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# Rings of Skew Polynomials and Gel'fand–Kirillov Conjecture for Quantum Groups

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**Abstract:** We introduce and study action of quantum groups on skew polynomial rings and related rings of quotients. This leads to a "$q$-deformation" of the Gel'fand–Kirillov conjecture which we partially prove. We propose a construction of automorphisms of certain non-commutative rings of quotients coming from complex powers of quantum group generators; this is applied to explicit calculation of singular vectors in Verma modules over $U_q(\mathfrak{sl}_{n+1})$. We finally give a definition of a $q$-connection with coefficients in a ring of skew polynomials and study the structure of quantum group modules twisted by a $q$-connection.

## 1. Introduction

This work was mainly inspired by Feigin's construction which associates to an element of the Weyl group $w \in W$ an associative algebra homomorphism of a "nilpotent part" of a quantum group to an appropriate algebra of skew polynomials:

$$\Phi(w)\colon U_q^-(\mathfrak{g}) \to C[X]\,, \tag{1}$$

where $X$ stands for $X_1, \ldots, X_l$, $l$ is a length of $w$ *and* $X_j X_i = q^{\alpha_{ij}} X_i X_j$ for some $\alpha_{ij} \in Z$, $1 \leqq i, j \leqq l$. The main topics treated in the work are as follows.

1. *Realizations of Lie algebras and quantum groups and Gel'fand-Kirillov conjecture.* The fact that a Lie algebra of an algebraic group ("algebraic Lie algebra") can be realized in differential operators acting on a suitable manifold is, probably, more fundamental than the notion of a Lie algebra. Explicit formulas for such a realization in the case when the algebra is simple, the manifold is a big cell of a flag

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manifold have become especially popular recently because of their relation to the free field approach to 2-dimensional conformal field theory ([F-F, B-McC-P1]). An important property of the realization was discovered by Gel'fand and Kirillov [G-K1] long ago and in a remarkable generality. Their observation is that however complicated classification of algebraic Lie algebras may be, equivalence classes of rings of quotients of universal enveloping algebras are labelled by pairs of positive integers: a ring of quotients of a universal enveloping algebra is isomorphic with a ring of quotients of a ring of differential operators on $n$ variables with polynomial coefficients trivially extended by a $k$-dimensional center, where $k$ is a dimension of a generic orbit in the coadjoint representation and $2n+k$ is equal to the dimension of the algebra. (This conjecture has been proven by themselves and others in many cases [G-K1, J, McC].)

A natural class of rings suitable for formation of rings of quotients is provided by the so-called *Ore domains*. Besides above mentioned universal enveloping algebras of finite dimensional Lie algebras and rings of differential operators the class of Ore domains comprises (deformed) enveloping algebras of affine Lie algebras, rings of skew polynomials and $q$-difference operators. We prove that Feigin's morphism $\Phi(w_0)$ associated to the longest Weyl group element provides an isomorphism of rings of quotients $Q(U_q^-(\mathfrak{sl}_{n+1}) \approx Q(\mathbf{C}[X])$. This isomorphism allows us to equip $Q(\mathbf{C}[X])$ with a structure of $U_q(\mathfrak{sl}_{n+1})$-module. More precisely we define an $n$-parameter family of associative algebra homomorphisms from $U_q(\mathfrak{sl}_{n+1})$ to an algebra of $q$-difference operators with coefficients in $Q(\mathbf{C}[X])$. We conjecture that this provides an isomorphism of $Q(U_q(\mathfrak{sl}_{n+1}))$ with an $n$-dimensional central extension of the algebra of $q$-difference operators. We prove this conjecture for $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$ in a slightly weaker form.

2. *Complex powers, automorphisms and screening operators.* A remarkable observation made in early works on Kac–Moody algebras [L-W, F-K] is that affine Lie algebras, like finite dimensional simple ones, are also realized in differential operators, though on infinitely many variables. A family of such realizations depending on a highest weight $\lambda$ was constructed by Wakimoto [*W*] for $\widehat{\mathfrak{sl}_2}$ and by Feigin and Frenkel [F-F] for all non-twisted affine Lie algebras. Thus obtained modules are now known as *Wakimoto modules* $F(\lambda)$. The main ingredient of the 2-dimensional conformal field theory associated to an affine algebra $\mathfrak{g}$ is a 2-sided complex consisting of direct sums of Wakimoto modules

$$\cdots \to F^{(-1)} \to F^{(0)} \to F^{(1)} \to \cdots$$

such that its homology is concentrated in the 0-th dimension and is equal to an irreducible highest weight module over $\hat{\mathfrak{g}}$ (BRST *resolution*).

Bouwknegt, McCarthy and Pilch revealed a quantum group structure hidden in the differential of the BRST resolution. Recall that $U_q(\mathfrak{g})$-morphisms of a Verma module $M(\lambda)$ into a Verma module $M(\mu)$ are in 1-1 correspondence with *singular vectors* of the weight $\lambda$ in the latter (the correspondence is established by assigning to a morphism an image of the vacuum vector under this morphism). Denote by $\mathrm{Sing}_\lambda(M(\mu))$ the set of singular vectors of the weight $\lambda$ in $M(\mu)$. It is argued in [M-McC-P1, B-McC-P2] that there is a linear map

$$\mathrm{Sing}_\lambda(M(\mu)) \to \mathrm{Hom}_{\hat{\mathfrak{g}}}(F(\lambda), F(\mu)) , \tag{2}$$

and that conjecturally this map is an isomorphism.

Singular vectors in Verma modules over quantum groups related to an arbitrary Kac–Moody algebra were found in the form [M-F-F, M]

$$F_{i_l}^{s_l} \cdots F_{i_1}^{s_1} F_{i_0}^{N} F_{i_1}^{t_1} \cdots F_{i_l}^{t_l}, \tag{3}$$

where $s_i, t_i, 1 \leqq i \leqq l$ are appropriate complex numbers, $F_i, 1 \leqq i \leqq n$ are canonical Cartan generators of $U_q^-(\mathfrak{g})$, $N \in \mathbf{N}$. Here we carry out an explicit calculation of (3), i.e. rewrite it in the form containing only natural powers of $F$'s. Observe that the map (2) is determined by assinging to each $F_i$ what is known as a *screening operator*.

One of the consequences of the prescription how to choose powers in (3) is that $s_i + t_i$, $1 \leqq i \leqq l$ are all non-negative integers. More generally, one may consider a map

$$U_q^-(\mathfrak{g}) \ni p \mapsto F_i^{\beta} p F_i^{-\beta}, \quad \beta \in \mathbf{C}.$$

A simple calculation using the notion of the *q-commutator* shows that this map extends to an automorphism of the quotient ring $Q(U_q^-(\mathfrak{g}))$. Therefore a singular vector is, roughly speaking, obtained by a sequence of automorphisms applied to a Cartan generator.

Similarly one may consider an operator of conjugation by a complex power of a linear form

$$\mathbf{C}[x_1, \ldots, x_k] \ni p \mapsto (x_{i_1} + \cdots x_{i_l})^{\beta} p (x_{i_l} + \cdots x_{i_l})^{-\beta},$$

acting on a certain completion of a ring of skew polynomials $\mathbf{C}[x_1, \ldots, x_k]$. Simple but nice calculation based on the $q$-binomial theorem shows that this map actually determines an automorphism of the ring of quotients $Q[\mathrm{C}[x_1, \ldots, x_k])$. This construction may be interesting in its own right: unlike the things in commutative realm, the very existence of (non-trivial) automorphisms of $Q(\mathbf{C}[x_1, \ldots, x_k])$ is not quite obvious. Combined with Feigin's morphism, this construction answers an informal question: "how does it happen that complex powers in the singular vector formula cancel out?" Another application of these automorphisms is that they produce natural examples of $q$-connections with coefficients in skew polynomials.

3. *Quantum group modules twisted by q-connections.* It has been realized [A-Y, F-G-P-P] that the "singular vector decoupling condition" makes it necessary to consider non-bounded–neither highest nor lowest weight – modules in 2-dimensional conformal field theory at a rational level. On the other hand the singular vector formula (3) makes it natural to consider an extension of a Verma module by complex powers of generators, which transparently produces non-bounded modules. It was shown in [F-M] that the duals to such modules are realized in multi-valued functions on a flag manifold or, in other words, in modules twisted by connections; in particular a family of integral intertwining operators acting among such modules was constructed. (One may also find in [F-M] and in the forthcoming paper [I-M] integral formulas for solutions to Knizhnik–Zamolodchikov equations with coefficients in non-bounded modules.)

Here we adjust the definition of a $q$-connection given by Aomoto and Kato [A-K] in the commutative case to the case of skew polynomials. This definition identifies $q$-connections with the cohomology group $H^1(\mathbf{Z}^k, Q(\mathbf{C}[x_1, \ldots, x_k]))$. We also produce a family of elements of $H^1(\mathbf{Z}^k, Q(\mathbf{C}[x_1, \ldots, x_k]))$ associated with complex powers of linear forms, all this being independent of quantum groups. In the case when the ring of skew polynomials is the one coming from Feigin's morphism, the twisting by such a $q$-connection is nothing but a passage from a Verma module to its extension by complex powers of generators. This allows to construct $q$-analogues of the intertwiners of [F-M], which in the quantum case may be thought of as right multiplications by certain complex powers of linear forms. We also find out what the $U_q(\mathfrak{g})$-module structure of a Verma module extended by a complex power of only 1 generator is. Such a module can be viewed as a module induced from a non-bounded module over a parabolic subalgebra. It turns out that its structure is formally close to that of a Verma module. In particular, the notion of a singular vector is naturally replaced with that of a *singular chain* and a singular chain encodes information on a family of singular vectors.

## 2. Main Definitions Related to Quantum Groups

The material of this section is fairly standard. The usual reference is the work [DC-K].

1. Let, as usual, $A=(a_{ij})$, $1\leqq i,j\leqq n$ stand for a generalized symmetrizable Cartan matrix, symmetrized by non-zero relatively prime integers $d_1,\ldots,d_n$ such that $d_i a_{ij}=d_j a_{ji}$ for all $i,j$. A Kac–Moody Lie algebra $\mathfrak{g}$ attached to $A$ is an algebra on generators $E_i, F_i, H_i$, $1\leqq i\leqq n$ and well-known relations explicitly depending on entries of $A$ (see [K]). Among the structures related to $\mathfrak{g}$ we shall use the following:

the triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$;
the dual space $\mathfrak{h}^*$; elements of $\mathfrak{h}^*$ will be referred to as weights;
the root space decomposition $\mathfrak{n}_\pm=\bigoplus_{\alpha\in\Delta_\pm}\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha_i}=\mathbf{C}E_i$;
the root lattice $Q\in\mathfrak{h}^*$, $\{\alpha_1,\ldots,\alpha_n\}\subset\Delta_+\subset\mathfrak{h}^*$ being the set of simple roots;
the invariant bilinear form $Q\times Q\to\mathbf{Z}$ defined by $(\alpha_i,\alpha_j)=d_i a_{ij}$.

2. For $q\in\mathbf{C}$, $d\in\mathbf{Z}$ set:

$$[n]_d=\frac{1-q^{2nd}}{1-q^{2d}},$$

$$[n]_d!=[n]_d\cdots[1]_d,$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_d=\frac{[n]_d\cdots[n-j+1]_d}{[j]_d!},$$

omitting the subscript if $d=1$.

Suppose $\mathfrak{g}$ is a Kac–Moody Lie algebra attached to $A$. *The Drinfeld–Jimbo quantum group* $U_q(\mathfrak{g})$, $q\in\mathbf{C}$ is said to be a hopf algebra with antipode $S$, comultiplication $\Delta$ and 1 on generators $E_i, F_i, K_i, K_i^{-1}$, $1\leqq i\leqq n$ and defining relations

$$K_iK_i^{-1}=K_i^{-1}K_i=1,\quad K_iK_j=K_jK_i\,, \tag{4}$$

$$K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,\quad K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,\quad q_i=q^{d_i}\,, \tag{5}$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i},\quad q_i=q^{d_i}\,, \tag{6}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu q_i^{\nu(\nu-1+a_{ij})} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{d_i} E_i^{1-a_{ij}-\nu} E_j E_i^\nu = 0 ,$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu q_i^{\nu(\nu-1+a_{ij})} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{d_i} F_i^{1-a_{ij}-\nu} F_j E_i^\nu = 0 \quad (i \neq j) , \tag{7}$$

the comultiplication being given by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i , \tag{8}$$

and antipode – by

$$SE_i = -K_i^{-1} E_i, \quad SF_i = -F_i K_i, \quad SK_i = K_i^{-1} . \tag{9}$$

The relations admit the **C**-algebra anti-automorphism $\omega$,

$$\omega E_i = F_i, \quad \omega F_i = E_i, \quad \omega K_i = K_i . \tag{10}$$

Set $U_q^+(\mathfrak{g})(U_q^-(\mathfrak{g}))$ equal to the subalgebra, generated by $E_i(F_i$ resp.) $(1 \leqq i \leqq n)$ and $U_q^0(\mathfrak{g}) = \mathbf{C}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$. We will sometimes reduce these notations to $U_q^+$, $U_q^-$, $U_q^0$ if this does not lead to confusion. One may check that the multiplication induces an isomorphism of linear spaces

$$U_q(\mathfrak{g}) \approx U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}) . \tag{11}$$

Set $U_q^{\geqq} = U_q^0 U_q^+$. From now on unless otherwise stated $A$ is assumed to be of finite type.

3. For any $Q$-graded associative algebra $\mathscr{A} = \bigoplus_{\beta \in Q} \mathscr{A}_\beta$ define a $q$-commutator, which associates to any homogeneous $b \in \mathscr{A}_\beta$ a mapping

$$ad_q b \colon \mathscr{A} \to \mathscr{A}$$

of degree $\beta$ determined by

$$ad_q b(c) = bc - q^{(\beta,\gamma)} cb, \quad \text{if } c \in \mathscr{A}_\gamma . \tag{12}$$

One deduces that the $q$-bracket is a $q$-derivation of $\mathscr{A}$, meaning that

$$ad_q a(bc) = (ad_q a(b))c + q^{(\alpha,\beta)} b\, ad_q a(c) \quad \text{if } a \in \mathscr{A}_\alpha, b \in \mathscr{A}_\beta, c \in \mathscr{A}_\gamma . \tag{13}$$

Using (13) one proves the following useful formula:

$$b^n c = q^{n(\beta,\gamma)} cb^n + \sum_{j=1}^{n} q^{(n-j)(\beta,\gamma)} \begin{bmatrix} n \\ j \end{bmatrix}_{(\beta,\beta)/2} (ad_q b)^j (c) b^{n-j} , \tag{14}$$

its Lie algebra analogue being

$$b^n c = cb^n + \sum_{j=1}^{n} \binom{n}{j} (ad\, b)^j (c) b^{n-j} . \tag{15}$$

In the case $\mathscr{A} = U_q^{\pm}(\mathfrak{g})$ one realizes that the relations (7) simply mean that

$$(ad_q E_i)^{-a_{ij}+1}(E_j) = (ad_q F_i)^{-a_{ij}+1}(F_j) = 0, \quad \text{if } i \neq j . \tag{16}$$

The following observation will be used below in the discussion of rings of quotients:

The relations (13, 14, 16) imply that for any $F_i, b\in U_q^-(\mathfrak{g})$ one has

$$F_i^N b\in U_q^-(\mathfrak{g})F_i\,, \tag{17}$$

for all sufficiently large $N$.

4. Following Lusztig [L] introduce the following automorphisms $R_i$, $1\leqq i\leqq n$ of the algebra $U(\mathfrak{g})$:

$$R_iE_i=-F_iK_i,\quad R_iE_j=\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q_i^{-s}\frac{E_i^{-a_{ij}-s}}{[-a_{ij}-s]_{d_i}!}E_j\frac{E_i^s}{[s]_{d_i}!} \tag{18}$$

if $i\neq j$,

$$R_iF_i=-K_i^{-1}F_i,\quad R_iF_j=\sum_{s=0}^{-a_{ij}}(-1)^{s-a_{ij}}q_i^{s}\frac{F_i^s}{[s]_{d_i}!}E_j\frac{E_i^{-a_{ij}-s}}{[-a_{ij}-s]_{d_i}!} \tag{19}$$

if $i\neq j$,

$$R_iK_j=K_jK_i^{-a_{ij}}\,. \tag{20}$$

Fix a reduced decomposition $r_{i_1}r_{i_2}\cdots r_{i_N}$ of the longest element of the Weyl group $W$. This gives an ordering of the set of positive roots:

$$\beta_1=\alpha_{i_1},\quad \beta_2=r_{i_1}\alpha_{i_2},\ldots,\quad \beta_N=r_{i_1}\cdots r_{i_{N-1}}\alpha_{i_N}\,.$$

One introduces root vectors [L]

$$E_{\beta_s}=R_{i_1}\cdots R_{i_{s-1}}E_{i_s}\,, \tag{21}$$

$$F_{\beta_s}=R_{i_1}\cdots R_{i_{s-1}}F_{i_s}\,. \tag{22}$$

For $k=(k_1,\ldots,k_N)\in\mathbf{Z}_+^N$ set $E^k=E_{\beta_1}^{k_1}\cdots E_{\beta_N}^{k_N}$, $F^k=\omega E^k$.

**Proposition 2.1.** (i) [L] *Elements* $E^k$($F^k$ resp.), $k\in\mathbf{Z}_+^N$, *form a basis of* $U_q^+(\mathfrak{g})(U_q^-(\mathfrak{g})$ resp.) *over* $\mathbf{C}$.

(ii) [*DC-K*] *The algebra* $U_q(\mathfrak{g})$ *affords a structure of a* $\mathbf{Z}_+^{2N+1}$*-filtered algebra, so that the associated graded algebra* $\mathrm{Gr}(U_q(\mathfrak{g}))$ *is an associative algebra over* $\mathbf{C}$ *on generators* $E_\alpha, F_\alpha$ $(\alpha\in\Delta^+)K_i^{\pm}$ $(0\leqq i\leqq n)$ *subject to the following relations*:

$$K_iK_j=K_jK_i,\quad K_iK_i^{-1}=1,\quad E_\alpha F_\beta=F_\beta E_\alpha\,,$$

$$K_iE_\alpha=q^{(\alpha,\alpha_i)}E_\alpha K_i,\quad K_iF_\alpha=q^{-(\alpha,\alpha_i)}F_\alpha K_i\,,$$

$$E_\alpha E_\beta=q^{(\alpha,\beta)}E_\beta E_\alpha, F_\alpha F_\beta=q^{(\alpha,\beta)}F_\beta F_\alpha,\quad \textit{if}\ \alpha>\beta\,.$$

Recall that an algebra $\mathbf{C}^s[x_1,\ldots,x_k]$ on generators $x_1,\ldots,x_k$ and defining relations $x_ix_j=\lambda_{ij}x_jx_i$ for $i>j$, where $\lambda_{ij}\in\mathbf{C}^*$, is called *an algebra of skew polynomials.* Therefore, the item (ii) of Proposition 2.1 asserts that $\mathrm{Gr}(U_q(\mathfrak{g}))$ is a skew polynomial algebra. The "classical" analogue of this is the fact that $\mathrm{Gr}(U(\mathfrak{g}))$ is a symmetric algebra $S(\mathfrak{g})$.

An algebra of skew polynomials has no zero divisors, therefore, the same is true for $U_q(\mathfrak{g})$ [DC-K].

## 3. Rings of Quotients Associated to (Deformed) Enveloping Algebras

### *3.1. Gelfand–Kirillov Conjecture and Feigin's Construction.*

1. A (non-commutative) ring $\mathscr{A}$ with no zero divisors is called an *Ore domain* if any 2 elements of $\mathscr{A}$ have a common right and a common left multiple. A class of examples of Ore domains is provided by the rings of polynomial growth. We shall be calling an N-filtered ring $\mathscr{A}=\bigcup_{i\geq 1}\mathscr{A}^{(i)}$ a ring of polynomial growth if $\dim\mathscr{A}^{(i)}$ is equivalent to a certain polynomial as $i\to\infty$.

**Lemma 3.1.** *A ring of polynomial growth with no zero divisors is an Ore domain.*

*Proof.* Assume that $\dim\mathscr{A}^{(i)}\sim a_0 i^k$, as $i\to\infty$. Then for any ideal $I$ on 1 generator one has: $\dim I^{(i)}\sim a_0 i^k$, $I^{(i)}=I\cap\mathscr{A}^{(i)}$. If 2 ideals $I_1, I_2$ on 1 generator have zero intersection then $(I_1+I_2)^{(i)}\sim 2a_0 i^k$, contradicting the assumption. □

The simplest examples of rings of polynomial growth are, therefore, algebras of (skew) polynomials. Further, affine and finite-dimensional Lie algebras are distinguished among Kac–Moody algebras as algebras of polynomial growth [K]. This combined with Proposition 2.1 implies that $U(\mathfrak{g})$, if $\mathfrak{g}$ is of either affine or finite type, and $U_q(\mathfrak{g})$, if $\mathfrak{g}$ is of finite type, are Ore domains, as well as the corresponding $U^{\pm}(\mathfrak{g})(U_q^{\pm}(\mathfrak{g}))$.

2. An Ore domain $\mathscr{A}$ is a suitable object for formation of a ring of quotients. Consider expressions of the form $ab^{-1}$, $b^{-1}a$, $a, b\in\mathscr{A}$ called right and left (resp.) quotients. Introduce a relation $\approx$ by saying that

$$\text{(i)}\ ab^{-1}\approx c^{-1}d \Leftrightarrow ca=db\ ,$$

(ii) 2 right (left) quotients are in relation $\approx$ if and only if they are in relation $\approx$ to one and the same left (right) quotient.

The Ore domain conditions imply that $\approx$ is an equivalence relation. Denote the set of equivalence classes of $\approx$ by $Q(\mathscr{A})$. One more application of the Ore domain conditions gives that each equivalence class contains left and right quotients and that any 2 left (right) quotients are equivalent to left (right) quotients with one and the same denominator. This allows to define operations of addition and multiplication (in the most natural way), which completes the definition of the ring of quotients $Q(\mathscr{A})$.

A definition of a ring of quotients $\mathscr{A}[S^{-1}]$ with respect to $S\subset\mathscr{A}$ is a more subtle matter because due to the noncommutativity of $\mathscr{A}$ it is not clear what can really appear as a denominator. However in the case when $\mathscr{A}$ is a (quantized) enveloping algebra one can say more. It follows from (14) that

$$F_i^{-n}F_j=q^{-n(\alpha_i,\alpha_j)}F_jF_i^{-n}+\sum_{j=1}^{\infty}q^{(-n-j)(\alpha_i,\alpha_j)}\begin{bmatrix}-n\\ j\end{bmatrix}_{(\alpha_i,\alpha_i)/2}(ad_qF_i)^j(F_j)F_i^{-n-j}\ ,$$

if $i\neq j$, $n>0$.

(Observe that (16) implies that only finite number of terms in the right-hand side of the above formula can be non-zero and, therefore, it makes sense as an element of $U_q^-(\mathfrak{g})$.) Therefore, the result of commuting negative powers of a Cartan generator to the right is negative powers of the same generator on the right.

One also deduces from (17) that 2 words $F_{i_1}^{s_1} \cdots F_{i_m}^{s_m}, F_{j_1}^{t_1} \cdots F_{i_l}^{t_l}$ have a common right multiple of the form

$$F_{i_1}^{N_1} \cdots F_{i_m}^{N_m} F_{j_1}^{t_1} \cdots F_{i_l}^{t_l},$$

If $N_1, \ldots, N_m$ are sufficiently large. Now, if $S \subset U_q^-(\mathfrak{g})$ is a multiplicatively closed subset multiplicatively generated by $F_{i_1}, \ldots, F_{i_k}$, one defines $U_q^-(\mathfrak{g})[S^{-1}]$ as a subset of $Q(U_q^-(\mathfrak{g}))$ consisting of classes of quotients of the form $ab^{-1}$, $b \in S$. The above discussion shows that $U_q^-(\mathfrak{g})[S^{-1}]$ is a subring. We shall sometimes denote $U_q^-(\mathfrak{g})[S^{-1}]$ by $U_q^-(\mathfrak{g})[F_{i_1}^{-1}, F_{i_2}^{-1}, \ldots, F_{i_k}^{-1}]$.

It is easy to see that the same goes through with $F$'s replaced with $E$'s or $U_q^-(\mathfrak{g})$ replaced with $U_q(\mathfrak{g})$, as well as with everything replaced with its classical ($q \to 1$) analogues. Further, even though a ring of quotients is not defined for an arbitrary Kac–Moody algebra, this discussion shows that a ring of quotients $U(\mathfrak{g})[S^{-1}]$ is well-defined if $S$ is multiplicatively generated by a (sub)set of real root vectors. Actually, formulas (13, 15) provide an algorithm of carrying out operations of multiplication and addition on elements of $U(\mathfrak{g})[S^{-1}]$.

3. It often happens that rings of quotients of universal enveloping algebras of different finite-dimensional Lie algebras are isomorphic with each other. Denote by $D_{nk}$ an algebra on generators $a_1, \ldots, a_n, a_1^*, \ldots, a_n^*, c_1, \ldots, c_k$ and defining relations

$$[a_i, a_j^*] = \delta_{ij}, \quad [c_i, a_j] = [c_i, a_j^*] = [a_i, a_j] = [a_i^*, a_j^*] = 0, \quad \text{for all } i, j.$$

$D_{nk}$ can, of course, be viewed as an algebra of differential operators on $n$ variables trivially extended by $k$-dimensional center.

**Conjecture 3.2** (*Gelfand–Kirillov* [G-K1]). *If* $\mathfrak{g}$ *is an algebraic Lie algebra then* $Q(U(\mathfrak{g}))$ *is isomorphic with* $Q(D_{nk})$ *for* $k$ *equal to the dimension of a generic* $\mathfrak{g}$*-orbit in the coadjoint represenation and* $n = (\dim \mathfrak{g} - k)/2$.

This conjecture has been proven in many cases [G-K1, J, McC].

4. It seems that the following construction (due to Feigin [F]) is relevant to a proper $q$-deformation of the Gelfand–Kirillov's conjecture. For a pair of $Q$-graded associative algebras $\mathscr{A}, \mathscr{B}$ define a *$q$-twisted tensor product* as an algebra $\mathscr{A} \otimes_q \mathscr{B}$ isomorphic with $\mathscr{A} \otimes \mathscr{B}$ as a linear space and with the multiplication given by $a_1 \otimes b_1 \cdot a_2 \otimes b_2 = q^{(\alpha, \beta)} a_1 a_2 \otimes b_1 b_2$ if $a_2 \in \mathscr{A}^{(\alpha)}, b_1 \in \mathscr{B}^{(\beta)}$. Evidently, $\mathscr{A} \otimes_q \mathscr{B}$ is a $Q$-graded algebra.

**Proposition 3.1** [F]. *For any Kac–Moody algebra* $\mathfrak{g}$ *the map*

$$\tilde{\Delta}: U_q^{\pm}(\mathfrak{g}) \to U_q^{\pm}(\mathfrak{g}) \otimes_q U_q^{\pm}(\mathfrak{g}),$$

$$\tilde{\Delta}: 1 \mapsto 1 \otimes 1,$$

$$\tilde{\Delta}: E_i \mapsto E_i \otimes 1 + 1 \otimes E_i (F_i \to F_i \otimes 1 + 1 \otimes F_i \text{ resp.}), \quad 1 \leqq i \leqq n$$

*is a homomorphism of associative algebras.*

*Remark.* It is known that the map $U_q^{\pm}(\mathfrak{g}) \to U_q^{\pm}(\mathfrak{g}) \otimes U_q^{\pm}(\mathfrak{g})$ does not exist in the category of associative algebras.

Iterating $\Delta$ one obtains a sequence of maps

$$\tilde{\Delta}^m\colon U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})^{\otimes m}, \quad m=2,3,\ldots,$$

determined by $\tilde{\Delta}^2=\tilde{\Delta}$, $\tilde{\Delta}^m=(\tilde{\Delta}\otimes \mathrm{id})\circ\tilde{\Delta}^{m-1}$.

For any simple root $\alpha_i$ consider a ring of polynomials on 1 variable $\mathbf{C}[X_i]$, which we regard as $Q$-graded by setting $\deg X_i=\alpha_i$. There arises a morphism of $Q$-graded associative algebras

$$\rho_i\colon U_q^-(\mathfrak{g}) \to \mathbf{C}[X_i]\,,$$
$$F_j \mapsto \delta_{ij} x_i\,.$$

Now, for any sequence of simple roots $\alpha_{i_1},\ldots,\alpha_{i_k}$ there arises a morphism of $Q$-graded associative algebras:

$$(\rho_{i_1}\otimes\cdots\otimes\rho_{i_k})\circ\tilde{\Delta}^k\colon U_q^-(\mathfrak{g})\to\mathbf{C}[X_{1i_1}]\otimes_q\cdots\otimes_q\mathbf{C}[X_{ki_k}]\,.$$

(The double indexation of $X$'s is necessary because some number can appear in the sequence $i_1,\ldots,i_k$ more than once but the corresponding indeterminates have to be regarded as different.)

Evidently, $\mathbf{C}[X_{1i_1}]\otimes_q\cdots\otimes_q\mathbf{C}[X_{ki_k}]$ is an algebra of skew polynomials $\mathbf{C}[X_{1i_1}\ldots X_{ki_k}]$, satisfying the relations $X_{si_s}X_{ti_t}=q^{(\alpha_{i_s},\,\alpha_{i_t})}X_{ti_t}X_{si_s}$, $s>t$. Therefore, we have constructed a family of morphisms of a "maximal nilpotent subalgebra" of a quantum group associated to an arbitrary Kac–Moody algebra to algebras of skew polynomials. It is interesting that a proper classical analogue of this construction is not so obvious and is best understood in the framework of rings of quotients (see below).

We now assume that $\mathfrak{g}$ is a simple finite-dimensional Lie algebra. Let $w_0=r_{i_1}\cdots r_{i_N}\in W$ be a reduced decomposition of the element of maximal length. Set $\Phi(i_1,\ldots,i_N)=(\rho_{i_1}\otimes\cdots\otimes\rho_{i_N})\circ\tilde{\Delta}^N$.

**Conjecture 3.3** [F].

(i) $\Phi(i_1,\ldots,i_N)$ *is an embedding.*

(ii) $\Phi(i_1,\ldots,i_N)$ *extends-at least for a special choice of a reduced decomposition* $w_0=r_{i_1}\ldots r_{i_N}$*-to an isomorphism of* $Q(U_q^-(\mathfrak{g}))$ *with* $Q(\mathbf{C}[X_{1i_1}\ldots X_{Ni_N}])$.

5. *Example*: $\mathfrak{g}=\mathfrak{sl}_{n+1}$. From the abstract point of view $\mathfrak{g}=\mathfrak{sl}_{n+1}$ is an algebra related to the Cartan matrix $(a_{ij})$, where

$$a_{ij}=\begin{cases} 2 \text{ if } i=j\\ -1 \text{ if } |i-j|=1\\ 0 \text{ if } |i-j|>1\end{cases}.$$

Choose a reduced decomposition of the longest Weyl group element to be $w_0=r_1r_2\cdots r_nr_1r_2\cdots r_{n-1}\cdots\cdots r_1r_2r_1$. Denote by $\mathbf{C}[X]$ the skew polynomial ring on generators $X_{ij}$ labelled by all pairs $i,j$ satisfying $1\leqq j\leqq n$, $1\leqq i\leqq n-i+1$ and defining relations

$$X_{ij}X_{rs}=p_{rs}^{ij}X_{rs}X_{ij}\,,$$

where

$$p_{rs}^{ij}\doteq\begin{cases} q^2 & \text{if } \ i>r, j=s\\ q & \text{if } \ i\leqq r, j=s-1\\ q^{-1} & \text{if } \ i>r, j=s-1\\ 1 & \text{if } \qquad j<s-1\,.\end{cases}$$

In this case the map $\Phi=\Phi(w_0)$ (here $w_0$ stands for the reduced decomposition $w_0=r_1r_2\cdots r_nr_1r_2\cdots r_{n-1}\cdots r_1\,r_2r_1$) acts as follows:

$$\Phi(w_0)(F_i)=X_{1i}+X_{2i}+\cdots X_{n+1-ii} \quad 1\leqq i\leqq n\,. \tag{23}$$

One solves (23) as a system of equations $X_{ij}$, $1\leqq i<j\leqq n$ with coefficients in $Q(\Phi(U_q^-(\mathfrak{sl}_{n+1})))$.

**Lemma 3.4.** *The following formulas hold*

$$X_{1n-1}=\frac{q}{q-q^{-1}}[\Phi(F_{n-1}),\Phi(F_n)]_q\Phi(F_n)^{-1}\,,$$

$$X_{2n-1}=\frac{q}{q-q^{-1}}[\Phi(F_n),\Phi(F_{n-1})]_q\Phi(F_n)^{-1}\,,$$

$$X_{1i}=\frac{q}{q-q^{-1}}[\Phi(F_i),X_{1i+1}]_qX_{1i+1}^{-1},\quad 1\leqq i\leqq n-2\,,$$

$$X_{n-i+1i}=\frac{q}{q-q^{-1}}[X_{n-ii+1},\Phi(F_i)]_qX_{n-ii+1}^{-1},\quad 1\leqq i\leqq n-2\,,$$

$$X_{ji}=\frac{1}{q-q^{-1}}(X_{j-1i+1}\Phi(F_i)X_{j-1i+1}^{-1}-X_{ji+1}\Phi(F_i)X_{ji+1}^{-1}),\quad 2\leqq j\leqq n-i\,.$$

One uses this lemma to prove that the Conjecture 3.3 is true.

**Theorem 3.5** (i) *The map $\Phi$ is an embedding.*

(ii) *The embedding $\Phi$: $U_q^-(\mathfrak{sl}_{n+1})\to\mathbf{C}[X]$ induces an isomorphism $Q(U_q^-(\mathfrak{sl}_{n+1})\approx Q(\mathbf{C}[X])$.*

*Proof.* Lemma 3.4 actually shows that $Q(\Phi(U_q^{-1}(\mathfrak{sl}_{n+1})))\approx Q(\mathbf{C}[X])$. It is, therefore, enough to prove that $\Phi$ is injective. In [G-K1] Gel'fand and Kirillov associated a number to an arbitrary algebra $\mathscr{A}$ which is now known as the *Gel'fand–Kirillov dimension* $\dim_{G-K}\mathscr{A}$. For example, the Gel'fand–Kirillov dimension of a polynomial ring on $n$ variables, as well as that of the corresponding ring of quotients, is equal to $n$. One of results of [G-K1] is that if an algebra $\mathscr{A}$ has a filtration such that the associated graded algebra $\mathrm{Gr}\mathscr{A}$ is isomorphic with a polynomial ring on $n$ variables then $\dim_{G-K}\mathscr{A}=\dim_{G-K}Q(\mathscr{A})=n$. Regarding $q$ as an indeterminate and introducing filtration by powers of $q-1$ one derives from the mentioned results of [G-K1] their "$q$-analogues": dimension of a ring of skew polynomials on $n$ indeterminates is equal to $n$ and if $\mathrm{Gr}\mathscr{A}$ is isomorphic with a ring of skew polynomials on $n$ indeterminates then $\dim_{G-K}\mathscr{A}=n$. It follows from Proposition 2.1 that $\dim_{G-K}U_q^-(\mathfrak{sl}_{n+1})=\dim_{G-K}\Phi(U_q^-(\mathfrak{sl}_{n+1}))=n(n+1)/2$ and, therefore, $\Phi$ is injective. $\square$

*3.2. Complex Powers, Automorphisms and Singular Vectors.*

*3.2.1. Construction of automorphisms of quotient rings of (deformed) universal enveloping algebras and algebras of skew polynomials.*

1. For any $k\in\mathbf{N}$ the map

$$\mathscr{C}_i^k\colon Q(U_q^-(\mathfrak{g})),\to Q(U_q^-(\mathfrak{g})),\ x\mapsto F_i^kxF_i^{-k}$$

is an automorphism. Clearly, $\mathscr{C}_i^{k_1+k_2}=\mathscr{C}_i^{k_1}\circ\mathscr{C}_i^{k_2}$. Formulas (14, 16) imply that $\mathscr{C}_i^k(x)$ is a polynomial function of $k$. For example, in the $\mathfrak{sl}_{n+1}$-case one has

$$\mathscr{C}_{i+1}^k(F_i)=F_{i+1}^k F_i F_{i+1}^{-k}=\{k\}F_{i+1}F_iF_{i+1}^{-1}+\{1-k\}F_i\,, \tag{24}$$

where we have used "symmetric" $q$-numbers: $\{k\}=\dfrac{q^k-q^{-k}}{q-q^{-1}}$. Using this we define an automorphism $\mathscr{C}_i^k$ by analytic continuation. Therefore, with every word $F_{i_l}^{\beta_l}\cdots F_{i_1}^{\beta_1}$ we have associated an automorphism $\mathscr{C}_{i_l}^{\beta_l}\cdots\mathscr{C}_{i_1}^{\beta_1}$ of $Q(U_q^-(\mathfrak{g}))$.

2. Since in the case $\mathfrak{g}=\mathfrak{sl}_{n+1}$ the rings $Q(U_q^-(\mathfrak{g}))$ and $Q(\mathbf{C}[X])$ are isomorphic with each other (Theorem 3.5) the above provides the family of automorphisms – also denoted by $\mathscr{C}_{i_l}^{\beta_l}\cdots\mathscr{C}_{i_1}^{\beta_1}$ – of $Q(\mathbf{C}[X])$. Moreover, the last assertion is valid for any $\mathfrak{g}$ regardless of Conjecture 3.3. In reality, there is a construction of automorphisms of a ring of skew polynomials which has nothing to do with quantum groups.

Consider for simplicity the ring $\mathbf{C}[x]=\mathbf{C}[x_1,\ldots,x_m]$, $x_jx_i=q^2x_ix_j$ if $j>i$. To proceed we need a *q-commutative version of the q-binomial theorem*:

$$(x_1+\cdots+x_m)^n=\sum_{i_1+\cdots+i_m=n}\frac{[n]!}{[i_1]!\cdots[i_m]!}x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}\quad n\in\mathbf{N}\,. \tag{25}$$

For $\beta\in\mathbf{C}$ we set

$$(x_1+\cdots+x_m)^\beta=\sum_{j=0}^{\infty}\sum_{j_1+\cdots+j_{m-1}=j}\frac{[\beta][\beta-1]\cdots[\beta-j+1]}{[j_1]!\cdots[j_{m-1}]!}$$
$$\times x_1^{j_1}\cdots x_{r-1}^{j_{r-1}}x_r^{\beta-j}x_{r+1}^{j_r}\cdots x_m^{j_{m-1}}\,, \tag{26}$$

for some $1\leqq r\leqq m$, thus making sense out of $(x_1+\cdots+x_m)^\beta$ as an element of a certain completion of $\mathbf{C}[X]$ consisting basically of formal power series (there are exactly $m$ different ways to do that).

Obviously the map $p\mapsto(x_1+\cdots+x_m)^\beta p(x_1+\cdots+x_m)^{-\beta}$ is an automorphism of the above-mentioned completion. An explicit calculation (see below) shows that

$$Q(\mathbf{C}[x])\ni p\Rightarrow(x_1+\cdots+x_m)^\beta p(x_1+\cdots+x_m)^{-\beta}\in Q(\mathbf{C}[x])\,. \tag{27}$$

Note that the same is true for $(x_1+\cdots+x_m)$ replaced with $(x_{i_1}+\cdots+x_{i_k})$, $1\leqq i_1<\cdots<i_k$ and – with minor restrictions – for $\mathbf{C}[x]$ replaced with an arbitrary ring of skew polynomials. In particular, in the case of the ring $\mathbf{C}[X]$ related to $U_q(\mathfrak{g})$ by Feigin's construction one obtains automorphisms

$$\mathscr{C}_i^\beta p=(\Phi(F_i))^\beta p(\Phi(F_i))^{-\beta}\,.$$

*Remarks.*

(i). It is natural to set $\log F_i=\left.\dfrac{d}{dk}\right|_{k=0}\mathscr{C}_i^k$. By definition $\log F_i$ is a differentiation of $Q(U_q^-(\mathfrak{g}))$ as well as of $Q(\mathbf{C}[X])$. It is easy to see that, moreover, this is an exterior differentiation. *Problem*: Classify non-trivial (exterior modulo inner) automorphisms of $Q(U_q^-(\mathfrak{g}))$, $Q(\mathbf{C}[X])$.

(ii) The set of words $F_{i_l}^{\beta_l}\cdots F_{i_1}^{\beta_1}$, $\beta_1,\ldots,\beta_l\in\mathbf{C}$ is naturally equipped with a group structure. With each such word one may associate an infinite series: its expansion over a "Poincaré–Birkhoff–Witt type basis" $F^k$, where complex powers of $F_i$ are allowed. (For details in classical setting see [M-F-F].) Thus we have identified this group with a subgroup of a certain infinite-dimensional group with a non-trivial topology. In [Kh-Z] a similar group was considered in the classical case of differential operators on the line. In particular, it was shown that this is a Poisson–Lie group.

3. *Calculation of* $(x_1+x_2)^\beta x_2(x_1+x_2)^{-\beta}$. It is easy to see that the proof of (27) reduces to the case $m=2$, $p=x_2$. One has

$$(x_1+x_2)^\beta x_2(x_1+x_2)^{-\beta}=x_2(q^{-1}x_1+x_2)^\beta(x_1+x_2)^{-\beta}.$$

The $q$-commutative version of the binomial theorem gives

$$(q^{-2}x_1+x_2)^\beta=q^{-2\beta}x_1^\beta\sum_{i=0}^{\infty}\frac{(q^{-2\beta})_i}{(q^2)_i}\left\{(-q^{2(\beta+1)}x_1^{-1}x_2)^i\right.,\tag{28}$$

$$(x_1+x_2)^{-\beta}=\left\{\sum_{i=0}^{\infty}\frac{(q^{2\beta})_i}{(q^2)_i}(-x_1^{-1}x_2)^i\right\}x_1^{-\beta},\tag{29}$$

where as usual $(a)_i=(1-a)(1-aq^2)\cdots(1-aq^{2(i-1)})$.

In order to show that in the product of the left-hand sides of (28–29) almost everything cancels out we employ a *commutative version of the q-binomial theorem* [G-R] which reads as follows:

$$\sum_{i=0}^{\infty}\frac{(a)_i}{(q^2)_i}z^i=\frac{(az)_\infty}{(z)_\infty},\quad z\in\mathbf{C},\tag{30}$$

where $(a)_\infty=\prod_{i\geqq 0}(1-aq^{2i})$.

(Although we are in the non-commutative realm the usage of (30) makes sense for the right-hand sides of (28–29) basically involve only one "variable" $x_1^{-1}x_2$.)

By (30) the equalities (28–29) are rewritten as follows:

$$(q^{-2}x_1+x_2)^\beta=q^{-2\beta}x_1^\beta\frac{(-q^2x_1^{-1}x_2)_\infty}{(-q^{2(\beta+1)}x_1^{-1}x_2)_\infty},\tag{31}$$

$$(x_1+x_2)^{-\beta}=\frac{(-q^{2\beta}x_1^{-1}x_2)_\infty}{(x_1^{-1}x_2)_\infty}x_1^{-\beta}.\tag{32}$$

Carrying out the multiplication one observes that almost all factors of infinite products cancel out:

$$\begin{aligned}(x_1+x_2)^\beta x_2(x_1+x_2)^{-\beta}&=q^{-2\beta}x_2x_1^\beta(1+q^{2\beta}x_1^{-1}x_2)(1+x_1^{-1}x_2)^{-1}x_1^{-\beta}\\&=q^{-2\beta}x_2(1+x_1^{-1}x_2)(1+q^{-2\beta}x_1^{-1}x_2)^{-1},\end{aligned}\tag{33}$$

which completes the proof.

*3.2.2 Application to singular vectors in Verma modules*

It follows from Sect. 3.2.1 that elements of the form

$$F_{i_l}^{s_l}\cdots F_{i_1}^{s_1}F_i^N F_{i_1}^{t_1}\cdots F_{i_l}^{t_l}$$

belong to $Q(U_q^-(\mathfrak{g}))$ if $N\in\mathbf{Z}$, $s_i+t_i\in\mathbf{Z}$, $1\leqq i\leqq l$. It was shown in [M-F-F, M] that such expressions are relevant to singular vectors in Verma modules. Here we explicitly calculate them in the case $\mathfrak{g}=\mathfrak{sl}_{n+1}$.

Recall that a Verma module $M(\lambda)$, $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbf{C}^n$ is said to be a $U_q(\mathfrak{g})$-module on one generator $v_\lambda$ and the following defining relations:

$$U_q^+(\mathfrak{g})v_\lambda=0,\ K_i v_\lambda=q_i^{\lambda_i}v_\lambda \quad i=1,\ldots,n\ .$$

It is easy to see that $M(\lambda)$ is reducible if and only if it contains a singular vector, i.e. a non-zero vector different from $v_\lambda$ and annihilated by $U_q^+(\mathfrak{g})$. The reducibility criterion is the same as in the classical case [K-K, DC-K] and for $U_q(\mathfrak{sl}_{n+1})$ reads as follows:

*$M(\lambda)$ is reducible if and only if for some $1\leqq i<j\leqq n$, $N\in\mathbf{N}$,*

$$\lambda_i+\lambda_{i+1}+\cdots\lambda_j+j-i+1=N\ . \tag{34}$$

It is known that for a generic point $\lambda$ on the hyperplane determined by (34) there is a unique (up to proportionality) singular vector in $M(\lambda)$. This means that there is a function sending a point $\lambda$ on the hyperplane to $S_{ij}^N(\lambda)\in U_q^-(\mathfrak{sl}_{n+1})$ so that the vector $S_{ij}^N(\lambda)v_\lambda$ is singular. We are going to evaluate $S_{ij}^N(\lambda)$.

Equation (34) can be rewritten in the following parametric form:

$$\begin{aligned}\lambda_j&=N-t_j-1\ ,\\ \lambda_{j-1}&=t_j-t_{j-1}-1\ ,\\ \lambda_{j-2}&=t_{j-1}-t_{j-2}-1\ ,\\ &\cdots\\ \lambda_{i+1}&=t_{i+2}-t_{i+1}-1\ ,\\ \lambda_i&=t_{i+1}-1\ .\end{aligned}$$

It follows from [M] that

$$S_{ij}^N(t)=F_j^{t_j}\cdots F_{i+1}^{t_{i+1}}F_i^N F_{i+1}^{N-t_{i+1}}\cdots F_j^{N-t_j}\ . \tag{35}$$

Though (35) is not quite explicit it is sometimes most convenient for derivation of properties of singular vectors. For example, playing with complex powers one proves that singular vectors related to $N>1$ are expressed in terms of singular vectors related to $N=1$. One has

$$\begin{aligned}S_{ii+1}^N(t)&=F_{i+1}^t F_i^N F_{i+1}^{N-t}\\ &=F_{i+1}^t F_i F_{i+1}^{1-t}F_{i+1}^{t-1}F_iF_{i+1}^{2-t}\cdots F_{i+1}^{t-N+1}F_iF_{i+1}^{N-t}\\ &=S_{ii+1}^1(t)S_{ii+1}^1(t-1)\cdots S_{ii+1}^1(t-N+1)\ .\end{aligned} \tag{36}$$

Arguing by induction one proves that likewise

$$S_{ij}^N(t)=S_{ij}^1(t)S_{ij}^1(t-1)\cdots S_{ij}^1(t-N+1)\ . \tag{37}$$

Therefore, it is enough to calculate $S_{ij}^1(t)$. It follows from (24) that

$$S_{ii+1}^1(t)=\{t\}F_{i+1}F_i+\{1-t\}F_iF_{i+1}\ . \tag{38}$$

Using (38) several times one reduces (35) to a form containing only natural powers of generators. Denote by $\mathscr{P}$ the set of all sequences $\varepsilon=(\varepsilon_{i+1},\ldots,\varepsilon_j)$, where each $\varepsilon_m$ is either 0 or 1. For each $\varepsilon\in\mathscr{P}$ fix a bijection $k_\varepsilon\colon\{i+1,\ldots,j\}\to\{i+1,\ldots,j\}$ satisfying

$$k_\varepsilon^{-1}(m)<k_\varepsilon^{-1}(m-1)\quad\text{if } \varepsilon_m=1\,,$$
$$k_\varepsilon^{-1}(m)>k_\varepsilon^{-1}(m-1)\quad\text{if } \varepsilon_m=0\,.$$

(Such a bijection obviously exists, though is not unique. However, the final result is independent of a choice.) Further, with each $\varepsilon\in\mathscr{P}$ associate a number $A_\varepsilon$, given by

$$A_\varepsilon=\prod_{m=1}^{j-i}\{t_{m,\varepsilon}\}\,,$$

where

$$t_{m,\varepsilon}=\begin{cases} t_m & \text{if } \varepsilon_{m+i}=1\\ 1-t_m & \text{if } \varepsilon_{m+i}=0\end{cases}.$$

**Theorem 3.6.**

$$S_{ij}^1(t)=\sum_{\varepsilon\in\mathscr{P}}A_\varepsilon F_{k_\varepsilon(i)}F_{k_\varepsilon(i+1)}\cdots F_{k_\varepsilon(j)}\,,$$
$$S_{ij}^N(t)=S_{ij}^1(t)S_{ii+1}^1(t-1)\cdots S_{ij}^1(t-N+1)\,.$$

## 4. $U_q(\mathfrak{g})$-Modules and q-Connections

*4.1. Modules* $U_q^-[S^{-1}]v_\lambda$. Let $S\subset U_q^-$ consist of homogeneous elements, and such that $U_q[S^{-1}]$ is well-defined. A typical example of $S$ is a multiplicative span of an arbitrary subset of $\{F_1,\ldots,F_n\}$. The following isomorphism of vector spaces is an analogue of the triangular decomposition:

$$U_q[S^{-1}]\approx U_q^{-1}[S^{-1}]\otimes U_q^{\geqq}\,.$$

(Existence of this isomorphism follows from the relation $[E_i,S_j^{-1}]=-S_j^{-1}[E_i,S_j]S_j^{-1}$, which allows to commute $E$'s to the right.) Denote by $\mathbf{C}_\lambda$ a character of $U_q^{\geqq}$ defined by $E_i\to 0$, $K_i\to q^{\lambda_i}$; $1\leqq i\leqq n$. A Verma module over $U_q[S^{-1}]$ is said to be $U_q[S^{-1}]\otimes_{U_q^{\geqq}}\mathbf{C}_\lambda$. Denote by $v_\lambda$ the image of $1\otimes 1$ in $U_q[S^{-1}]\otimes_{U_q^{\geqq}}\mathbf{C}_\lambda$. Clearly, $U_q[S^{-1}]\otimes_{U_q^{\geqq}}\mathbf{C}_\lambda$ is a free $U_q^-[S^{-1}]$-module generated by $v_\lambda$. We shall be interested in the restriction of $U_q[S^{-1}]\otimes_{U_q^{\geqq}}\mathbf{C}_\lambda$ to $U_q(\mathfrak{g})$. Due to the lack of better notation $U_q^-[S^{-1}]_{v_\lambda}$ will stand for this restriction.

Note that the module $U_q^-[S^{-1}]v_\lambda$ is always reducible for it contains a Verma module $M(\lambda)=U_q^-v_\lambda$. Though its structure is unknown in general, we are able to describe it in the simplest case when $S$ is multiplicatively generated by one of $F$'s, say, $F_i$.

It is easy to see that

$$E_jF_i^m v_\lambda=\delta_{ij}\{m\}F_i^{m-1}\frac{q_i^{\lambda_i-m+1}-q_i^{-\lambda_i+m-1}}{q_i-q_i^{-1}}v_\lambda,\quad m\in\mathbf{Z}\,. \tag{39}$$

One realizes that $U_q^-[F_i^{-1}]v_\lambda$ is a module induced from the representation of the parabolic subalgebra generated by $E_1,\ldots,E_n,F_i$ in the space $\oplus_{m\in\mathbf{Z}}F_i^m v_\lambda$. Further, if $\lambda_i$ is not in $\{-2,-3,\ldots\}$, then (39) implies that $E_i$ acts freely on the quotient module $U_q^-[F_i^{-1}]v_\lambda/M(\lambda)$. It is now easy to show that $U_q^-[F_i^{-1}]v_\lambda/M(\lambda)$

is a Verma module related to a Borel subalgebra $R_i U_q^{\geq}$ twisted by the Lusztig's automorphism (see (18, 19, 20)) and the highest weight $\lambda+\alpha_i$. If, however, $\lambda_i$ does belong to $\{-2,-3,\dots\}$ then, as (39) implies, the vector $F_i^{\lambda_i+1}$ is singular. This means that there arises a chain of submodules $M(\lambda)\subset M(\lambda+(\lambda_i+1)\alpha_i)\subset U_q^-[F_i^{-1}]v_\lambda$. As above one shows that the quotient module $U_q^-[F_i^{-1}]v_\lambda/M(\lambda+(\lambda_i+1)\alpha_i)$ is a Verma module related to a twisted Borel subalgebra and the highest weight $(\lambda+(\lambda_i+2)\alpha_i)$.

For the sake of brevity, denote by ${}^{R_i}M_q(\lambda)$ an $U_q(\mathfrak{g})$-module, isomorphic to $M(\lambda)$ as a vector space with the action being twisted by $R_i$:

$$U_q(\mathfrak{g})\ni x\mapsto R_i x\mapsto \mathrm{End}(M(\lambda)) .$$

We have obtained

**Proposition 4.1.** *If* $\lambda_i$ *is not in* $\{-2,-3,\dots\}$ *then* $U_q^-[F_q^{-1}]v_\lambda/M(\lambda)\approx {}^{R_i}M(\lambda+\alpha_i)$.

*If* $\lambda_i\in\{-2,-3,\dots\}$ *then there exists a chain of submodules* $M(\lambda)\subset M(\lambda+(\lambda_i+1)\alpha_i)\subset U_q^-[F_i^{-1}]v_\lambda$ *and* $U_q^-[F_i^{-1}]v_\lambda/M(\lambda+(\lambda_i+1)\alpha_i)\approx {}^{R_i}M(\lambda+(\lambda_i+2)\alpha_i)$.

Observe that Proposition 4.1 along with its proof carries over to the case of a quantum group attached to an arbitrary symmetrizable Cartan matrix $A$.

### *4.2. Modules realized in Skew Polynomials*

1. The Feigin's embedding $U_q^-\to \mathbf{C}^s[X]$ makes the latter into a $U_q^-$-module, action being defined by means of the left multiplication. One may want to extend this to an action of the entire $U_q$. It is straightforward in view of the results of the previous section if $\mathfrak{g}=\mathfrak{sl}_{n+1}$ for in this case $Q(\mathbf{C}[X])\approx Q(U_q)$ (Theorem 3.5) and one obtains a family of modules $Q(\mathbf{C}[X])v_\lambda(=Q(U_q)v_\lambda)$. The module $Q(\mathbf{C}[X])v_\lambda$ is definitely too big and it is natural to confine to the smallest submodule containing $\mathbf{C}[X]v_\lambda$. This module is still always reducible, for example, it contains a Verma module $M_q(\lambda)$ – the one generated by $X_{1i}+\cdots+X_{n-i+1,i}$, $1\leqq i\leqq n$ – and $U_q^-[F_n^{-1}]v_\lambda$ – the one generated by $X_{1i}+\cdots+X_{n-i+1,i}$, $1\leqq i\leqq n-1$, $X_{1n}^{\pm 1}$. Though we do not have explicit description of this module in general we are able to consider the case of $\mathfrak{sl}_3$ in full detail.

**Proposition 4.2.** *For generic* $\lambda U_q(\mathfrak{sl}_3)\cdot\mathbf{C}[X_{11},X_{21},X_{12}]v_\lambda=\mathbf{C}[X_{11},X_{21},X_{12}^{\pm 1}]v_\lambda$. *For any* $\lambda\mathbf{C}[X_{11},X_{21},U_{12}^{\pm 1}]v_\lambda\approx U_q^-(\mathfrak{sl}_3)[F_2^{-1}]v_\lambda$.

Proposition 4.1, therefore, determines the structure of $\mathbf{C}[X_{11},X_{21},X_{12}^{\pm}]v_\lambda$.

As to the general case, the module in question should also be isomorphic to a module $U_q^-[S^{-1}]v_\lambda$ for an appropriate set $S$ determined by formula of Lemma 3.4.

2. The above is relevant to the Gel'fand–Kirillov conjecture for $U_q(\mathfrak{sl}_{n+1})$. Denote by $\mathscr{D}[X]$ an algebra of $q$-difference operators acting on $Q(\mathbf{C}[X])$. In other words, $\mathscr{D}[X]$ is an algebra generated by $Q(\mathbf{C}[X])$ viewed as operators of left multiplication and $T_{ij}$, $1\leqq j\leqq n$, $1\leqq i\leqq n-j+1$, where

$$T_{ij}\colon X_{rs}\mapsto\begin{cases} qX_{rs} & \text{if } (i,j)=(rs)\\ X_{rs} & \text{if } (i,j)\neq(rs)\,. \end{cases}$$

Denote by $\mathscr{D}[X,\lambda]$ the trivial central extension of $\mathscr{D}[X]$ by commuting variables $q^{\lambda_1},\ldots,q^{\lambda_n}$, where $\lambda=(\lambda_1,\ldots,\lambda_n)$ is understood as a highest weight. The definition of the module $Q(\mathbf{C}[X])v_\lambda$ implies that there exists a family of embeddings – parametrized by $\lambda$ – of $U_q(\mathfrak{sl}_{n+1})$ into $\mathscr{D}[X]$ or, equivalently, an embedding

$$\rho\colon U_q(\mathfrak{sl}_{n+1})\to\mathscr{D}[X,\lambda]\,. \tag{40}$$

**Conjecture 4.1.** *$\rho$ provides an isomorphism of a quotient field a certain algebraic extension $U_q(\mathfrak{sl}_{n+1})$ of $U_q(\mathfrak{sl}_{n+1})$ with a quotient field of a certain subalgebra of $\mathscr{D}[X,\lambda]$.*

Construction of $\hat{U}_q(\mathfrak{sl}_{n+1})$, which goes back to Gel'fand and Kirillov [G-K2], is as follows:

Identify $U_q(\mathfrak{sl}_{n+1})$ with $\rho(U_q(\mathfrak{sl}_{n+1}))\subset\mathscr{D}[X,\lambda]$ and define $\hat{U}_q(\mathfrak{sl}_{n+1})$ to be the subalgebra of $\mathscr{D}[X,\lambda]$ generated by $U_q(\mathfrak{sl}_{n+1})$, $q^{\lambda_1},\ldots,q^{\lambda_n}$ and $K^{\omega_1},\ldots,K^{\omega_n}$, where $\omega_i$, $1\leqq i\leqq n$ are dual fundamental weights, i.e. $\alpha_j(\omega_i)=\delta_{ji}$. (It is meant that $K^\omega E_i=q^{\alpha_i(\omega)i}E_iK^\omega$.) Note that elements $q^{\lambda_1},\ldots,q^{\lambda_n}$ generate a certain finite algebraic extension of the center of $U_q(\mathfrak{sl}_{n+1})$. (Description of the center of $U_q(\mathfrak{sl}_{n+1})$ may be found in [DC-K].)

We have been able to verify the conjecture in the cases of $\mathfrak{sl}_2$, $\mathfrak{sl}_3$ by straightforward calculation of $\rho^{-1}$, which is simple in the $\mathfrak{sl}_2$-case and rather tiresome in the $\mathfrak{sl}_3$-case.

**Proposition 4.3.**

$$\text{(i)}\quad Q(\hat{U}_q(\mathfrak{sl}_2))\approx\mathscr{D}[X,\lambda]\,.$$

*(In this case $X$ stands for $X_{11}$.)*

(ii) *$Q(\hat{U}_q(\mathfrak{sl}_3))$ is isomorphic with the quotient field of the subalgebra of $\mathscr{D}[X,\lambda]$ generated by $T_{11}^2, T_{21}^2, T_{11}, T_{21}, T_{12}$; $X_{11}, X_{21}, X_{12}$; $q^{\lambda_1}, q^{\lambda_2}$.*

*4.3. $U_q(\mathfrak{g})$ – Modules and q-Connections.*

*4.3.1. A q-connection with coefficients in a ring of skew polynomials.* Let $\mathbf{C}[x]:=\mathbf{C}[x_1,\ldots,x_n]$, $x_jx_i=q^2x_ix_j$, $i<j$ be a ring of skew polynomials (as yet it has nothing to do with quantum groups) and the corresponding ring of $q$-difference operators $\mathscr{D}[x]$.

By a *quantum line bundle* we mean a free rank 1 module over $\mathbf{C}[x]$ or, more generally, $\mathbf{C}[x][S^{-1}]$ for a suitable $S$. Sections of a quantum line bundle, i.e. elements of $\mathbf{C}[x][S^{-1}]$, thereforee become a $\mathscr{D}[x]$-module.

By a *q-connection* with coefficients in a quantum line bundle we mean an associative algebra homomorphism $\nabla\colon\mathscr{D}[x]\to\mathscr{D}[x]$ such that $\nabla(x_i)=x_i$, $\nabla(T_i)=R(b_i(x))T_i$, $1\leqq i\leqq n$, where $b_i(x)\in Q(\mathbf{C}[x])$ and $R(b_i(x))$ stands for the operator of the right multiplication by $b_i(x)$.

The same can be equivalently described in terms of cohomology. For $\chi=(\chi_1,\ldots,\chi_n)\in\mathbf{Z}^n$ set $T^\chi=T_1^{\chi_1}\circ\cdots\circ T_n^{\chi_n}$ and $\nabla_\chi=\nabla(T^\chi)$. Obviously, $\nabla_\chi=R(b_\chi(x))T^\chi$ for some $b_\chi(x)\in Q(C[x])$. The associative algebra homomorphism condition reads as

$$b_{\chi_1+\chi_2}(x)=(T^{\chi_1}b_{\chi_2}(x))\,b_{\chi_1}(x)\,.$$

The last equality simply means that the map $\mathbf{Z}^n\ni\chi\mapsto b_\chi(x)\in Q(\mathbf{C}[x])$ is a 1-cocyle of an abelian group $\mathbf{Z}^n$ with coefficients in $Q(\mathbf{C}[x])$. It is natural to say that a cocyle is trivial if it is given by $b_\chi(x)=(T^\chi r(x))r^{-1}(x)$ for some $r(x)\in Q(\mathbf{C}[x])$. Indeed the cocycle $\chi\mapsto(T^\chi r(x))r^{-1}(x)$ makes into the cocycle $\chi\mapsto 1$ by "the change of trivialization": $f(x)\mapsto f(x)r(x)^{-1}$. The cocycle $\chi\mapsto(T^\chi r(x))r^{-1}(x)$ is a coboundary of the 0-cocycle $r(x)$. Therefore, we have established 1-1 correspondence between non-trivial $q$-connections and elements of $H^1(\mathbf{Z}^n, Q(\mathbf{C}[x]))$.

To produce a construction of some elements of $H^1(\mathbf{Z}^n, Q(\mathbf{C}[x]))$ fix arbitrary subsets $J_1,\dots,J_l$ of $\{1,\dots,m\}$ and set

$$r_i=\sum_{j\in J_i} x_j\,.$$

Let $\Psi=r_l^{\beta_1}r_2^{\beta_2}\cdots r_l^{\beta_l}$ for some $\beta_1,\dots,\beta_l\in\mathbf{C}$ (see Sect. 3.2.1).

**Lemma 4.2.** *The correspondence $\chi\mapsto\Psi_\chi=(T^\chi\Psi)\Psi^{-1}$ represents an element of $H^1(\mathbf{Z}^n, Q(\mathbf{C}[X]))$.*

*Proof.* The fact that $\Psi_\chi=(T^\chi\Psi)\Psi^{-1}$ is a 1-cocycle is obvious for this is a coboundary of $\Psi$. (One may also think of it as the "local" change of trivialization $f(x)\mapsto f(x)\,\Psi(x)$ in the bundle with the trivial $q$-connection.) What has to be proven is that $\Psi_\chi\in Q(\mathbf{C}[X])$ for any $\chi$. To do this observe that

$$x_i^{-1}\frac{1-T_i^2}{1-q^2}(x_1+\cdots+x_m)^\beta=[\beta](q^{-2}x_1+\cdots+q^{-2}x_{i-1}+x_i+\cdots+x_m)^{\beta-1}\,.$$

The calculation as in (31, 32, 33) shows that $(q^{-2}x_1+\cdots+q^{-2}x_{i-1}+x_i+\cdots+x_m)^{\beta-1}=p(x_1+\cdots+x_m)^\beta$, $p\in Q(\mathbf{C}[x])$. It implies that

$$T_i^2(x_1+\cdots+x_m)^\beta=\{1-(1-q^2)[\beta]x_ip\}\{x_1+\cdots+x_m)^\beta\,.$$

Therefore, a $q$-difference operator makes $\Psi$ into $p_1r_1^{\beta_1}\cdots p_lr_l^{\beta_l}$ for some $p_1,\dots,p_l\in Q(\mathbf{C}[x])$. To rewrite the latter in the form $p\Psi$, $p\in Q(\mathbf{C}[x])$ one wants to move each $p_i$ to the left. This can be done by using the automorphisms

$$Q(\mathbf{C}[x])\ni q\mapsto r_i^\beta qr_i^{-\beta}\,,$$

see (27). One has

$$p_1r_1^{\beta_1}\cdots p_lp_lr_l^{\beta_l}=p_1\tilde{p}_2\cdots\tilde{p}_l\Psi\,,$$

where $\tilde{p}_j=r_1^{\beta_1}\cdots r_{j-1}^{\beta_{j-1}}p_jr_{j-1}^{-\beta_{j-1}}\cdots p_1^{-\beta_1}$, $2\leqq j\leqq l$. □

We will denote by $\nabla(\Psi)$ the connection $\chi\mapsto\Psi_\chi=(T^\chi\Psi)\Psi^{-1}$ given by Lemma 4.2.

In the classical case tensor product of a pair of trivial line bundles with flat connections is a trivial line bundle equipped with a canonical flat connection. This gives an operation on connections. In the classical case connections are also identified with a certain 1st cohomology group and this operation happens to be simply an addition. Though we are unable to carry out the same in full generality in the $q$-commutative realm, we can produce the following non-commutative operation on the $q$-connections of the form $\nabla(\Psi)$:

$$\nabla(\Psi_1)_q\otimes\nabla(\Psi_2)=\nabla(\Psi_1\Psi_2)\,.$$

This operation is obviously a $q$-analogue of an addition of 1-cocyles in the classical setting.

*Remark*: Our approach here is a $q$-commutative version of that of Aomoto and Kato in [A-K]. In particular the construction of cocycles in Lemma 4.2 has its commutative counterpart, which is claimed to possess a sort of universality of property. The same may be true – with minor modifications – in our case.

*4.3.2. $U_q(\mathfrak{g})$-modules twisted by a q-connection.*

1. *Intertwining operators.* Of course everything written in the previous section applies to more general rings of skew polynomials provided one takes more care about the choice of elements $r_1, \ldots, r_l$. For example, in the case of an algebra $\mathbf{C}[X]$, coming from the Feigin's morphism $\Phi$: $U_q^- \to \mathbf{C}[X]$ (see Sect. 3.1), a natural choice is $r_j = \Phi(F_{i_j})$ for an arbitrary sequence $i_1, \ldots, i_l$. (There are some others which one can easily think of.) In the case of $\mathfrak{g} = \mathfrak{sl}_{n+1}$ the $U_q(\mathfrak{sl}_{n+1})$-module structure on $Q(\mathbf{C}[X])$ implies a homomorphism (see (40)):

$$\rho_\lambda\colon U_q(\mathfrak{sl}_{n+1}) \to \mathscr{D}[X] \ . \tag{41}$$

Given a $q$-connection $\nabla$ one twists this $U_q(\mathfrak{sl}_{n+1})$-module structure by

$$\nabla \circ \rho_\lambda\colon U_q(\mathfrak{sl}_{n+1}) \to \mathscr{D}[X] \ . \tag{42}$$

As above denote by $\nabla(\Psi)$ the $q$-conection coming from $(T^\chi \Psi)\Psi^{-1} \in H^1(\mathbf{Z}^N, Q(\mathbf{C})[X]))$, $\Psi = (\Phi(F_{i_1}))^{\beta_1} \cdots (\Phi(F_{i_l}))^{\beta_l}$ for some $\beta_1, \ldots, \beta_l \in \mathbf{C}$. Let $r_i \in W$ be a reflection at the simple root $\alpha_i$. Set

$$\beta_j = \frac{2(r_{i_{l+2-j}} \cdots r_{i_l} \cdot \lambda, \alpha_{i_{l+1-j}})}{(\alpha_{i_{l+1-j}}, \alpha_{i_{l+1-j}})} + 1 \ , \tag{43}$$

where $r_i \cdot \lambda$ stands for the shifted action of the Weyl group. Set $\nabla(r_{i_1} \cdots r_{i_l}; \lambda) = \nabla(\Psi)$ if $\beta_1, \ldots, \beta_l$ are given by (43).

**Proposition 4.4.** *There is a $U_q(\mathfrak{sl}_{n+1})$-linear map of the module related to $\nabla(\Psi) \circ \rho_{\omega \cdot \lambda}$ into the one related to $\nabla(\Psi)_q \otimes \nabla(r_{i_1} \cdots r_{i_l}; \lambda) \circ \rho_\lambda$, where $\omega = r_{i_1} \cdots r_{i_l}$.*

*Proof.* Passage from $\rho_\mu$ to $\nabla(\Psi) \circ \rho_\mu$ means that one replaces $v_\mu$ (i.e. unit of $\mathbf{C}[X]$) with $F_{i_1}^{\beta_1} \cdots F_{i_l}^{\beta_l} v_\mu$. (See the proof of Lemma 4.2.) Formula (39) implies that under the choice (43) the vector $F_{i_1}^{\beta_1} \cdots F_{i_l}^{\beta_l} v_\lambda$ is singular of the weight $r_{i_1} \cdots r_{i_l} \cdot \lambda$ and, therefore, satisfies all the conditions imposed on $v_{\omega \cdot \lambda}$. $\square$

Observe that having looked over the definitions one can make precise sense out of the statement: the module related to $\nabla(\Psi) \circ \rho_{\omega \cdot \lambda}$ embeds into the one related to $\nabla(\Psi)_q \otimes \nabla(r_{i_1} \cdots r_{i_l}; \lambda) \circ \rho_\lambda$ as a space of sections satisfying a certain regularity condition.

2. *Structure of modules $U_q^- [F_i^{-1}) F_i^\beta v_\lambda$.* Here we obtain the structure description of the modules twisted by a $q$-connection in the simplest case of $\nabla = \nabla((\Psi(F_i))^\beta)$, $\beta \in \mathbf{C}$. This module always contains a submodule generated by $1 \in \mathbf{C}[X]$ and, as one easily sees, isomorphic with the following extension of a Verma module: $U_q^- [F_i^{-1}] F_i^\beta v_\lambda$. The $U_q(\mathfrak{g})$-module structure on the latter is defined as follows:

(i) $F_1, \ldots, F_n$ act by left multiplication;
(ii) action of $E_1, \ldots, E_n$ is determined by setting (cf. (39))

$$E_j F_i^\beta v_\lambda = \delta_{ij} \{\beta\} \ F_i^{\beta - 1} \frac{q_i^{\lambda_i - \beta + 1} - q_i^{-\lambda_i + \beta - 1}}{q_i - q_i^{-1}} v_\lambda, \quad \beta \in \mathbf{C} \ . \tag{44}$$

Denote by ${}^iU_q^+$ the parabolic subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n; K_1^{\pm 1}, \ldots, K_n^{\pm 1}; F_i$. Equation (44) determines a structure of ${}^iU_q^+$-module on the space spanned by $F_i^{\beta+k}$, $k \in \mathbf{Z}$. Denote a module obtained in this way by $\mathscr{V}_\lambda^i$. Clearly, $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ is isomorphic with the induced module $\mathrm{Ind}_{{}^iU_q^+}^{U_q} \mathscr{V}_\lambda^i$. This isomorphism provides a precise analogy between $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ and a Verma module $U_q^- v_\lambda$: one is obtained from another by replacing the vacuum vector $v_\lambda$ with the *vacuum chain* $\mathscr{V}_\lambda^i$. This analogy can be pushed further by remarking that $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ is reducible if and only if it contains a *singular chain* in much the same way as a Verma module is reducible if and only if it contains a singular vector. Here by a singular chain we naturally mean a non-zero ${}^iU_q^+$-linear map $\mathscr{V}_\mu^i \subset U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ different from $\mathscr{V}_\lambda^i \subset U_q^-[F_i^{-1}]F_i^\beta v_\lambda$. A weight lattice of a singular chain $\mathscr{V}_\mu^i$ is of the form $\mu + \mathbf{Z}\alpha_i$. By the weight of a singular chain $\mathscr{V}_\mu^i$ we mean an element $\bar{\mu} \in \mathfrak{h}^*/\mathbf{Z}\alpha_i$, where $\bar{\mu}$ stands for an image of $\mu$ under the natural projection $\mathfrak{h}^* \to \mathfrak{h}^*/\mathbf{Z}\alpha_i$. The following partially relies on Sect. 4.1.

**Theorem 4.3.** (i) *If $\beta \in \mathbf{Z}$ or $\beta \in \lambda(H_i) + \mathbf{Z}$ then $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ is isomorphic with either $U_q^-[F_i^{-1}]v_\lambda$ or $U_q^-[F_i^{-1}]v_{r_i \cdot \lambda}$* (resp.); *see Proposition* 4.1.

(ii) *Otherwise $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ is reducible $\Leftrightarrow$ it contains a singular chain of the weight $\overline{\lambda - N\alpha}$ for some $\alpha \in \Delta_+$, $\alpha \neq \alpha_i$, $N \in \mathbf{N} \Leftrightarrow$ there is $j \in \mathbf{Z}$, $N \in \mathbf{N}$ such that*

$$(\lambda + \rho, \alpha + j\alpha_i) = \frac{N}{2}(\alpha + j\alpha_i, \alpha + j\alpha_i), \ \alpha + j\alpha_i \in \Delta_+ ,$$

*where $\rho \in \mathfrak{h}^*$ is determined by $\rho(H_k) = 1$, $1 \leqq k \leqq n$.*

*Remark*: The Verma module $M(\lambda)$ contains a singular vector of the weight $\lambda - N\alpha$, $\alpha \in \Delta_+$ if and only if $\lambda$ belongs to the Kac–Kazhdan hyperplane ([K-K], see also Sect. 3.2.2) related to the pair $(\alpha, N)$:

$$(\lambda + \rho, \alpha) = \frac{N}{2}(\alpha, \alpha), \quad \alpha \in \Delta_+ . \tag{45}$$

Item (ii) of Theorem 4.3 claims that $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ contains a singular chain of the weight $\overline{\lambda - N\alpha}$ if and only if $\lambda$ belongs to the union of Kac–Kazhdan hyperplanes related to all pairs $(\alpha + j\alpha_i, N)$, $\alpha + j\alpha_i \in \Delta_+$. Therefore a singular chain encodes information on a collection of singular vectors in a Verma module.

*Proof of Theorem 4.3.* Item (i) immediately follows from definitions and (44). As to (ii), fix root vectors $E_\alpha$, $\alpha \in \Delta_+$ as in (21). It is clear that the space spanned by all singular chains coincides with the space of solutions of the following systems of linear equation

$$E_\alpha \omega = 0 \quad \text{for all } \alpha \neq \alpha_i . \tag{46}$$

(This system should be regarded as restricted to each weight space of the module.)

One deduces from Proposition 2.1 the space of solutions to (46) is a $\langle F_i, K_i^{\pm 1}, E_i \rangle$-module from which one easily extracts a singular chain.

All the vector spaces $U_q^-[F_i^{-1}]F_i^\beta v_\lambda$ parametrized by $\beta \in \mathbf{C}$ are naturally isomorphic with each other and with the space $U_q^-[F_i^{-1}]$. Therefore (46) may be regarded as a family of systems of linear equations on $U_q^-[F_i^{-1}]$ polynomially depending on $\beta$. For a fixed weight space of $U_q^-[F_i^{-1}]$ existence of solutions to (46) lying in this space is equivalent to vanishing of a certain polynomial depending on $\lambda$ and $\beta$. It is easy to deduce, however, that under the assumptions of (ii) once there is a solution for $\beta=\beta_0$ then there are solutions for infinitely many values $\beta\in\beta_0+\mathbf{Z}$. Therefore the mentioned polynomial is actually independent of $\beta$. Now we may set $\beta=0$ without lack of generality. But then it is easy to see that – again under the assumptions of (ii) – the same system (46) gives reducibility criterion for the submodule $M(\lambda)$ of $U_q^-[F_i^{-1}]v_\lambda$, see Proposition 4.1. The proof now follows from the Kac–Kazhdan equations, see the above Remark. □

The Kac–Kazhdan reducibility criterion (45) was carried over to the case of a quantum group attached to an arbitrary symmetrizable Cartan matrix $A$ in [M]. It follows that Theorem 4.3 remains valid in this general setting.

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