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Zero Measure Spectrum for the Almost Mathieu Operator

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Abstract: We study the almost Mathieu operator: $(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n)$, on $l^2(Z)$, and show that for all λ, θ, and (Lebesgue) a.e. α, the Lebesgue measure of its spectrum is precisely $|4 - 2|\lambda||$. In particular, for $|\lambda| = 2$ the spectrum is a zero measure cantor set. Moreover, for a large set of irrational α's (and $|\lambda| = 2$) we show that the Hausdorff dimension of the spectrum is smaller than or equal to 1/2.

1. Introduction

In this paper, we study the almost Mathieu (also called Harper's) operator on $l^2(Z)$. This is the (bounded, self adjoint) operator $H_{\alpha,\lambda,\theta}$, defined by:

$$H_{\alpha,\lambda,\theta} = H_0 + V_{\alpha,\lambda,\theta}\,, \qquad (H_0u)(n) = u(n+1) + u(n-1)\,,$$
$$(V_{\alpha,\lambda,\theta}u)(n) = \lambda\cos(2\pi\alpha n + \theta)u(n)\,, \tag{1.1}$$

where $\alpha, \lambda, \theta \in R$.

$H_{\alpha,\lambda,\theta}$ is a tight binding model for the Hamiltonian of an electron in a one dimensional lattice, subject to a commensurate (if α is rational) or incommensurate (if α is irrational) potential. It is also related to the Hamiltonian of an electron in a two dimensional lattice, subject to a perpendicular magnetic field [11, 13] (in which case the relevant energy spectrum is the union over θ of the energy spectra of $H_{\alpha,\lambda,\theta}$).

The almost Mathieu operator has been studied by many authors [1–13, 15, 17–24, 26], and many of its spectral characteristics are known. Our main result in this paper is:

Theorem 1. *If α is an irrational, for which there is a sequence of rationals $\{p_n/q_n\}$ obeying:*

$$\lim_{n\to\infty} {q_n}^2 \left|\alpha - \frac{p_n}{q_n}\right| = 0\,,$$

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then for every $\lambda, \theta \in R$*:*

$$|\sigma(\alpha, \lambda, \theta)| = |4 - 2|\lambda||\,,$$

where $\sigma(\alpha, \lambda, \theta)$ *is the spectrum of* $H_{\alpha,\lambda,\theta}$*, and* $|\cdot|$ *denotes Lebesgue measure.*

Remarks. 1) The set of irrationals characterized in Theorem 1 is precisely the set of irrationals having unbounded continued fraction expansions. This set is known to have full Lebesgue measure [16].
2) The θ independence part of Theorem 1 is immediate, since, for irrational α, $\sigma(\alpha, \lambda, \theta)$ itself is known to be independent of θ [9].

The equality $|\sigma(\alpha, \lambda, \theta)| = |4 - 2|\lambda||$ was conjectured by Aubry and Andre [1] to hold for every irrational α. It was later studied by Thouless [21], and by Avron, van Mouche, and Simon [3], who established the inequality $|\sigma(\alpha, \lambda, \theta)| \geq |4 - 2|\lambda||$ (for every α, λ, θ).

For $|\lambda| \neq 2$, Theorem 1 has already been proved in [17]. The main theme of the current paper is the handling of the case $|\lambda| = 2$, for which we prove:

Lemma 1. *Let* $p, q \in N$ *be relatively prime, and denote:*

$$S(\alpha, \lambda) \equiv \bigcup_{\theta} \sigma(\alpha, \lambda, \theta)\,;$$

then:

$$\frac{2(\sqrt{5} + 1)}{q} < |S(p/q, 2)| < \frac{8e}{q}$$

(where $e \equiv \exp(1) = 2.71\ldots$*).*

It should be remarked that a similar (though somewhat weaker) lower bound on $|S(p/q, 2)|$ was already established in [18]. It is the upper bound in Lemma 1, which is the main new result of the current paper and from which the completion of the proof of Theorem 1 follows.

Lemma 1 is strongly related to a conjecture of Thouless [21–24], which says:

$$\lim_{q \to \infty} q|S(p/q, 2)| = \text{const} = 9.32\ldots.$$

This conjecture was found numerically for sequences with p, q relatively prime and $q \to \infty$. It was derived analytically only for some sequences with fixed p and $q \to \infty$ [22, 23, 26]. Some (nonrigorous) analytical argumentation that it should also hold for more general cases was given in [18].

It is interesting to note that since $\sigma(\alpha, \lambda, \theta)$ has no isolated points [9], the vanishing of its measure for $|\lambda| = 2$ also implies:

Corollary 1.1. *For irrational* α *as in Theorem 1,* $\sigma(\alpha, 2, \theta)$ *is a (zero measure) Cantor set (i.e. a closed, nowhere dense set, with no isolated points).*

Moreover, if α is an irrational which is very well approximated by rationals, we will show that Lemma 1 implies an upper bound on the Hausdorff dimension of $\sigma(\alpha, 2, \theta)$, namely:

Theorem 2. *If* α *is an irrational obeying:*

$$q_n^4 \left| \alpha - \frac{p_n}{q_n} \right| < C\,,$$

for some constant C, and a sequence of rationals $\{p_n/q_n\}$ with $q_n \to \infty$, then:

$$\dim_H(\sigma(\alpha, 2, \theta)) \leq \tfrac{1}{2},$$

where $\dim_H(\cdot)$ denotes Hausdorff dimension.

Remark. The set of irrationals characterized in Theorem 2 has zero Lebesgue measure, but it contains a dense G_δ set, which makes it "generic" in the commonly used topological sense.

The analysis leading to our results is based on previous findings of Avron, van Mouche, and Simon [3], and this paper is, to a large extent, a continuation of their work. While some parts of this analysis were already carried out in [17] and in [18], for the reader's convenience, we shall repeat the relevant derivations of those papers.

In Sect. 2 we describe some preliminaries and previously obtained results. In Sect. 3 we prove Lemma 1, and in Sect. 4 we prove Theorem 1. Finally, in Sect. 5, we prove Theorem 2.

2. Preliminaries

We begin this section with a remark about the considered ranges of α, λ, θ. Since $H_{\alpha,\lambda,\theta}$ is invariant under: $\alpha \to \alpha \pm 1$, $\theta \to \theta \pm 2\pi$, we may always assume: $\alpha \in [0, 1]$, $\theta \in [0, 2\pi]$. This has no effect on the correctness of our results for more general values of α and θ. Moreover, a sign change of λ ($\lambda \to -\lambda$) is equivalent to a translation of θ by π. Thus, any quantity or result which is independent of θ must be invariant under a sign change of λ, and throughout the rest of the paper we will usually assume: $\lambda \geq 0$.

In what follows we will be largely concerned with the spectral analysis of the almost Mathieu operator at rational frequencies. That is, we will consider $H_{\alpha,\lambda,\theta}$, where $\alpha = p/q$, $p, q \in N$, and we assume throughout that p and q are relatively prime (i.e. they have no common divisor other than 1). In this case $\sigma(\alpha, \lambda, \theta)$ does depend on θ, and we will also be interested in the two spectral sets:

$$\begin{aligned} S(\alpha, \lambda) &\equiv \bigcup_\theta \sigma(\alpha, \lambda, \theta), \\ S_-(\alpha, \lambda) &\equiv \bigcap_\theta \sigma(\alpha, \lambda, \theta). \end{aligned} \tag{2.1}$$

These sets are also well define for irrational α, but in this case: $S_-(\alpha, \lambda) = S(\alpha, \lambda) = \sigma(\alpha, \lambda, \theta)$. As we shall see later, the set $S(\alpha, \lambda)$ has good continuity properties (in α), and our results for irrational α are essentially based on the study of $S(\alpha, \lambda)$ for rational α.

A central role in the spectral analysis of $H_{p/q,\lambda,\theta}$ is played by the discriminant $D_{p/q,\lambda,\theta}(E)$, defined by:

$$\begin{aligned} D_{p/q,\lambda,\theta}(E) \equiv \text{Trace} \Bigg[&\begin{pmatrix} E - V(1) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - V(2) & -1 \\ 1 & 0 \end{pmatrix} \cdots \\ &\begin{pmatrix} E - V(q) & -1 \\ 1 & 0 \end{pmatrix} \Bigg], \end{aligned} \tag{2.2}$$

where $V(n) \equiv \lambda \cos(2\pi(p/q)n + \theta)$. $D_{p/q,\lambda,\theta}(E)$ is a polynomial of order q (in E) having the following properties (see e.g. [25]):

(i) $D_{p/q,\lambda,\theta}(E)$ has q real simple zeroes.
(ii) $D_{p/q,\lambda,\theta}(E)$ is larger than or equal to 2 at all its maxima points, and it is smaller than or equal to -2 at all its minima points.

The spectrum $\sigma(p/q,\lambda,\theta)$ is precisely the inverse image under $D_{p/q,\lambda,\theta}(E)$ of the interval $[-2,2]$ (i.e. it is precisely the set of E's for which: $-2 \le D_{p/q,\lambda,\theta}(E) \le 2$). Thus, from the properties of $D_{p/q,\lambda,\theta}(E)$ it is seen that $\sigma(p/q,\lambda,\theta)$ is made of q bands (closed intervals), such that $D_{p/q,\lambda,\theta}(E)$ is strongly monotone on each band. A remarkable formula, originally due to Chambers [6] (also see [5] for a proof), gives the θ dependence of $D_{p/q,\lambda,\theta}(E)$:

Proposition 2.1. *If p, q are relatively prime, then:*

$$D_{p/q,\lambda,\theta}(E) = \Delta_{p/q,\lambda}(E) - 2\left(\frac{\lambda}{2}\right)^q \cos\theta q\,,$$

where $\Delta_{p/q,\lambda}(E) \equiv D_{p/q,\lambda,\pi/29}(E)$.

Proposition 2.1 implies that $S(p/q,\lambda)$ is precisely the inverse image under $\Delta_{p/q,\lambda}(E)$ of the interval $[-2-2(\lambda/2)^q,\ 2+2(\lambda/2)^q]$. Moreover, it shows that if $\lambda > 2$ then $S_{-}(p/q,\lambda) = \emptyset$, and if $\lambda \le 2$ then $S_{-}(p/q,\lambda)$ is the inverse image under $\Delta_{p/q,\lambda}(E)$ of the interval $[-2+2(\lambda/2)^q,\ 2-2(\lambda/2)^q]$. We remark that from the fact that the above properties (i) and (ii) of $D_{p/q,\lambda,\theta}(E)$ hold for every θ, and from Proposition 2.1, it follows that $\Delta_{p/q,\lambda}(E)$ is larger than or equal to $2+2(\lambda/2)^q$ at all its maxima points, and it is smaller than or equal to $-2-2(\lambda/2)^q$ at all its minima points. Moreover, each of the sets $S(p/q,\lambda)$ and $S_{-}(p/q,\lambda)$ (when it is not empty) is made of q bands, such that $\Delta_{p/q,\lambda}(E)$ is strongly monotone on each band.

An important property of $H_{\alpha,\lambda,\theta}$ is the Aubry duality [1], which allows relating eigenfunctions and spectra of $H_{\alpha,\lambda,\theta}$ to those of $H_{\alpha,4/\lambda,\theta}$. The following version of this duality was rigorously proven by Avron and Simon [2]:

Proposition 2.2. *For every real α:*

$$S(\alpha,\lambda) = \frac{\lambda}{2}\,S(\alpha,4/\lambda)\,.$$

Thus, it is sufficient to study $S(\alpha,\lambda)$ for $0 \le \lambda \le 2$, since, for $\lambda > 2$, $S(\alpha,\lambda)$ is obtained by Proposition 2.2. from the $\lambda < 2$ case.

Avron, van Mouche, and Simon [3] proved the following:

Proposition 2.3. *For $0 \le \lambda \le 2$ and p, q relatively prime:*

(i) $$|S_{-}(p/q,\lambda)| = 4 - 2\lambda\,.$$

(ii) $$4 - 2\lambda \le |S(p/q,\lambda)| \le 4 - 2\lambda + 4\pi\left(\frac{\lambda}{2}\right)^{q/2}.$$

In particular, Proposition 2.3 shows that if $0 \le \lambda < 2$, and if $p_n/q_n \to \alpha$, where the p_n/q_n's are rationals, and α is irrational, then $|S(p_n/q_n,\lambda)| \to 4-2\lambda$. If $\lambda = 2$ then statement (ii) of Proposition 2.3 becomes useless; but, it was shown in [18] that, in this case, the remarkable exact equality for $|S_{-}(p/q,\lambda)|$ (statement (i)) translates to an exact equality involving the slopes of $\Delta_{p/q,\lambda}(E)$ at its zero crossings. Namely:

Proposition 2.4. *If p, q are relatively prime, then:*

$$\sum_{\nu=1}^{q} \frac{1}{|\Delta'_{p/q,2}(E_\nu)|} = \frac{1}{q},$$

where $\Delta'_{p/q,\lambda}(E) \equiv \frac{d}{dE} \Delta_{p/q,\lambda}(E)$, and $E_1, E_2, \ldots, E_q$ are the q zeroes of $\Delta_{p/q,2}(E)$.

Proof. Since $S_-(p/q, \lambda)$ is the inverse image under $\Delta_{p/q,\lambda}(E)$ of the interval $[-2 + 2(\lambda/2)^q, 2 - 2(\lambda/2)^q]$, and since $\Delta_{p/q,\lambda}(E)$ is also a polynomial in λ, we have, for $\lambda < 2$, in the limit $\lambda \to 2$:

$$|S_-(p/q, \lambda)| \sim \sum_{\nu=1}^{q} \frac{4 - 4(\lambda/2)^q}{|\Delta'_{p/q,\lambda}(E_\nu)|}. \tag{2.3}$$

Thus, from statement (i) of Proposition 2.3 we obtain:

$$\sum_{\nu=1}^{q} \frac{1}{|\Delta'_{p/q,2}(E_\nu)|} = \lim_{\lambda \nearrow 2} \frac{|S_-(p/q, \lambda)|}{4 - 4(\lambda/2)^q} = \lim_{\lambda \to 2} \frac{4 - 2\lambda}{4 - 4(\lambda/2)^q} = \frac{1}{q}. \quad \bigcirc \tag{2.4}$$

Proposition 2.4 is in the heart of Lemma 1 that we prove in the next section.

For every $\alpha, \lambda \in R$, the set $S(\alpha, \lambda)$ is compact, and, therefore, it has definite edges: $\max S(\alpha, \lambda), \min S(\alpha, \lambda) \in S(\alpha, \lambda)$. The complement of $S(\alpha, \lambda)$ in the interval $[\min S(\alpha, \lambda), \max S(\alpha, \lambda)]$ is open, and it is therefore a union of countably many (finite) open intervals. We shall refer to such open intervals, when they are chosen to have maximal length, as gaps in $S(\alpha, \lambda)$, and we shall denote their union by $G(\alpha, \lambda)$. That is:

$$G(\alpha, \lambda) \equiv [\min S(\alpha, \lambda), \max S(\alpha, \lambda)] \backslash S(\alpha, \lambda), \tag{2.5}$$

and so we have:

$$|S(\alpha, \lambda)| = \max S(\alpha, \lambda) - \min S(\alpha, \lambda) - |G(\alpha, \lambda)|. \tag{2.6}$$

When α is rational ($\alpha = p/q$) we have seen that $S(\alpha, \lambda)$ is made of q bands. Thus, it has at most $q - 1$ gaps. When α is irrational $S(\alpha, \lambda)$ may have an infinite number of gaps.

We conclude this section by quoting another result of Avron, van Mouche, and Simon [3], this time regarding the continuity properties of $S(\alpha, \lambda)$:

Proposition 2.5. *For every $\lambda > 0$, there is a constant C, such that if $|\alpha - \alpha'| < C$ (for any $\alpha, \alpha' \in R$), then for every $E \in S(\alpha, \lambda)$, there is $E' \in S(\alpha', \lambda)$ with:*

$$|E - E'| < 6(\lambda|\alpha - \alpha'|)^{1/2}.$$

Proposition 2.5 has the immediate corollary:

Corollary 2.1. (i) *If $|\alpha - \alpha'| < C$, then for every gap in $S(\alpha, \lambda)$ with midpoint E_g, and measure $|g|$ larger than $12(\lambda|\alpha - \alpha'|)^{1/2}$, there is a corresponding (containing E_g) gap in $S(\alpha', \lambda)$ with measure larger than: $|g| - 12(\lambda|\alpha - \alpha'|)^{1/2}$.*
(ii) *The same continuity as in* (i) *also holds for the extreme edges of $S(\alpha, \lambda)$, namely, for $|\alpha - \alpha'| < C$:*

$$\left| \begin{matrix}\max \\ \min\end{matrix} S(\alpha, \lambda) - \begin{matrix}\max \\ \min\end{matrix} S(\alpha', \lambda) \right| < 6(\lambda|\alpha - \alpha'|)^{1/2}.$$

3. Proof of Lemma 1

In this section we consider $\lambda = 2$ and a fixed rational p/q, where p and q are relatively prime. For simplicity of notation we denote: $\Delta(E)$ for $\Delta_{p/q,2}(E)$ and S for $S(p/q,2)$.
(i) *Proof of the upper bound.* Consider a nonexternal band of $S: I_\nu = [E_1^\nu, E_2^\nu]$, and suppose that $\Delta(E)$ is increasing on I_ν. Denote by E_ν the zero of $\Delta(E)$ inside I_ν, and by E_0^ν the maximum of $\Delta(E)$ just above E_2^ν. (In principle, we can have $E_0^\nu = E_2^\nu$, but typically $E_0^\nu > E_2^\nu$, and E_0^ν is inside the gap just above I_ν). Define:

$$f(E) \equiv \frac{d}{dE}(\ln(\Delta(E)))\,. \tag{3.1}$$

Since $\Delta(E)$ can be expressed as:

$$\Delta(E) = \prod_{j=1}^{q}(E - E_j)\,, \tag{3.2}$$

$f(E)$ can be written as:

$$f(E) = \sum_{j=1}^{q} \frac{1}{E - E_j}\,, \tag{3.3}$$

and we have:

$$f'(E) \equiv \frac{d}{dE} f(E) = -\sum_{j=1}^{q} \frac{1}{(E - E_j)^2}\,. \tag{3.4}$$

From (3.4) we see that:

$$f'(E) < \frac{-1}{(E - E_\nu)^2}\,, \tag{3.5}$$

and since E_0^ν is a zero of $f(E)$, we have for every $E \in (E_\nu, E_0^\nu)$:

$$f(E) = -\int_E^{E_0^\nu} f'(E')dE' > \int_E^{E_0^\nu} \frac{dE'}{(E' - E_\nu)^2} = \frac{1}{E - E_\nu} - \frac{1}{E_0^\nu - E_\nu}\,. \tag{3.6}$$

Now, consider $E \in (E_\nu, E_2^\nu)$. Since $\Delta(E_2^\nu) = 4$, we have:

$$\ln\frac{4}{\Delta(E)} = \ln\Delta(E_2^\nu) - \ln\Delta(E) = \int_E^{E_2^\nu} f(E')dE'\,, \tag{3.7}$$

and by using (3.7) and the estimate (3.6), we obtain:

$$\ln\frac{4}{\Delta(E)} > \ln\left(\frac{E_2^\nu - E_\nu}{E - E_\nu}\right) - 1\,. \tag{3.8}$$

Equation (3.8) implies:

$$\frac{4}{\Delta(E)} > \frac{1}{e}\left(\frac{E_2^\nu - E_\nu}{E - E_\nu}\right), \tag{3.9}$$

which can also be written as:

$$E_2^\nu - E_\nu < 4e\,\frac{E - E^\nu}{\Delta(E)}\,. \tag{3.10}$$

Since $\Delta(E_\nu) = 0$, we obtain from (3.10), by letting $E \to E_\nu$:

$$E_2^\nu - E_\nu < \frac{4e}{\Delta'(E_\nu)} . \tag{3.11}$$

Clearly, a similar calculation can also be carried for the lower part of the band, by integrating $f'(E)$ from the minimum of $\Delta(E)$ just below (or at) E_1^ν. Thus, we also have:

$$E_\nu - E_1^\nu < \frac{4e}{\Delta'(E_\nu)} , \tag{3.12}$$

which (together with (3.11)) implies:

$$|I_\nu| = E_2^\nu - E_1^\nu < \frac{8e}{|\Delta'(E_\nu)|} . \tag{3.13}$$

It's easy to see that (3.13) could be obtained with a similar calculation, also if $\Delta(E)$ was decreasing on I_ν. Thus, (3.13) clearly holds for all of the nonexternal bands. If I_ν is an external band, we can still make a similar calculation to (3.1)–(3.11) for the "less external" part of this band, and obtain either (3.11) (for the lowest band) or (3.12) (for the highest band). But, since $|\Delta'(E)|$ is monotone on an external band, the "more external" part of such a band must be smaller than its "less external" part. Thus, (3.13) holds for every band, and from Proposition 2.4 we obtain:

$$|S| = \sum_{\nu=1}^{q} |I_\nu| < \frac{8e}{q} . \quad \bigcirc \tag{3.14}$$

(ii) *Proof of the lower bound.* For each band of $S : I_\nu = [E_1^\nu, E_2^\nu]$, we denote the two parts of the band by:

$$b_1^\nu \equiv [E_1^\nu, E_\nu] , \qquad b_2^\nu \equiv [E_\nu, E_2^\nu] . \tag{3.15}$$

Since $\Delta'(E)$ is a polynomial of order $q - 1$, which has $q - 1$ distinct real zeroes, $|\Delta'(E)|$ has a single maximum between every two consecutive zeroes of $\Delta'(E)$, and it is monotone above and below the extreme zeroes of $\Delta'(E)$. Thus, $|\Delta'(E)|$ has a single maximum on each band (which may be at the edge of the band), and it is monotone on every subinterval of the band which does not contain this maximum. This implies that for each band, either for all $E \in b_1^\nu$ (if the maximum is on b_2^ν), or for all $E \in b_2^\nu$ (if the maximum is on b_1^ν), we have:

$$|\Delta'(E)| \leq |\Delta'(E_\nu)| . \tag{3.16}$$

Since:

$$\int_{E_1^\nu}^{E_\nu} |\Delta'(E)| dE = |\Delta(E_\nu) - \Delta(E_1^\nu)| = 4 , \tag{3.17}$$

and also

$$\int_{E_\nu}^{E_2^\nu} |\Delta'(E)| dE = |\Delta(E_2^\nu) - \Delta(E^\nu)| = 4 , \tag{3.18}$$

we have, either for $i = 1$, or for $i = 2$:

$$|b_i^\nu| > \frac{4}{|\Delta'(E_\nu)|} \equiv l_\nu . \tag{3.19}$$

In [18], (3.19) has been used to obtain: $|S| > 4/q$. We shall now improve this bound.

Suppose that I_ν is a nonextermal band and that $\Delta(E)$ is increasing on I_ν, and consider the polynomial:

$$G(E) \equiv \Delta(E) + 4 , \tag{3.20}$$

which has a zero at E_1^ν. A similar estimate to (3.1)–(3.6) shows that for every $E \in (E_1^\nu, E_0^\nu)$:

$$\frac{G'(E)}{G(E)} = \frac{d}{dE} \ln G(E) > \frac{1}{E - E_1^\nu} - \frac{1}{E_0^\nu - E_1^\nu} . \tag{3.21}$$

By taking $E = E_\nu$ in (3.21), we obtain:

$$\frac{\Delta'(E_\nu)}{4} = \frac{G'(E_\nu)}{G(E_\nu)} > \frac{1}{|b_1^\nu|} - \frac{1}{|b_1^\nu| + |b_2^\nu|} = \frac{|b_2^\nu|}{|b_1^\nu|(|b_1^\nu| + |b_2^\nu|)} , \tag{3.22}$$

which implies:

$$|b_1^\nu| > \frac{l_\nu |b_2^\nu|}{|b_1^\nu| + |b_2^\nu|} . \tag{3.23}$$

Clearly, by considering: $H(E) \equiv \Delta(E) - 4$ instead of $G(E)$, and integrating from the minimum of $H(E)$ just below E_1^ν, we can similarly obtain (3.23) with b_1^ν and b_2^ν interchanged, namely:

$$|b_2^\nu| > \frac{l_\nu |b_1^\nu|}{|b_1^\nu| + |b_2^\nu|} . \tag{3.24}$$

We have seen that either $|b_1^\nu| > l_\nu$ or $|b_2^\nu| > l_\nu$. Suppose that $|b_2^\nu| > l_\nu$, then (3.23) implies:

$$|b_1^\nu| > \frac{{l_\nu}^2}{|b_1^\nu| + l_\nu} , \tag{3.25}$$

which can be rewritten as:

$$|b_1^\nu|^2 + |b_1^\nu| l_\nu - {l_\nu}^2 > 0 . \tag{3.26}$$

Solving the appropriate quadratic equation shows that (3.26) implies:

$$|b_1^\nu| > \frac{\sqrt{5} - 1}{2} l_\nu . \tag{3.27}$$

Similarly, if $|b_1^\nu| > l_\nu$, (3.24) would imply:

$$|b_2^\nu| > \frac{\sqrt{5} - 1}{2} l_\nu , \tag{3.28}$$

so in either case we have:

$$|I_\nu| = |b_1^\nu| + |b_2^\nu| > \frac{\sqrt{5} + 1}{2} l_\nu , \tag{3.29}$$

and it is clear that (3.29) holds for every nonextermal band. For the case of an extermal band, only one of the inequalities (3.23) or (3.24) can be obtained. But, in this case, due to the monotonicity of $|\Delta'(E)|$, we also know which one of the $|b_i^\nu|$ $(i = 1, 2)$ is larger and obeys: $|b_i^\nu| > l_\nu$. It is easy to verify that $|b_2^\nu| > l_\nu$ corresponds to the case where (3.23) holds, and that the other case corresponds to (3.24). Thus, we obtain (3.29) for all the bands, and from Proposition 2.4 we get:

$$|S| > \frac{2(\sqrt{5}+1)}{q}. \quad \bigcirc \tag{3.30}$$

4. Proof of Theorem 1

Lemma 4.1. *For every $\lambda \in R$, and a sequence of rationals $\{p_n/q_n\}$, with p_n, q_n relatively prime, and $q_n \to \infty$:*

$$\lim_{n\to\infty} |S(p_n/q_n, \lambda)| = |4 - 2|\lambda||.$$

Proof. Combining statement (ii) of Proposition 2.3 and Lemma 1 we obtain for every $0 \le \lambda \le 2$: $|S(p_n/q_n, \lambda)| \to 4 - 2\lambda$. From that and from Proposition 2.2 the lemma follows. $\bigcirc$

Proposition 4.1 (Thouless [21]; Avron, van Mouche, and Simon [3]). *For any irrational α, and $\lambda, \theta \in R$:*

$$|\sigma(\alpha, \lambda, \theta)| \ge |4 - 2|\lambda||.$$

Proof. Let $\{g_j(\alpha, \lambda)\}_{j=1}^\infty$ be the gaps in $S(\alpha, \lambda)$ (ordered somehow), and pick some $\varepsilon > 0$. Since $\sum_{j=1}^{\infty} |g_j(\alpha, \lambda)| = |G(\alpha, \lambda)|$, there is a finite J_ε such that $\sum_{j=1}^{J_\varepsilon} |g_j(\alpha, \lambda)| > |G(\alpha, \lambda)| - \varepsilon$. Now, consider a sequence of rationals: $p_n/q_n \to \alpha$. From statement (i) of Corollary 2.1 we have:

$$\liminf_{n\to\infty} |G(p_n/q_n, \lambda)| \ge \sum_{j=1}^{J_\varepsilon} |g_j(\alpha, \lambda)| > |G(\alpha, \lambda)| - \varepsilon, \tag{4.1}$$

and from statement (ii):

$$\lim_{n\to\infty} \begin{matrix}\max\\ \min\end{matrix} S(p_n/q_n, \lambda) = \begin{matrix}\max\\ \min\end{matrix} S(\alpha, \lambda). \tag{4.2}$$

Thus, from (2.6) we obtain:

$$|S(\alpha, \lambda)| > \limsup_{n\to\infty} |S(p_n/q_n, \lambda)| - \varepsilon, \tag{4.3}$$

which by Lemma 4.1 implies:

$$|S(\alpha, \lambda)| > |4 - 2|\lambda|| - \varepsilon. \tag{4.4}$$

Since $S(\alpha, \lambda) = \sigma(\alpha, \lambda, \theta)$ and since ε is arbitrary this completes the proof. $\bigcirc$

Proof of Theorem 1. Let α be an appropriate irrational, and let $\{p_n/q_n\}$ be a sequence of rationals obeying: $\lim_{n\to\infty} {q_n}^2|\alpha - p_n/q_n| = 0$. Obviously, $q_n \to \infty$, and we can assume p_n, q_n to be relatively prime. Since there are at most $q_n - 1$ gaps in $S(p_n/q_n, \lambda)$, we obtain from statement (i) of Corollary 2.1 (for $|\alpha - p_n/q_n| < C$):

$$|G(\alpha, \lambda)| > |G(p_n/q_n, \lambda)| - 12(q_n - 1)(\lambda|\alpha - p_n/q_n|)^{1/2}. \tag{4.5}$$

By (2.6) and statement (ii) of Corollary 2.1 this implies:

$$|S(\alpha,\lambda)| < |S(p_n/q_n,\lambda)| + 12q_n(\lambda|\alpha - p_n/q_n|)^{1/2} . \tag{4.6}$$

As $n \to \infty$, we have from Lemma 4.1: $|S(p_n/q_n,\lambda)| \to |4 - 2|\lambda||$, and by our assumption on $\{p_n/q_n\}$: $q_n|\alpha - p_n/q_n|^{1/2} \to 0$. Thus, (4.6) implies:

$$|\sigma(\alpha,\lambda,\theta)| = |S(\alpha,\lambda)| \le |4 - 2|\lambda|| , \tag{4.7}$$

which together with Proposition 4.1 completes the proof. ○

5. Proof of Theorem 2

Lemma 5.1. *Let $S \subset R$, and suppose that S has a sequence of covers: $\{S_n\}_{n=1}^{\infty}$, $S \subset S_n$, such that each S_n is a union of q_n intervals, $q_n \to \infty$ as $n \to \infty$, and for each n:*

$$|S_n| < \frac{C}{q_n^{\beta}} ,$$

where β and C are positive constants; then:

$$\dim_H(S) \le \frac{1}{1+\beta} .$$

Proof. Let $S_n = \bigcup_{\nu=1}^{q_n} b_\nu^n$, where each b_ν^n is an interval. Without loss, we can assume that the b_ν^n's are disjoint. Let $t = 1/(1+\beta)$, then:

$$\frac{1}{q_n}\sum_{\nu=1}^{q_n} |b_\nu^n|^t \le \left(\frac{1}{q_n}\sum_{\nu=1}^{q_n} |b_\nu^n|\right)^t , \tag{5.1}$$

which implies:

$$\begin{aligned}\sum_{\nu=1}^{q_n} |b_\nu^n|^t &\le {q_n}^{1-t}\left(\sum_{\nu=1}^{q_n} |b_\nu^n|\right)^t = {q_n}^{1-t}|S_n|^t \\ &\le {q_n}^{1-t}\left(\frac{C}{{q_n}^{\beta}}\right)^t = {q_n}^{1-(1+\beta)t}C^t = C^t .\end{aligned} \tag{5.2}$$

Recall that the Hausdorff dimension of S is given by (see e.g. [14]):

$$\dim_H(S) = \inf\left\{ t \in R^+ \;\middle|\; \lim_{\delta\to 0}\ \inf_{\delta-\text{covers}} \sum_\nu |b_\nu|^t < \infty \right\}, \tag{5.3}$$

where a δ-cover is a cover of S: $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that each b_ν is an interval, and $|b_\nu| < \delta$. Thus, since $q_n \to \infty$ as $n \to \infty$, (5.2) implies: $\dim_H(S) \le 1/(1+\beta)$. ○

Proof of Theorem 2. Let α be an appropriate irrational, and let $\{p_n/q_n\}$ be a sequence of rationals obeying: $q_n \to \infty$ as $n \to \infty$, and ${q_n}^4|\alpha - p_n/q_n| < C$. Clearly, we can assume that p_n and q_n are relatively prime. For each n, $S(p_n/q_n, 2)$ is made of q_n bands:

$$S(p_n/q_n, 2) = \bigcup_{\nu=1}^{q_n} [E_1^{\nu,n}, E_2^{\nu,n}], \tag{5.4}$$

and by Proposition 2.5, we have (for sufficiently large n):

$$S(\alpha, 2) \subset \bigcup_{\nu=1}^{q_n} [E_1^{\nu,n} - 6(2|\alpha - p_n/q_n|)^{1/2},\ E_2^{\nu,n} + 6(2|\alpha - p_n/q_n|)^{1/2}] \equiv S_n\,. \tag{5.5}$$

S_n is a cover of $S(\alpha, 2)$ by q_n intervals, and from Lemma 1 and our assumptions on $\{p_n/q_n\}$ we have:

$$\begin{aligned} |S_n| &\le |S(p_n/q_n, 2)| + 12q_n(2|\alpha - p_n/q_n|)^{1/2} \\ &< \frac{8e}{q_n} + 12q_n\left(\frac{2C}{q_n{}^4}\right)^{1/2} = \frac{8e + 12(2C)^{1/2}}{q_n}\,. \end{aligned} \tag{5.6}$$

Thus, Lemma 5.1 implies: $\dim_H(S(\alpha, 2)) \le 1/2$. ○

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