules which can be expressed as an integral over an auxiliary space which includes the dilaton and the marginal fields as propagating degrees of freedom and whose vertices are the correlation functions of the lower genus and the propagators are made of a *canonical*[41] prepotential, for the anti-topological theory, and its derivatives. This fixes the genus g partition function up to a holomorphic modular form, which are typically finite in number and thus reduces the computation of the partition function to fixing the coefficients of these functions. In concrete examples using mirror symmetry these coefficients can also be fixed, at least for low genera.

We discussed the realization of $N = 2$ SCFT's in terms of sigma–models on Calabi–Yau manifolds. There are two different ways to twist such theories, the A–twist (the Kähler twist) or the B–twist (the complex twist). In the case of A–twist a particular limit of the topological string theory computes the number of holomorphic maps (or an appropriate Euler character on the moduli space of holomorphic maps) from the Riemann surfaces to the Calabi–Yau. In the case of the B–twist the target space theory of the topological theory may be described as an ordinary field theory, actually a topological field theory, which quantizes the complex structures on Calabi–Yau, which we called the Kodaira–Spencer theory of gravity.

We also found an interpretation of the computations of the topological partition function F_g as the genus–g correction to four dimensional low energy lagrangian generating superpotential terms that arise upon compactification of superstrings on internal theory with $\hat{c} = 3$ from 10 to 4 dimensions. In the open string case this term will be relevant for the gaugino condensates. Also we related F_1, the genus one partition function, to the threshold corrections for gauge group couplings for heterotic strings in the 'standard' compactification scenario (identifying the gauge connection with the spin connection of the Calabi–Yau). This shows a surprisingly universal *exponential* dependence of the effective GUT scale with respect to the volume of the Calabi–Yau manifold.

This was a basic summary of some of the main results. Let us now discuss some directions for future research. One of the most significant aspects of the master anomaly equation is that it captures the anomaly to all orders in string perturbation theory and is thus a way even to proceed towards non–perturbative formulation of it. It would be interesting to compare how the non-perturbative aspects of the topological strings discussed here compare with those of some other string theories discussed in [63]. A first step in this direction is to find exact solutions to the anomaly equation to all orders. In this paper we saw how we can do it order by order in perturbation theory (using the Feynman graph technique discussed in the text) but we did not manage to find a simple closed form for any example which would be valid to all orders. The simplest example to consider in this connection is the toroidal example discussed

[41] Which is different from the ones previously used in the literature.

in Sect. 7. One might well imagine that a theta function like solution may exist to the master anomaly equation, though we were not able to find one. The attempt is complicated by the fact that not only one wishes to find a solution to the master equation, but a solution which satisfies the correct boundary conditions (dictated by the genus–1 answer). Of course finding a solution to the master anomaly equation, even if it satisfies the correct boundary condition is no guarantee to be the correct amplitudes given by the string amplitudes, just because the anomaly equation only captures the anti-holomorphic dependence of the partition function on the moduli. One still has the freedom to correct it order by order by addition of holomorphic terms. Indeed changing the holomorphic dependence at a given genus will affect even the non-holomorphic dependence for any higher genus computation. Finding a nice way to fix the holomorphic dependence, even though it just means fixing a finite number of coefficients at each order, is a major challenge. The most logical way to proceed, in the case of the B-model is to study the Kodaira–Spencer perturbation theory which naturally will also give the holomorphic part as well as the anomalous part of the amplitudes. Otherwise we have rather limited resources to fix the holomorphic part of the amplitudes. Mirror symmetry helps, as it did in the examples considered in Sect. 7, in fixing some of the low genus answers by relating it to counting holomorphic maps of genus g to a target space. But even in these examples the attempt was complicated by the fact that for a given genus g the lower genus holomorphic maps may contribute as a kind of degenerate contribution to the genus g amplitude (which in the case of genus 0 contribution of degree one is called the 'bubbling'). A deeper understanding of these general bubbling phenomena would be greatly helpful in fixing the holomorphic ambiguity of the solution to the anomaly equation. The situation in understanding these contributions can significantly improve through collaboration between algebraic geometers and physicists. Some discussions of the bubbling phenomena appear in Appendix A.

One of the most mysterious aspects which emerged in the course of solving the anomaly equation (see Sect. 6) was the appearance of Feynman rules involving propagation of massless modes and the dilaton. This was rather unexpected and needs to be understood better. In a sense it seems to suggest that effectively we can add the massless modes as dynamical fields to the string field theory despite the fact that we had to delete them in order to write the string field action in Sect. 5. In this interpretation putting back the massless fields in the theory is effectively a way to restore background independence and so would suggest that including the massless modes would simply lead to answers which are independent of $t, \bar{t}$ thus explaining the Feynman graph rules we found for computing F_g. In fact the propagator we have for the marginal fields, which is formally identified with $b_0\bar{b}_0/L_0$ and is ill-defined $0/0$ is effectively 'regularized' by the propagator S^{ij} introduced in Sect. 6. In fact one can 'formally' derive the defining property of S^{ij} from this definition using a tt^*-type argument. It would be interesting to develop this further as well as see how the propagators involving the dilaton field will appear. At any rate demystification of the Feynman rules that we found is a very important hint in progress in a better understanding of these theories.

Perhaps the most important aspect of the present work is the discovery of a new topological gravity theory in six dimensions, the Kodaira–Spencer theory. It is *topological in the sense that it is independent of the metric of the Calabi–Yau manifold*, though it depends on the complex structure chosen. This topological theory is the target space description of a topological worldsheet theory on a Calabi–Yau. The fact that there is a string theory description of this theory makes us believe that the ultraviolet divergencies of the KS theory are not a real obstacle to its existence and strings can be viewed as effectively giving a 'nice regularization' of the theory (deforming it from the manifold space to the loop space). Nevertheless it should be interesting to regularize the KS theory using the more standard regularization techniques of field theories. In particular it should be possible to derive the holomorphic anomaly master equation directly in this field theory set up for all loops. The one–loop version of the anomaly was checked explicitly to agree with the field theory one using the zeta function regularization techniques (which were used in [41]).

In more than one way the KS theory in 3 complex dimensions mirrors its cousin the Chern–Simons theory in 3 real dimensions. It is a closed string version of Chern–Simons theory. Thus just as one has interesting topological invariants in the Chern–Simons theory, giving link invariants on three manifolds, one also expects the same here in the context of invariants associated to Calabi–Yau 3–folds (or more abstractly, classification question of variation of Hodge structures which arise in superconformal theories). This aspect is worth more thought. Also the open string version of strings on Calabi–Yau is a mirror to ordinary Chern–Simons theory. So in this setup the coupling of this mirror theory to KS theory is interesting to study. In particular the holomorphic anomalies in the open string sector discussed in this paper should be the mirror transformed versions of (a certain limit of) Chern-Simons theory's anomalous dependence on the metric of the 3 manifold which has been studied recently [64] to all loops (for the study of Chern-Simons perturbation theory see also [65]). It would be interesting to work out the detail of the anomaly equation for the open string case which, except for the one–loop case which we computed in detail, we just briefly discussed in this paper. This is more urgent in view of the fact that gaugino condensates which are believed to be a mechanism to break supersymmetry in string theories will be strongly affected by such terms. This aspect of the present work, which may have potential relevance in questions of phenomenology, i.e., the fact that topological partition functions may also be viewed as particular computations in certain string models compactified on the corresponding topological theory we find rather significant. Not only topological theories can be used to compute some amplitudes in ordinary strings, but the amplitudes that they compute are the most interesting ones to compute, i.e. the superpotential terms. This opens the door to exact computations in string theories using topological techniques. Amplitudes which are computable, at least in the context of open superstrings, will be of interest also in connection with gaugino condensates which has been proposed as a mechanism to break supersymmetry. In fact it would be quite satisfactory that deep facts such as supersymmetry breaking be linked to very natural topological computations. It would be nice to extend these computations to the heterotic case in view of the potential phenomenological implications. The fact that heterotic strings morally should behave like the open strings suggests that even in this case the topologically formulated heterotic string should compute similar superpotential terms, as would be interesting in questions of gaugino condensates. At any rate it would be very important to determine the consequences of such terms in the supersymmetry breaking scenarios in string theory.

The crucial link needed to establish between topological string theories with conventional superstrings was the observation that basically the twisting of an ordinary superstring is equivalent to insertion of an appropriate number of FMS spin operators which twists the field measure. Even though topological amplitudes correspond to very special amplitudes in string theories it is natural to ask whether one can formulate *arbitrary* amplitudes in superstring theories using the twisted topological models by inclusion of non–topological operators (including conjugate FMS spin operators to untwist the measure). If such a formulation can be done it would be a step forward in that one would not have to deal with issues of summing over spin structures or the question of splitness of supermoduli space, both of which are naturally absent in the topological theory because the spin of all the fields are integral. In fact results of [66] suggest that this should be possible.

Another aspect of the present work was the fact that in all the examples studied, the large area behavior of the genus g partition function of the topological theory on a Calabi–Yau 3–fold is in the same universality class as the $c = 1$ theory coupled to gravity (i.e. has the same exponents). This result shows that the identification of the $c = 1$ theory coupled to gravity with a particular supersymmetric coset representation of black hole with $\hat{c} = 3$, discovered in [53] which is a non-unitary $N = 2$ twisted model, is actually only the tip of the iceberg. Indeed what we have found seems to strongly suggest that the universality class of $c = 1$ strings is the same as that of topologically twisted $\hat{c} = 3$ theories. It would be interesting to study this connection further. In particular for each Calabi–Yau manifold M the large 'worldsheet' area $A \gg 1$ behaviour should go like

$$a_g(M)\, A^{2g-3}\log^{2-2g}(A)\exp[b(M)A]\ .$$

It would be interesting to compute $a_g(M)$ for all g and for all Calabi–Yau manifolds ($b(M)$ can be computed from the genus zero result if one knows the mirror manifold). For a fixed g how does the number $a_g(M)$ depend on M? Also for a fixed M how do the numbers $a_g(M)$ depend on g? Do they satisfy recursion relations of the type encountered in topological theory coupled to gravity?

There are even more connections with $c = 1$. Indeed as pointed out in [67] the target space physics of $c = 1$ strings has the symmetry of the volume preserving diffeomorphism. As discussed in Sect. 5 this is precisely the gauge symmetry of the Kodaira–Spencer theory which is the target space physics of the critical topological strings. This relation is also worth further investigation and is suggestive of the universal relation between $c = 1$ strings and $\hat{c} = 3$ topologically twisted theories.

The special status the $\hat{c} = 3$ topological string enjoys among more general topologically twisted theories, is very much analogous to the special status $c = 1$ strings enjoy among all the theories with $c \leq 1$ coupled to gravity. It is natural to ask if what we have been discussing in connection with unitary $\hat{c} = 3$ twisted theories has any bearing on the more general classes of possibly non–unitary theories (as is the case with the theory discussed in [53]) or twisted theories with $\hat{c} < 3$ (as is the case for the minimal $N = 2$ twisted theories which is related to the $(1, p)$ theories coupled to gravity [23, 22]). The central question is whether the anomaly equation should exist in these cases. In fact morally it should be true but to make it precise a few technical obstacles should be overcome: In the context of non–unitary theories one has to argue that the cohomology elements are the only ones that contribute for long tubes (this is no longer guaranteed in the non-unitary case). In the context of $\hat{c} < 3$ models one has to recall that in order to get non–zero amplitudes one will have to perturb the corresponding conformal theory in two directions: The massive direction, as well as turning on the gravitational (or topological) descendants. Turning on relevant perturbations which makes the theory massive raises the question of whether we can still integrate over conformally inequivalent classes of metric. Even if this can be done, we will have to know the analog of the Zamolodchikov metric for these massive theories. One would imagine that the analog of tt^* equations which is also known for the massive [2] case should be relevant (in fact the results of [68] suggest that the one–loop partition function should be related to the tau-function). However it is not completely straightforward because as we discussed in Sect. 2 the Zamolodchikov connection and tt^* connection differ by a term involving the connection on the line bundle $\mathcal{L}$. Unfortunately in the massive case the line bundle $\mathcal{L}$ is not a holomorphic sub-bundle of the vacuum bundle and so this prevents one from constructing canonical connections on it[42]. This will have to be better understood. Another direction of perturbation is turning on the gravitational descendants (which are in particular needed for a non–vanishing amplitude at higher genus for the twisted minimal models coupled to gravity). The correlations involving topological descendants can typically be viewed as boundary contributions to the amplitudes [22]. Thus one would expect an interesting mixture with the anomaly discussed in this paper. In this connection the Landau-Ginzburg formulation of the descendants may be particularly useful [69].

Typically string theories have infinitely many particles. However there are some cases known where string theory has only a finite number of particles. Precisely in these cases the string theory seems also to be related to topological theories both in the sense of world sheet *and* in the sense of target theory. Let us summarize some of the known examples and speculate on the relation between them.

Let us summarize some of the most important known topological field theories: Apart from the 6-dimensional one that we discovered in this paper, and its open string analog [24], there are two important topological theories in 4-dimensions, topological gravity and topological Yang-Mills theory (Donaldson theory) [70], in 3-dimensions one has the Chern–Simons theory [71] and in 2-dimensions one has topological sigma–models and topological gravity theories, and topological Yang-Mills theories [4]. Amazingly enough *almost all* of these theories seem to be describing the target space physics of some string theory: The 6-dimensional KS theory is the target space physics of critical topological strings as we have discussed in this paper. The 4-dimensional topological theories seem also to be related to target space of $N = 2$ strings [37] in that the relevant target space geometry in both cases involves self-dual geometries[43]. The 3-dimensional CS theory is equivalent to open string topological theory [24]. Finally the 2d topological YM theories, which are equivalent to ordinary 2d YM theories may also be viewed as a string theory using the results of [72] which could also be viewed even as a topological string theory (a deformed topological sigma–model coupled to gravity [20, 73]). So it seems that many of these topological field theories are string field theories of string theories which themselves are topological (i.e. are coupling of 2d topological theories to topological gravity). The completion of this picture suggests that the $N = 2$ strings should have a reformulation as a topological theory in the worldsheet sense, and should also be able to obtain topological sigma–models and topological gravity theories in 2 dimensions as effective target space theories for a topological world sheet theory. Having such a unified picture also raises the question of what are the relations between various topological theories, and also their relation to integrable theories. In a sense the six–dimensional

[42] In the context of integrable massive perturbations $\mathcal{L}$ is typically a sub-bundle because of discrete symmetries and one can define the corresponding line bundle connection. It would be interesting to study this particular class further.

[43] This connection needs to be clarified further.

topological theories should play a key role in connecting them. In particular self-dual geometries arise naturally from considerations of holomorphic vector bundles in six dimensions, through twistor transform. One could speculate whether this formulation can be used to connect it to the topological theory describing the open strings on Calabi–Yau [24] which has as a solution an arbitrary holomorphic vector bundle in six dimensions. Similarly one may expect that the KS theory which characterizes the complex structures of a six dimensional space be related to the topological gravity theories in 4d, which characterize self-dual geometries. Clearly a lot more work remains to be done. We hope to have taken one small step which may be helpful in the final emergence of a unified picture.

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10. Appendix A. The Bubbled Torus

In this appendix we shall rederive from a physical viewpoint the result by Katz (see the appendix of [20]) on the bubbling of spheres in genus 1. Some preliminary computation for $g > 1$ is also presented.

As before, ψ denotes a fermi field which is a section[44] of $K \otimes f^* T_M^*$, whereas χ is a section of $f^* T_M$.

The basic degenerate instanton in genus one is given by (here $D \subset M$ is a rational curve rigid in M and C is the world–sheet torus)

$$\Gamma_{pq} \subset C \times D \subset C \times M \ ,$$

given by

$$(C \times \{q\}) \cup (\{p\} \times D) \ . \tag{A.1}$$

As always, the χ zero–modes are in one–to–one correspondence with the 'collective coordinates' describing the given family of instantons. In the present case we have just two of them, corresponding to the freedom of choosing the two points p and q in C and D, respectively. We are also interested in finding the zero–modes of ψ in this configuration. From the viewpoint of an 'observer' in a 'generic' point of the torus[45], the situation looks as follows. The torus C gets mapped into the point $q \subset D \subset M$. Then — as for any constant map — the pullback of T_M to C is trivial, and $\dim H^1(C, T_M) = 3$. However, there are not really three obstructions, since the deformation of q in the direction tangent to D is 'not obstructed.' So we remain with

[44] Strictly speaking, this description is adequate for *non–degenerate* instantons only.

[45] I.e. not at the point p where the 'bubble' is attached.

the two zero–modes which are orthogonal to D at q, in agreement with the index theorem which predicts an equal number of χ and ψ zero–modes. Our observer on C understands this as follows. For him the instanton Γ_{pq} arises by the following limiting process: One constructs an *approximate* solution mapping the world–sheet to the (rational) curve D by taking an usual instanton on the plane, of scale[46] a much smaller than the periods of the torus C and 'gluing' it at the point p. Then letting $a \rightarrow 0$ we get a true solution which — to our observer — looks like a 'delta–function' instanton centered at p. To be specific, let us identify D with $\mathbf{P}^1$ in such a way that the point q is taken as the origin. Then our approximate instanton reads (for $w \sim p$)

$$f(w) = \frac{a}{w-p} \ .$$

The pullback Kähler form for $w \sim p$ reads (say, taking D to be a line, and the metric to be the one induced by Fubini–Study)[47]

$$\left. \frac{a^2 dz \wedge d\bar{z}}{(2\pi i)(|z|^2 + a^2)^2} \right|_{a \rightarrow 0} = \frac{1}{i} \delta(z)\, \delta(\bar{z}) dz \wedge d\bar{z} \ ,$$

where $z \equiv w - p$. Of course this is just the statement that we have a δ–function instanton. Clearly if D is a degree k rational curve this generalizes to

$$\begin{pmatrix} \text{the pulled back Kahler class} \\ \text{as seen by our observer on } C \end{pmatrix} = -ik\, \delta(z)\, \delta(\bar{z}) dz \wedge d\bar{z} \ . \tag{A.2}$$

From the viewpoint of this observer, as $a \rightarrow 0$ the instanton disappears, leaving a local operator inserted at the point p. This operator implements a boundary condition at p for the ψ zero modes; it is this condition that gets rid of the tangent component of ψ leaving just the two 'normal' components.

To see the nature of the above boundary condition we have to discuss the situation from the viewpoint of a second observer on D. From the point of view of this observer, the limit $a \rightarrow 0$ is accompanied by a compensating conformal rescaling by a^{-1}, so that to him the instanton looks to have a finite size in the limit. However, at $a = 0$ he happens to be in a different $2d$ 'universe' with respect to the other guy (i.e. on D). For this observer the pulled back Kähler form is

$$\begin{pmatrix} \text{the pulled back Kahler class} \\ \text{as seen by our observer on } D \end{pmatrix} = i^* \omega_M \ , \tag{A.3}$$

where ω_M is the Kähler form for M and i: $D \rightarrow M$ is the embedding. Putting together the two 'universes' one gets

$$\text{the pulled back Kahler class} = -ik\, \delta(z)\, \delta(\bar{z}) dz \wedge d\bar{z} + i^* \omega_M \ , \tag{A.4}$$

which should be compared with Katz's result, i.e. $E_1 + h$ (see the appendix of [20]).

From the viewpoint of the second observer, there are no zero–modes for ψ, since on the sphere

$$H^1(T_M) \simeq H^0(K \otimes t^* T^*_M) = 0 \ .$$

[46] By conformal invariance, there are instanton of any arbitrary small scale.
[47] The normalization of the δ's is such that the integral of the r.h.s. is just 1.

In order for a zero–mode to be regarded as vanishing by the observer on D, it should have a vanishing invariant norm as $z \longrightarrow p$. Let $\psi_=$ be the tangent would–be zero–mode. Near p its norm reads

$$\|\psi_=\|^2 = \left(1 + \frac{a^2}{|z|^2}\right)^2 |\psi_=|^2$$

which is divergent as $z \longrightarrow 0$, unless $\psi_=(0) = 0$. But a holomorphic function[48] vanishing at one point vanishes everywhere. Instead for the 'normal' zero modes[49]

$$\|\psi_\perp\|^2 = \frac{|\psi_\perp|^2}{(1 + a^2/|z|^2)} ,$$

which vanishes at $z = 0$ *for any* $\psi_\perp$. This shows that we have just two ψ zero modes (as required by the index theorem). The structure of these zero–modes is as predicted by Katz.

The moduli space of the above configuration is $\mathcal{M}_{1,2} \times \mathbf{P}^1$ (here we identify $D \simeq \mathbf{P}^1$). As we vary the point in the moduli space, the zero modes $\psi_{\imath a}$ ($a = 1, 2$) will also vary, giving a bundle $\mathcal{B}$ over the above moduli space. By construction

$$\mathcal{B} = \pi_1^* \mathcal{H} \otimes \pi_2^* \mathcal{N} ,$$

where $\mathcal{H}$ is the Hodge (line) bundle over $\mathcal{M}_{1,1}$ whose fiber is spanned by the holomorphic one–forms for the corresponding elliptic curve, and $\mathcal{N}$ is the normal bundle to D in X. The curvature of $\mathcal{B}$ has the structure $1 \otimes P + \tilde{R} \otimes 1$, where P is the Hodge bundle curvature as computed in §.1.1, and $\tilde{R}$ is the curvature of the normal bundle $\mathcal{N} = T_M/T_D$. As it is well known

$$\tilde{R} = R\Big|_{\mathcal{N}} - \theta \wedge \theta^\dagger , \qquad (A.5)$$

where θ is the 2nd–fundamental form of

$$0 \longrightarrow T_D \longrightarrow T_M \longrightarrow \mathcal{N} \longrightarrow 0 .$$

Thus, in general it is not true that the curvature of the normal bundle is the restriction of the curvature of the tangent bundle to the normal directions. However, if D is a rigid sphere on some Calabi–Yau 3–fold M,

$$T_M \simeq \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) ,$$

and hence the bundle splits and the extra term in (A.5) should be an exact form. Then the curvature is[50]

$$\left(\mathcal{R}_{\mathcal{B}}\right)^i{}_j = \delta^i{}_j N^{-1} P dy \wedge d\bar{y} + R^i{}_{jk\bar{l}} dx^k \wedge d\bar{x}^{\bar{l}} + \dots , \qquad (A.6)$$

where $\dots$ means exact terms and x are the coordinates of the point q in M.

48 Recall that from the viewpoint of the first observer T_X is trivial and so is K.

49 We use that

$$T_X|_D = \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) .$$

50 We do not write indices for N and P since in the present case the indices can take only one value.

Let us summarize: there are 2 zero–modes for χ corresponding to the deformations of the point p and q, and two zero modes for ψ_i. Then consider the quantity

$$-\int\limits_{\mathcal{M}_{1,1}} \left\langle \left(dy \int\limits_C \mu\psi_i\partial X^i\right) \bigwedge \left(d\bar{y} \int\limits_C \bar{\mu}\bar{\psi}_{\bar{j}}\bar{\partial}\bar{X}^{\bar{j}}\right) \left(k_{m\bar{l}}\chi^m\bar{\chi}^{\bar{l}}\right) \int\limits_C R_{k\bar{h}}{}^{n\bar{p}}\chi^k\bar{\chi}^{\bar{h}}\psi_n\bar{\psi}_{\bar{p}} \right\rangle ,$$

i.e. the $g = 1$ one–point function. We wish to compute the contribution of the (single) degenerate instanton to the above quantity. The subtle identity is (A.4). It means that we have the replacement

$$k_{i\bar{j}}(0)\chi^i\bar{\chi}^{\bar{j}} \mapsto \deg(D)\ \delta(z)\delta(\bar{z})\chi^z\bar{\chi}^{\bar{z}} + \tilde{k}\chi^x\bar{\chi}^{\bar{x}} , \qquad (A.7)$$

where χ^z (resp. χ^x) is the zero mode associated to the variation of the coordinate z (resp. x) of the point p (resp. q) on C (resp. D), and $-i\tilde{k}dx \wedge d\bar{x} \equiv i^*\omega_M$.

Integrating away the χ's and the ψ's (using the same formulae as in Sect. 1.2) we reduce to an integration over the boson zero–modes, of the expression

$$\frac{\deg(D)}{(2\pi i)^2} \int\limits_{\mathcal{M}_{1,2}\otimes D} \delta(z)\delta(\bar{z})dz \wedge d\bar{z}\, \det[\mathcal{R}_{\mathcal{B}}] ,$$

where $\mathcal{R}_{\mathcal{B}}$ is given in (A.6). This can be rewritten as

$$\deg(D) \int\limits_{\mathcal{M}_{1,1}\otimes D} c_2(\mathcal{H} \otimes \mathcal{N}) .$$

A simple computation gives (recall $\mathcal{N} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$)

$$c_2(\mathcal{H} \otimes \mathcal{N}) = 2\, c_1(\mathcal{H})\, c_1(\mathcal{O}_D(-1)) .$$

Finally from

$$2\deg(D) \int\limits_{\mathcal{M}_{1,1}} c_1(\mathcal{H}) \int\limits_D c_1(\mathcal{O}(-1)) =$$

$$= 2\deg(D)\deg(\mathcal{O}(-1))\,\chi(\mathcal{M}_{1,1}) = \frac{2}{12}\deg(D) .$$

Preliminary Considerations for Genus $g > 1$. If M is simply–connected it is also algebraic. Then let ω_0 be the Kähler form induced by the imbedding of M inside $\mathbf{P}^N$. By degree of a curve C lying on M we mean $\int_C \omega_0$. Then a curve of degree one is a line in $\mathbf{P}^N$ and hence it is necessarily rational. Therefore for all g the $O(e^{-t})$ contribution to F_g should arise from maps of the form

$$\Sigma_g \xrightarrow{f} D \overset{i}{\hookrightarrow} M ,$$

where D is a degree 1 rational curve on M, i the inclusion and f some degree 1 holomorphic map. However, for $g > 0$ there is no such a thing as a degree 1 meromorphic function. Thus at first sight, it may seem that for $g > 0$ the $O(e^{-t})$ term in F_g should vanish. However, it is not so as was shown explicitly in ref. [20]

for $g = 1$. The point is that although there is no smooth instanton in this topological class, we can construct an approximate solution mapping Σ_g to D by taking an usual instanton on the plane for the $\mathbf{P}^1$ sigma–model, of scale[51] a much smaller than the periods of the curve Σ_g and 'gluing' it at the point $p \in \Sigma_g$. The approximation gets better and better as $a \to 0$. In the limit we get a solution which looks like a 'delta–function' instanton centered at p. However, there is a better viewpoint. By conformal invariance, while we let $a \to 0$ we can do a compensating scale transformation in a neighborhood of p such that the instanton remains of a finite scale in the limit. In this picture, as $a \to 0$ a sphere will 'bubble off' the world–sheet. In terms of the graph Γ of the map $\Sigma_g \to D$, the resulting degenerate instanton will be (q is a point in D)

$$\Gamma_{pq} \subset \Sigma_g \times D \subset \Sigma_g \times M ,$$

given by

$$(\Sigma_g \times \{q\}) \cup (\{p\} \times D) . \tag{A.8}$$

That such singular instantons like Γ_{pq} should be taken into account, follows from Gromov's theory of symplectic invariants; as $\bar{t} \to \infty$ the functional measure gets concentrated on the critical points only if the integration space (the 'space of all maps') is compactified. Otherwise the instanton may 'escape to infinity.' Now, these configurations (A.8) belong to the Gromov compactification of the 'space of all maps.'

Comparing with the genus one case, it appears that the following computation should be relevant for the higher genus bubbling:

$$\int_{\mathcal{M}_{g,1}\otimes D} C_{3g-1}(\mathcal{B}) , \tag{A.9}$$

where

$$\mathcal{B} = \pi_1^*\mathcal{H} \otimes \pi_2^*\mathcal{N}^* \oplus \pi_1^*\tilde{\mathcal{H}} \otimes \pi_2^* T_D^* .$$

Here $\mathcal{N}$ is the normal bundle to D in M and $\mathcal{H}$ is the Hodge vector bundle as before. The fiber of $\mathcal{H}$ is $H^0(\Sigma_g, K)$. Instead $\tilde{\mathcal{H}}$ is the bundle with fiber $\Gamma(\mathcal{O}(-p) \otimes K)$ — that is the holomorphic one–forms vanishing at $p \in \Sigma_g$. Obviously we have the following exact sequence:

$$0 \to \tilde{\mathcal{H}} \to \mathcal{H} \to L \to 0 , \tag{A.10}$$

where L is the line bundle over $\mathcal{M}_{g,1}$ whose fiber is T_p^*.

From the definition of $\mathcal{B}$ one has

$$c(\mathcal{B}) = c(\mathcal{H} \otimes \mathcal{N}^*)\, c(\tilde{\mathcal{H}} \otimes T_D^*) ,$$

and then

$$c_{3g-1}(\mathcal{B}) = c_{2g}(\mathcal{H} \otimes \mathcal{N}^*)\, c_{g-1}(\tilde{\mathcal{H}} \otimes T_D^*) .$$

Since D is one–dimensional

$$c_{2g}(\mathcal{H} \otimes \mathcal{N}) = c_g(\mathcal{H})^2 - 2c_1(\mathcal{N})\, c_g(\mathcal{H})\, c_{g-1}(\mathcal{H}) = 2c_1(\mathcal{N})\, c_g(\mathcal{H})\, C_{g-1}(\mathcal{H}) ,$$
$$c_{g-1}(\tilde{\mathcal{H}} \otimes T_D) = c_{g-1}(\tilde{\mathcal{H}}) - c_{g-2}(\tilde{\mathcal{H}})\, c_1(T_D) ,$$

where, in the first line we used (5.53). Now we see that

$$c_1(\mathcal{N}) = 2\, c_1(\mathcal{O}_D(-1)), \qquad c_1(T_D) = c_1(\mathcal{O}_D(2)) .$$

[51] By conformal invariance, there are instanton of any arbitrary small scale.

On the other hand, from (A.10)

$$c(\mathcal{H}) = c(\tilde{\mathcal{H}})\, c(L) = c(\tilde{\mathcal{H}})\left(1 + c_1(L)\right) ,$$

or equivalently,

$$c(\tilde{H}) = c(\mathcal{H})\left(1 + \sum_{k=1}^{g} (-1)^k c_1(L)^k\right) ,$$

which in particular gives

$$c_{g-1}(\tilde{\mathcal{H}}) = \sum_{k=0}^{g} (-1)^k c_{g-k-1}(\mathcal{H})\, c_1(L)^k .$$

Then

$$\begin{aligned}\int_D c_{3g-1}(\mathcal{B}) &= \int_D \left[c_{2g}(\mathcal{H}\otimes\mathcal{N}^*)\, c_{g-1}(\tilde{\mathcal{H}}\otimes T_D^*)\right] = \\ &= 4\, c_g(\mathcal{H})\, c_{g-1}(\mathcal{H}) \sum_{k=0}^{g} (-1)^k c_1(L)^k\, c_{g-k-1}(\mathcal{H}) .\end{aligned}$$

The term with $k = 0$ vanishes, since the integrand is the pull–back of a $(3g-2)$–form on $\mathcal{M}_{g,0}$. Also the term with $k = g$ vanishes for trivial reasons. Comparing with [48] this can be rewritten in terms of Mumford classes as

$$4\sum_{k=1}^{g-1} (-1)^k \int_{\mathcal{M}_g} \lambda_g \wedge \lambda_{g-1} \wedge \lambda_{g_k-1} \wedge \kappa_{k-1} .$$

It remains to understand the precise relation between this Chern class computation and the actual bubbling coefficient.

11. Appendix B. Further Analysis on the Master Anomaly Equation

In Sect. 3, we found that the generating function

$$W(\lambda, x; t, \bar{t}) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{2g-2} C^{(g)}_{i_1\cdots i_n} x^{i_1}\cdots x^{i_n} + \left(\frac{\chi}{24} - 1\right)\log\lambda \qquad (B.1)$$

is characterized by the two equations,

$$\begin{aligned}&\frac{\partial}{\partial \bar{t}^{i}} \exp(W) = \\ &= \left[\frac{\lambda^2}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \left(\lambda\frac{\partial}{\partial\lambda} + x^k \frac{\partial}{\partial x^k}\right)\right]\exp(W)\end{aligned} \qquad (B.2)$$

and

$$\begin{aligned}&\left[\frac{\partial}{\partial t^i} + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} + \partial_i K\left(\frac{\chi}{24} - 1 - \lambda\frac{\partial}{\partial\lambda}\right)\right]\exp(W) = \\ &= \left(\frac{\partial}{\partial x^i} - \partial_i F_1 - \frac{1}{2\lambda^2} C_{ijk} x^j x^k\right)\exp(W) .\end{aligned} \qquad (B.3)$$

The first equation (B.3) summarizes the holomorphic anomaly equations for $C^{(g)}_{i_1\cdots i_n}$, and the second equation (B.3) implies that $C^{(g)}_{i_1\cdots i_n}$ in $W(\lambda, x; t, \bar{t})$ are given by derivatives of the partition function F_g. In Sect. 6, we developed a method to solve the holomorphic anomaly equation order by order in g. In this appendix, we analyze the two Eqs. (B.2) and (B.3) directly to all order in g. We hope that the method presented here would be useful to understand non–pertubative aspects of the string theory.

Let us first solve the anomaly equation (B.2) without imposing (B.3). This turned out to be possible by the Borel transformation in the string coupling constant λ and by the Fourier transformation in x^i,

$$\exp(W(\lambda, x; t, \bar{t})) = \int dp\, dq \exp(-\lambda^{-1} q + i\lambda^{-1} x^i p_i + \Gamma(q, p; t, \bar{t})) \,. \tag{B.4}$$

The anomaly equation (B.2) for W is transformed into the following first-order linear differential equation:

$$\left(\frac{\partial}{\partial \bar{t}^{\bar{i}}} + iG_{\bar{i}j} q \frac{\partial}{\partial p_j}\right) \Gamma = -\frac{1}{2} \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} p_j p_k \,. \tag{B.5}$$

A special solution to this equation is easily found as

$$\Gamma_0(q, p; t, \bar{t}) = -\frac{1}{2} S^{ij} p_i p_j + iS^i p_i q + Sq^2 \,.$$

This satisfies (B.5) by the definitions of S^{ij}, S^i and S

$$S^{ij} = \overline{C}_{\bar{i}\bar{j}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \,, \quad S^i = \overline{C}_{\bar{i}} e^{2K} G^{i\bar{i}} \,, \quad S = \overline{C} e^{2K}$$
$$\overline{C}_{\bar{i}\bar{j}\bar{k}} = D_{\bar{i}} \overline{C}_{\bar{j}\bar{k}} \,, \quad \overline{C}_{\bar{j}\bar{k}} = D_{\bar{j}} \overline{C}_{\bar{k}} \,, \quad \overline{C}_{\bar{k}} = D_{\bar{k}} \overline{C} \,.$$

Since (B.5) is first-order and linear, its general solution can be expressed as

$$\exp(\Gamma(q, p; t, \bar{t})) = \vartheta(q, p - iq\partial \log(e^K |f|^2); t) \exp(\Gamma_0(q, p; t, \bar{t})) \,, \tag{B.6}$$

where $f(t)$ is a meromorphic section of $\mathcal{L}$, and ϑ does not depend on $\bar{t}$ except through e^K in the second argument.

The general solution (B.6), after the Borel transformation, does not necessarily have the form (B.1). So we need to impose the second equation (B.3). After the Borel transformation (B.4), (B.3) becomes

$$\left[\frac{\partial}{\partial t^i} - \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j} - \Gamma^j_{ij} - \partial_i K \left(q\frac{\partial}{\partial q} + p_j \frac{\partial}{\partial p_j} + n + 2 - \frac{\chi}{24}\right)\right] \exp(\Gamma) =$$
$$= \left(ip_i \frac{\partial}{\partial q} - \partial_i F_1 + \frac{1}{2} C_{ijk} \frac{\partial^2}{\partial p_j \partial p_k}\right) \exp(\Gamma) \,,$$

where n is the dimensions of the moduli space of the $N = 2$ theory. Substituting (B.6) in the above, we obtain a differential equation for ϑ as

$$\begin{aligned} &\left(\frac{\partial}{\partial t^i} + f^k_{ij}(t) \tilde{p}_k \frac{\partial}{\partial \tilde{p}_j} - if_{ij}(t) \tilde{q} \frac{\partial}{\partial \tilde{p}_j} - i\tilde{p}_i \frac{\partial}{\partial \tilde{q}}\right) \vartheta = \\ &= \left(\frac{1}{2} \tilde{C}_{ijk}(t) \frac{\partial^2}{\partial \tilde{p}^j \partial \tilde{p}^k} - \frac{1}{2} e^{jk}_i(t) \tilde{p}_j \tilde{p}_k + ie^j_i(t) \tilde{p}_j \tilde{q} + e_i(t) \tilde{q}^2 + h_i(t)\right) \vartheta \,, \end{aligned} \tag{B.7}$$

where

$$\tilde{p}_i = f^{-1}\left(p_i - iq\partial_i \log(e^K|f|^2)\right), \quad \tilde{q} = f^{-1}q, \quad \tilde{C}_{ijk} = f^{-2}C_{ijk}\,.$$

Due to the special geometry relation, the coefficients

$$\begin{aligned}
f_{ij}^k(t) =& C_{ijl}S^{lk} - \Gamma_{ij}^k - \delta_i^k\partial_j \log(e^K|f|^2) - \delta_j^k\partial_k \log(e^K|f|^2)\,,\\
f_{ij}(t) =& C_{ijk}\left[S^k - S^{kl}\partial_l \log(e^K|f|^2)\right] + D_i\partial_j \log(e^K|f|^2) + \\
&+ \partial_i \log(e^K|f|^2)\partial_j \log(e^K|f|^2)
\end{aligned}$$

are holomorphic in t, and so are

$$\begin{aligned}
e_i^{jk}(t) =& f^{-2}\left[D_iS^{jk} + C_{imn}S^{mj}S^{nk} - \delta_i^jS^k - \delta_i^kS^j\right],\\
e_i^j(t) =& f^{-2}\Big[D_iS^j + C_{imn}S^mS^{nj} - 2S\delta_i^j + \\
&+ (2S^j\delta_i^k - D_iS^{jk} - C_{imn}S^{mj}S^{nk})\partial_k \log(e^K|f|^2)\Big]\,,\\
e_i(t) =& f^{-2}\Big[D_iS - \frac{1}{2}C_{ijk}S^jS^k - \\
&- (D_iS^j + C_{imn}S^mS^{nj} - 2S\delta_i^j)\partial_j \log(e^K|f|^2) + \\
&+ (\frac{3}{2}D_iS^{jk} + \frac{1}{2}C_{imn}S^{mj}S^{nk} - S^j\delta_i^k)\times \\
&\times \partial_j \log(e^K|f|^2)\partial_k \log(e^K|f|^2)\Big]\,.
\end{aligned}$$

Due to the genus-1 anomaly equation, $h_i(t)$ given by

$$h_i(t) = \partial_iF_1 + \frac{1}{2}C_{ijk}S^{jk} - \partial_iK\left(n + 2 - \frac{\chi}{24}\right) - \Gamma_{ij}^j$$

is also holomorphic.

For each t^i, Eq. (B.7) is of the form of the Schrödinger equation for a particle moving in an $(n+1)$-dimensional space of $\tilde{p}^i$ and $\tilde{q}$ in a t-dependent harmonic oscillator potential and a t-dependent constant magnetic field. Since it is a first order differential equation in t, we can solve it uniquely once we know ϑ at a particular value of t. The situation is similar to the case of the Wess-Zumino-Witten (WZW) model on a Riemann surface where the partition function satisfies the heat equation with the moduli of the surface Σ being time-like variables and the moduli of the holomorphic vector bundle on Σ being space-like variables. It is known that the WZW model is related to the three-dimensional Chern–Simons (CS) theory, and the heat equation in the WZW model is identified as the physical state condition for a wave–function in the CS theory. The similarity between the WZW model and the Kodaira–Spencer theory suggests that the Schrödinger type equation (B.7) for the (Borel–transformed) generating function ϑ may also be derived as a physical state condition of some higher-dimensional system. It would be very interesting to identify such a system. This would also explain the origin of the finite dimensional quantum system discussed in [28].

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